symmetry
MDPI

Article

Algebraic Bethe Ansatz for the Trigonometric $s\ell(2)$ Gaudin Model with Triangular Boundary

Nenad Manojlović [1,†] and **Igor Salom** [2,*,†]

1 Departamento de Matemática, F. C. T., Universidade do Algarve, Campus de Gambelas, PT-8005-139 Faro, Portugal; nmanoj@ualg.pt
2 Institute of Physics, University of Belgrade, P.O. Box 57, 11080 Belgrade, Serbia
* Correspondence: isalom@ipb.ac.rs
† These authors contributed equally to this work.

Received: 28 January 2020; Accepted: 19 February 2020; Published: 1 March 2020

Abstract: In this paper we deal with the trigonometric Gaudin model, generalized using a nontrivial triangular reflection matrix (corresponding to non-periodic boundary conditions in the case of anisotropic XXZ Heisenberg spin-chain). In order to obtain the generating function of the Gaudin Hamiltonians with boundary terms we follow an approach based on Sklyanin's derivation in the periodic case. Once we have the generating function, we obtain the corresponding Gaudin Hamiltonians with boundary terms by taking its residues at the poles. As the main result, we find the generic form of the Bethe vectors such that the off-shell action of the generating function becomes exceedingly compact and simple. In this way—by obtaining Bethe equations and the spectrum of the generating function—we fully implement the algebraic Bethe ansatz for the generalized trigonometric Gaudin model.

Keywords: Gaudin model; Algebraic Bethe Ansatz; non-unitary r-matrix

1. Introduction

The so-called rational $s\ell(2)$ Gaudin model was first introduced in [1] as a model of "long-range" interacting spins in a chain. Having non-trivial long-range (pairwise) interactions and yet being fully integrable, the model was of clear potential interest in many areas of physics. Naturally, the most promising were its applications in condensed matter physics, where the need for exactly (or even only quasi-exactly) solvable interacting many-body models was maybe the most acute, and where the generalizations of Gaudin algebra arguably play a significant role [2]. However, the potential physical significance of Gaudin model is not confined to this area; for example, its connection with Wess–Zumino–Novikov–Witten model was pointed out in [3], more recently it was also related to AGT correspondence [4], and applied to obtain classical integrable field theories [5].

Therefore it is not surprising that many generalizations ensued soon after the Gaudin's original paper, along various directions: to other simple Lie algebras [6,7], in the context of the quantum inverse scattering method [8–10], to other cases where skew-symmetric r-matrix fulfills the classical Yang-Baxter equation [11,12]. In general, not only Gaudin models corresponding to the classical r-matrices of simple Lie algebras [3,13–17], but also those of Lie superalgebras [18–22] attracted considerable attention.

One particular approach to the relation between Heisenberg spin-chains and Gaudin models was due to Hikami, Kulish and Wadati, who showed that the Gaudin Hamiltonians can be obtained by making the so-called quasi-classical expansion of the transfer matrix of the periodic chain [23,24]. This was soon demonstrated also for cases with non-periodic boundary conditions [25]. The progress paved the way for further studies of open Gaudin models, and algebraic Bethe ansatz (ABA) was soon applied to open Gaudin model in the context of Lie superalgebras [26] and of the the Vertex-IRF correspondence [27–29]. Open Gaudin models were also studied in the light of the classical reflection equation [30–32] and, more recently, an approach utilizing non-unitary r-matrices was demonstrated [33,34]. A more detailed review of the open Gaudin model can be found in [35].

Due to the close mathematical connection between Heisenberg spin-chains and Gaudin models, development in one area inevitably led also to the progress in the other. Further generalizations in applications of ABA to spin-chains with non-periodic boundary conditions [36–53] recently influenced study of the corresponding Gaudin model [54,55]. A new method to obtain the eigenvalues of the Gaudin Hamiltonians and the corresponding Bethe ansatz equations was derived in [56], based on T–Q approach to implementation of Bethe ansatz [41,42]. Also, in [53], by taking the so-called quasi-classical limit we obtained the off-shell action of the generating function on the Bethe vectors for the trigonometric Gaudin model with boundary terms.

While in [53] we considered expansion of the XXZ spin-chain expressions to obtain Bethe vectors for the Gaudin model (i.e., by exploiting the mathematical relation between the two models), the open trigonometric Gaudin model can be treated in its own right, by fully implementing the algebraic Bethe ansatz for this case. This is the essential goal of the present paper.

The first important goal of an independent treatment of the Gaudin model is to obtain the generating function of the Gaudin Hamiltonians. Similarly like in the rational case [55], we will follow the approach based on Sklyanin's method in the periodic case [10,57]. Once we have the generating function, taking its residues at the poles will yield the Gaudin Hamiltonians with the boundary terms.

The next step is to establish the algebra of Bethe operators. This will be accomplished by first constructing non-unitary classical r-matrix (which satisfies the generalized classical Yang-Baxter equation) and the corresponding modified Lax matrix—both of which depend on the reflection K-matrix (following the analogy with the spin-chain case, we may say that the K-matrix encodes non-period boundary conditions). These two entities are mutually related via linear bracket, which is anti-symmetric, obeys the Jacobi identity and will lead us to algebraic relations between Bethe operators (matrix elements of the Lax matrix). Additionally, as an important nontrivial step, we will demonstrate how an appropriate change of generator basis can result in significant simplification of the initial Bethe algebra relations.

The most difficult part of the ABA implementation is to find the Bethe vectors and the corresponding off-shell action of the generating function. After explicitly solving the first few particular cases, we will conjecture the general form of a Bethe vector using a family of suitably defined creation operators. Such Bethe vectors will turn out to yield strikingly simple off-shell action of the generating function—so simple that it is hardly any more complex than the corresponding formula when the boundary matrix is diagonal [25]. As usual, off-shell action of the generating function provides us both with the spectrum of the system and with the corresponding Bethe equations (the latter are the necessary conditions to eliminate the unwanted terms and promote Bethe vectors into true solutions of the eigenproblem).

This paper is organized as follows. In Section 2 we establish the fundamentals of the framework, defining the reflection matrix, Lax matrix and r-matrix that correspond to the Gaudin model with nontrivial boundary. The generating function, as well as the Gaudin Hamiltonians are also derived in this section. Section 3 deals with the algebraic Bethe ansatz of the model: we derive the relevant algebraic structure and proceed to solve the generating function eigenproblem by finding the appropriate Bethe vectors and the formula for the off-shell action. We summarize our results in the Section 4. In Appendix A we provide proof for the essential commutativity property of the generating function of the Gaudin Hamiltonians, while the Appendix B contains some explicit formulas regarding the Bethe vector $\varphi_3(\mu_1, \mu_2, \mu_3)$.

2. Trigonometric Gaudin Model with Boundary

Lax matrix of the trigonometric Gaudin model with periodic boundary conditions [20] is of the following form [53,57]:

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\sigma_0^3 \otimes \cosh(\lambda - \alpha_m) S_m^3 + \frac{1}{2}\left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right)}{\sinh(\lambda - \alpha_m)}, \tag{1}$$

where spin operators S_m^α, $\alpha = +, -, 3$, living in the product Hilbert space

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}, \tag{2}$$

satisfy the usual commutation relations:

$$[S_m^3, S_n^\pm] = \pm S_m^\pm \, \delta_{mn}, \quad [S_m^+, S_n^-] = 2 S_m^3 \, \delta_{mn}. \tag{3}$$

The corresponding classical r-matrix is given by:

$$r(\lambda) = \frac{-1}{2\sinh(\lambda)} \left(\cosh(\lambda)(\mathbb{1} \otimes \mathbb{1} + \sigma^3 \otimes \sigma^3) + \frac{1}{2}\left(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+\right) \right). \tag{4}$$

The r-matrix (4) satisfies the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0, \tag{5}$$

and also has the following unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda). \tag{6}$$

Crucially, Lax matrix and r-matrix satisfy the so-called Sklyanin linear bracket relation:

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]. \tag{7}$$

The Sklyanin linear bracket (7) obeys the Jacobi identity and is also anti-symmetric. From here it follows that the entries of the Lax matrix (1) generate a Lie algebra (Gaudin algebra), which in this case corresponds to the trigonometric Gaudin model with periodic boundary conditions [20].

The Gaudin model described by the above Lax matrix (1) and r-matrix (4) is mathematically tightly related to the trigonometric spin-chain model with periodic boundary: the former can be obtained from the latter by the so-called quasi-classical expansion [23,24]. This can be verified by considering linear terms in the η expansion of the XXZ Heisenberg spin-chain Lax operator and R-matrix, taken from [53], in a full analogy with the rational case [55]:

$$\begin{aligned} \mathbb{L}_{0m}(\lambda - \alpha_m) = \mathbb{1}_0 \otimes \mathbb{1}_m + \eta \left(\sigma_0^3 \otimes \coth(\lambda - \alpha_m)\, S_m^3 + \frac{1}{2\sinh(\lambda - \alpha_m)} \left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right) \right) \\ + \frac{\eta^2}{2} \mathbb{1}_0 \otimes \left(S_m^3\right)^2 + \mathcal{O}(\eta^3); \end{aligned} \tag{8}$$

and

$$\frac{1}{\sinh(\lambda)} R(\lambda) = \mathbb{1} - \eta r(\lambda) + \mathcal{O}(\eta^2). \tag{9}$$

To extend the generality of the model it is possible to introduce the reflection K-matrix, in a mathematically similar manner as when the periodic Heisenberg spin-chain model is extended to account for nontrivial boundary conditions. K-matrix must then satisfy the classical reflection equation [30,31,55]:

$$\begin{aligned} & r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu)+K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)= \\ & =K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)+K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu). \end{aligned} \tag{10}$$

In the spin chain case, parameters of K^- and K^+ matrices (encoding, respectively, boundary conditions on the left and on the right end of the chain) are allowed to be different. However, in the Gaudin model case we must impose the following additional condition on the reflection matrices [35,55]:

$$\lim_{\eta\to 0}\left(K^+(\lambda)K^-(\lambda)\right)=\left(\kappa^2\sinh(\xi-\lambda)\sinh(\xi+\lambda)-\phi\psi\sinh^2(\lambda)\right)\mathbb{1}. \tag{11}$$

In turn, this implies that parameters of the K^- and K^+ cannot be mutually independent, and effectively, we cannot speak of two reflection matrices but of a single one. Intuitively, this is not surprising, since the long-range Gaudin interactions actually do not single out any nodes as boundary nodes and thus, physically, K-matrices cannot be literally interpreted as describing any boundary conditions but merely as parameters that provide further generalization of the model (this can be best inferred from the Hamiltonian Expression (23) below). In spite of this, we will say for such a Gaudin model—incorporating a nontrivial K-matrix—that it satisfies nontrivial boundary conditions (or denote it as "open"), simply due to the analogy and direct relation with the corresponding spin-chain model (the strict mathematical connection can be again established by the quasi-classical expansion procedure).

The solutions for $K^-(\lambda)$ and $K^+(\lambda)$ can be thus given in terms of a single K-matrix, and they take the following form [58,59]:

$$K^-(\lambda)\equiv K(\lambda)=\begin{pmatrix}\kappa\sinh(\xi+\lambda) & \psi\sinh(2\lambda)\\ \phi\sinh(2\lambda) & \kappa\sinh(\xi-\lambda)\end{pmatrix} \tag{12}$$

and

$$K^+(\lambda)=K(-\lambda-\eta)=\begin{pmatrix}\kappa\sinh(\xi-\lambda-\eta) & -\psi\sinh\left(2(\lambda+\eta)\right)\\ -\phi\sinh\left(2(\lambda+\eta)\right) & \kappa\sinh(\xi+\lambda+\eta)\end{pmatrix}. \tag{13}$$

Moreover, it is straightforward to check the following useful identities

$$K(-\lambda)K(\lambda)=\det\left(K(\lambda)\right)\mathbb{1}, \tag{14}$$

$$K(-\lambda)=\operatorname{tr}K(\lambda)-K(\lambda). \tag{15}$$

Now, the new Gaudin Lax matrix generalized by the K-matrix, is given by

$$\mathcal{L}_0(\lambda)=L_0(\lambda)-K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda), \tag{16}$$

and the corresponding non-unitary r-matrix is given by:

$$r^K_{00'}(\lambda,\mu)=r_{00'}(\lambda-\mu)-K_{0'}(\mu)r_{00'}(\lambda+\mu)K^{-1}_{0'}(\mu). \tag{17}$$

It is not difficult to check that this r-matrix satisfies the classical Yang-Baxter equation

$$[r^K_{32}(\lambda_3,\lambda_2),r^K_{13}(\lambda_1,\lambda_3)]+[r^K_{12}(\lambda_1,\lambda_2),r^K_{13}(\lambda_1,\lambda_3)+r^K_{23}(\lambda_2,\lambda_3)]=0 \tag{18}$$

and that the linear bracket for the modified Lax operator is now preserved:

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right]. \tag{19}$$

This linear bracket is also anti-symmetric. Since the r-matrix (17) satisfies the classical Yang-Baxter equation, it additionally obeys the Jacobi identity.

We have now set the stage to introduce the generating function of the Gaudin Hamiltonians with boundary terms:

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda) \tag{20}$$

The essential property of $\tau(\lambda)$ is that it commutes for different values of the spectral parameter:

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{21}$$

The proof of this relation is given in the Appendix A.

Next, we obtain the Gaudin Hamiltonians with the boundary terms by taking the residues of the generating function (20) at poles $\lambda = \pm\alpha_m$:

$$\mathop{\mathrm{Res}}_{\lambda=\alpha_m} \tau(\lambda) \;=\; 4\, H_m \qquad \text{and} \qquad \mathop{\mathrm{Res}}_{\lambda=-\alpha_m} \tau(\lambda) \;=\; (-4)\, H_m \tag{22}$$

where

$$\begin{aligned} H_m = {} & \sum_{n\neq m}^{N} \left(\coth(\alpha_m - \alpha_n)\, S^3_m S^3_n + \frac{S^+_m S^-_n + S^-_m S^+_n}{2\sinh(\alpha_m - \alpha_n)} \right) + \sum_{n=1}^{N} \coth(\alpha_m + \alpha_n)\, \frac{S^3_m S^3_n + S^3_n S^3_m}{2} \\ & + \frac{\psi}{\kappa}\, \frac{\sinh(2\alpha_m)}{\sinh(\xi+\alpha_m)} \sum_{n=1}^{N} \frac{S^3_m S^+_n + S^+_n S^3_m}{2\sinh(\alpha_m+\alpha_n)} + \frac{\sinh(\xi-\alpha_m)}{2\sinh(\xi+\alpha_m)} \sum_{n=1}^{N} \frac{S^-_m S^+_n + S^+_n S^-_m}{2\sinh(\alpha_m+\alpha_n)} \\ & - \frac{\psi}{\kappa}\, \frac{\sinh(2\alpha_m)}{\sinh(\xi-\alpha_m)} \sum_{n=1}^{N} \coth(\alpha_m+\alpha_n)\, \frac{S^+_m S^3_n + S^3_n S^+_m}{2} + \frac{\sinh(\xi+\alpha_m)}{2\sinh(\xi-\alpha_m)} \sum_{n=1}^{N} \frac{S^+_m S^-_n + S^-_n S^+_m}{2\sinh(\alpha_m+\alpha_n)} \\ & - \frac{\psi^2}{\kappa^2}\, \frac{\sinh^2(2\alpha_m)}{2\sinh(\xi-\alpha_m)\sinh(\xi+\alpha_m)} \sum_{n=1}^{N} \frac{S^+_m S^+_n + S^+_n S^+_m}{2\sinh(\alpha_m+\alpha_n)}. \end{aligned} \tag{23}$$

In the next section we will deal with the arduous task of obtain the spectrum and the corresponding Bethe vectors of the generating function.

3. Algebraic Bethe Ansatz

Implementation of the algebraic Bethe ansatz requires triangularity of the K-matrix (12). As opposed to the rational case [55] where the triangularity of the K-matrix can be guaranteed by the similarity transformation independent of the spectral parameter, in the present case there is no, in general, $U(1)$ symmetry transformation that can bring the reflection matrix to the upper triangular form. For this reason, we are forced to impose an extra condition on the parameters of $K(\lambda)$. By setting

$$\phi = 0$$

the reflection matrix becomes upper triangular

$$K(\lambda) = \begin{pmatrix} \kappa\sinh(\xi+\lambda) & \psi\sinh(2\lambda) \\ 0 & \kappa\sinh(\xi-\lambda) \end{pmatrix}. \tag{24}$$

The inverse matrix has the following form:

$$K^{-1}(\lambda) = \frac{1}{\kappa^2 \sinh(\xi+\lambda)\sinh(\xi-\lambda)} \begin{pmatrix} \kappa \sinh(\xi-\lambda) & -\psi \sinh(2\lambda) \\ 0 & \kappa \sinh(\xi+\lambda) \end{pmatrix}. \tag{25}$$

By substituting these formulas into (16),

$$\mathcal{L}_0(\lambda) = \begin{pmatrix} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{pmatrix} = L_0(\lambda) - K_0(\lambda) L_0(-\lambda) K_0^{-1}(\lambda), \tag{26}$$

we obtain local realisation for the entries of the Lax matrix of the following form:

$$E(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^+}{\sinh(\lambda-\alpha_m)} + \frac{\sinh(\xi-\lambda)\, S_m^+}{\sinh(\xi+\lambda)\sinh(\lambda+\alpha_m)} \right), \tag{27}$$

$$H(\lambda) = \sum_{m=1}^{N} \left(\coth(\lambda-\alpha_m)\, S_m^3 + \coth(\lambda+\alpha_m)\, S_m^3 + \frac{\psi \sinh(2\lambda)\, S_m^+}{\kappa \sinh(\xi+\lambda)\sinh(\lambda+\alpha_m)} \right), \tag{28}$$

$$\begin{aligned} F(\lambda) = \sum_{m=1}^{N} \Bigg(& \frac{S_m^-}{\sinh(\lambda-\alpha_m)} + \frac{\sinh(\xi+\lambda)\, S_m^-}{\sinh(\xi-\lambda)\sinh(\lambda+\alpha_m)} - \frac{2\psi \sinh(2\lambda)}{\kappa \sinh(\xi-\lambda)} \coth(\lambda+\alpha_m)\, S_m^3 \\ & - \frac{\psi^2 \sinh^2(2\lambda)\, S_m^+}{\kappa^2 \sinh(\xi-\lambda)\sinh(\xi+\lambda)\sinh(\lambda+\alpha_m)} \Bigg). \end{aligned} \tag{29}$$

Similarly, by using (4), (24), (25) and (17) we obtain explicit expression for $r^K_{00'}(\lambda,\mu)$. This non-unitary, classical r-matrix together with the Lax matrix (26) defines the Lie algebra relevant for the open trigonometric Gaudin model. By explicitly rewriting the relation (19) we find the following commutation relations for the generators $E(\lambda)$, $H(\lambda)$ and $F(\lambda)$:

$$[E(\lambda), E(\mu)] = 0, \tag{30}$$

$$[H(\lambda), E(\mu)] = \frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \left(\sinh(2\lambda)\, E(\mu) - \frac{\sinh(\xi+\lambda)}{\sinh(\xi+\mu)} \sinh(2\mu)\, E(\lambda) \right), \tag{31}$$

$$\begin{aligned} [E(\lambda), F(\mu)] = {} & \frac{2\psi}{\kappa} \coth(\lambda+\mu) \frac{\sinh(2\mu)}{\sinh(\xi-\mu)}\, E(\lambda) + \frac{2}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \times \\ & \times \left(\frac{\sinh(\xi+\mu)}{\sinh(\xi+\lambda)} \sinh(2\lambda)\, H(\mu) - \frac{\sinh(\xi-\lambda)}{\sinh(\xi-\mu)} \sinh(2\mu)\, H(\lambda) \right), \end{aligned} \tag{32}$$

$$[H(\lambda), H(\mu)] = \frac{-\psi}{\kappa \sinh(\lambda+\mu)} \left(\frac{\sinh(2\lambda)}{\sinh(\xi+\lambda)}\, E(\mu) - \frac{\sinh(2\mu)}{\sinh(\xi+\mu)}\, E(\lambda) \right), \tag{33}$$

$$\begin{aligned} [H(\lambda), F(\mu)] = {} & -\frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \left(\sinh(2\lambda)\, F(\mu) - \frac{\sinh(\xi-\lambda)}{\sinh(\xi-\mu)} \sinh(2\mu)\, F(\lambda) \right) \\ & + \frac{2\psi \sinh(2\lambda)}{\kappa \sinh(\lambda+\mu)\sinh(\xi+\lambda)}\, H(\mu) - \frac{\psi^2 \sinh^2(2\mu)}{\kappa^2 \sinh(\lambda+\mu)\sinh(\xi-\mu)\sinh(\xi+\mu)}\, E(\lambda), \end{aligned} \tag{34}$$

$$\begin{aligned} [F(\lambda), F(\mu)] = {} & \frac{2\psi}{\kappa} \coth(\lambda+\mu) \left(\frac{\sinh(2\lambda)}{\sinh(\xi-\lambda)}\, F(\mu) - \frac{\sinh(2\mu)}{\sinh(\xi-\mu)}\, F(\lambda) \right) \\ & - \frac{2\psi^2}{\kappa^2 \sinh(\lambda+\mu)} \left(\frac{\sinh^2(2\lambda)}{\sinh(\xi-\lambda)\sinh(\xi+\lambda)}\, H(\mu) - \frac{\sinh^2(2\mu)}{\sinh(\xi-\mu)\sinh(\xi+\mu)}\, H(\lambda) \right). \end{aligned} \tag{35}$$

In terms of the entries of the Lax matrix, the generating function of the Gaudin Hamiltonians (20) has the following form:

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda) = 2H^2(\lambda) + 2F(\lambda)E(\lambda) + [E(\lambda), F(\lambda)], \tag{36}$$

where, from (32) we evaluate the last term to be

$$[E(\lambda), F(\lambda)] = 2\frac{\cosh(2\xi)\cosh(2\lambda) - 1}{\sinh(2\lambda)\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\, H(\lambda) - 2H'(\lambda) + \frac{2\psi\cosh(2\lambda)}{\kappa\sinh(\xi-\lambda)}\, E(\lambda), \tag{37}$$

and thus the final expression becomes

$$\tau(\lambda) = 2\left(H^2(\lambda) + \frac{\cosh(2\xi)\cosh(2\lambda) - 1}{\sinh(2\lambda)\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\, H(\lambda) - H'(\lambda)\right) + \left(2F(\lambda) + \frac{2\psi\cosh(2\lambda)}{\kappa\sinh(\xi-\lambda)}\right) E(\lambda). \tag{38}$$

In order to simplify commutation relations (30)–(35) we switch to new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$, defined as linear combinations of the original ones:

$$e(\lambda) = \frac{\sinh(\xi+\lambda)}{\sinh(2\lambda)} E(\lambda) = \sum_{m=1}^{N} \frac{\sinh(\xi+\alpha_m)\, S_m^+}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}, \tag{39}$$

$$h(\lambda) = \frac{1}{\sinh(2\lambda)}\left(H(\lambda) - \frac{\psi\sinh(\lambda)}{\kappa\sinh(\xi)}\, E(\lambda)\right) = \sum_{m=1}^{N} \frac{S_m^3 - \dfrac{\psi\sinh(\alpha_m)}{\kappa\sinh(\xi)}\, S_m^+}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}, \tag{40}$$

$$f(\lambda) = \frac{1}{\sinh(2\lambda)}\left(\sinh(\xi-\lambda)F(\lambda) + \frac{\psi}{\kappa}\sinh(2\lambda)H(\lambda)\right) = \sum_{m=1}^{N} \frac{\sinh(\xi-\alpha_m)\, S_m^- + \dfrac{\psi}{\kappa}\sinh(2\alpha_m)\, S_m^3}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}. \tag{41}$$

The essential property of the new basis operators is:

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = [f(\lambda), f(\mu)] = 0. \tag{42}$$

Therefore there are only three remaining nontrivial commutation relations

$$[h(\lambda), e(\mu)] = \frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\, (e(\mu) - e(\lambda)), \tag{43}$$

$$\begin{aligned}[h(\lambda), f(\mu)] = {} & \frac{-1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\, (f(\mu) - f(\lambda)) + \frac{2\psi\coth(\xi)}{\kappa\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \times \\ & \times \left(\sinh^2(\mu)\, h(\mu) - \sinh^2(\lambda)\, h(\lambda)\right) + \frac{2\psi^2}{\kappa^2\sinh(\lambda-\mu)\sinh(\lambda+\mu)\sinh^2(\xi)} \times \\ & \times \left(\sinh^2(\mu)\, e(\mu) - \sinh^2(\lambda)\, e(\lambda)\right), \end{aligned} \tag{44}$$

$$\begin{aligned}[e(\lambda), f(\mu)] = {} & \frac{-2\psi\coth(\xi)}{\kappa\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \left(\sinh^2(\mu)\, e(\mu) - \sinh^2(\lambda)\, e(\lambda)\right) + \frac{2}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \times \\ & \times (\sinh(\xi-\mu)\sinh(\xi+\mu)\, h(\mu) - \sinh(\xi-\lambda)\sinh(\xi+\lambda)\, h(\lambda)). \end{aligned} \tag{45}$$

Lie algebra (42)–(45) will be the basis of our implementation of the algebraic Bethe ansatz. The first step is to find the expression for the generating function $\tau(\lambda)$ as a function of the new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$. To this end, we will invert the relations (39)–(41)

$$E(\lambda) = \frac{\sinh(2\lambda)}{\sinh(\xi+\lambda)}\, e(\lambda), \tag{46}$$

$$H(\lambda) = \sinh(2\lambda)\left(h(\lambda) + \frac{\psi\sinh(\lambda)}{\kappa\sinh(\xi)\sinh(\xi+\lambda)}\, e(\lambda)\right), \tag{47}$$

$$F(\lambda) = \frac{\sinh(2\lambda)}{\sinh(\xi-\lambda)}\left(f(\lambda) - \frac{\psi\sinh(2\lambda)}{\kappa}\, h(\lambda) - \frac{\psi^2\sinh(\lambda)\sinh(2\lambda)}{\kappa^2\sinh(\xi)\sinh(\xi+\lambda)}\, e(\lambda)\right). \tag{48}$$

In particular, we find

$$\begin{aligned}H^2(\lambda) &= \sinh^2(2\lambda)\left(h^2(\lambda) + \frac{\psi\sinh(\lambda)}{\kappa\sinh(\xi)\sinh(\xi+\lambda)}\left(2h(\lambda)e(\lambda) - [h(\lambda),e(\lambda)]\right)\right.\\ &\left.+\frac{\psi^2\sinh^2(\lambda)}{\kappa^2\sinh^2(\xi)\sinh^2(\xi+\lambda)}\,e^2(\lambda)\right)\\ &= \sinh^2(2\lambda)\left(h^2(\lambda) + \frac{\psi\sinh(\lambda)}{\kappa\sinh(\xi)\sinh(\xi+\lambda)}\left(2h(\lambda)e(\lambda) + \frac{e'(\lambda)}{\sinh(2\lambda)}\right)\right.\\ &\left.+\frac{\psi^2\sinh^2(\lambda)}{\kappa^2\sinh^2(\xi)\sinh^2(\xi+\lambda)}\,e^2(\lambda)\right).\end{aligned} \tag{49}$$

Substitution of (46)–(49) into (38) yields the desired expression for the the generating function

$$\begin{aligned}\tau(\lambda) &= 2\sinh^2(2\lambda)\left(h^2(\lambda) + \frac{h(\lambda)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \frac{h'(\lambda)}{\sinh(2\lambda)}\right) + \frac{2\sinh^2(2\lambda)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\times\\ &\times\left(f(\lambda) - 2\frac{\psi}{\kappa}\coth(\xi)\sinh^2(\lambda)\,h(\lambda) - \frac{\psi^2\sinh^2(\lambda)}{\kappa^2\sinh^2(\xi)}\,e(\lambda) + \frac{\psi}{\kappa}\coth(\xi)\right)e(\lambda).\end{aligned} \tag{50}$$

In every factor-space $V_m = \mathbb{C}^{2s+1}$ from the Hilbert space $\mathcal{H}$ (2) there exists a vector $\omega_m \in V_m$ such that

$$S^3_m\omega_m = s_m\omega_m \quad \text{and} \quad S^+_m\omega_m = 0. \tag{51}$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{52}$$

The action of the generators $e(\lambda)$ and $h(\lambda)$ on the vector Ω_+ can be obtained from the definitions above and the formulas (27)–(28) and (39)–(41):

$$e(\lambda)\Omega_+ = 0 \;\text{ and }\; h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \;\text{ with }\; \rho(\lambda) = \sum_{m=1}^{N}\frac{s_m}{\sinh(\lambda+\alpha_m)\sinh(\lambda-\alpha_m)}. \tag{53}$$

Of a crucial importance in what follows is to note that the vector Ω_+ (52) is an eigenvector of the generating function $\tau(\lambda)$. This can be shown by using (53)

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\sinh^2(2\lambda)\left(\rho^2(\lambda) + \frac{\rho(\lambda)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \frac{\rho'(\lambda)}{\sinh(2\lambda)}\right)\Omega_+. \tag{54}$$

By using the explicit expression for the function $\rho(\lambda)$ (53) the eigenvalue $\chi_0(\lambda)$ can be also written as

$$\begin{aligned}\chi_0(\lambda) &= 2\sinh^2(2\lambda)\left(\sum_{m=1}^{N}\frac{s_m(s_m+1)}{\sinh^2(\lambda+\alpha_m)\sinh^2(\lambda-\alpha_m)} + \sum_{m=1}^{N}\frac{s_m}{\sinh(\lambda+\alpha_m)\sinh(\lambda-\alpha_m)}\times\right.\\ &\left.\times\left(\frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} + \sum_{n>m}^{N}\frac{2s_n}{\sinh(\lambda+\alpha_n)\sinh(\lambda-\alpha_n)}\right)\right).\end{aligned} \tag{55}$$

The essential goal of the implementation of the algebraic Bethe ansatz is to find the corresponding Bethe vectors. Due to the existence of Bethe conditions, it turns out that their form is not unique, and we seek the solution that renders the off-shell action of the generating function of the Gaudin

Hamiltonians as simple as possible. We proceed by demonstrating that the Bethe vector $\varphi_1(\mu)$ has the form

$$\varphi_1(\mu) = \left(f(\mu) + c_1^{(1)}(\mu)\right)\Omega_+, \tag{56}$$

where $c_1(\mu)$ is given by

$$c_1^{(1)}(\mu) = \frac{\psi}{\kappa}\left(1 + \left(e^{-2\xi} - \cosh(2\mu)\right)\,\rho(\mu)\right). \tag{57}$$

We proceed by explicit calculation to find

$$\tau(\lambda)\varphi_1(\mu) = [\tau(\lambda), f(\mu)]\,\Omega_+ + \chi_0(\lambda)\varphi_1(\mu), \tag{58}$$

where the commutator in the first term of (58) becomes

$$\begin{aligned}
[\tau(\lambda), f(\mu)]\,\Omega_+ = &-\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\left(2\rho(\lambda) + \frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\right)\varphi_1(\mu) \\
&+ \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\frac{\sinh(\xi+\mu)\sinh(\xi-\mu)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\times \\
&\times\left(2\rho(\mu) + \frac{1}{\sinh(\xi+\mu)\sinh(\xi-\mu)}\right)\varphi_1(\lambda).
\end{aligned} \tag{59}$$

Hence the action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is

$$\begin{aligned}
\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) &+ \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\frac{\sinh(\xi+\mu)\sinh(\xi-\mu)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\times \\
&\times\left(2\rho(\mu) + \frac{1}{\sinh(\xi+\mu)\sinh(\xi-\mu)}\right)\varphi_1(\lambda),
\end{aligned} \tag{60}$$

with

$$\chi_1(\lambda,\mu) = \chi_0(\lambda) - \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\left(2\rho(\lambda) + \frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\right). \tag{61}$$

We can make the unwanted term in (60) vanish in the standard manner, i.e., by imposing the following Bethe equation:

$$2\rho(\mu) + \frac{1}{\sinh(\xi+\mu)\sinh(\xi-\mu)} = 0. \tag{62}$$

Therefore, we have shown that $\varphi_1(\mu)$ (56) is indeed a Bethe vector of the generating function $\tau(\lambda)$ with the eigenvalue $\chi_1(\lambda,\mu)$ (61).

Next, we seek the Bethe vector $\varphi_2(\mu_1,\mu_2)$ in the form of the following symmetric function

$$\varphi_2(\mu_1,\mu_2) = f(\mu_1)f(\mu_2)\Omega_+ + c_2^{(1)}(\mu_2;\mu_1)f(\mu_1)\Omega_+ + c_2^{(1)}(\mu_1;\mu_2)f(\mu_2)\Omega_+ + c_2^{(2)}(\mu_1,\mu_2)\Omega_+. \tag{63}$$

We now proceed to show that a proper solution for the scalar coefficients $c_2^{(1)}(\mu_1;\mu_2)$ and $c_2^{(2)}(\mu_1,\mu_2)$ is

$$c_2^{(1)}(\mu_1;\mu_2) = \frac{\psi}{\kappa}\left(1 + \left(e^{-2\xi} - \cosh(2\mu_1)\right)\left(\rho(\mu_1) - \frac{1}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}\right)\right), \tag{64}$$

$$\begin{aligned}
c_2^{(2)}(\mu_1,\mu_2) = \frac{\psi^2}{\kappa^2}\Bigg(&3 + \left(e^{-2\xi} - \cosh(2\mu_1)\right)\left(e^{-2\xi} - \cosh(2\mu_2)\right)\rho(\mu_1)\rho(\mu_2) + \\
&+ \frac{2e^{-4\xi} + 2e^{-2\xi}\left(\cosh(2\mu_1) - 3\cosh(2\mu_2)\right) - \left(3 + \cosh(4\mu_1) - 6\cosh(2\mu_1)\cosh(2\mu_2)\right)}{4\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}\rho(\mu_1) \\
&+ \frac{2e^{-4\xi} + 2e^{-2\xi}\left(\cosh(2\mu_2) - 3\cosh(2\mu_1)\right) - \left(3 + \cosh(4\mu_2) - 6\cosh(2\mu_2)\cosh(2\mu_1)\right)}{4\sinh(\mu_2-\mu_1)\sinh(\mu_2+\mu_1)}\rho(\mu_2)\Bigg).
\end{aligned} \tag{65}$$

The action of $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ can be written as

$$\begin{aligned}\tau(\lambda)\varphi_2(\mu_1,\mu_2) &= \left[\left[\tau(\lambda), f(\mu_1)\right], f(\mu_2)\right]\Omega_+ + \left(f(\mu_2)+c_2^{(1)}(\mu_2;\mu_1)\right)\left[\tau(\lambda), f(\mu_1)\right]\Omega_+ \\ &+ \left(f(\mu_1)+c_2^{(1)}(\mu_1;\mu_2)\right)\left[\tau(\lambda), f(\mu_2)\right]\Omega_+ + \chi_0(\lambda)\varphi_2(\mu_1,\mu_2).\end{aligned} \quad (66)$$

Then, we already have the Expression (59) for the second and third term above, and we use the relations

$$\begin{aligned}&\left(f(\mu_1)+c_2^{(1)}(\mu_1;\mu_2)\right)\varphi_1(\mu_2) = \varphi_2(\mu_1,\mu_2) - \frac{\psi}{\kappa}\,\frac{e^{-2\xi}-\cosh(2\mu_2)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}\varphi_1(\mu_1) \\ &- \left(c_2^{(2)}(\mu_1,\mu_2) - c_1^{(1)}(\mu_1)c_1^{(1)}(\mu_2) + \frac{\psi}{\kappa}\,\frac{\left(e^{-2\xi}-\cosh(2\mu_1)\right)c_1^{(1)}(\mu_2) - \left(e^{-2\xi}-\cosh(2\mu_2)\right)c_1^{(1)}(\mu_1)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}\right)\Omega_+,\end{aligned} \quad (67)$$

$$\begin{aligned}&\left(f(\mu_1)+c_2^{(1)}(\mu_1;\mu_2)\right)\varphi_1(\lambda) = \varphi_2(\mu_1,\lambda) - \frac{\psi}{\kappa}\,\frac{e^{-2\xi}-\cosh(2\lambda)}{\sinh(\lambda-\mu_1)\sinh(\lambda+\mu_1)}\varphi_1(\mu_1) \\ &+ \left(c_2^{(1)}(\mu_1;\mu_2) - c_2^{(1)}(\mu_1;\lambda)\right)\varphi_1(\lambda) \\ &- \left(c_2^{(2)}(\mu_1,\lambda) - c_1^{(1)}(\mu_1)c_1^{(1)}(\lambda) + \frac{\psi}{\kappa}\,\frac{\left(e^{-2\xi}-\cosh(2\lambda)\right)c_1^{(1)}(\mu_1) - \left(e^{-2\xi}-\cosh(2\mu_1)\right)c_1^{(1)}(\lambda)}{\sinh(\lambda-\mu_1)\sinh(\lambda+\mu_1)}\right)\Omega_+,\end{aligned} \quad (68)$$

which follow from the definition (63). After expressing appropriately the first term on the right-hand side of (66) and using twice the expression for the action of the commutator of $\tau(\lambda)$ with the generator $f(\lambda)$ on the vector Ω_+ (59) as well as the identities (67) and (68), a straightforward calculation shows that the off-shell action of the generating function $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ is given by

$$\begin{aligned}\tau(\lambda)\varphi_2(\mu_1,\mu_2) &= \chi_2(\lambda,\mu_1,\mu_2)\varphi_2(\mu_1,\mu_2) + \sum_{i=1}^{2}\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)}\,\frac{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\times \\ &\times\left(2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \frac{2}{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i})}\right)\varphi_2(\lambda,\mu_{3-i}),\end{aligned} \quad (69)$$

with the eigenvalue

$$\begin{aligned}\chi_2(\lambda,\mu_1,\mu_2) &= \chi_0(\lambda) - \sum_{i=1}^{2}\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)}\times \\ &\times\left(2\rho(\lambda) + \frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \frac{1}{\sinh(\lambda+\mu_{3-i})\sinh(\lambda-\mu_{3-i})}\right).\end{aligned} \quad (70)$$

Again, we can take care of the two unwanted terms in (69) by imposing the Bethe equations on the parameters μ_1 and μ_2:

$$2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \frac{2}{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i})} = 0, \quad (71)$$

with $i=1,2$. Thus, we have established that $\varphi_2(\mu_1,\mu_2)$ is the Bethe vector of the generating function of the Gaudin Hamiltonians corresponding to the eigenvalue $\chi_2(\lambda,\mu_1,\mu_2)$.

The form of the Bethe vector $\varphi_3(\mu_1, \mu_2, \mu_3)$ is given explicitly (as a symmetric function of these parameters) in the Appendix B. By a straightforward (but lengthy) computation one can show that the action of the generating function $\tau(\lambda)$ on $\varphi_3(\mu_1, \mu_2, \mu_3)$ has the following form:

$$\begin{aligned}
&\tau(\lambda)\varphi_3(\mu_1, \mu_2, \mu_3) = \chi_3(\lambda, \mu_1, \mu_2, \mu_3)\varphi_3(\mu_1, \mu_2, \mu_3) \\
&+ \sum_{i=1}^{3} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)} \frac{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} \times \\
&\times \left(2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \sum_{j\neq i}^{3} \frac{2}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j)}\right) \varphi_3(\lambda, \{\mu_j\}_{j\neq i}),
\end{aligned} \tag{72}$$

where the eigenvalue is

$$\begin{aligned}
\chi_3(\lambda, \mu_1, \mu_2, \mu_3) &= \chi_0(\lambda) - \sum_{i=1}^{3} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)} \times \\
&\times \left(2\rho(\lambda) + \frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \sum_{j\neq i}^{3} \frac{1}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)}\right).
\end{aligned} \tag{73}$$

The three unwanted terms in (72) vanish upon imposing the following Bethe conditions on the parameters μ_i:

$$2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \sum_{j\neq i}^{3} \frac{2}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j)} = 0, \tag{74}$$

with $i = 1, 2, 3$.

Instead of processing further in a brute-force manner to find the form of $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$, it turns out that it is possible to unify the obtained expressions for φ_1, φ_2 and φ_3 by introducing a family of operators

$$\begin{aligned}
\mathcal{C}_K(\mu) &= f(\mu) + \frac{\psi}{\kappa}\left((2K-1) + \left(e^{-2\xi} - \cosh(2\mu)\right) h(\mu)\right) + \frac{\psi^2}{\kappa^2}\frac{e^{-\xi}}{2\sinh(\xi)} \times \\
&\times \left(e^{-2\xi} + 1 - 2\cosh(2\mu)\right) e(\mu),
\end{aligned} \tag{75}$$

for any natural number K. Now, it can be shown by a direct calculation that the Bethe vectors (56), (63) and (A8) can be expressed as

$$\varphi_1(\mu) = \mathcal{C}_1(\mu)\Omega_+, \quad \varphi_2(\mu_1, \mu_2) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\Omega_+ \quad \text{and} \quad \varphi_3(\mu_1, \mu_2, \mu_3) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\mathcal{C}_3(\mu_3)\Omega_+. \tag{76}$$

Although in general the operators $\mathcal{C}_K(\mu)$ (75) do not commute, it is easy to verify that the Bethe vector $\varphi_2(\mu_1, \mu_2)$ is a symmetric function

$$\varphi_2(\mu_1, \mu_2) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\Omega_+ = \mathcal{C}_1(\mu_2)\mathcal{C}_2(\mu_1)\Omega_+ = \varphi_2(\mu_2, \mu_1). \tag{77}$$

Analogously, it is straightforward to check that the Bethe vector $\varphi_3(\mu_1, \mu_2, \mu_3)$ is a symmetric function of its arguments

$$\varphi_3(\mu_1, \mu_2, \mu_3) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\mathcal{C}_3(\mu_3)\Omega_+ = \mathcal{C}_1(\mu_2)\mathcal{C}_2(\mu_1)\mathcal{C}_3(\mu_3)\Omega_+ = \varphi_3(\mu_2, \mu_1, \mu_3), \tag{78}$$

etc. Moreover using the formulae above (76) for the Bethe vectors it is somewhat simpler to calculate the off-shell action of the generating function. Evidently,

$$[\tau(\lambda), \mathcal{C}_1(\mu)]\,\Omega_+ = [\tau(\lambda), f(\mu)]\,\Omega_+,$$

and consequently, the action (60) of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_1(\mu)$ follows directly from (59). In order to show (69), we calculate

$$\begin{aligned}
&[[\tau(\lambda), \mathcal{C}_1(\mu_1)]\,, \mathcal{C}_2(\mu_2)]\,\Omega_+ = \frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\varphi_2(\mu_1,\mu_2) \\
&-\sum_{i=1}^{2}\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)}\,\frac{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\,\frac{2}{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i})}\varphi_2(\lambda,\mu_{3-i}) \\
&+\frac{\psi}{\kappa}\frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\left(2\rho(\lambda)+\frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\right)(\varphi_1(\mu_1)-\varphi_1(\mu_2)) \\
&-\frac{\psi}{\kappa}\frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\sinh(\xi+\mu_1)\sinh(\xi-\mu_1)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\left(2\rho(\mu_1)+\frac{1}{\sinh(\xi+\mu_1)\sinh(\xi-\mu_1)}\right)\times \\
&\times(\varphi_1(\lambda)-\varphi_1(\mu_2)) \\
&-\frac{\psi}{\kappa}\frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\left(2\rho(\lambda)+\frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\right)\varphi_1(\mu_1) \\
&+\frac{\psi}{\kappa}\frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\,\frac{\sinh(\xi+\mu_2)\sinh(\xi-\mu_2)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\left(2\rho(\mu_2)+\frac{1}{\sinh(\xi+\mu_2)\sinh(\xi-\mu_2)}\right)\varphi_1(\mu_1)
\end{aligned} \tag{79}$$

and use (59) appropriately. Finally, the action (72) of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_3(\mu_1,\mu_2,\mu_3)$ can be obtained by expressing $[[[\tau(\lambda), \mathcal{C}_1(\mu_1)]\,, \mathcal{C}_2(\mu_2)]\,, \mathcal{C}_3(\mu_3)]\,\Omega_+$ conveniently and using (59) and (79) adequately.

We proceed in a natural way and conjecture that the Bethe vector $\varphi_4(\mu_1,\ldots,\mu_4)$ can be written in the form

$$\varphi_4(\mu_1,\ldots,\mu_4) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\mathcal{C}_3(\mu_3)\mathcal{C}_4(\mu_4)\Omega_+. \tag{80}$$

With the aim of calculating the action of the generating function of the Gaudin Hamiltonians on the vector above we calculate $[[[[\tau(\lambda), \mathcal{C}_1(\mu_1)]\,, \mathcal{C}_2(\mu_2)]\,, \mathcal{C}_3(\mu_3)]\,, \mathcal{C}_4(\mu_4)]\,\Omega_+$, expressing it appropriately as a linear combination of all the previous Bethe vectors. This formula is very long and cumbersome and for this reason, is not presented in the text. Using this result it is possible to obtain the desired off-shell action in the following form

$$\begin{aligned}
&\tau(\lambda)\varphi_4(\mu_1,\mu_2,\mu_3,\mu_4) = \chi_4(\lambda,\mu_1,\mu_2,\mu_3,\mu_4)\varphi_4(\mu_1,\mu_2,\mu_3,\mu_4) \\
&+\sum_{i=1}^{4}\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)}\,\frac{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\times \\
&\times\left(2\rho(\mu_i)+\frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}-\sum_{j\neq i}^{4}\frac{2}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j)}\right)\varphi_4(\lambda,\{\mu_j\}_{j\neq i}),
\end{aligned} \tag{81}$$

where the eigenvalue is

$$\begin{aligned}
&\chi_4(\lambda,\mu_1,\mu_2,\mu_3,\mu_4) = \chi_0(\lambda)-\sum_{i=1}^{4}\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)}\times \\
&\times\left(2\rho(\lambda)+\frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}-\sum_{j\neq i}^{4}\frac{1}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)}\right).
\end{aligned} \tag{82}$$

This result we have confirmed also by computer algorithms for symbolical calculation. Upon putting constrains on parameters μ_i in the form of the following Bethe equations:

$$2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \sum_{j\neq i}^{4} \frac{2}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j)} = 0, \tag{83}$$

with $i = 1,2,3,4$, the four unwanted terms vanish in (81).

We readily proceed to define $\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)$, for an arbitrary positive integer M,

$$\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\cdots\mathcal{C}_M(\mu_M)\Omega_+, \tag{84}$$

and the operators $\mathcal{C}_K(\mu)$ are given in (75). Although the operators $\mathcal{C}_K(\mu)$ do not commute, the Bethe vector $\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)$ is nonetheless a symmetric function of its arguments, since these operators satisfy the following identity,

$$\mathcal{C}_K(\mu)\mathcal{C}_{K+1}(\widetilde{\mu}) - \mathcal{C}_K(\widetilde{\mu})\mathcal{C}_{K+1}(\mu) = 0, \tag{85}$$

for $K = 1,\ldots,M-1$. It can be confirmed by explicit calculation that the off-shell action of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)$, is given by

$$\begin{aligned}
&\tau(\lambda)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) = \chi_M(\mu_1,\mu_2,\ldots,\mu_M)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)\\
&+\sum_{i=1}^{M} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)} \frac{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\times\\
&\times\left(2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \sum_{j\neq i}^{M} \frac{2}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j)}\right)\varphi_M(\lambda,\{\mu_j\}_{j\neq i}),
\end{aligned} \tag{86}$$

with the eigenvalue

$$\begin{aligned}
\chi_M(\mu_1,\mu_2,\ldots,\mu_M) = \chi_0(\lambda) - \sum_{i=1}^{M} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)}\times\\
\times\left(2\rho(\lambda) + \frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \sum_{j\neq i}^{M} \frac{1}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)}\right).
\end{aligned} \tag{87}$$

Imposing the following Bethe equations on parameters μ_i:

$$2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \sum_{j\neq i}^{M} \frac{2}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j)} = 0, \tag{88}$$

with $i = 1,2,\ldots,M$, results in vanishing of M unwanted terms in (86).

The obtained formula (86) for the action of the generating function $\tau(\lambda)$ has a strikingly compact form. This simplicity stems from our suitable definition of the Bethe vector $\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)$ (84) and of the corresponding creation operators $\mathcal{C}_K(\mu)$ (75). In this sense we have successfully implemented the algebraic Bethe ansatz for the trigonometric Gaudin model, with triangular K-matrix (24). The implementation was based on the non-unitary classical r-matrix (17) and the corresponding linear bracket (19).

4. Conclusions

Our first step was to derive the generating function of the Gaudin Hamiltonians with boundary terms. We followed the approach based on Sklyanin's method in the periodic case, just as we previously did in the rational case. Having obtained the generating function, we could calculate its residues at poles and find the corresponding Gaudin Hamiltonians with boundary terms.

Our next step was the implementation of the algebraic Bethe ansatz for the trigonometric Gaudin model with triangular reflection matrix (24). To this end we have introduced the non-unitary classical r-matrix (17), which satisfies the generalized classical Yang-Baxter Equation (18), as well as the modified Lax matrix (16). Together they define the linear bracket (19), which is obviously anti-symmetric and obeys the Jacobi identity. As a consequence, it follows that the entries of the modified Lax matrix generate an infinite dimensional Lie algebra, which is the basis of the open trigonometric Gaudin model. A suitable set of generators (39)–(41) simplifies the commutation relations (42)–(45) and therefore facilitates the algebraic Bethe ansatz. Another crucial observation for the implementation of the algebraic Bethe ansatz was the existence of the so-called pseudo-vacuum or the reference state Ω_+ (52) (see also (53) and (54)). The simplest way to define the relevant Bethe vectors turned out to be by using the family of the creation operators $\mathcal{C}_K(\mu)$ (75). Obtained Bethe vectors $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ (84) are symmetric functions of their arguments and they result in exceedingly simple and compact form of off-shell action of the generating function. In this sense, we have fully implemented the algebraic Bethe ansatz: we have obtained the spectrum of the generating function and found the corresponding Bethe equations.

Having in mind the already discussed range of potential applications that Gaudin model and its generalizations have in various areas of physics (from condensed matter physics to field theory [2–5]), we believe that the inclusion of nontrivial boundary conditions while retaining the integrability of the model—demonstrated here—also has its share of significance. In this regard, it would be of further considerable interest to establish a relationship between the presented Bethe vectors of the trigonometric Gaudin model and the solutions to the related generalized Knizhnik-Zamolodchikov equations, analogously as we did it for the rational case in [60]. These results will be reported elsewhere. Also, we intend to give a mathematical completion of the work presented here by providing a strict analytical proof (omitted here primarily due to its length) of the general Expression (86) for the off-shell action.

Author Contributions: Investigation, N.M. and I.S; methodology N.M.; software, I.S.; writing–original draft preparation, N.M.; writing–review and editing, I.S. All authors have read and agreed to the published version of the manuscript.

Funding: I.S. was supported in part by the Serbian Ministry of Science and Technological Development under grant number ON 171031.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Commutativity of the Generating Function

There are multiple ways to prove the commutation relation (21). Here we will employ the relation of the present Gaudin model with the XXZ Heisenberg spin-chain model with boundary, explored in detail in [53]. Thus, in this appendix, we will extensively reference expressions from that paper.

We will begin by computing the expansion (with respect to the quasi-classical parameter η) of the monodromy matrix taken from [53]:

$$\begin{aligned}T(\lambda) &= \mathbb{1} + \eta L_0(\lambda) + \frac{\eta^2}{2}\mathbb{1}_0 \otimes \sum_{m=1}^{N}\left(S_m^3\right)^2 \\ &+ \frac{\eta^2}{2}\sum_{\substack{n,m=1\\ n\neq m}}^{N} \frac{\mathbb{1}_0 \otimes \left(\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)\right)}{\sinh(\lambda-\alpha_m)\ \sinh(\lambda-\alpha_n)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n<m}^{N} \frac{\sigma_0^3 \otimes (S_m^- S_n^+ - S_m^+ S_n^-) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\sinh(\lambda-\alpha_m)\ \sinh(\lambda-\alpha_n)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n<m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^-\right)}{2\sinh(\lambda-\alpha_m)\ \sinh(\lambda-\alpha_n)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n>m}^{N} \frac{\sigma_0^3 \otimes (S_n^- S_m^+ - S_n^+ S_m^-) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\sinh(\lambda-\alpha_n)\ \sinh(\lambda-\alpha_m)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n>m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^-\right)}{2\sinh(\lambda-\alpha_n)\ \sinh(\lambda-\alpha_m)} + \mathcal{O}(\eta^3).\end{aligned} \tag{A1}$$

Analogously, it is straightforward to obtain the expansion of the $\widetilde{T}(\lambda)$ monodromy matrix from [53] in the powers the quasi-classical parameter η

$$\begin{aligned}\widetilde{T}(\lambda) &= \mathbb{1} - \eta L_0(-\lambda) + \frac{\eta^2}{2}\sum_{m=1}^{N}\left(\mathbb{1}_0 \otimes (S_m^3)^2 - \frac{2\left(\sigma_0^3 \otimes S_m^3 + \frac{1}{2}\cosh(\lambda+\alpha_m)\left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right)\right)}{\sinh^2(\lambda+\alpha_m)}\right) \\ &+ \frac{\eta^2}{2}\sum_{\substack{n,m=1\\ n\neq m}}^{N} \frac{\mathbb{1}_0 \otimes \left(\cosh(\lambda+\alpha_m)\cosh(\lambda+\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)\right)}{\sinh(\lambda+\alpha_m)\ \sinh(\lambda+\alpha_n)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n<m}^{N} \frac{\sigma_0^3 \otimes (S_m^- S_n^+ - S_m^+ S_n^-) + \sigma_0^+ \otimes \left(\cosh(\lambda+\alpha_m) S_m^3 S_n^- - \cosh(\lambda+\alpha_n) S_m^- S_n^3\right)}{2\sinh(\lambda+\alpha_m)\ \sinh(\lambda+\alpha_n)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n<m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda+\alpha_m) S_m^3 S_n^-\right)}{2\sinh(\lambda+\alpha_m)\ \sinh(\lambda+\alpha_n)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n>m}^{N} \frac{\sigma_0^3 \otimes (S_n^- S_m^+ - S_n^+ S_m^-) + \sigma_0^+ \otimes \left(\cosh(\lambda+\alpha_n) S_n^3 S_m^- - \cosh(\lambda+\alpha_m) S_n^- S_m^3\right)}{2\sinh(\lambda+\alpha_n)\ \sinh(\lambda-\alpha_m)} \\ &+ \frac{\eta^2}{2}\sum_{m=1}^{N}\sum_{n>m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda+\alpha_m) S_n^+ S_m^3 - \cosh(\lambda+\alpha_n) S_n^3 S_m^-\right)}{2\sinh(\lambda+\alpha_n)\ \sinh(\lambda+\alpha_m)} + \mathcal{O}(\eta^3).\end{aligned} \tag{A2}$$

Using these formulas, as well as the first three terms in the power series of the K-matrix (13), we can deduce the expansion of the transfer matrix of the chain $t(\lambda)$ in powers of η. We similarly obtain the expansion of the so-called Sklyanin determinant $\Delta\left[\mathcal{T}(\lambda)\right]$ in powers of η. However these formulas are long and cumbersome, therefore we will not present them here. Instead we will give the expansion of the difference between the transfer matrix of the chain and the Sklyanin determinant:

$$t(\lambda)-\frac{\Delta\left[\mathcal{T}(\lambda)\right]}{\sinh(2\lambda)}=\frac{1}{2}\mathrm{tr}_0K_0(\lambda)K_0(-\lambda)+\eta\left(\mathrm{tr}_0K_0'(-\lambda)K_0(\lambda)+\mathrm{tr}_{00'}P_{00'}^{-}K_0(\lambda)r_{00'}(2\lambda)K_{0'}(\lambda)\right)$$
$$+\eta^2\left(\mathrm{tr}_0K_0'(-\lambda)\mathcal{L}_0(\lambda)K_0(\lambda)+\mathrm{tr}_{00'}P_{00'}^{-}\left(\mathcal{L}_0(\lambda)K_0(\lambda)r_{00'}(2\lambda)K_{0'}(\lambda)+K_0(\lambda)r_{00'}(2\lambda)\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\right)$$
$$-\eta^2\,\mathrm{tr}_{00'}P_{00'}^{-}\mathcal{L}_0(\lambda)K_0(\lambda)\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)+\frac{\eta^2}{2}\Bigg(\mathrm{tr}_0K_0''(-\lambda)K_0(\lambda)-\frac{1}{4}\,\mathrm{tr}_0K_0''(\lambda)K_0(-\lambda)$$
$$+\frac{1}{2}\,\mathrm{tr}_{00'}P_{00'}^{-}K_0'(\lambda)K_{0'}'(\lambda)-\frac{1}{\sinh(2\lambda)}\,\mathrm{tr}_{00'}P_{00'}^{-}K_0(\lambda)\partial_\eta^2R_{00'}(2\lambda)\big|_{\eta=0}K_{0'}(\lambda)\Bigg)+\mathcal{O}(\eta^3). \tag{A3}$$

Note the explicit appearance of the K-modified Lax matrix (16) in the above result.

Actually, a straightforward calculation shows that the terms in the second line of the expression above vanish

$$\mathrm{tr}_0K_0'(-\lambda)\mathcal{L}_0(\lambda)K_0(\lambda)+\mathrm{tr}_{00'}P_{00'}^{-}\left(\mathcal{L}_0(\lambda)K_0(\lambda)r_{00'}(2\lambda)K_{0'}(\lambda)+K_0(\lambda)r_{00'}(2\lambda)\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)=0. \tag{A4}$$

Also, it is important to notice that using the following identity

$$\mathcal{L}_0(\lambda)K_0(\lambda)-\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0=K_0(-\lambda)\mathcal{L}_0(\lambda), \tag{A5}$$

the first term in the third line of (A3) can be simplified

$$\mathrm{tr}_0\,K_0(-\lambda)\mathcal{L}_0(\lambda)\mathcal{L}_0(\lambda)K_0(\lambda)=\det K_0(\lambda)\;\mathrm{tr}_0\,\mathcal{L}_0^2(\lambda). \tag{A6}$$

Finally, the Expansion (A3) reads

$$t(\lambda)-\frac{\Delta\left[\mathcal{T}(\lambda)\right]}{\sinh(2\lambda)}=\det K_0(\lambda)+\eta\left(\mathrm{tr}_0K_0'(-\lambda)K_0(\lambda)+\mathrm{tr}_{00'}P_{00'}^{-}K_0(\lambda)r_{00'}(2\lambda)K_{0'}(\lambda)\right)$$
$$+\frac{\eta^2}{2}\det K_0(\lambda)\;\mathrm{tr}_0\,\mathcal{L}_0^2(\lambda)+\frac{\eta^2}{2}\Bigg(\mathrm{tr}_0K_0''(-\lambda)K_0(\lambda)-\frac{1}{4}\,\mathrm{tr}_0K_0''(\lambda)K_0(-\lambda)$$
$$+\frac{1}{2}\,\mathrm{tr}_{00'}P_{00'}^{-}K_0'(\lambda)K_{0'}'(\lambda)-\frac{1}{\sinh(2\lambda)}\,\mathrm{tr}_{00'}P_{00'}^{-}K_0(\lambda)\partial_\eta^2R_{00'}(2\lambda)\big|_{\eta=0}K_{0'}(\lambda)\Bigg)+\mathcal{O}(\eta^3). \tag{A7}$$

Since both the transfer matrix $t(\lambda)$ and the Sklyanin determinant $\Delta\left[\mathcal{T}(\lambda)\right]$ commute (as well as their difference) for different values of the spectral parameter, the result (21) directly follows from the previous relation.

Appendix B. Bethe Vector $\varphi_3(\mu_1,\mu_2,\mu_3)$

Here we provide explicit formulas of the Bethe vector $\varphi_3(\mu_1,\mu_2,\mu_3)$:

$$\varphi_3(\mu_1,\mu_2,\mu_3)=f(\mu_1)f(\mu_2)f(\mu_3)\Omega_++c_3^{(1)}(\mu_1;\mu_2,\mu_3)f(\mu_2)f(\mu_3)\Omega_++c_3^{(1)}(\mu_2;\mu_3,\mu_1)f(\mu_3)f(\mu_1)\Omega_+$$
$$+c_3^{(1)}(\mu_3;\mu_1,\mu_2)f(\mu_1)f(\mu_2)\Omega_++c_3^{(2)}(\mu_1,\mu_2;\mu_3)f(\mu_3)\Omega_++c_3^{(2)}(\mu_2,\mu_3;\mu_1)f(\mu_1)\Omega_+$$
$$+c_3^{(2)}(\mu_3,\mu_1;\mu_2)f(\mu_2)\Omega_++c_3^{(3)}(\mu_1,\mu_2,\mu_3)\Omega_+, \tag{A8}$$

where the three scalar coefficients above are given by

$$c_3^{(1)}(\mu_1;\mu_2,\mu_3) = \frac{\psi}{\kappa}\left(1+\left(\rho(\mu_1) - \frac{1}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)} - \frac{1}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3)}\right)\times \right.$$
$$\left. \times \left(e^{-2\xi} - \cosh(2\mu_1)\right)\right), \tag{A9}$$

$$c_3^{(2)}(\mu_1,\mu_2;\mu_3) = \frac{\psi^2}{\kappa^2}\left(3+\left(e^{-2\xi}-\cosh(2\mu_1)\right)\left(e^{-2\xi}-\cosh(2\mu_2)\right)\times \right.$$
$$\times\left(\rho(\mu_1) - \frac{1}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3)}\right)\left(\rho(\mu_2) - \frac{1}{\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3)}\right)$$
$$+\frac{2e^{-4\xi}+2e^{-2\xi}\left(\cosh(2\mu_1)-3\cosh(2\mu_2)\right)-\left(3+\cosh(4\mu_1)-6\cosh(2\mu_1)\cosh(2\mu_2)\right)}{4\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}$$
$$\times\left(\rho(\mu_1) - \frac{1}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3)}\right)$$
$$+\frac{2e^{-4\xi}+2e^{-2\xi}\left(\cosh(2\mu_2)-3\cosh(2\mu_1)\right)-\left(3+\cosh(4\mu_2)-6\cosh(2\mu_2)\cosh(2\mu_1)\right)}{4\sinh(\mu_2-\mu_1)\sinh(\mu_2+\mu_1)}$$
$$\left.\times\left(\rho(\mu_2) - \frac{1}{\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3)}\right)\right), \tag{A10}$$

$$c_3^{(3)}(\mu_1,\mu_2,\mu_3) = \frac{\psi^3}{\kappa^3}\left(15+\left(e^{-2\xi}-\cosh(2\mu_1)\right)\left(e^{-2\xi}-\cosh(2\mu_2)\right)\left(e^{-2\xi}-\cosh(2\mu_3)\right)\rho(\mu_1)\rho(\mu_2)\rho(\mu_3)\right.$$
$$+\left(\left(5-2\coth(\xi)\left(\frac{\sinh^2(\mu_1)}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3)}+\frac{\sinh^2(\mu_2)}{\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3)}\right)\right)\left(e^{-2\xi}-\cosh(2\mu_1)\right)\right.$$
$$\times\left(e^{-2\xi}-\cosh(2\mu_2)\right) - \frac{e^{-\xi}}{\sinh(\xi)}\left(\frac{\sinh(\xi-\mu_1)\sinh(\xi+\mu_1)}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3)}\left(e^{-2\xi}+1-2\cosh(2\mu_1)\right)\times\right.$$
$$\left.\left.\times\left(e^{-2\xi}-\cosh(2\mu_2)\right)+\frac{\sinh(\xi-\mu_2)\sinh(\xi+\mu_2)}{\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3)}\left(e^{-2\xi}-\cosh(2\mu_1)\right)\left(e^{-2\xi}+1-2\cosh(2\mu_2)\right)\right)\right)$$
$$\times\rho(\mu_1)\rho(\mu_2)$$
$$+\left(\left(5-2\coth(\xi)\left(\frac{\sinh^2(\mu_1)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}+\frac{\sinh^2(\mu_3)}{\sinh(\mu_3-\mu_2)\sinh(\mu_3+\mu_2)}\right)\right)\left(e^{-2\xi}-\cosh(2\mu_1)\right)\right.$$
$$\times\left(e^{-2\xi}-\cosh(2\mu_3)\right) - \frac{e^{-\xi}}{\sinh(\xi)}\left(\frac{\sinh(\xi-\mu_1)\sinh(\xi+\mu_1)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)}\left(e^{-2\xi}+1-2\cosh(2\mu_1)\right)\times\right.$$
$$\left.\left.\times\left(e^{-2\xi}-\cosh(2\mu_3)\right)+\frac{\sinh(\xi-\mu_3)\sinh(\xi+\mu_3)}{\sinh(\mu_3-\mu_2)\sinh(\mu_3+\mu_2)}\left(e^{-2\xi}-\cosh(2\mu_1)\right)\left(e^{-2\xi}+1-2\cosh(2\mu_3)\right)\right)\right)$$
$$\times\rho(\mu_1)\rho(\mu_3)$$

$$
\begin{aligned}
&+\Bigg(\Bigg(5-2\coth(\xi)\left(\frac{\sinh^2(\mu_2)}{\sinh(\mu_2-\mu_1)\sinh(\mu_2+\mu_1)}+\frac{\sinh^2(\mu_3)}{\sinh(\mu_3-\mu_1)\sinh(\mu_3+\mu_1)}\right)\Bigg)\left(e^{-2\xi}-\cosh(2\mu_2)\right)\\
&\times\left(e^{-2\xi}-\cosh(2\mu_3)\right)-\frac{e^{-\xi}}{\sinh(\xi)}\Bigg(\frac{\sinh(\xi-\mu_2)\sinh(\xi+\mu_2)}{\sinh(\mu_2-\mu_1)\sinh(\mu_2+\mu_1)}\left(e^{-2\xi}+1-2\cosh(2\mu_2)\right)\times\\
&\times\left(e^{-2\xi}-\cosh(2\mu_3)\right)+\frac{\sinh(\xi-\mu_3)\sinh(\xi+\mu_3)}{\sinh(\mu_3-\mu_1)\sinh(\mu_3+\mu_1)}\left(e^{-2\xi}-\cosh(2\mu_2)\right)\left(e^{-2\xi}+1-2\cosh(2\mu_3)\right)\Bigg)\Bigg)\\
&\times\rho(\mu_2)\rho(\mu_3)\\
&+\Big(8e^{-6\xi}+4e^{-4\xi}\left(4\cosh(2\mu_1)-5\left(\cosh(2\mu_2)+\cosh(2\mu_3)\right)\right)+2e^{-2\xi}\left(-5+7\cosh(4\mu_1)+30\cosh(2\mu_2)\times\right.\\
&\left.\times\cosh(2\mu_3)-10\cosh(2\mu_1)\left(\cosh(2\mu_2)+\cosh(2\mu_3)\right)\right)-3\cosh(6\mu_1)+10\left(3+\cosh(4\mu_1)\right)\times\\
&\times\left(\cosh(2\mu_2)+\cosh(2\mu_3)\right)-5\cosh(2\mu_1)\left(5+12\cosh(2\mu_2)\cosh(2\mu_3)\right)\Big)\times\\
&\times\frac{\rho(\mu_1)}{16\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2)\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3)}\\
&+\Big(8e^{-6\xi}+4e^{-4\xi}\left(4\cosh(2\mu_2)-5\left(\cosh(2\mu_1)+\cosh(2\mu_3)\right)\right)+2e^{-2\xi}\left(-5+7\cosh(4\mu_2)+30\cosh(2\mu_1)\times\right.\\
&\left.\times\cosh(2\mu_3)-10\cosh(2\mu_2)\left(\cosh(2\mu_1)+\cosh(2\mu_3)\right)\right)-3\cosh(6\mu_2)+10\left(3+\cosh(4\mu_2)\right)\times\\
&\times\left(\cosh(2\mu_1)+\cosh(2\mu_3)\right)-5\cosh(2\mu_2)\left(5+12\cosh(2\mu_1)\cosh(2\mu_3)\right)\Big)\times\\
&\times\frac{\rho(\mu_2)}{16\sinh(\mu_2-\mu_1)\sinh(\mu_2+\mu_1)\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3)}\\
&+\Big(8e^{-6\xi}+4e^{-4\xi}\left(4\cosh(2\mu_3)-5\left(\cosh(2\mu_1)+\cosh(2\mu_2)\right)\right)+2e^{-2\xi}\left(-5+7\cosh(4\mu_3)+30\cosh(2\mu_1)\times\right.\\
&\left.\times\cosh(2\mu_1)-10\cosh(2\mu_3)\left(\cosh(2\mu_1)+\cosh(2\mu_2)\right)\right)-3\cosh(6\mu_3)+10\left(3+\cosh(4\mu_3)\right)\times\\
&\times\left(\cosh(2\mu_1)+\cosh(2\mu_2)\right)-5\cosh(2\mu_3)\left(5+12\cosh(2\mu_1)\cosh(2\mu_2)\right)\Big)\times\\
&\times\frac{\rho(\mu_3)}{16\sinh(\mu_3-\mu_1)\sinh(\mu_3+\mu_1)\sinh(\mu_3-\mu_2)\sinh(\mu_3+\mu_2)}\Bigg)
\end{aligned}
\tag{A11}
$$

References

1. Gaudin, M. Diagonalisation d'une classe d'hamiltoniens de spin. *J. Phys.* **1976**, *37*, 1087–1098. [CrossRef]
2. Ortiz, G.; Somma, R.; Dukelsky, J.; Rombouts, S. Exactly-solvable models derived from a generalized Gaudin algebra. *Nuclear Phys. B* **2005**, *707*, 421–457. [CrossRef]
3. Feigin, B.; Frenkel, E.; Reshetikhin, N. Gaudin model, Bethe ansatz and correlation functions at the critical level. *Commun. Math. Phys.* **1994**, *166*, 27–62. [CrossRef]
4. Mironov, A.; Morozov, A.; Runov, B.; Zenkevich, Y.; Zotov, A. Spectral Duality between Heisenberg Chain and Gaudin Model. *Lett. Math. Phys.* **2013**, *103*, 299–329. [CrossRef]
5. Delduc, F.; Lacroix, S.; Magro, M.; Vicedo, B. Assembling integrable sigma-models as affine Gaudin models. *J. High Energy Phys.* **2019**, *2019*, 17. [CrossRef]
6. Gaudin, M. *La Fonction D'onde de Bethe*; Masson: Paris, France, 1983.
7. Gaudin, M. *The Bethe Wavefunction*; Cambridge University Press: Cambridge, UK, 2014.
8. Takhtajan, L.A.; Faddeev, L.D. The quantum method for the inverse problem and the *XYZ* Heisenberg model. *Uspekhi Mat. Nauk* **1979**, *34*, 13–63; translation in *Russ. Math. Surv.* **1979**, *34*, 11–68. (In Russian) [CrossRef]
9. Kulish, P.P.; Sklyanin, E.K. Quantum spectral transform method. Recent developments. *Lect. Notes Phys.* **1982**, *151*, 61–119.
10. Sklyanin, E.K. Separation of variables in the Gaudin model. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **1987**, *164*, 151–169; translation in *J. Soviet Math.* **1989**, *47*, 2473–2488. [CrossRef]
11. Sklyanin, E.K.; Takebe, T. Algebraic Bethe ansatz for the XYZ Gaudin model. *Phys. Lett. A* **1996**, *219*, 217–225. [CrossRef]

12. Semenov-Tian-Shansky, M.A. Quantum and classical integrable systems. *Integr. Nonlinear Syst.* **1997**, *495*, 314–377.
13. Jurčo, B. Classical Yang-Baxter equations and quantum integrable systems. *J. Math. Phys.* **1989**, *30*, 1289–1293. [CrossRef]
14. Jurčo, B. Classical Yang-Baxter equations and quantum integrable systems (Gaudin models). *Lect. Notes Phys.* **1990**, *370*, 219–227.
15. Babujian, H.M.; Flume, R. Off-shell Bethe ansatz equation for Gaudin magnets and solutions of Knizhnik-Zamolodchikov equations. *Mod. Phys. Lett. A* **1994**, *9*, 2029–2039. [CrossRef]
16. Reshetikhin, N.; Varchenko, A. Quasiclassical asymptotics of solutions to the KZ equations. In *Geometry, Topology and Physics. Conf. Proc. Lecture Notes Geom. Topology IV*; Internat. Press: Cambridge, MA, USA, 1995; pp. 293–322.
17. Wagner, F.; Macfarlane, A.J. Solvable Gaudin models for higher rank symplectic algebras. Quantum groups and integrable systems (Prague, 2000). *Czechoslovak J. Phys.* **2000**, *50*, 1371–1377. [CrossRef]
18. Brzezinski, T.; Macfarlane, A.J. On integrable models related to the osp(1,2) Gaudin algebra. *J. Math. Phys.* **1994**, *35*, 3261–3272. [CrossRef]
19. Kulish, P.P.; Manojlović, N. Creation operators and Bethe vectors of the $osp(1|2)$ Gaudin model. *J. Math. Phys.* **2001**, *42*, 4757–4778. [CrossRef]
20. Kulish, P.P.; Manojlović, N. Trigonometric $osp(1|2)$ Gaudin model. *J. Math. Phys.* **2003**, *44*, 676–700. [CrossRef]
21. Lima-Santos, A.; Utiel, W. Off-shell Bethe ansatz equation for $osp(2|1)$ Gaudin magnets. *Nucl. Phys. B* **2001**, *600*, 512–530. [CrossRef]
22. Kurak, V.; Lima-Santos, A. $sl(2|1)^{(2)}$ Gaudin magnet and its associated Knizhnik-Zamolodchikov equation. *Nuclear Phys. B* **2004**, *701*, 497–515. [CrossRef]
23. Hikami, K.; Kulish, P.P.; Wadati, M. Integrable Spin Systems with Long-Range Interaction. *Chaos Solitons Fractals* **1992**, *2*, 543–550. [CrossRef]
24. Hikami, K.; Kulish, P.P.; Wadati, M. Construction of Integrable Spin Systems with Long-Range Interaction. *J. Phys. Soc. Jpn.* **1992**, *61*, 3071–3076. [CrossRef]
25. Hikami, K. Gaudin magnet with boundary and generalized Knizhnik-Zamolodchikov equation. *J. Phys. A Math. Gen.* **1995**, *28*, 4997–5007. [CrossRef]
26. Lima-Santos, A. The $sl(2|1)^{(2)}$ Gaudin magnet with diagonal boundary terms. *J. Stat. Mech.* **2009**. [CrossRef]
27. Yang, W.L.; Sasaki, R.; Zhang, Y.Z. $\mathbb{Z}_n$ elliptic Gaudin model with open boundaries. *J. High Energy Phys.* **2004**, *9*, 046. [CrossRef]
28. Yang, W.L.; Sasaki, R.; Zhang, Y.Z. A_{n-1} Gaudin model with open boundaries. *Nuclear Phys. B* **2005**, *729*, 594–610. [CrossRef]
29. Hao, K.; Yang, W.-L.; Fan, H.; Liu, S.Y.; Wu, K.; Yang, Z.Y.; Zhang, Y.Z. Determinant representations for scalar products of the XXZ Gaudin model with general boundary terms. *Nuclear Phys. B* **2012**, *862*, 835–849. [CrossRef]
30. Sklyanin, E.K. Boundary conditions for integrable equations. *Funktsional. Anal. Prilozhen.* **1987**, *21*, 86–87; translation in *Funct. Anal. Its Appl.* **1987**, *21*, 164–166. (In Russian) [CrossRef]
31. Sklyanin, E.K. Boundary conditions for integrable systems. In Proceedings of the VIIIth International Congress on Mathematical Physics, Marseille, France, 16–25 July 1986; World Sci. Publishing: Singapore, 1987; pp. 402–408.
32. Sklyanin, E.K. Boundary conditions for integrable quantum systems. *J. Phys. A Math. Gen.* **1988**, *21*, 2375–2389. [CrossRef]
33. Skrypnyk, T. Non-skew-symmetric classical r-matrix, algebraic Bethe ansatz, and Bardeen-Cooper-Schrieffer-type integrable systems. *J. Math. Phys.* **2009**, *50*, 033540. [CrossRef]
34. Skrypnyk, T. "Z2-graded" Gaudin models and analytical Bethe ansatz. *Nuclear Phys. B* **2013**, *870*, 495–529. [CrossRef]
35. Cirilo António, N.; Manojlović, N.; Nagy, Z. Trigonometric $s\ell(2)$ Gaudin model with boundary terms. *Rev. Math. Phys.* **2013**, *25*, 1343004. [CrossRef]
36. Cao, J.; Lin, H.; Shi, K.; Wang, Y. Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields. *Nucl. Phys. B* **2003**, *663*, 487–519. [CrossRef]

37. Nepomechie, R.I. Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms. *J. Phys. A* **2004**, *37*, 433–440. [CrossRef]
38. Arnaudon, D.; Doikou, A.; Frappat, L.; Ragoucy, E.; Crampé, N. Analytical Bethe ansatz in $gl(N)$ spin chains. *Czechoslovak J. Phys.* **2006**, *56*, 141–148. [CrossRef]
39. Melo, C.S.; Ribeiro, G.A.P.; Martins, M.J. Bethe ansatz for the XXX-S chain with non-diagonal open boundaries. *Nuclear Phys. B* **2005**, *711*, 565–603. [CrossRef]
40. Frappat, L.; Nepomechie, R.I.; Ragoucy, E. A complete Bethe ansatz solution for the open spin-s XXZ chain with general integrable boundary terms. *J. Stat. Mech.* **2007**, 2007, P09009. [CrossRef]
41. Cao, J.; Yang, W.-L.; Shi, K.; Wang, Y. Off-diagonal Bethe ansatz solution of the XXX spin chain with arbitrary boundary conditions. *Nuclear Phys. B* **2013**, *875*, 152–165. [CrossRef]
42. Cao, J.; Yang, W.-L.; Shi, K.; Wang, Y. Off-diagonal Bethe ansatz solutions of the anisotropic spin-1/2 chains with arbitrary boundary fields. *Nuclear Phys. B* **2013**, *877*, 152–175. [CrossRef]
43. Ragoucy, E. Coordinate Bethe ansätze for non-diagonal boundaries. *Rev. Math. Phys.* **2013**, *25*, 1343007. [CrossRef]
44. Belliard, S.; Crampé, N.; Ragoucy, E. Algebraic Bethe ansatz for open XXX model with triangular boundary matrices. *Lett. Math. Phys.* **2013**, *103*, 493–506. [CrossRef]
45. Belliard, S.; Crampé, N. Heisenberg XXX model with general boundaries: Eigenvectors from algebraic Bethe ansatz. *SIGMA Symm. Integr. Geom. Methods Appl.* **2013**, *9*, 072. [CrossRef]
46. Pimenta, R.A.; Lima-Santos, A. Algebraic Bethe ansatz for the six vertex model with upper triangular K-matrices. *J. Phys. A* **2013**, *46*, 455002. [CrossRef]
47. Belliard, S. Modified algebraic Bethe ansatz for XXZ chain on the segment—I: Triangular cases. *Nuclear Phys. B* **2015**, *892*, 1–20. [CrossRef]
48. Belliard, S.; Pimenta, R.A. Modified algebraic Bethe ansatz for XXZ chain on the segment—II—General cases. *Nuclear Phys. B* **2015**, *894*, 527–552. [CrossRef]
49. Avan, J.; Belliard, S.; Grosjean, N.; Pimenta, R.A. Modified algebraic Bethe ansatz for XXZ chain on the segment—III—Proof. *Nuclear Phys. B* **2015**, *899*, 229–246. [CrossRef]
50. Gainutdinov, A.M.; Nepomechie, R.I. Algebraic Bethe ansatz for the quantum group invariant open XXZ chain at roots of unity. *Nuclear Phys. B* **2016**, *909*, 796–839. [CrossRef]
51. Zhang, X.; Li, Y.-Y.; Cao, J.; Yang, W.-L.; Shi, K.; Wang, Y. Bethe states of the XXZ spin-$\frac{1}{2}$ chain with arbitrary boundary fields. *Nuclear Phys. B* **2015**, *893*, 70–88. [CrossRef]
52. Cirilo António, N.; Manojlović, N.; Salom, I. Algebraic Bethe ansatz for the XXX chain with triangular boundaries and Gaudin model. *Nuclear Phys. B* **2014**, *889*, 87–108. [CrossRef]
53. Manojlović, N.; Salom, I. Algebraic Bethe ansatz for the XXZ Heisenberg spin chain with triangular boundaries and the corresponding Gaudin model. *Nuclear Phys. B* **2017**, *923*, 73–106. [CrossRef]
54. Lukyanenko, I.; Isaac, P.S.; Links, J. On the boundaries of quantum integrability for the spin-1/2 Richardson-Gaudin system. *Nuclear Phys. B* **2014**, *886*, 364–398. [CrossRef]
55. Cirilo António, N.; Manojlović, N.; Ragoucy, E.; Salom, I. Algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with boundary. *Nuclear Phys. B* **2015**, *893*, 305–331. [CrossRef]
56. Hao, K.; Cao, J.; Yang, T.; Yang, W.-L. Exact solution of the XXX Gaudin model with the generic open boundaries. *Ann. Phys.* **2015**, *354*, 401–408. [CrossRef]
57. Manojlović, N.; Nagy, Z.; Salom, I. Derivation of the trigonometric Gaudin Hamiltonians. In Proceedings of the 8th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics, Belgrade, Serbia, 24–31 August 2014; Institute of Physics: Belgrade, Serbia, 2015; pp. 127–135.
58. De Vega, H.J.; González Ruiz, A. Boundary K-matrices for the XYZ, XXZ, XXX spin chains. *J. Phys. A Math. Gen.* **1994**, *27*, 6129–6137. [CrossRef]
59. Ghoshal, S.; Zamolodchikov, A.B. Boundary S-matrix and boundary state in two-dimensional integrable quantum field theory. *Int. J. Modern Phys. A* **1994**, *9*, 3841–3885; Errata in **1994**, *9*, 4353. [CrossRef]
60. Salom, I.; Manojlović, N.; Cirilo António, N. Generalized $s\ell(2)$ Gaudin algebra and corresponding Knizhnik-Zamolodchikov equation. *Nuclear Phys. B* **2019**, *939*, 358–371. [CrossRef]