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Behavior of Non-Oscillatory Solutions of Fourth-Order Neutral Differential Equations

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Abstract: In this paper, we deal with the asymptotics and oscillation of the solutions of fourth-order neutral differential equations of the form $(r(t)(z'''(t))^\alpha)' + q(t)x^\alpha(g(t)) = 0$, where $z(t) := x(t) + p(t)x(\delta(t))$. By using a generalized Riccati transformation, we study asymptotic behavior and derive some new oscillation criteria. Our results extend and improve some well-known results which were published recently in the literature. Symmetry ideas are often invisible in these studies, but they help us decide the right way to study them, and to show us the correct direction for future developments. An example is given to illustrate the importance of our results.

Keywords: fourth-order differential equations; neutral delay; oscillation

1. Introduction

To date, the oscillatory behavior of the solutions to differential equations has been discussed in many papers. Among them, there are many papers about the oscillation of the solutions to functional differential equations. In a related field, the asymptotic behavior of the solutions to delay and neutral delay differential equations were discussed in many works, and there have been very fruitful achievements see [1–28].

In this paper, our focus is on improving the criteria of oscillation of fourth-order neutral equations

$$(r(t)(z'''(t))^\alpha)' + q(t)x^\alpha(g(t)) = 0, \tag{1}$$

where $t \geq t_0$ and $z(t) := x(t) + p(t)x(\delta(t))$. In this work, we assume:

Hypothesis 1. $\alpha \in \{a/b : a, b \in \mathbb{Z}^+\}$, $r \in C^1([t_0, \infty))$, $r(t) > 0$, $r'(t) \geq 0$ and

$$\theta(t_0) := \int_{t_0}^{\infty} r^{-1/\alpha}(s) ds < \infty; \tag{2}$$

Hypothesis 2. $p, q \in C([t_0, \infty))$, $q(t) > 0$, $0 \leq p(t) < p_0 < \infty$,

Hypothesis 3. $\delta \in C^1([t_0, \infty))$, $g \in C([t_0, \infty))$, $\delta'(t) > 0$, $\delta(t) \leq t$ and $\lim_{t \rightarrow \infty} \delta(t) = \lim_{t \rightarrow \infty} g(t) = \infty$.

By a solution of (1) we mean a function $x \in C^3([t_x, \infty))$, $t_x \geq t_0$, which has the property $(r(t)(z'''(t))^\alpha) \in C^1([t_x, \infty))$, and satisfies (1) on $[t_x, \infty)$. We consider only those solutions x of (1)

which satisfy $\sup\{|x(t)| : t \geq T\} > 0$, for all $T \geq t_x$. A solution x of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.

Delay differential equations are often studied in one of two cases

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s) ds = \infty \quad (3)$$

or (2) which it is said to be in canonical or noncanonical. For canonical, Moaaz et al. [21] proved that (1) is oscillatory if

$$\liminf_{t \rightarrow \infty} \int_{\delta^{-1}(\eta(t))}^t \left(\frac{(\delta^{-1}(\eta(s)))^{n-1}}{r^{1/\alpha}(\delta^{-1}(\eta(s)))} \right)^\alpha q(s) P_n^\alpha(g(s)) ds > \frac{((n-1)!)^\alpha}{e} \quad (4)$$

and

$$\liminf_{t \rightarrow \infty} \int_{\delta^{-1}(\zeta(t))}^t \delta^{-1}(\zeta(s)) R_{n-3}(s) ds > \frac{1}{e}, \quad (5)$$

where $P_n(t) = 1/p(\delta^{-1}(t)) \left(1 - \left((\delta^{-1}(\delta^{-1}(t)))^{n-1} / (\delta^{-1}(t))^{k-1} p(\delta^{-1}(\delta^{-1}(t))) \right) \right)$.

In [25], the authors proved that (1) is oscillatory if the first-order differential equation

$$y'(t) + q(t) (1 - p(\delta(t)))^\alpha \left(\frac{\lambda \delta^{n-1}(t)}{(n-1)! r^{1/\alpha}(\delta(t))} \right)^\alpha y(\delta(t)) = 0,$$

is oscillatory, also

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left(q(s) (1 - p(\delta(s)))^\alpha \left(\frac{\lambda_1 \delta^{n-2}(s) \delta(s)}{(n-2)!} \right)^\alpha - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1} \delta(s) r^{1/\alpha}(s)} \right) ds = \infty$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left(q(s) \left(\frac{\delta^{n-2}(s) \delta(s)}{(n-2)!} \right)^\alpha - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1} \delta(s) r^{1/\alpha}(s)} \right) ds = \infty.$$

Now, we state some lemmas that will be useful in establishing our main results:

Lemma 1 ([18]). *If the function x satisfies $x^{(i)}(t) > 0$, $i = 0, 1, \dots, n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 2 ([6] (Lemma 2.2.3)). *Let $x \in C^n([t_0, \infty), (0, \infty))$. Assume that $x^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$ and that there exists a $t_1 \geq t_0$ such that $x^{(n-1)}(t) x^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t \rightarrow \infty} x(t) \neq 0$, then for every $\mu \in (0, 1)$ there exists $t_\mu \geq t_1$ such that*

$$x(t) \geq \frac{\mu}{(n-1)!} t^{n-1} |x^{(n-1)}(t)| \text{ for } t \geq t_\mu.$$

Lemma 3 ([10]). *Let β be a ratio of two odd numbers, $C > 0$ and D are constants. Then*

$$Dx - Cx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{D^{\beta+1}}{C^\beta}.$$

In this work, we obtain some new oscillation criteria for (1). The paper is organized as follows. Firstly, we study the behavior of non-oscillatory solutions of (1) and we obtain the sufficient conditions which guarantee that every non-oscillatory solution of (1) tends to zero. Secondly, we will use the

Riccati transformation technique to give some conditions for the oscillation of (1). Finally, an example is provided to illustrate the main results.

2. The Behavior of Non-Oscillatory Solutions

In this section, we study the behavior of non-oscillatory solutions of (1) when $p_0 \in (0, 1)$. We use an approach that leads to only three independent conditions, but we obtain sufficient conditions which guarantee that every non-oscillatory solution of (1) tends to zero.

Definition 1. A solution x of (1) is said to be non-oscillatory if it is positive or negative; otherwise, it is said to be oscillatory.

Lemma 4. Assume that x is an eventually positive solution of (1). Then, $r(t)(z'''(t))^\alpha$ is non-increasing. Moreover, we have the following cases:

$$\begin{aligned} (S_1) : & \quad z'(t) > 0, z''(t) > 0, z'''(t) > 0 \text{ and } z^{(4)}(t) < 0; \\ (S_2) : & \quad z'(t) > 0, z''(t) < 0, z'''(t) > 0 \text{ and } z^{(4)}(t) < 0; \\ (S_3) : & \quad z'(t) > 0, z''(t) > 0 \text{ and } z'''(t) < 0; \\ (S_4) : & \quad z'(t) < 0, z''(t) > 0 \text{ and } z'''(t) < 0. \end{aligned}$$

Lemma 5. Let x be a positive solution of (1) with property (S_1) or (S_2) . Then the equation

$$w'(t) + (1 - p_0)^\alpha \frac{q(t)}{r(g(t))} \left(\frac{\mu}{6} g^3(t)\right)^\alpha w(g(t)) = 0, \quad (6)$$

has a non-oscillatory solution.

Proof. Suppose the x is a positive solution of (1) with property (S_1) or (S_2) . Then, we have that

$$z'(t) > 0, z'''(t) > 0 \text{ and } z^{(4)}(t) < 0.$$

Thus, from Lemma 2, we obtain

$$z(t) \geq \frac{\mu}{6} t^3 z'''(t). \quad (7)$$

From definition of z , we see that $x(t) \geq (1 - p_0)z(t)$, which with (1) gives

$$\left(r(t)(z'''(t))^\alpha\right)' + (1 - p_0)^\alpha q(t) z^\alpha(g(t)) \leq 0. \quad (8)$$

Hence, from (7), if we set $w := r(z''')^\alpha > 0$, then the differential inequality

$$w'(t) + (1 - p_0)^\alpha \frac{q(t)}{r(g(t))} \left(\frac{\mu}{6} g^3(t)\right)^\alpha w(g(t)) \leq 0.$$

From [4] (Corollary 1), we have that (6) also has a positive solution, and this completes the proof. \square

Lemma 6. Let x be a positive solution of (1) with property (S_3) . Then the equation

$$\left(r(t)(\omega'(t))^\alpha\right)' + (1 - p_0)^\alpha q(t) \left(\frac{\mu}{2} g^2(t)\right)^\alpha \omega^\alpha(t) = 0, \quad (9)$$

has a non-oscillatory solution.

Proof. Suppose the x is a positive solution of (1) with property (S_3) . Using Lemma 2, we obtain

$$z(t) \geq \frac{\mu}{2} t^2 z''(t). \quad (10)$$

As in the proof of Lemma 6, we can obtain that (8). Next, if we set $G := r(z''' / z'')^\alpha < 0$, then we get

$$G'(t) \leq -(1 - p_0)^\alpha q(t) \frac{z^\alpha(g(t))}{(z''(t))^\alpha} - \alpha r^{-1/\alpha}(t) G^{1+1/\alpha}(t).$$

Hence, from the fact that $z''' < 0$ and (10), we find

$$G'(t) + (1 - p_0)^\alpha q(t) \left(\frac{\mu}{2} g^2(t)\right)^\alpha + \alpha r^{-1/\alpha}(t) G^{1+1/\alpha}(t) \leq 0. \quad (11)$$

Therefore, there exists a function $G \in C^1([t_0, \infty), \mathbb{R})$ such that (11) holds. It follows from [1] that (9) has a non-oscillatory solution, and this completes the proof. \square

Theorem 1. Assume that the differential equations (6) and (9) are oscillatory. Then every non-oscillatory solution of (1) tends to zero if

$$\int_{t_0}^{\infty} \left(\frac{1}{r(u)} \int_{t_0}^u q(s) ds \right)^{1/\alpha} du = \infty. \quad (12)$$

Proof. Assume the contrary that x is a positive solution of (1) with property $\lim_{t \rightarrow \infty} x(t) \neq 0$. From Lemma 4, we have cases $(S_1) - (S_4)$. Using Lemmas 5 and 6 with the fact that the differential Equations (6) and (9) are oscillatory, we conclude that x satisfies case (S_4) . Then, since z is a positive decreasing function, we get that $\lim_{t \rightarrow \infty} z(t) = c \geq 0$. Suppose the contrary that $c > 0$. Thus, for all $\varepsilon > 0$ and t enough large, we have $c \leq z(t) < c + \varepsilon$. Choosing $\varepsilon < (1 - p_0)(c/p_0)$, we obtain

$$\begin{aligned} x(t) &= z(t) - p_0(t)x(\delta(t)) > c - p_0z(\delta(t)) \\ &> L(\gamma + \varepsilon) > Lz(t), \end{aligned} \quad (13)$$

where $L = (c - p_0(c + \varepsilon)) / (c + \varepsilon) > 0$. Hence, from (1), we have

$$\begin{aligned} \left(r(t) (z'''(t))^\alpha \right)' &= -q(t) x^\alpha(g(t)) \leq -L^\alpha q(t) z^\alpha(g(t)) \\ &\leq -L^\alpha \varepsilon^\alpha q(t). \end{aligned}$$

Integrating this inequality from t_1 to t , we get

$$z'''(t) \leq -L\varepsilon \left(\frac{1}{r(t)} \int_{t_1}^t q(s) ds \right)^{1/\alpha}.$$

By integrating from t_1 to t , we obtain

$$z''(t) \leq z''(t_1) - L\varepsilon \int_{t_1}^t \left(\frac{1}{r(u)} \int_{t_1}^u q(s) ds \right)^{1/\alpha} du.$$

Letting $t \rightarrow \infty$ and taking into account (12), we get that $\lim_{t \rightarrow \infty} z''(t) = -\infty$. This contradicts the fact that $z''(t) > 0$. Therefore, $c = 0$; moreover the fact $x(t) \leq z(t)$ implies $\lim_{t \rightarrow \infty} x(t) = 0$, a contradiction. This completes the proof. \square

Corollary 1. Assume that (12) holds. Then every non-oscillatory solution of (1) tends to zero if $\int_{t_0}^{\infty} q(s) ds = \infty$,

$$\liminf_{t \rightarrow \infty} \int_{g(t)}^t \frac{q(s) g^{3\alpha}(s)}{r(g(s))} ds > \frac{6^\alpha}{e\mu^\alpha (1 - p_0)^\alpha} \quad (14)$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left((1 - p_0)^\alpha \theta^\alpha(s) q(s) \left(\frac{\mu}{2} g^2(s) \right)^\alpha - \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} \frac{1}{r^{1/\alpha}(s) \theta(s)} \right) ds > 0. \quad (15)$$

Proof. It is well-known from [3] (Theorem 2) and [2] (Corollary 2.8) that (14) and (15) imply oscillation of (6) and (9), respectively. \square

Lemma 7. Assume that x is an eventually positive solution of (1). If z is an increasing and

$$p(t) \sum_{k=0}^{(n-1)/2} \prod_{r=1}^{2k} p(\delta^r(t)) < 1, \quad (16)$$

then

$$x(t) \geq (1 - \widehat{p}(t)) z(t), \quad (17)$$

for any odd positive integer n , where

$$\widehat{p}(t) := p(t) \sum_{k=0}^{(n-1)/2} \prod_{r=1}^{2k} p(\delta^r(t)).$$

Proof. From the definition of $z(t)$, we obtain

$$\begin{aligned} x(t) &= z(t) - p(t) x(\delta(t)) \\ &= z(t) - p(t) z(\delta(t)) + p(t) p(\delta(t)) x(\delta^2(t)) \\ &= z(t) - p(t) z(\delta(t)) - p(t) p(\delta(t)) p(\delta^2(t)) z(\delta^3(t)) + p(t) p(\delta(t)) p(\delta^2(t)) p(\delta^3(t)) x(\delta^4(t)) \\ &\geq z(t) - \sum_{k=0}^{(n-1)/2} \prod_{r=0}^{2k} p(\delta^r(t)) z(\delta^{2k+1}(t)) + \prod_{r=0}^n p(\delta^r(t)) x(\delta^{n+1}(t)) \\ &\geq z(t) - \sum_{k=0}^{(n-1)/2} \prod_{r=0}^{2k} p(\delta^r(t)) z(\delta^{2k+1}(t)), \end{aligned} \quad (18)$$

for $t \geq t_2$, where $t_2 \geq t_0$ sufficiently large, and any odd positive integer n . Since $\delta^{2k+1}(t) \leq \delta^{2k}(t)$, we find

$$z(\delta^i(t)) \leq z(t), \quad \text{for } i = 0, 1, \dots, n,$$

which with (18) gives

$$x(t) \geq \left(1 - \sum_{k=0}^{(n-1)/2} \prod_{r=0}^{2k} p(\delta^r(t)) \right) z(t).$$

The proof is complete. \square

By replacing $\widehat{p}(t)$ instead of p in the previous results, we can get the following corollary:

Corollary 2. Assume that (12) holds. Then every non-oscillatory solution of (1) tends to zero if $\int_{t_0}^{\infty} q(s) ds = \infty$,

$$\liminf_{t \rightarrow \infty} \int_{g(t)}^t (1 - \widehat{p}(g(s)))^\alpha \frac{q(s) g^{3\alpha}(s)}{r(g(s))} ds > \frac{6^\alpha}{\mu^\alpha e}$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left((1 - \widehat{p}(g(s)))^\alpha \theta^\alpha(s) q(s) \left(\frac{\mu}{2} g^2(s) \right)^\alpha - \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} \frac{1}{r^{1/\alpha}(s) \theta(s)} \right) ds > 0.$$

3. New Oscillation Criteria

For convenience, we denote:

$$P_k(t) : = \frac{1}{p(\delta^{-1}(t))} \left(1 - \frac{\left((\delta^{-1}(t))^{k-1} p(\delta^{-1}(\delta^{-1}(t))) \right)^{-1}}{(\delta^{-1}(\delta^{-1}(t)))^{1-k}} \right), \text{ for } k = 2, n,$$

$$\Theta(s) : = \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{r^\alpha(\delta^{-1}(g(s)))}{r^{1/\alpha}(s)\theta(s)\left((\delta^{-1}(g(s)))'\right)^\alpha}, \theta(t) = \int_{t_0}^\infty r^{-1/\alpha}(s) ds$$

and

$$\tilde{\Theta}(s) = \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{2^\alpha r^\alpha(\delta^{-1}(g(s)))}{r^{1/\alpha}(s)\theta(s)\mu_1\left((\delta^{-1}(g(s)))'(\delta^{-1}(g(s)))^2\right)^\alpha}.$$

Also, we define the Riccati substitutions

$$\omega(t) := \frac{r(t)(z'''(t))^\alpha}{(z''(\delta^{-1}(g(t))))^\alpha} \tag{19}$$

and

$$\xi(t) := \frac{r(t)(z'''(t))^\alpha}{z^\alpha(\delta^{-1}(g(t)))}. \tag{20}$$

At studying the asymptotic behavior of positive solutions, there are three Cases $(S_1) - (S_4)$. We recall an existing criterion for Cases (S_1) and (S_2) in the following lemma:

Lemma 8 ([21]). Assume that x be an eventually positive solution of (1). If (4) and (5) hold, then z is neither satisfied (S_1) nor (S_2) .

Lemma 9. Assume that x be an eventually positive solution of (1) and

$$\left(\delta^{-1}(\delta^{-1}(t))\right)^{n-1} < \left(\delta^{-1}(t)\right)^{n-1} p(\delta^{-1}(\delta^{-1}(t))). \tag{21}$$

Then

$$x(t) \geq \frac{z(\delta^{-1}(t))}{p(\delta^{-1}(t))} - \frac{1}{p(\delta^{-1}(t))} \frac{z(\delta^{-1}(\delta^{-1}(t)))}{p(\delta^{-1}(\delta^{-1}(t)))}. \tag{22}$$

Proof. Assume that x be an eventually positive solution of (1) on $[t_0, \infty)$. From the definition of $z(t)$, we see that

$$p(t)x(\delta(t)) = z(t) - x(t)$$

and so

$$p(\delta^{-1}(t))x(t) = z(\delta^{-1}(t)) - z(\delta^{-1}(t)).$$

Repeating the same process, we obtain

$$x(t) = \frac{1}{p(\delta^{-1}(t))} \left(z(\delta^{-1}(t)) - \left(\frac{z(\delta^{-1}(\delta^{-1}(t)))}{p(\delta^{-1}(\delta^{-1}(t)))} - \frac{x(\delta^{-1}(\delta^{-1}(t)))}{p(\delta^{-1}(\delta^{-1}(t)))} \right) \right),$$

which yields

$$x(t) \geq \frac{z(\delta^{-1}(t))}{p(\delta^{-1}(t))} - \frac{1}{p(\delta^{-1}(t))} \frac{z(\delta^{-1}(\delta^{-1}(t)))}{p(\delta^{-1}(\delta^{-1}(t)))}.$$

Thus, (22) holds. This completes the proof. \square

Lemma 10. Assume that x is an eventually positive solution of (1) and

$$\left(r(t) (z'''(t))^\alpha \right)' \leq -q(t) P_1^\alpha(g(t)) z^\alpha(\delta^{-1}(g(t))), \text{ if } z \text{ satisfies } (S_3) \quad (23)$$

and

$$\left(r(t) (z'''(t))^\alpha \right)' + q(t) P_2^\alpha(g(t)) z^\alpha(\delta^{-1}(g(t))) \leq 0, \text{ if } z \text{ satisfies } (S_4). \quad (24)$$

Proof. Let case (S_3) holds. From Lemma 1, we have $z(t) \geq \frac{t}{2} z'(t)$ and hence the function $t^{-1}z(t)$ is nonincreasing, which with the fact that $\delta(t) \leq t$ gives

$$\left(\delta^{-1}(t) \right) z \left(\delta^{-1} \left(\delta^{-1}(t) \right) \right) \leq \left(\delta^{-1} \left(\delta^{-1}(t) \right) \right) z \left(\delta^{-1}(t) \right). \quad (25)$$

Combining (22) and (25), we see that

$$\begin{aligned} x(t) &\geq \frac{1}{p(\delta^{-1}(t))} \left(1 - \frac{(\delta^{-1}(\delta^{-1}(t)))}{\delta^{-1}(t) p(\delta^{-1}(\delta^{-1}(t)))} \right) z(\delta^{-1}(t)) \\ &= P_2(t) z(\delta^{-1}(t)). \end{aligned} \quad (26)$$

From (1) and (26), we obtain

$$\left(r(t) (z'''(t))^\alpha \right)' \leq -q(t) P_n^\alpha(g(t)) z^\alpha(\delta^{-1}(g(t))). \quad (27)$$

Thus, (23) holds. Assume that Case (S_4) holds. Since $\delta^{-1}(t) \leq \delta^{-1}(\delta^{-1}(t))$. From (22), we see that

$$\begin{aligned} x(t) &\geq \frac{1}{p(\delta^{-1}(t))} \left(1 - \frac{1}{p(\delta^{-1}(\delta^{-1}(t)))} \right) z(\delta^{-1}(t)) \\ &= P_2(t) z(\delta^{-1}(t)), \end{aligned} \quad (28)$$

which with (1) yields

$$\left(r(t) (z'''(t))^\alpha \right)' + q(t) P_2^\alpha(g(t)) z^\alpha(\delta^{-1}(g(t))) \leq 0.$$

Thus, (24) holds. This completes the proof. \square

Lemma 11. Assume that x be an eventually positive solution of (1) and (S_3) holds. If we have the function $\omega \in C^1[t, \infty)$ defined as (19), then

$$\omega'(t) \leq -q(t) P_1^\alpha(g(t)) \left(\frac{\lambda}{2} (\delta^{-1}(g(t)))^2 \right)^\alpha - \alpha \frac{(\delta^{-1}(g(t)))'}{r^{1/\alpha}(t) r(\delta^{-1}(g(t)))} \omega^{\frac{\alpha+1}{\alpha}}(t), \quad (29)$$

for all $t > t_1$, where t_1 large enough.

Proof. Let x is an eventually positive solution of (1). From Lemma (2), we get

$$z(\delta^{-1}(g(t))) \geq \frac{\lambda}{2} (\delta^{-1}(g(t)))^2 z''(\delta^{-1}(g(t))), \quad (30)$$

for every $\lambda \in (0, 1)$ and all sufficiently large t . Recalling that $r(t) (z'''(t))^\alpha$ is decreasing, we get

$$r(\delta^{-1}(g(t))) \left(z'''(\delta^{-1}(g(t))) \right)^\alpha \geq r(t) (z'''(t))^\alpha.$$

This yields

$$\left(z'''(\delta^{-1}(g(t)))\right)^\alpha \geq \frac{r(t)}{r(\delta^{-1}(g(t)))} (z'''(t))^\alpha. \tag{31}$$

From the definition of $\omega(t)$, we see that $\omega(t) < 0$ for $t \geq t_1$. By differentiating, we find

$$\omega'(t) = \frac{(r(t)(z'''(t))^\alpha)'}{(z''(\delta^{-1}(g(t))))^\alpha} - \alpha \frac{r(t)(z'''(t))^\alpha z'''(\delta^{-1}(g(t))) (\delta^{-1}(g(t)))'}{(z''(\delta^{-1}(g(t))))^{\alpha+1}}.$$

From (19), (30) and (31), we get

$$\begin{aligned} \omega'(t) &\leq -q(t) P_1^\alpha(g(t)) \frac{z^\alpha(\delta^{-1}(g(t)))}{(z''(\delta^{-1}(g(t))))^\alpha} - \alpha \frac{r(t)(\delta^{-1}(g(t)))'}{r(\delta^{-1}(g(t)))} \frac{(z'''(t))^{\alpha+1}}{(z''(\delta^{-1}(g(t))))^{\alpha+1}} \\ &\leq -q(t) P_1^\alpha(g(t)) \left(\frac{\lambda}{2}(\delta^{-1}(g(t)))^2\right)^\alpha - \alpha \frac{(\delta^{-1}(g(t)))'}{r^{1/\alpha}(t)r(\delta^{-1}(g(t)))} \omega^{\frac{\alpha+1}{\alpha}}(t). \end{aligned}$$

The proof is complete. \square

Lemma 12. Assume that x be an eventually positive solution of (1) and (S_4) holds. If we have the function $\zeta \in C^1[t, \infty)$ defined as (20), then

$$\zeta'(t) \leq -q(t) P_2^\alpha(g(t)) - \alpha \frac{\mu_1(\delta^{-1}(g(t)))'(\delta^{-1}(g(t)))^2}{2r^{1/\alpha}(\delta^{-1}(g(t)))} \zeta^{\alpha+1}(t), \tag{32}$$

for all $t > t_1$, where t_1 large enough.

Proof. Let x is an eventually positive solution of (1). From the definition of $\zeta(t)$, we see that $\zeta(t) < 0$ for $t \geq t_1$. By differentiating, we find

$$\zeta'(t) \leq -q(t) P_2^\alpha(g(t)) - \alpha \frac{r(t)(z'''(t))^\alpha (\delta^{-1}(g(t)))' z'(\delta^{-1}(g(t)))}{z^{\alpha+1}(\delta^{-1}(g(t)))}. \tag{33}$$

From Lemma 2 and (31), we get

$$z'(\delta^{-1}(g(t))) \geq \frac{\mu_1}{2} (\delta^{-1}(g(t)))^2 \left(\frac{r(t)}{r(\delta^{-1}(g(t)))}\right)^{1/\alpha} z'''(t), \tag{34}$$

for all $\mu_1 \in (0, 1)$ and every sufficiently large t . Thus, by (20), (33) and (34), we get

$$\zeta'(t) \leq -q(t) P_2^\alpha(g(t)) - \alpha \frac{\mu_1(\delta^{-1}(g(t)))'(\delta^{-1}(g(t)))^2}{2r^{1/\alpha}(\delta^{-1}(g(t)))} \zeta^{\alpha+1}(t).$$

The proof is complete. \square

Theorem 2. Assume that (4) and (5) hold. If

$$\int_{t_0}^\infty \left(q(s) P_1^\alpha(g(s)) \left(\frac{\lambda}{2}(\delta^{-1}(g(s)))^2\right)^\alpha \theta^\alpha(s) ds - \Theta(s)\right) ds = \infty \tag{35}$$

and

$$\int_{t_0}^\infty \left(q(s) P_2^\alpha(g(s)) \theta^\alpha(s) ds - \tilde{\Theta}(s)\right) ds = \infty, \tag{36}$$

for some $\mu, \lambda \in (0, 1)$, then every solution of (1) is oscillatory.

Proof. Assume the contrary that x is a positive solution of (1). From Lemma 4, we have cases $(S_1) - (S_4)$. From Lemmas 8, z is neither satisfied (S_1) nor (S_2) . Suppose that case (S_3) holds. From Lemma 11, we get that (29) holds. Multiplying this inequality by $\theta^\alpha(t)$ and integrating the resulting inequality from t_1 to t , we get

$$\begin{aligned} & \theta^\alpha(t) \omega(t) - \theta^\alpha(t_1) \omega(t_1) + \alpha \int_{t_1}^t r^{\frac{-1}{\alpha}}(s) \theta^{\alpha-1}(s) \omega(s) ds \\ & \leq - \int_{t_1}^t q(s) P_1^\alpha(g(s)) \left(\frac{\lambda}{2} (\delta^{-1}(g(t)))^2 \right)^\alpha \theta^\alpha(s) ds \\ & \quad - \alpha \int_{t_1}^t \frac{\theta^\alpha(s) (\delta^{-1}(g(s)))' g'(s)}{r^{1/\alpha}(s) r (\delta^{-1}(g(s)))} \omega^{\frac{\alpha+1}{\alpha}}(s) ds. \end{aligned} \tag{37}$$

We set

$$D := r^{\frac{-1}{\alpha}}(s) \theta^{\alpha-1}(s), C := \frac{\theta^\alpha(s) (\delta^{-1}(g(s)))' g'(s)}{r^{1/\alpha}(s) r (\delta^{-1}(g(s)))}, y := -\omega(s).$$

Using Lemma 12, we find

$$\begin{aligned} & r^{\frac{-1}{\alpha}}(s) \theta^{\alpha-1}(s) \omega(s) - \frac{\theta^\alpha(s) (\delta^{-1}(g(s)))' g'(s)}{r^{1/\alpha}(s) r (\delta^{-1}(g(s)))} \omega^{\frac{\alpha+1}{\alpha}} \\ & \leq \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{r^\alpha (\delta^{-1}(g(s)))}{r^{1/\alpha}(s) \theta(s) ((\delta^{-1}(g(s)))')^\alpha}. \end{aligned}$$

From (37), for every $\lambda \in (0, 1)$, and all sufficiently large t , we obtain

$$\int_{t_1}^t \left(q(s) P_1^\alpha(g(s)) \left(\frac{\lambda}{2} (\delta^{-1}(g(t)))^2 \right)^\alpha \theta^\alpha(s) ds - \Theta(s) \right) ds \leq \theta^\alpha(t_1) g(t_1) + 1,$$

but this contradicts (35). The proof is complete. Let case (S_3) holds. Using Lemma 12, we have that (32) holds. Multiplying this inequality by $\theta^\alpha(t)$ and integrating the resulting inequality from t_1 to t , we get

$$\begin{aligned} & \theta^\alpha(t) \zeta(t) - \theta^\alpha(t_1) \zeta(t_1) + \alpha \int_{t_1}^t r^{\frac{-1}{\alpha}}(s) \theta^{\alpha-1}(s) \zeta(s) ds \\ & \leq - \int_{t_1}^t q(s) P_2^\alpha(g(s)) \theta^\alpha(s) ds - \alpha \int_{t_1}^t \frac{\mu_1 \theta^\alpha(s) (\delta^{-1}(g(s)))' g'(s) (\delta^{-1}(g(s)))^2}{2r^{1/\alpha}(s) r (\delta^{-1}(g(s)))} \zeta^{\frac{\alpha+1}{\alpha}}(s) ds. \end{aligned}$$

We set

$$D := r^{\frac{-1}{\alpha}}(s) \theta^{\alpha-1}(s), C := \frac{\mu_1 \theta^\alpha(s) (\delta^{-1}(g(s)))' g'(s) (\delta^{-1}(g(s)))^2}{2r^{1/\alpha}(s) r (\delta^{-1}(g(s)))}, y := -\zeta(s).$$

Applying Lemma 3, for every $\mu_1 \in (0, 1)$, we obtain

$$\begin{aligned} & r^{\frac{-1}{\alpha}}(s) \theta^{\alpha-1}(s) \zeta(s) - \frac{\mu_1 \theta^\alpha(s) (\delta^{-1}(g(s)))' g'(s) (\delta^{-1}(g(s)))^2}{2r^{1/\alpha}(s) r (\delta^{-1}(g(s)))} \zeta^{\frac{\alpha+1}{\alpha}} \\ & \leq \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{2^\alpha r^\alpha (\delta^{-1}(g(s)))}{r^{1/\alpha}(s) \theta(s) \mu_1 ((\delta^{-1}(g(s)))' (\delta^{-1}(g(s)))^2)^\alpha}, \end{aligned}$$

which imply that

$$\int_{t_1}^t \left(q(s) P_2^\alpha(g(s)) \theta^\alpha(s) ds - \tilde{\Theta}(s) \right) ds \leq \theta^\alpha(t_1) g(t_1) + 1,$$

but this contradicts (36). The proof is complete. \square

Example 1. Consider the equation

$$\left(t^2 \left(x(t) + 16x \left(\frac{t}{2} \right) \right)''' \right)' + q_0 x \left(\frac{t}{2} \right) = 0, \quad (38)$$

where $t \geq 1$, $q_0 > 0$. We note that $r(t) = t^2$, $p(t) = 16$, $\delta(t) = g(t) = 1/2t$ and $q(t) = q_0$. Moreover, we get

$$P_1(t) = \frac{7}{128}, P_2(t) = \frac{1}{32}, \theta(t) = \frac{1}{t}, \Theta(t) = \frac{t}{4}$$

and

$$\tilde{\Theta}(t) = \frac{1}{2t}.$$

Thus, we find

$$\begin{aligned} & \int_{t_0}^{\infty} \left(q(s) P_1^\alpha(g(s)) \left(\frac{\lambda}{2} (\delta^{-1}(g(t)))^2 \right)^\alpha \theta^\alpha(s) ds - \Theta(s) \right) ds \\ &= \left(\frac{7q_0}{256} - \frac{1}{4} \right) \int_{t_0}^{\infty} s ds \\ &= \infty \text{ if } q_0 > 9.14 \end{aligned}$$

and

$$\begin{aligned} & \int_{t_0}^{\infty} \left(q(s) P_2^\alpha(g(s)) \theta^\alpha(s) ds - \tilde{\Theta}(s) \right) ds \\ &= \left(\frac{q_0}{32} - \frac{1}{2} \right) \int_{t_0}^{\infty} \frac{1}{s} ds \\ &= \infty \text{ if } q_0 > 16. \end{aligned}$$

Therefore, applying Theorem 2, we have that every solution of (38) is oscillatory if $q_0 > 16$.

Example 2. Consider the equation

$$\left(t^2 \left(x(t) + 4x \left(\frac{t}{2} \right) \right)''' \right)' + q_0 x \left(\frac{t}{2} \right) = 0, \quad (39)$$

where $t \geq 1$, $q_0 > 0$. We note that $r(t) = t^2$, $p(t) = 4$, $\delta(t) = g(t) = 1/2t$ and $q(t) = q_0$. Thus, it's easy to see that (4) and (5) are satisfied. Moreover, we have

$$P_1(t) = \frac{1}{8}, P_2(t) = \frac{3}{16}.$$

Hence, Condition (35) and (36) become

$$q_0 > 4 \quad (40)$$

and

$$q_0 > \frac{8}{3}, \quad (41)$$

respectively. It's easy to see that (40) implies (41). Therefore, by Theorem 2, we conclude that (39) is oscillatory if (40) holds.

Remark 1. The results of this paper can be extended to the more general equation of the form

$$\left(r(t) \left(z^{(n-1)}(t) \right)^\alpha \right)' + q(t) x^\beta(g(t)) = 0.$$

The statement and the formulation of the results are left to the interested reader.

Remark 2. One can easily see that the results obtained in [25] cannot be applied to Theorem 2, so our results are new.

4. Conclusions

This paper is concerned with oscillatory behavior of a class of fourth-order delay differential equations. Using a Riccati transformation, a new asymptotic criterion for (1) is presented. In future work, we will aim to present a new comparison theorem that compares the higher-order Equation (1) with first-order equations. There are numerous results concerning the oscillation criteria of first order equations, which include various forms of criteria such as Hille/Nehari, Philos, etc. This allows us to obtain various criteria for the oscillation of (1). Further, we can try to get some oscillation criteria of (1) if $z(t) := x(t) - p(t)x(\delta(t))$.

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