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Optimal System and Invariant Solutions of a New AKNS Equation with Time-Dependent Coefficients

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Abstract: The Lie point symmetries are reported by performing the Lie symmetry analysis to the Ablowitz-Kaup-Newell-Suger (AKNS) equation with time-dependent coefficients. In addition, the optimal system of one-dimensional subalgebras is constructed. Based on this optimal system, several categories of similarity reduction and some new invariant solutions for the equation are obtained, which include power series solutions and travelling and non-traveling wave solutions.

Keywords: AKNS equation with time-dependent coefficients; Lie symmetry analysis; Optimal system; Invariant solutions

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1. Introduction

The term nonlinear partial differential equation (NLPDE) is broadly utilized as a model in order to represent actual phenomena that occur many science areas, particularly in plasma physics, optical fields, and fluid mechanics. It is well known that many physical phenomena are described by NPDEs with variable coefficients in light of the fact that the vast majority of genuine nonlinear physical conditions have variable coefficients. On the one hand, many types of exact solutions have also been constructed to explain complex physical phenomena, such as solitary wave solutions [1], doubled Wronskian solutions [2], multiple rogue wave solutions [3], and localized excitation solutions [4]; on the other hand, many powerful methods have been developed to construct solutions of NLPDEs, such as the Hirota method [5–7], the generalized Darboux transformation [8–10], the extended tanh method [11,12], the generalized Jacobi elliptic functions technique [13], numerical method [14], and the Lie group method [15–17].

As well as we know, Lie symmetry analysis is a powerful and prolific method for constructing exact solutions for NLPDEs with constant variable [18–20]. Recently, the Lie symmetry analysis is extended to find exact solutions of fractional and variable coefficient NLPDEs, such as Time-Fractional Boussinesq-Burgers [21], Gardner equations [22], coupled short pulse equation [23] and so on [24–26].

Recently, Zhang et al. [27] studied the multi-soliton solutions of the following Ablowitz- Kaup-Newell-Suger (AKNS) equation

$$\begin{aligned} q_t &= \alpha_3(t)(q_{xxx} - 6qrq_x) + \alpha_2(t)(-q_{xx} + 2q^2r) + \alpha_1(t)q_x - \alpha_0(t)q, \\ r_t &= \alpha_3(t)(r_{xxx} - 6qrr_x) + \alpha_2(t)(r_{xx} - 2r^2q) + \alpha_1(t)r_x + \alpha_0(t)r, \end{aligned} \tag{1}$$

which is a particular example at $m = 3$ of the generalized AKNS hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{i=0}^{m} \alpha_i(t) L^i \begin{pmatrix} -q \\ r \end{pmatrix}, (m = 1, 2, \ldots),$$

where the recursive operator is being utilized, as follows

$$L = \sigma\,\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r,q), \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \partial = \frac{\partial}{\partial x}, \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} dx - \int_{x}^{\infty} dx\right).$$

We note that system (1) includes a lot of famous NLPDEs as its special cases. For example, if $\alpha_0(t) = \alpha_1(t) = \alpha_2(t) = 0$, $\alpha_3(t) = -1$ and $r = -1$, then system (1) is the KdV equation

$$q_t + q_{xxx} + 6qq_x = 0.$$

If $\alpha_0(t) = \alpha_1(t) = \alpha_2(t) = 0$, $\alpha_3(t) = -1$ and $r = -q$, then system (1) is the mKdV equation

$$q_t + q_{xxx} + 6q^2q_x = 0.$$

If $\alpha_0(t) = \alpha_1(t) = 0$, $\alpha_2(t) = i$, $\alpha_3(t) = -1$ and $r = -q$, then system (1) is the mKdV-NLS equation

$$q_t + q_{xxx} + 6q^2q_x + i(q_{xx} + 2q^3) = 0.$$

If $\alpha_0(t) = \alpha_1(t) = \alpha_3(t) = 0$ and $\alpha_2(t) = i$, then system (1) is the second order AKNS coupled system [28,29]

$$iq_t = q_{xx} - 2q^2r,$$
$$ir_t = -r_{xx} + 2r^2q.$$

To our knowledge, the AKNS equation with time-dependent coefficients has not been studied via Lie symmetry analysis. The aim of the present paper is to construct optimal system and invariant solutions to (1) based on Lie point symmetries. The rest of this paper is organized, as follows. In Section 2, the Lie point symmetries of (1) are obtained by utilizing Lie symmetry analysis. In Section 3, we construct the optimal system of one-dimensional subalgebras of Lie algebra spanned by $V_1 - V_3$. In Section 4, several types of similarity reduction and some invariant solutions are discussed on the optimal system. In Section 5, we conclude this paper.

2. Symmetry Analysis

In this section, our aim is to obtain the symmetry algebra of the AKNS Equation (1) while using the Lie symmetry analysis [15–17]. Suppose that the associated vector field of system (1) is as follows:

$$V = \xi(t,x,q,r)\frac{\partial}{\partial x} + \eta(t,x,q,r)\frac{\partial}{\partial t} + Q(t,x,q,r)\frac{\partial}{\partial q} + R(t,x,q,r)\frac{\partial}{\partial r}, \quad (2)$$

where $\xi(t,x,q,r)$, $\eta(t,x,q,r)$, $Q(t,x,q,r)$, and $R(t,x,q,r)$ are unknown functions that need to be determined.

If vector field (2) generates a symmetry of system of Equation (1), then V must satisfy the symmetry condition

$$pr^{(3)}V(\Delta_1)\big|_{\Delta_1} = 0,$$
$$pr^{(3)}V(\Delta_2)\big|_{\Delta_2} = 0,$$

where $\Delta_1 = \alpha_3(t)(q_{xxx} - 6qrq_x) + \alpha_2(t)(-q_{xx} + 2q^2r) + \alpha_1(t)q_x - \alpha_0(t)q - q_t$, $\Delta_2 = \alpha_3(t)(r_{xxx} - 6qrr_x) + \alpha_2(t)(r_{xx} - 2r^2q) + \alpha_1(t)r_x + \alpha_0(t)r - r_t$.

The infinitesimals ξ, η, Q and R must satisfy the following invariant conditions

$$\begin{aligned} Q^t = &\ \alpha_3{}'(t)\eta(q_{xxx} - 6qrq_x) + \alpha_3(t)(Q^{xxx} - 6Qrq_x - 6qRq_x - 6qrQ^x) \\ &+\alpha_2{}'(t)\eta(-q_{xx} + 2q^2r) + \alpha_2(t)(-Q^{xx} + 4qQr + 2q^2R) \\ &+\alpha_1{}'(t)\eta q_x + \alpha_1(t)Q^x - \alpha_0{}'(t)\eta q - \alpha_0(t)Q, \\ R^t = &\ \alpha_3{}'(t)\eta(r_{xxx} - 6qrr_x) + \alpha_3(t)(R^{xxx} - 6Qrr_x - 6qRr_x - 6qrR^x) \\ &+\alpha_2{}'(t)\eta(r_{xx} - 2r^2q) + \alpha_2(t)(R^{xx} - 4rRq - 2r^2Q) \\ &+\alpha_1{}'(t)\eta r_x + \alpha_1(t)R^x + \alpha_0{}'(t)\eta r + \alpha_0(t)R, \end{aligned} \tag{3}$$

where

$$\begin{aligned} R^t &= D_t(R - \xi r_x - \eta r_t) + \xi r_{xt} + \xi r_{tt}, \\ R^x &= D_x(R - \xi r_x - \eta r_t) + \xi r_{xx} + \xi r_{xt}, \\ R^{xx} &= D_{xx}(R - \xi r_x - \eta r_t) + \xi r_{xxx} + \xi r_{xxt}, \\ R^{xxx} &= D_{xxx}(R - \xi r_x - \eta r_t) + \xi r_{xxxx} + \xi r_{xxxt}, \\ Q^t &= D_t(Q - \xi q_x - \eta q_t) + \xi q_{xt} + \xi q_{tt}, \\ Q^x &= D_x(Q - \xi q_x - \eta q_t) + \xi q_{xx} + \xi q_{xt}, \\ Q^{xx} &= D_{xx}(Q - \xi q_x - \eta q_t) + \xi q_{xxx} + \xi q_{xxt}, \\ Q^{xxx} &= D_{xxx}(Q - \xi q_x - \eta q_t) + \xi q_{xxxx} + \xi q_{xxxt}. \end{aligned} \tag{4}$$

Substituting (4) into system (3), we obtain a large number of determining equations

$$\begin{gathered} \xi_t = 0, \xi_{xx} = 0, Q_r = 0, Q_{qq} = 0, R_q = 0, R_{rr} = 0, \\ \alpha_{1t}\eta + \alpha_1\eta_t - \alpha_1\xi_x = 0, \alpha_{2t}\eta + \alpha_2\eta_t - \alpha_2\xi_x = 0, \alpha_{3t}\eta + \alpha_3\eta_t - 3\alpha_3\xi_x = 0, \\ \alpha_3\xi_x qr - \alpha_3\eta_t qr - \alpha_{3t}\eta qr - \alpha_3 qR - \alpha_3 rQ = 0, \\ \alpha_2\eta_t q^2 r + \alpha_{2t}\eta q^2 r + \alpha_2 q^2 R - \alpha_2 Q_q q^2 r + 2\alpha_2 qrQ = 0, \\ \alpha_{0t}\eta q + \alpha_0\eta_t q - \alpha_0 qQ_q + \alpha_0 Q + Q_t = 0, \\ \alpha_{0t}\eta r + \alpha_0\eta_t r - \alpha_0 rR_r + \alpha_0 R - R_t = 0. \end{gathered} \tag{5}$$

Solving the system, one can get

$$\begin{gathered} \xi = c_1 x + c_2, \eta = \frac{1}{\alpha_3}\Big(3c_1\int \alpha_3 dt + c_3\Big), \\ Q = \Big(-\frac{3c_1\alpha_0}{\alpha_3}\int \alpha_3 dt - \frac{c_3\alpha_0}{\alpha_3} - c_1\Big)q,\ R = \Big(\frac{3c_1\alpha_0}{\alpha_3}\int \alpha_3 dt + \frac{c_3\alpha_0}{\alpha_3} - c_1\Big)r, \end{gathered} \tag{6}$$

where c_1, c_2,, and c_3 are arbitrary constants, and two coefficient functions α_1 and α_2 are determined by

$$\eta_t\alpha_1 + \eta\alpha_{1t} - c_1\alpha_1 = 0,\ \eta_t\alpha_2 + \eta\alpha_{2t} - 2c_1\alpha_2 = 0. \tag{7}$$

The Lie algebra of infinitesimal symmetries of system (1) is generated by the three vector fields:

$$\begin{gathered} V_1 = x\frac{\partial}{\partial x} + \Big(\frac{3}{\alpha_3}\int \alpha_3 dt\Big)\frac{\partial}{\partial t} - \Big(\frac{3\alpha_0}{\alpha_3}\int \alpha_3 dt + 1\Big)q\frac{\partial}{\partial q} + \Big(\frac{3\alpha_0}{\alpha_3}\int \alpha_3 dt - 1\Big)r\frac{\partial}{\partial r}, \\ V_2 = \frac{\partial}{\partial x}, \\ V_3 = \frac{1}{\alpha_3}\frac{\partial}{\partial t} - \Big(\frac{\alpha_0}{\alpha_3}q\Big)\frac{\partial}{\partial q} + \Big(\frac{\alpha_0}{\alpha_3}r\Big)\frac{\partial}{\partial r}. \end{gathered} \tag{8}$$

Table 1 presents the commutator table.

Table 1. Table of Lie brackets.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	$-V_2$	$-3V_3$
V_2	V_2	0	0
V_3	$3V_3$	0	0

3. Optimal System of Subalgebras

In present work, we shall construct the optimal system of one-dimensional subalgebra of the Lie algebra L_3 for AKNS Equation (1) by the method proposed in [19,30,31].

An arbitrary operator $V \in L_3$ is written in the form

$$V = l^1 V_1 + l^2 V_2 + l^3 V_3. \tag{9}$$

The following generators are used in order to find the linear transformations of the vector $l = \left(l^1, l^2, l^3\right)$,

$$E_i = c_{ij}^{\tau} l^j \frac{\partial}{\partial l^{\lambda}}, i = 1,2,3, \tag{10}$$

where c_{ij}^{τ} is defined by $[V_i, V_j] = c_{ij}^{\tau} V_{\tau}$. According to Equation (10) and Table 1, E_1, E_2, and E_3 are

$$\begin{aligned} E_1 &= -l^2 \tfrac{\partial}{\partial l^2} - 3l^3 \tfrac{\partial}{\partial l^3}, \\ E_2 &= l^1 \tfrac{\partial}{\partial l^2}, \\ E_3 &= 3l^1 \tfrac{\partial}{\partial l^3}. \end{aligned} \tag{11}$$

For the generators E_1, E_2, and E_3, the Lie equations with parameters a_1, a_2, and a_3 with the initial condition $\bar{l}\Big|_{a_i=0} = l, i = 1,2,3$ are written as

$$\frac{d\bar{l}^1}{da_1} = 0, \frac{d\bar{l}^2}{da_1} = -\bar{l}^2, \frac{d\bar{l}^3}{da_1} = -3\bar{l}^3, \tag{12}$$

$$\frac{d\bar{l}^1}{da_2} = 0, \frac{d\bar{l}^2}{da_2} = \bar{l}^1, \frac{d\bar{l}^3}{da_2} = 0, \tag{13}$$

$$\frac{d\bar{l}^1}{da_3} = 0, \frac{d\bar{l}^2}{da_3} = 0, \frac{d\bar{l}^3}{da_3} = 2\bar{l}^1. \tag{14}$$

The solutions of Equations (12)–(14) provide the transformation

$$T_1 : \bar{l}^1 = l^1, \bar{l}^2 = e^{-a_1} l^2, \bar{l}^3 = e^{-2a_1} l^3, \tag{15}$$

$$T_2 : \bar{l}^1 = l^1, \bar{l}^2 = a_2 l^1 + l^2, \bar{l}^3 = l^3, \tag{16}$$

$$T_3 : \bar{l}^1 = l^1, \bar{l}^2 = l^2, \bar{l}^3 = 2a_3 l^1 + l^3. \tag{17}$$

The method of constructing an optimal system needs a simplification of the vector

$$l = \left(l^1, l^2, l^3\right), \tag{18}$$

By means of the transformation $T_1 - T_3$. Our aim is to find the simplest representative of each class of similar vectors (18). The construction will be carried out under the following cases.

Case 1. $l^1 \neq 0$

By taking $a_2 = -\frac{l^2}{l^1}$ in the transformation T_2, $a_3 = -\frac{l^3}{2l^1}$ in the transformation T_3, we obtain $\bar{l}^2 = 0$, $\bar{l}^3 = 0$. Thus, vector (18) can be reduced to the form

$$l = \left(l^1, 0, 0\right). \tag{19}$$

This case gives the operator:

$$V_1.$$

Case 2. $l^1 = 0$

3.1. $l^2 \neq 0$

The vector (18) can be reduced to the form

$$l = \left(0, l^2, l^3\right). \tag{20}$$

Using all of the possible combinations, this case give rise to following operators:

$$V_2, V_2 + V_3, V_2 - V_3.$$

3.2. $l^2 = 0$

The vector (22) is reduced to the form

$$l = \left(0, 0, l^3\right). \tag{21}$$

Thus, we have the operator

$$V_3.$$

Theorem 1. *The optimal system of one-dimensional subalgebras of the Lie algebra is spanned by V_1, V_2, V_3 of Equation (1), as given by*

$$V_1, V_2, V_3, V_2 + V_3, V_2 - V_3. \tag{22}$$

4. Symmetry Reductions and Exact Solutions

By virtue of the optimal system (22), we will deal with the similarity reductions and group invariant solutions to the AKNS equation with time-dependent coefficients.

4.1. Solutions through V_1

The characteristic equations of the generator V_1 can be written as

$$\frac{dx}{x} = \frac{dt}{\frac{3}{\alpha_3}\int \alpha_3 dt} = \frac{dq}{-\left(\frac{3\alpha_0}{\alpha_3}\int \alpha_3 dt + 1\right)q} = \frac{dr}{\left(\frac{3\alpha_0}{\alpha_3}\int \alpha_3 dt - 1\right)r}. \tag{23}$$

Solving these equations yields the three similarity variables

$$\xi = x\left(\int \alpha_3 dt\right)^{-\frac{1}{3}}, q = e^{-\int \alpha_0 dt \cdot \left(\int \alpha_3 dt\right)^{-\frac{1}{3}}} F(\xi), r = e^{\int \alpha_0 dt \cdot \left(\int \alpha_3 dt\right)^{-\frac{1}{3}}} H(\xi), \tag{24}$$

and solving the constrained conditions (7), we get

$$\alpha_1 = \frac{1}{3}k_1\alpha_3\left(\int \alpha_3 dt\right)^{-\frac{2}{3}}, \alpha_2 = \frac{1}{3}k_2\alpha_3\left(\int \alpha_3 dt\right)^{-\frac{1}{3}},$$

where k_1 and k_2 are arbitrary constants and the AKNS Equation (1) is reduced to the following nonlinear coupled ordinary differential equations (ODEs):

$$\begin{aligned} -\tfrac{1}{3}F - \tfrac{1}{3}\xi F' &= F''' - 6FHF' - \tfrac{k_2}{3}F'' + \tfrac{2}{3}k_2F^2H + \tfrac{k_1}{3}F', \\ -\tfrac{1}{3}H - \tfrac{1}{3}\xi H' &= H''' - 6FHH' + \tfrac{k_2}{3}H'' - \tfrac{2}{3}k_2H^2F + \tfrac{k_1}{3}H'. \end{aligned} \tag{25}$$

The solution for (25) in a power series can be found in the form [32]

$$F=\sum_{n=0}^{\infty}A_n\xi^n,\, H=\sum_{n=0}^{\infty}B_n\xi^n. \tag{26}$$

Substituting (26) into (25), we get

$$\begin{aligned}
-\tfrac{1}{3}A_0-\tfrac{1}{3}\sum_{n=1}^{\infty}A_n\xi^n-\tfrac{1}{3}\sum_{n=1}^{\infty}A_{n-1}\xi^n &= 6A_3+\sum_{n=1}^{\infty}(n+3)(n+2)(n+1)A_{n+3}\xi^n-6A_0A_1B_0\\
&-6\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(k-i+1)A_iA_{k-i+1}B_{n-k}\xi^n-\tfrac{2k_2}{3}A_2-\tfrac{k_2}{3}\sum_{n=1}^{\infty}(n+2)(n+1)A_{n+2}\xi^n\\
&+\tfrac{2k_2}{3}A_0^2B_0+\tfrac{2k_2}{3}\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}A_iA_{k-i}B_{n-k}\xi^n+\tfrac{k_1}{3}A_1+\tfrac{k_1}{3}\sum_{n=1}^{\infty}(n+1)A_{n+1}\xi^n,\\
-\tfrac{1}{3}B_0-\tfrac{1}{3}\sum_{n=1}^{\infty}B_n\xi^n-\tfrac{1}{3}\sum_{n=1}^{\infty}B_{n-1}\xi^n &= 6B_3+\sum_{n=1}^{\infty}(n+3)(n+2)(n+1)B_{n+3}\xi^n-6A_0B_0B_1\\
&-6\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(k-i+1)A_{n-k}B_iB_{k-i+1}\xi^n+\tfrac{2k_2}{3}B_2+\tfrac{k_2}{3}\sum_{n=1}^{\infty}(n+2)(n+1)B_{n+2}\xi^n\\
&-\tfrac{2k_2}{3}A_0B_0^2-\tfrac{2k_2}{3}\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}A_{n-k}B_iB_{k-i}\xi^n+\tfrac{k_1}{3}B_1+\tfrac{k_1}{3}\sum_{n=1}^{\infty}(n+1)B_{n+1}\xi^n.
\end{aligned} \tag{27}$$

Now from (27), comparing coefficients, for $n=0$, we get

$$\begin{aligned}
A_3&=\tfrac{1}{18}\Big(18A_0A_1B_0+2k_2A_2-2k_2A_0^2B_0-k_1A_1-A_0\Big),\\
B_3&=\tfrac{1}{18}\Big(18A_0B_0B_1-2k_2B_2+2k_2A_0B_0^2-k_1B_1-B_0\Big).
\end{aligned} \tag{28}$$

Generally, for $n\geq 1$, we obtain

$$\begin{aligned}
A_{n+3}&=\tfrac{1}{(n+3)(n+2)(n+1)}\Bigg[-\tfrac{1}{3}A_n-\tfrac{1}{3}A_{n-1}+6\sum_{k=0}^{n}\sum_{i=0}^{k}(k-i+1)A_iA_{k-i+1}B_{n-k}+\tfrac{k_2}{3}(n+2)(n+1)A_{n+2}\\
&\qquad-\tfrac{2k_2}{3}\sum_{k=0}^{n}\sum_{i=0}^{k}A_iA_{k-i}B_{n-k}-\tfrac{k_1}{3}(n+1)A_{n+1}\Bigg],\\
B_{n+3}&=\tfrac{1}{(n+3)(n+2)(n+1)}\Bigg[-\tfrac{1}{3}B_n-\tfrac{1}{3}B_{n-1}+6\sum_{k=0}^{n}\sum_{i=0}^{k}(k-i+1)A_{n-k}B_iB_{k-i+1}-\tfrac{k_2}{3}(n+2)(n+1)B_{n+2}\\
&\qquad+\tfrac{2k_2}{3}\sum_{k=0}^{n}\sum_{i=0}^{k}A_{n-k}B_iB_{k-i}-\tfrac{k_1}{3}(n+1)B_{n+1}\Bigg].
\end{aligned} \tag{29}$$

From (27) and (28), we can get all of the coefficients A_n, B_n $(n\geq 3)$ of the power series (25). Substituting (28), (29) into (26) and using similarity transformations (24), we can obtain the solutions of system (1).

4.2. Solutions through V_2

The similarity variables of this generator are

$$\xi=t,\, q=F(\xi),\, r=H(\xi), \tag{30}$$

and solving the constrained conditions (7), we get α_0, α_2 are arbitrary functions of t.

These reduce the system (1) to the following nonlinear coupled ODEs:

$$\begin{aligned}
F'&=2\alpha_2F^2H-\alpha_0F,\\
H'&=-2\alpha_2H^2F+\alpha_0H.
\end{aligned} \tag{31}$$

Solving Equation (31) and using the similarity transformations (30), we obtain the solution of system (1) is

$$q = e^{t+\int (2\alpha_2 - \alpha_0 - 1)dt},\qquad r = e^{-t-\int (2\alpha_2 - \alpha_0 - 1)dt}. \tag{32}$$

4.3. Solutions through V_3

The similarity variables of this generator are

$$\xi = x, q = e^{-\int \alpha_0 dt} F(\xi), r = e^{\int \alpha_0 dt} H(\xi), \tag{33}$$

and solving the constrained conditions (7), we get

$$\alpha_1 = k_1\alpha_3, \alpha_2 = k_2\alpha_3,$$

where k_1 and k_2 are arbitrary constants, and the AKNS Equation (1) is reduced to the following nonlinear coupled ODEs:

$$F''' - 6FHF' - k_2F'' + 2k_2F^2H + k_1F' = 0,\qquad H''' - 6FHH' + k_2H'' - 2k_2FH^2 + k_1H' = 0. \tag{34}$$

To obtain the solutions of the reduction (34), we shall use the $\left(\frac{G'}{G}\right)$ method, as described in [20,33]. Assume that the solution of (34) is given in a polynomial form, as follows:

$$F = \sum_{i=0}^{m} A_i \left(\frac{G'}{G}\right)^i, H = \sum_{i=0}^{n} B_i \left(\frac{G'}{G}\right)^i. \tag{35}$$

By balancing highest order derivative term and nonlinear term in (34), we get $m = n = 1$ and $G = G(\xi)$ satisfies second-order linear ordinary differential equation (LODE)

$$G'' + \lambda G' + \mu G = 0. \tag{36}$$

Substituting (35) into (34) and equating coefficients of $\left(\frac{G'}{G}\right)$ to 0, we obtain an algebraic system of equations in A_0, A_1, B_0, and B_1. With the help of Maple, we obtain

$$\lambda = \frac{A_1^2B_0^2 + \mu}{A_1B_0}, A_0 = \frac{\mu}{B_0}, B_1 = \frac{1}{A_1}, k_1 = \frac{2\mu A_1^2B_0^2 + \mu A_1B_0k_2 - A_1^4B_0^4 - A_1^3B_0^3k_2 - \mu^2}{A_1^2B_0^2}, \tag{37}$$

where A_1, B_0, k_2, and μ are the arbitrary constants.

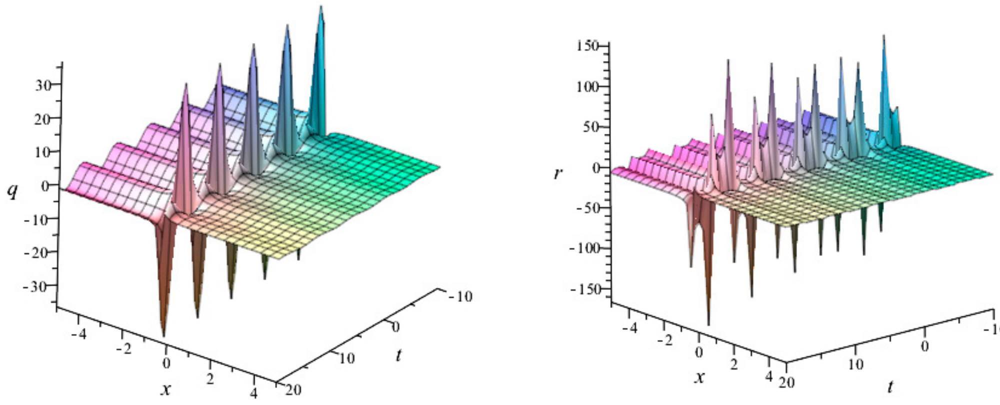
Substituting (37) into (35) and using similarity transformations (33), we obtain three types of solution of system (1), as follows:

When $\lambda^2 - 4\mu > 0$,

$$q = e^{-\int \alpha_0 dt}\left(\frac{A_1}{2}\sqrt{\lambda^2-4\mu} \times \left(\frac{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)}{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)}\right) - \frac{A_1\lambda}{2} + \frac{2\mu A_1}{\lambda\pm\sqrt{\lambda^2-4\mu}}\right),$$
$$r = e^{\int \alpha_0 dt}\left(\frac{1}{2A_1}\sqrt{\lambda^2-4\mu} \times \left(\frac{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)}{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}x\right)}\right) - \frac{\lambda}{2A_1} + \frac{\lambda\pm\sqrt{\lambda^2-4\mu}}{2A_1}\right), \tag{38}$$

where A_1, C_1, C_2, λ, and μ are arbitrary constants and $k_1 = \frac{2\left(\left(6\mu\lambda^2+4\mu\lambda k_2-8\mu^2-\lambda^4-k_2\lambda^3\right)\pm\left(4\mu\lambda+2\mu k_2-k_2\lambda^2-\lambda^3\right)\sqrt{\lambda^2-4\mu}\right)}{\left(\lambda\pm\sqrt{\lambda^2-4\mu}\right)^2}$.

When we take $A_1 = 1$, $C_1 = 2$, $C_2 = 1$, $\lambda = 3$, $\mu = 1$ and $\alpha_0 = \tan t$, the values of q and r are as illustrated in Figure 1, below.



(a) (b)

Figure 1. (**a**). Spatial structure of the exact solution q of (38) for Equation (1), with the parameters as $A_1 = 1$, $C_1 = 2$, $C_2 = 1$, $\lambda = 3$, $\mu = 1$, and $\alpha_0 = \tan t$. (**b**). Spatial structure of the exact solution r of (38), in which the parameters are the same as (**a**).

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} q &= e^{-\int \alpha_0 dt}\left(\frac{A_1}{2}\sqrt{4\mu-\lambda^2}\times\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)-C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)}\right)-\frac{A_1\lambda}{2}+\frac{2\mu A_1}{\lambda\pm i\sqrt{4\mu-\lambda^2}}\right), \\ r &= e^{\int \alpha_0 dt}\left(\frac{1}{2A_1}\sqrt{4\mu-\lambda^2}\times\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)-C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)}{C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}x\right)}\right)-\frac{\lambda}{2A_1}+\frac{\lambda\pm i\sqrt{4\mu-\lambda^2}}{2A_1}\right), \end{aligned} \tag{39}$$

where A_1, C_1, C_2, λ, and μ are arbitrary constants and $k_1 = 2\frac{4k_2\mu\lambda+6\mu\lambda^2-k_2\lambda^3-\lambda^4-8\mu^2\pm\left(2k_2\mu+4\mu\lambda-k_2\lambda^2-\lambda^3\right)i\sqrt{4\mu-\lambda^2}}{\left(\lambda\pm i\sqrt{4\mu-\lambda^2}\right)^2}$.

When $\lambda^2 - 4\mu = 0$,

$$\begin{aligned} q &= e^{-\int \alpha_0 dt}\left(A_1\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2x}\right)+\frac{2\mu A_1}{\lambda}\right), \\ r &= e^{\int \alpha_0 dt}\left(\frac{1}{A_1}\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2x}\right)+\frac{\lambda}{2A_1}\right), \end{aligned} \tag{40}$$

where A_1, C_1, C_2, λ, and μ are arbitrary constants and $k_1 = 0$.

4.4. Solutions through $V_2 + V_3$

The similarity variables of this generator are

$$\xi = \int \alpha_3 dt - x,\ q = e^{-\int \alpha_0 dt}F(\xi),\ r = e^{\int \alpha_0 dt}H(\xi), \tag{41}$$

and solving the constrained conditions (7), we get

$$\alpha_1 = k_1\alpha_3,\ \alpha_2 = k_2\alpha_3,$$

where k_1 and k_2 are arbitrary constants, and the AKNS Equation (1) is reduced to the following nonlinear coupled ODEs:

$$\begin{aligned} F' &= -F''' + 6FHF' - k_2F'' + 2k_2F^2H - k_1F', \\ H' &= -H''' + 6FHH' + k_2H'' - 2k_2H^2F - k_1H'. \end{aligned} \tag{42}$$

We shall use the simplest equation method described in [34] to obtain the solutions of reduction (42). Let us consider the solutions of (42), as

$$F=\sum_{i=0}^{m}A_i\phi^i(\xi),\, H=\sum_{i=0}^{n}B_i\phi^i(\xi). \tag{43}$$

By balancing highest order derivative term and nonlinear term in (42), we get $m=n=1$ and $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi)=a\phi^2(\xi)+b\phi(\xi)+c. \tag{44}$$

The solutions of (44) can be written as

$$\phi(\xi)=-\frac{b}{2a}-\frac{\theta}{2a}\tanh\Big(\frac{1}{2}\theta(\xi+C)\Big), \tag{45}$$

and

$$\phi(\xi)=-\frac{b}{2a}-\frac{\theta}{2a}\tanh\Big(\frac{1}{2}\theta\xi\Big)+\frac{\sec h\big(\frac{\theta\xi}{2}\big)}{C\cosh\big(\frac{\theta\xi}{2}\big)-\frac{2a}{\theta}\sinh\big(\frac{\theta\xi}{2}\big)}, \tag{46}$$

where $\theta^2=b^2-4ac$.

Substituting (43) into (42), an algebraic system of equations in A_0, A_1, B_0, and B_1 can be obtained by equating the coefficients of the functions $\phi^i(\xi)$ to zero. With the aid of Maple, solution to this system can be obtained, as follows:

$$A_0=\frac{ac}{B_0},\, A_1=\frac{\big(b\pm\sqrt{\theta^2}\big)a}{2B_0},\, B_1=\frac{2B_0a}{b\pm\sqrt{\theta^2}},\, k_2=\frac{-\theta^2-k_1-1}{\pm\sqrt{\theta^2}}, \tag{47}$$

where B_0, k_1, a, b, and c are arbitrary constants.

Substituting (47) into (43) and using similarity transformations (41), we obtain a set of solutions of system (1) are

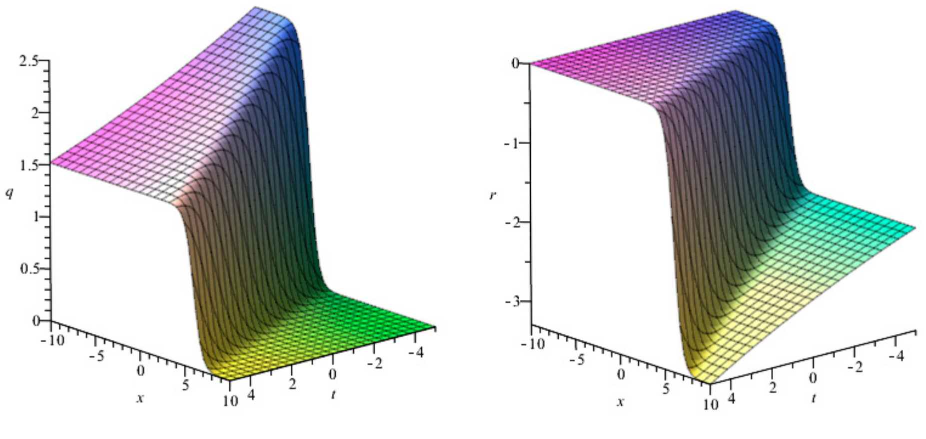
$$\begin{aligned} q&=e^{-\int\alpha_0dt}\Big\{\tfrac{ac}{B_0}+\tfrac{(b+\theta)a}{2B_0}\Big[-\tfrac{b}{2a}-\tfrac{\theta}{2a}\tanh\Big(\tfrac{1}{2}\theta(\xi+C)\Big)\Big]\Big\},\\ r&=e^{\int\alpha_0dt}\Big\{B_0+\tfrac{2B_0a}{b+\theta}\Big[-\tfrac{b}{2a}-\tfrac{\theta}{2a}\tanh\Big(\tfrac{1}{2}\theta(\xi+C)\Big)\Big]\Big\}, \end{aligned} \tag{48}$$

And

$$\begin{aligned} q&=e^{-\int\alpha_0dt}\left\{\tfrac{ac}{B_0}+\tfrac{(b+\theta)a}{2B_0}\left[-\tfrac{b}{2a}-\tfrac{\theta}{2a}\tanh\Big(\tfrac{1}{2}\theta\xi\Big)+\tfrac{\sec h\big(\frac{\theta\xi}{2}\big)}{C\cosh\big(\frac{\theta\xi}{2}\big)-\frac{2a}{\theta}\sinh\big(\frac{\theta\xi}{2}\big)}\right]\right\},\\ r&=e^{\int\alpha_0dt}\left\{B_0+\tfrac{2B_0a}{b+\theta}\left[-\tfrac{b}{2a}-\tfrac{\theta}{2a}\tanh\Big(\tfrac{1}{2}\theta\xi\Big)+\tfrac{\sec h\big(\frac{\theta\xi}{2}\big)}{C\cosh\big(\frac{\theta\xi}{2}\big)-\frac{2a}{\theta}\sinh\big(\frac{\theta\xi}{2}\big)}\right]\right\}, \end{aligned} \tag{49}$$

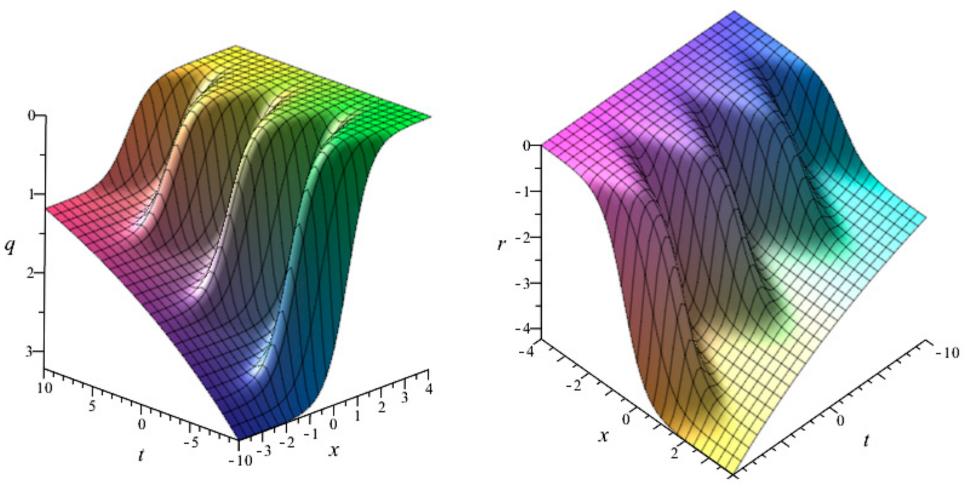
where $\xi=\int\alpha_3dt-x$.

We can choose different values of α_3 in solution (48) in order to construct travelling and non-travelling wave solutions of Equation (1). Figure 2 depicts the travelling wave solution, which is obtained by taking $\alpha_3=1$. Figure 3 displays the non-travelling wave solution by selecting $\alpha_3=\cos t$. Other parameters are selected as $B_0=-3$, $a=1$, $b=3$, $c=1$, and $\alpha_0=0.05$.



(a) (b)

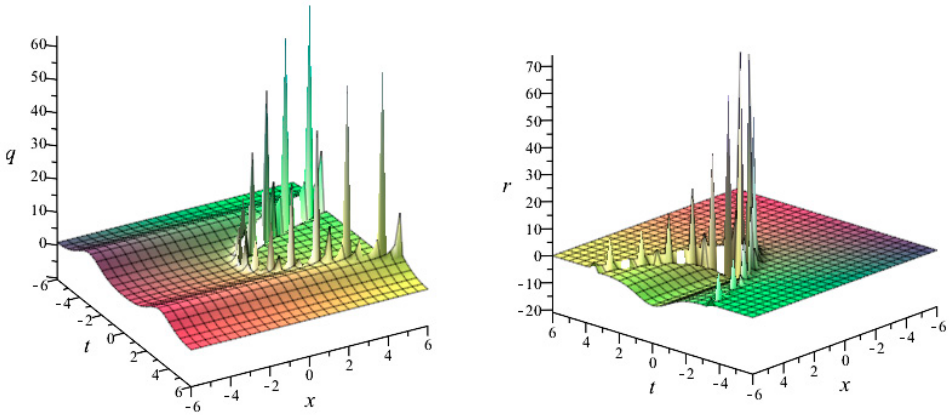
Figure 2. (**a**). Spatial structure of the exact solution q of (48) for Equation (1), with the parameters as $B_0 = -3$, $a = 1$, $b = 3$, $c = 1$, $\alpha_0 = 0.05$ and $\alpha_3 = 1$. (**b**). Spatial structure of the exact solution r of (48), in which the parameters are the same as (**a**).



(a) (b)

Figure 3. (**a**). Spatial structure of the exact solution q of (48) for Equation (1), with the parameters as $B_0 = -3$, $a = 1$, $b = 3$, $c = 1$, $\alpha_0 = 0.05$, and $\alpha_3 = \cos t$. (**b**). Spatial structure of the exact solution r of (48), in which the parameters are the same as (**a**).

When we take $B_0 = -3$, $a = 1$, $b = 3$, $c = 1$, $\alpha_0 = -\sin t$, and $\alpha_3 = t$ in solution (49), the shapes of non-travelling wave solutions of Equation (1) are displayed in Figure 4.



(a) (b)

Figure 4. (**a**). Spatial structure of the exact solution q of (49) for Equation (1), with the parameters as $B_0 = -3, a = 1, b = 3, c = 1, \alpha_0 = -\sin t$, and $\alpha_3 = t$. (**b**). Spatial structure of the exact solution r of (49), in which the parameters are the same as (**a**).

4.5. Solutions through $V_2 - V_3$

The similarity variables of this generator are

$$\xi = \int \alpha_3 dt + x, q = e^{-\int \alpha_0 dt} F(\xi), r = e^{\int \alpha_0 dt} H(\xi), \tag{50}$$

and solving the constrained conditions (7), we get

$$\alpha_1 = k_1 \alpha_3, \alpha_2 = k_2 \alpha_3,$$

where k_1 and k_2 are arbitrary constants, and the AKNS Equation (1) is reduced to the following nonlinear coupled ODEs:

$$\begin{aligned} F' &= F''' - 6FHF' - k_2 F'' + 2k_2 F^2 H + k_1 F', \\ H' &= H''' - 6FHH' + k_2 H'' - 2k_2 H^2 F + k_1 H'. \end{aligned} \tag{51}$$

We shall use the simplest equation method to obtain the solutions of reduction (51) [34]. For the Bernoulli equation

$$\phi'(\xi) = a\phi(\xi)^2 + b\varphi(\xi), \tag{52}$$

We use the following solution

$$\phi(\xi) = b\left\{\frac{\cosh[b(\xi + C)] + \sinh[b(\xi + C)]}{1 - a\cosh[b(\xi + C)] - a\sinh[b(\xi + C)]}\right\}.$$

The balancing procedure gives $m = n = 1$ and the solutions of (51), as

$$F = A_0 + A_1\varphi, H = B_0 + B_1\varphi. \tag{53}$$

Substitution of (53) into (51) yields

$$A_0 = 0, B_0 = \frac{ab}{A_1}, B_1 = \frac{a^2}{A_1}, k_1 = -b^2 + bk_2 + 1, \tag{54}$$

where A_1, k_2, a, and b are arbitrary constants.

Substituting (54) into (53) and using the similarity transformations (50), we obtain the solution of system (1), as

$$q = e^{-\int \alpha_0 dt} A_1 b \left\{ \frac{\cosh[b(\xi+C)] + \sinh[b(\xi+C)]}{1 - a\cosh[b(\xi+C)] - a\sinh[b(\xi+C)]} \right\},$$
$$r = e^{\int \alpha_0 dt} \left\{ \frac{ab}{A_1} + \frac{a^2}{A_1} b \left[\frac{\cosh(b(\xi+C)) + \sinh(b(\xi+C))}{1 - a\cosh(b(\xi+C)) - a\sinh(b(\xi+C))} \right] \right\}, \quad (55)$$

where $\xi = \int \alpha_3 dt + x$.

Figure 5 illustrates the travelling wave solutions of Equation (1) by taking $\alpha_3 = 1$, $A_1 = 1$, $a = -1$, $b = 1$, $C = 0$, and $\alpha_0 = \sin t$ in Equation (55). Figure 6 portrays the non-travelling wave solutions of Equation (1) by setting $\alpha_3 = t$, and the other parameters are the same as those in Figure 5.

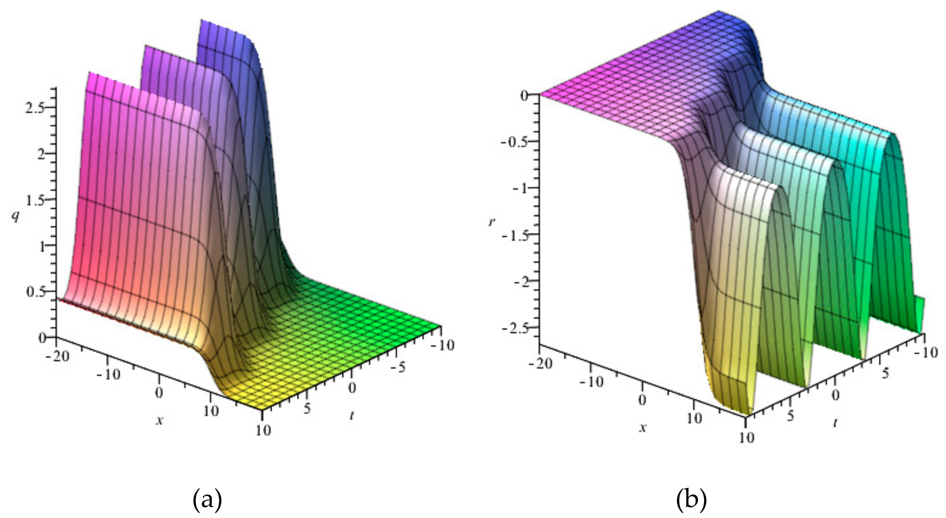


Figure 5. (**a**). Spatial structure of the exact solution q of (55) for Equation (1), with the parameters as $A_1 = 1, a = -1, b = 1, C = 0, \alpha_0 = \sin t$, and $\alpha_3 = 1$. (**b**). Spatial structure of the exact solution r of (55), in which the parameters are the same as (**a**).

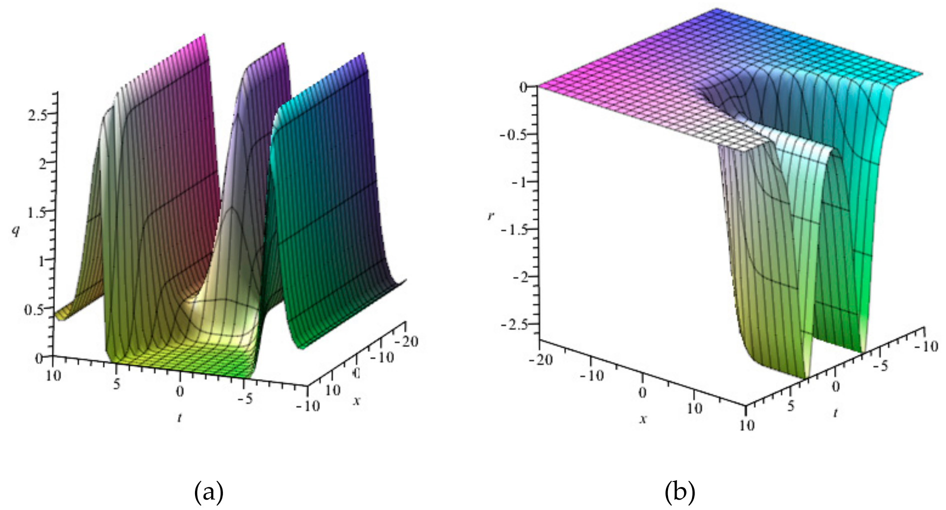


Figure 6. (**a**). Spatial structure of the exact solution q of (55) for Equation (1), with the parameters as $A_1 = 1, a = -1, b = 1, C = 0, \alpha_0 = \sin t$, and $\alpha_3 = t$. (**b**). Spatial structure of the exact solution r of (55), in which the parameters are the same as (**a**).

When compared with the results in the existing literature, we find that the obtained invariant solutions are different from those in Refs. [28,29], due to $\alpha_3(t) \neq 0$ in Equation (6). To the best of our knowledge, the obtained invariant solutions (38)–(40), (48), (49), and (55) are new, and they have not been reported in the literature.

5. Conclusions

In summary, by performing the Lie symmetry analysis on the AKNS Equation (1), Lie point symmetries of the AKNS equation are discussed. Moreover, we construct the optimal system of one-dimensional subalgebras of Lie algebra spanned by $V_1 - V_3$. Five types of similarity reduction are presented by using the optimal system. Meanwhile, some new exact solutions, such as power series solutions and travelling and non-traveling wave solutions are obtained for system (1).

It is easy to see that the obtained invariant solutions include coefficient functions α_0 and α_3, which provide enough freedom for us to construct travelling and non-travelling wave solutions for the AKNS Equation (1). This paper shows that the Lie symmetry analysis method is an effective mathematical tool for constructing travelling and non-traveling wave solutions of some other nonlinear PDEs with variable coefficients.

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