

Article

Scale Symmetry in the Universe

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Abstract: Scale symmetry is a fundamental symmetry of physics that seems however not to be fully realized in the universe. Here, we focus on the astronomical scales ruled by gravity, where scale symmetry holds and gives rise to a truly scale invariant distribution of matter, namely it gives rise to a fractal geometry. A suitable explanation of the features of the fractal cosmic mass distribution is provided by the nonlinear Poisson–Boltzmann–Emden equation. An alternative interpretation of this equation is connected with theories of quantum gravity. We study the fractal solutions of the equation and connect them with the statistical theory of random multiplicative cascades, which originated in the theory of fluid turbulence. The type of multifractal mass distributions so obtained agrees with results from the analysis of cosmological simulations and of observations of the galaxy distribution.

Keywords: scale symmetry; fractal geometry; gravitational clustering; cosmic web; turbulence

1. Introduction

The symmetry of the physical laws is probably the essential foundation of our current understanding of physics and the universe [1]. Symmetry principles are indeed essential in the formulation of quantum field theory, as one of the fundamental theories of physics [2]. The oldest and most common symmetries are the space-time symmetries, namely the symmetry of the physical laws under space or time translations and under space rotations, a symmetry that is enlarged to the Poincaré symmetry group by the theory of relativity. These symmetries induce very relevant conservation laws, namely the conservation of linear and angular momenta and the conservation of energy. When a theory of gravity is added to quantum field theory, the space-time symmetries become more involved, because they only hold locally in inertial frames. However, the relativistic theories of gravity can be formulated as *gauge* theories of the space-time symmetries [3]. In addition to the mentioned space-time symmetries, there is another transformation of space-time that is intuitively appealing and has had an important role in physics and other sciences, even though it is not necessarily a symmetry, namely the transformation of scale or dilatation. It closes, together with the space-time transformations, a group that can be enlarged to the *conformal group* of transformations, of ever increasing interest in theoretical physics [4].

Given the large range of sizes in the universe [5], it is surely an old question how to know what determines the relevant sizes, from micro to macro-physics. While micro-physics or human scale physics involve various physical constants and laws, macro-physics, especially the large scale structure of the universe, is just the realm of gravity. Gravity has no intrinsic length scale, so one can wonder why large astronomical objects have a given size. In fact, beyond galaxies, whose size is determined by both gravity and the electromagnetic interaction [6], there seems to be no way to construct objects of a given size. Simply put, if one finds an object of a given size, there must be similar objects of larger size (and possibly of smaller size, as long as one does not go to too small scales). For example, take a cluster of galaxies; there must be similar *superclusters* of every possible size. Not surprisingly, the idea of a scale invariant structure of the universe on large scales is old, but its modern formulation

had to await the advent of the appropriate mathematical description, namely fractal geometry [7]. Simple fractals are scale invariant and are indeed composed of clusters of clusters of . . . , down to the infinitesimally small. Naturally, in the universe, the self-similarity must stop at a scale about the size of galaxies, although it could be limitless towards the large scales, in principle.

However, the appealing idea of an infinite hierarchy of clusters of clusters of galaxies [7,8] clashes with the large scale homogeneity prescribed by the standard cosmological principle and embodied in the Friedmann–Lemaître–Robertson–Walker relativistic model of the universe [9,10]. Naturally, a compromise is possible: the universe is homogeneous on very large scales but is fractal on smaller yet large scales (in the so-called strong-clustering regime). In the intermediate range of scales, the structure of matter in the universe undergoes a transition from fractal to homogeneous. Therefore, there is a scale of transition to homogeneity, which admits several definitions that should give approximately the same value. However, despite the work of many researchers along several decades, the debate about the scale of transition to homogeneity is not fully settled and quite different values appear in the literature [9–15]. This situation is surely a consequence of the different definitions used and the limitations of the current methods of observation. At any rate, this mainly observational issue is not crucial for us and we are content to study the fractal structure of the universe without worrying about the definitive value of the scale of transition to homogeneity.

It is normal in physics that scale invariance holds in one range of scales and is lost in another range or changes to a different type of scale invariance (in a sense, the homogeneous state is trivially scale invariant). This situation is common, for example, in critical phenomena in statistical physics, in which it is called *crossover*. The theory of critical phenomena is actually a fruitful domain of application of the theory of scale invariance and furthermore of the full theory of conformal invariance [16]. The purpose of the present paper is to study the theory of the fractal structure of the universe with methods of statistical physics and field theory. There is a basic difference between the large mass fluctuations in fractal geometry and the more moderate fluctuations in the theory of critical phenomena [15], but statistical field theory methods can nonetheless be applied to fractal geometry. At any rate, the scale invariance and fractal nature of gravitational clustering is due to the form of the law of gravity. In fact, the statistical field theory of gravitational systems is peculiar.

The Newton law, with its inverse dependence on distance, does not fulfill the condition of short-rangedness that makes an interaction tractable in statistical physics, thus there can be no homogeneous equilibrium state (§74, [17]). This problem leads to the peculiar statistical and thermodynamic properties of many-body systems with long-range interactions, such as negative specific heats, ensemble nonequivalence, metastable states whose lifetimes diverge with the number of bodies, and spatial inhomogeneity, among others [18]. Fortunately, there are methods to study gravitationally interacting systems. In the mean field approximation for a system of N bodies in gravitational interaction, which becomes exact for $N \rightarrow \infty$, the equation that rules the distribution of the gravitational potential is a higher dimensional generalization of Liouville's equation, originally introduced to describe the conformal geometry of surfaces [19]. The gravitational equation is also old and has been given various names; it is called the Poisson–Boltzmann–Emden equation in Bavaud's review [20]. Its scale covariance is remarked by de Vega et al. [21] (who actually studied the fractal structure of the interstellar medium, on galactic scales below the cosmological scales).

Deep studies of the fractal geometry of mass distributions related to the Poisson–Boltzmann–Emden equation have been carried out in the theory of stochastic processes, namely in the theory of random multiplicative cascades [22–24]. Historically, this theory arose in relation to the *lognormal model* of turbulence [25,26]. Random multiplicative cascades give rise to multifractal distributions, which have applications in several areas [27,28]. Indeed, random multiplicative cascades are naturally applied to cosmology: while in models of turbulence the energy or vorticity “cascades” down towards smaller scales, in cosmology, mass undergoes successive gravitational collapses. The result of the infinite iteration of random mass condensations is a multifractal mass distribution, that is to say, a generalization of a simple fractal structure.

Gravity is the dominant force in the universe on large scales but it also dominates the very small scales near the Planck length, which is the domain of quantum gravity [29]. Field equations connected with the Poisson–Boltzmann–Emden equation appear in generalizations of the general theory of relativity that add a scalar field, such as the Dicke–Brans–Jordan theory [29]. Scalar–tensor theories of gravity are old but they gained popularity with the advent of string theory, a theory of quantum gravity in which a scalar field, the dilaton, naturally arises as a partner of the graviton (the metric tensor field quantum) [30,31]. The dilaton is essential in the understanding of the conformal symmetry of space-time [32]. However, the connection of the dilaton field with the gravitational potential in the Poisson–Boltzmann–Emden equation is, at best, indirect; except in two-dimensional relativistic gravity, which is defined in terms of only one scalar field [30]. Of course, the fractal geometry of the large scale structure of the universe is produced by three-dimensional gravity, but the study of lower dimensional models also has interest in cosmology.

We begin with a summary of fractal geometry, in particular, of multifractal geometry, orientated towards the description of the cosmic mass distribution. We proceed to study how to generate the fractal geometry of the large scale structure, especially combining the Vlasov dynamics and the Poisson–Boltzmann–Emden equation [20] with the Zel’dovich approximation and the adhesion model of the early stage of structure formation [33,34]. The fractal geometry of the web structure that arises in the cosmological evolution according to the adhesion model is thoroughly studied in [35]. In this paper, we take a further step in the attempt to describe analytically the fractal geometry of the universe, by means of simple models and taking advantage of the power of the scale symmetry.

2. Multifractal Geometry

Multifractal geometry is, of course, the geometry of multifractal mass distributions, which are a natural generalization of the concept of simple fractal set, such that a range of dimensions appear instead of just a single dimension. However, multifractal geometry can almost be defined as the geometry of *generic* mass distributions, because most of them are actually susceptible to a multifractal analysis. To be precise, despite common prejudices, generic mass distributions are *strictly singular*, that is to say, the mass density is not well defined and is in fact either zero or infinity at every point [36]. The singularities give rise to a spectrum of dimensions, as indeed happens in the mass distributions that appear in cosmology [35].

Actually, the formation of structure in the universe occurs in a definite way, from the growth of small density fluctuations in an initially homogeneous and isotropic universe, up to a size such that the increased gravitational force leads to a collapse of mass patches towards the formation of singularities. In the adhesion model of structure formation, the collapse initially leads to matter sheets (two-dimensional singularities), then to filaments (one-dimensional singularities), and finally to point-like singularities [33,34]. The web structure so formed is a particular example of multifractal mass distribution and is indeed a rough model of the actual cosmic structure, but it is not a very accurate model [35]. The problem is that the formation of matter filaments and especially point-like singularities, in Newtonian gravitation, can only occur after the dissipation of an infinite amount of gravitational energy, so that the process requires a fully relativistic treatment (Box 32.3, [29]), beyond the scope of the adhesion model.

At any rate, the adhesion model is not meant to describe accurately the formation of singularities, even within Newtonian gravitation. Gurevich and Zybin’s specific approach to this process (within Newtonian gravitation) [37] obtains different results, namely the formation of singular power-law mass concentrations instead of the collapse of a number of dimensions (one, two or three) to zero size. Singular power-law mass concentrations are the hallmark of typical multifractals. On the other hand, multifractals are the natural generalization of simple hierarchical mass distributions, namely simple fractal sets, also referred to as *monofractal* distributions. Therefore, we are led to study the geometry of multifractal mass distributions and the proper way to characterize them [35]. Examples of monofractal, multifractal and adhesion model web structures are shown in Figure 1.

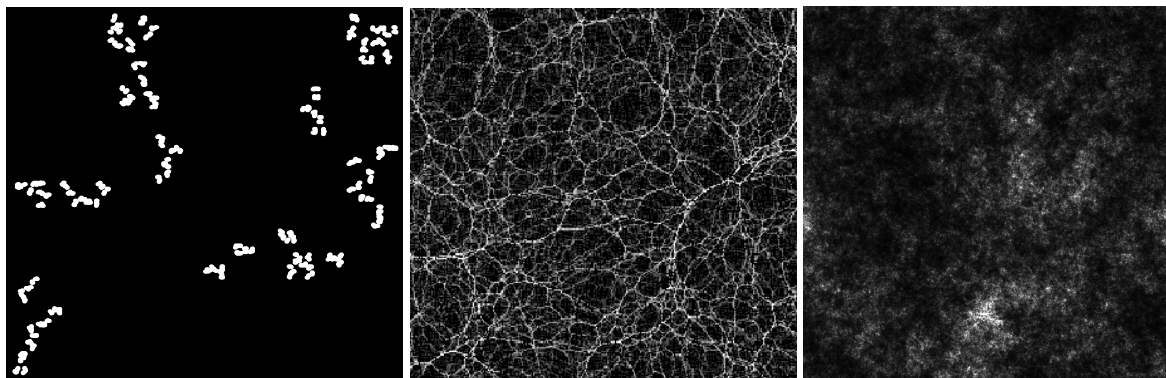


Figure 1. Three different types of fractal structure that can represent the cosmic mass distribution (from left to right): (i) monofractal cluster hierarchy, with $D = 1$ and therefore $m(r) \propto r$; (ii) cosmic web structure, rendered in two dimensions, showing one-dimensional filaments and zero-dimensional nodes organized in a self-similar structure; and (iii) multifractal lognormal mass distribution, with a range of dimensions α such that $m(r, x) \propto r^{\alpha(x)}$ from the point x .

The mathematical definition of multifractal can be found in books about fractal geometry [28,38]. It is to be remarked that the definition of multifractal, as well as the simpler definition of fractal set, are formulated in terms of limits for some vanishing length scale. In other words, the scale symmetry only needs to hold asymptotically for vanishing length scales. However, there is an important class of multifractals in which scale symmetry holds in a finite range of scales, namely the *self-similar* fractals or multifractals, generated by some iterative process [28,38]. These processes are especially interesting in the theory of cosmological structure formation. One reason for it is the symmetry of the initial state, which is strictly homogeneous and isotropic and is such that structure formation proceeds homogeneously in every part of the universe, defining a single scale of transition to homogeneity. Another reason, related to the preceding one, is that the standard cosmological principle admits a weaker, stochastic formulation, namely Mandelbrot's conditional cosmological principle, in which every *possible* observer is replaced by every observer located at a *material* point (§22, [7]). This principle implies the self-similarity of the strong-clustering regime.

As mentioned in the Introduction, good models of the strong-clustering regime of the cosmological evolution, in terms of successive gravitational collapses, are the random multiplicative cascades. Before we study in detail these cascade processes in Section 4, we prepare ourselves by studying methods of multifractal analysis, especially those that take into account the statistical homogeneity and isotropy of the mass distribution.

A statistically homogeneous, isotropic, and scale invariant mass distribution can be characterized by the probability of a density field $\rho(x)$ with those symmetries, that is to say, by a suitable functional of the function $\rho(x)$. If we forgo the scale invariance, the simplest functional is, of course, the Gaussian functional, which in Fourier space is simply a product of the respective Gaussian probability functions over all the Fourier modes [6]. However, in cosmology, it is normal to use, instead of the probability of $\rho(x)$, the N -point correlation functions of $\rho(x)$ or the method of *counts in cells* [10]. These counts are usually galaxy counts in a sphere of radius r placed at random across a fair sample of the process [10]. Therefore, the density refers to just the galaxy number density. This method of galaxy counts in cells has been applied to the study of scale invariance in the large scale structure of the universe [39]. However, galaxy counts do not consider the mass of galaxies. Nevertheless, in cosmological N -body simulations, which describe the mass distribution in terms of N bodies (or particles) of equal mass, number density equals mass density, so the density is the real mass density or, to be precise, a *coarse-grained* mass density. Furthermore, the good mass resolution of the simulations makes them suitable for the multifractal analysis [40–44]. The results of these analyses are summarized in Section 5.

It is often assumed that the full set of N -point correlation functions allow recovering the probability functional of $\rho(x)$ or that the full set of integral statistical moments of the probability distribution function of the coarse-grained density allow recovering the probability distribution function of the coarse-grained density. Both assumptions do not necessarily hold for random multifractals. Surely, this is a problem that affects the strictly singular distributions, as they have mass densities that are zero or infinity at every point. It is clear that this singular behavior makes the probability distribution function of the coarse-grained density singular for a vanishing *coarse-graining length* (e.g., for a vanishing radius r of the above-mentioned sphere): the probability distribution function gets concentrated on zero or infinity. In fact, the multifractal analysis can be made in terms of the probability distribution function of the coarse-grained density, analyzing *how* it becomes singular. This type of multifractal analysis is equivalent to the lattice or point centered based methods [28,38], according to the type of coarse-graining, if we take into account the homogeneity and isotropy of the mass distribution. Of course, the mass distribution must also be self-similar, namely scale invariant in some range of scales and not just asymptotically for vanishing coarse-graining length.

It has to be remarked here that self-similarity is not equivalent to multifractality. A pertinent example is the case of the adhesion model with scale invariant initial conditions [45]. The scale invariant initial conditions consist of a uniform density and a *fractional Brownian* velocity field. This type of field is self similar but not multifractal; in fact, it is a normal non-differentiable function (ch. IX, [7]) (ch. 16, [38]). However, in the adhesion model, the mass distribution becomes singular and multifractal as a consequence of the scale invariant initial conditions [35,45].

In a multifractal distribution, the point density $\rho(x)$ is singular: in mathematical terms, it does not exist as a function. However, the coarse-grained density is a standard function, namely the coarse graining is a *regularization* of $\rho(x)$. However, it is a function that depends on the coarse-graining length, say r , and on the precise method of coarse graining. Since the point density $\rho(x)$ is a random variable, thus is the coarse-grained density. The probability distribution of this variable is independent of the base point, by statistical homogeneity, and it is a suitable starting point for the multifractal analysis, as said above. Let us call ρ_r the density coarse grained with length r and $P(\rho_r)$ its probability distribution function. We are interested in the statistical moments of this probability distribution function. These moments are related to the density correlation functions: in fact, the statistical moment of order k can be expressed as the integral of the k -point correlation function over the k points in the coarse-graining volume. Furthermore, there is a relation between the probability of given counts in cells in some volume and the statistical moments of the probability distribution function of the number density in that volume [39,46]. In particular, the probability of the volume being empty, called the *void probability function*, is the Laplace transform of the probability distribution function of the density and can be taken as the generating function of the statistical moments of this probability distribution function, following standard methods of statistical mechanics. Naturally, these relations play a role in the study of *cosmic voids* [47]. Here, we focus on just the probability $P(\rho_r)$.

The multifractal analysis consists in finding how $P(\rho_r)$ depends on r through its *fractional* statistical moments. Within the *scaling regime*, every fractional statistical moment behaves as a power law of r , namely for $q \in \mathbb{R}$,

$$\mu_q = \frac{\langle \rho_r^q \rangle}{\langle \rho_r \rangle^q} = \int_0^\infty \left(\frac{\rho_r}{\langle \rho_r \rangle} \right)^q P(\rho_r) d\rho_r \propto r^{\tau(q)-3(q-1)}, \quad (1)$$

where the somewhat complicated exponent of r comes from the somewhat different definition of statistical moments that is standard in fractal geometry [35,44]. Notice that we assume that the multifractal has support in the full three-dimensional space, that is to say, we have a *nonlacunar* multifractal, with $\tau(0) = -3$, as is natural in Newtonian cosmology. The function $\tau(q)$ describes the singularity structure of any realization of $\rho(x)$, that is to say, the local behavior of the mass distribution at every singularity. When this description is complete the distribution is said to fulfill the multifractal

formalism [28]. The local behavior of a realization of the mass distribution at a singular point x is given by the *local dimension* $\alpha(x)$, which defines how the mass grows from that point, that is to say,

$$m(x, r) \sim r^{\alpha(x)}. \quad (2)$$

Naturally, $\alpha \geq 0$. Every set of points with a given local dimension α constitutes a fractal set with a dimension that depends on α , namely $f(\alpha)$. The multifractal formalism involves a relationship between $\tau(q)$ and the local behaviors given by α and $f(\alpha)$, in the form of a Legendre transform:

$$f(\alpha) = q\alpha - \tau(q), \quad (3)$$

and $\alpha(q) = \tau'(q)$. It is to be remarked that the multifractal formalism becomes trivial if there is only one dimension $\alpha = f(\alpha)$ (the case of a *monofractal* or *unifractal*); then, $\alpha = \tau(q)/(q - 1)$, as follows from Equation (3). The quotient $D_q = \tau(q)/(q - 1)$ is called, in general, the Rényi dimension and has an information-theoretic meaning [28]. In a monofractal, $D_q = \alpha$ is constant, but in a multifractal is a non-increasing function of q . A typical example of monofractal is a self-similar fractal *set* such that the mass is uniformly spread on it.

Before studying the theory of random multiplicative cascades, let us consider some simple examples of self-similar multifractals [7,28,38]. They are constructed as self-similar fractal sets with an irregular mass distribution on them. The simplest example is surely the “Cantor measure” (§1.2.1, [28]) (example 17.1, [38]). If the mass is distributed on the three subintervals instead of just two of them, we have the “Besicovitch weighted curdling”, an example of *nonlacunar fractal* (p. 377, [7]). A simpler nonlacunar fractal is obtained by using just two equal subintervals, so defining the *binomial multiplicative process* (§6.2, [48]). Its function $\tau(q)$ is very simple:

$$\tau(q) = -\log_2[p^q + (1 - p)^q], \quad (4)$$

where p is, say, the mass fraction in the left hand side subinterval. The generalization of these constructions leads to the general theory of deterministic self-similar multifractals (§17.3, [38]) (or Moran cascade processes (§6.2, [28])). Notice that the average in Equation (1), in the case of deterministic multifractals, is to be interpreted as a spatial average in a suitable domain. Random multiplicative cascades are just an elaboration of the deterministic cascades, in the setting of random processes. Only in this setting, we can achieve statistically homogeneous, isotropic and scale invariant mass distributions.

Several properties of the self-similar multifractals defined by an iterated function system and a set of weights for the redistribution of mass are relatively easy to prove (these multifractals are sometimes called *multinomial* because one iteration gives rise to a multinomial distribution) [28,38]. In a multifractal with support in \mathbb{R}^3 , $\tau(q)$ is a concave and increasing function, such that $\tau(0) = -3$ and $\tau(1) = 0$. Furthermore, $\tau(q)$ has definite asymptotes as $q \rightarrow \infty$ or $q \rightarrow -\infty$ (see the realistic example of Figure 2). The slopes can be computed and, in particular,

$$\lim_{q \rightarrow \infty} \frac{\tau(q)}{q} = \alpha_{\min}, \quad (5)$$

where α_{\min} is, of course, the minimum value of the local dimension and is smaller than 3 (or smaller than the ambient space dimension, in general). Therefore, the asymptotic value of the exponent of r in Equation (1) for $q \rightarrow \infty$ is $(\alpha_{\min} - 3)q < 0$. On the other hand, the exponent vanishes for $q = 0$ and $q = 1$. We deduce that the sequence of integral statistical moments has a reasonable behavior: as regards the r -dependence, the exponent in Equation (1) is negative, thus the moments decrease with r ; and as regards the asymptotic q -dependence, it is such that $P(\rho_r)$ is determined by the integral moments, according to the standard criteria for a probability distribution function to be determined

by its integral moments (p. 20, [49]). This property of determinacy cannot be generalized to arbitrary random multiplicative cascades.

A remark is in order here. People familiar with the theory of critical phenomena may find that the description of a statistically homogeneous, isotropic and scale invariant mass distribution in this section is strange or restrictive, because the same symmetries are generally present in the critical phenomena of statistical physics, yet the treatment is quite different and the concept of multifractal is not necessary. The crucial point is that we are demanding here the scale symmetry of the *full* mass distribution and not just the scale symmetry of the mass *fluctuations* about a homogeneous and isotropic mass distribution. The importance of this distinction in cosmology is discussed in [15].

3. Gravity and Scale Symmetry

Let us consider the universe on large scales as a system of bodies in gravitational interaction and with a moderate range of velocities (the bodies may be galaxies, but this is not important). From such a simple model, we can draw an interesting conclusion in Newtonian gravity (§9, [7]): the accumulated mass $m(r)$ inside a sphere of radius r centered on one body is proportional to r , that is to say, the system of bodies approaches, in a range of scales, a monofractal of dimension one [see Equation (2)]. The argument is also very simple: the average velocity of the bodies on the surface of the sphere of radius r is about $[Gm(r)/r]^{1/2}$, and this velocity must be almost constant. Of course, this argument ignores that any realization of a monofractal of dimension one has to be anisotropic. The relation $m(r) \propto r$ is anyway interesting and actually applies in various situations and goes beyond Newtonian gravity. For example, the mass of a black hole of radius r is proportional to r (the crucial velocity is then the velocity of light). A more interesting example is the singular solution of the relativistic Oppenheimer–Volkoff equation for isothermal hydrostatic equilibrium, which yields $m(r) \propto r$ (p. 320, [50]).

The quantity $Gm(r)/r$ represents minus the value of the gravitational potential on the surface of the sphere, assuming isotropy (if there is anisotropy, then consider the average over the surface of the sphere). In a multifractal distribution of mass in the universe, we may think that the condition that the gravitational potential is almost constant everywhere is too restrictive, but we should require that it be bounded. Under this relaxed condition, $\alpha = 1$ becomes a lower bound to the local dimension and this bound seems to agree with the analysis of data from cosmological simulations and observations of the galaxy distribution [35] (see the $\alpha = 1$ lower bound in Figure 2). If we further relax the condition of a bounded gravitational potential, the dimension one appears again: the set of points for which the potential diverges must have a (Hausdorff) dimension smaller than or equal to one (§18.2, [38]). In the cosmic web produced by the adhesion model, the set of filaments and point-like singularities is the set of a diverging potential and has precisely dimension equal to one. However, the analysis of data in [35] favors a mass distribution in which the potential is either finite everywhere or its set of singularities is much more reduced (the precise meaning of this is explained below).

The above preliminary considerations are interesting but have limited predictive power; unless one takes too seriously the model of bodies with a restricted range of velocities and concludes that the mass distribution must adjust to a monofractal that is irregular but one-dimensional. However, Mandelbrot (§9, [7]), who put forward the argument, also asked why the observational exponent of growth of $m(r)$ is larger than one (he quoted the value 1.23). In retrospect, we can affirm that Mandelbrot's assumption of a monofractal distribution was the main stumbling block for a better understanding of the issue. Indeed, in a multifractal, $m(r)$ has different exponents of growth at different points, and the analysis of the correlation function of galaxies, from which the value 1.23 was obtained, only yields a sort of average of them. There are reasons to expect multifractality, apart from that it realizes the most general form of scale symmetry. One reason for multifractality lies in the predictions of the adhesion model, which is a reasonably good model for the early formation of structure, including matter sheets and the voids they leave in between, as well as filaments and point-like singularities.

Nevertheless, the filaments and point-like singularities are more complex objects than the ones predicted by the adhesion model [35]. However, this pertains to Newtonian gravity, while the formation of matter filaments (*thread-like singularities*) and point-like singularities is feasible in General Relativity (Box 32.3, [29]). One may wonder why not consider the problem of structure formation in General Relativity from the start. General Relativity has no intrinsic length scale, similar to Newtonian gravity, and should also give rise to a fractal structure. Actually, General Relativity has indeed been employed in the study of large-scale structure formation. For example, in the old cosmology models with locally inhomogeneous but globally homogeneous spacetimes that everywhere satisfy Einstein's field equations [51–53] (modernly called “Swiss cheese” models). However, these models are far from realistic, because they disregard that the initial conditions of structure formation cannot lead to such a structure. In fact, large-scale structure formation can be studied within simple Newtonian gravity [6,9,10], that is to say, with the exception of zones with strong gravitational fields, where zero-size singularities can form. However, such zones have a very small size, in cosmic terms.

The adhesion model is a very simplified model of the action of gravity in structure formation, although it gives a rough idea of the type of early structures. Naturally, what we need for definite and accurate predictions of structure formation are dynamic models that are less simplified than the adhesion model. One can take the full set of cosmological equations of motion in the Newtonian limit [6,9,10]. They are applicable on scales small compared to the Hubble length and away from strong gravitational fields, but they are *nonlinear* and quite intractable. In fact, these equations bring in the classic and hard problem of *fluid turbulence*, with the added complication of the gravitational interaction [35]. In fact, methods of the theory of fluid turbulence can be applied in the theory of structure formation, and it happens that the peculiarities of the gravitational interaction can be useful as constraints.

For example, let us consider the *stable clustering* hypothesis, proposed by Peebles and collaborators for the strong clustering regime [9]. It says that the average relative velocity of pairs of bodies vanishes. This hypothesis arose in the search for a simplifying hypothesis to solve a statistical formulation of the cosmological equations, one of which was a scaling ansatz. Actually, the stable clustering hypothesis can be considered in its own right and is equivalent to the constancy in time of the average conditional density, namely the average density at distance r from an occupied point. In a monofractal, it is constant and equals the derivative of $m(r)$ divided by the area of a spherical shell of radius r [11,13]. However, just the constancy of the average conditional density does not necessarily imply scale symmetry. Nevertheless, the fact that the average conditional density is singular and indeed a power law of r can be argued on general grounds [35].

While the preceding arguments somewhat justify the scale symmetry of the mass distribution, they may not be cogent enough and do not predict a definite type of mass distribution. Of course, the problem is that we have hardly considered the consequences of the equations of motion. The statistical formulation of these equations is very complicated but it can be simplified in the *mean field limit*, valid for a large number of interacting bodies. In this limit, the one-particle distribution function suffices, and it fulfills the *collisionless Boltzmann equation* (or Vlasov equation) (§1.5, [6]) [20]. Although this equation is time-reversible, it embodies the nonlinear and chaotic nature of the gravitational dynamic and leads to *strong mixing*: the flow of matter becomes *multistreaming* on ever decreasing scales and eventually a state of dynamic equilibrium arises, in a coarse-grained approximation [37]. These dynamic equilibrium states fulfill the virial theorem, as is to be expected and is explicitly proved by the *Layzer–Irvine equation* (pp. 506–508, [10]). Naturally, the formation of dynamic equilibrium states can be considered as a concrete form of the stable clustering hypothesis.

An equilibrium state that fulfills the virial theorem is not necessarily a state of *thermodynamic equilibrium*. However, stationary solutions of the collisionless Boltzmann equation often mimic the properties of the states of thermodynamic equilibrium. As a case in point, let us take the state of

equilibrium of an isolated singularity with spherical symmetry, obtained by Gurevich and Zybin [37]. Its density is given by

$$\rho(r) \propto r^{-2} [\ln(1/r)]^{-1/3}. \quad (6)$$

This density profile can be compared with the density profile $\rho(r) \propto r^{-2}$ of the *singular isothermal sphere* (which is the asymptotic form of the density profile of any isothermal sphere for large r) [6,20]. The difference between the two profiles is very small. Let us remark that the density profile $\rho(r) \propto r^{-2}$ corresponds to the mass–radius relation $m(r) \propto r$ but is unrelated to any fractal property, because it applies to a smooth distribution with just one singularity at $r = 0$, unlike a fractal, in which every mass point is singular. A smooth distribution of matter, especially dark matter, with possibly a singularity at its center, is called in cosmology a *halo*, and it has been proposed that the large scale structure should be described in terms of halos (*halo models*) [54]. Certain distributions of halos can be considered as coarse-grained multifractals [43] and, in fact, halo models and fractal models have much in common [35].

Given the success of the assumption of thermodynamic equilibrium, we take the corresponding description of gravitationally bound states as a reasonable approach, although the temperature might have, in the end, a different meaning than it does in the thermodynamics of systems of particles that interact through a short range potential. The thermodynamic approach leads to the Poisson–Boltzmann–Emden equation:

$$\Delta\phi = 4\pi GA \exp[-\phi/T], \quad (7)$$

where ϕ is the gravitational potential, A is a normalizing constant, and T is the temperature in units of energy per unit mass (equal to one third of the mean square velocity of bodies). Several derivations of the equation appear in [20] (see also (§1.5, [6]), [21]). Probably, the simplest way of understanding this equation is to realize that it is simply the Poisson equation with a source $\rho = A \exp[-\phi/T]$ given by hydrostatic equilibrium in the gravitational field defined by ϕ itself (for the theory of thermodynamic equilibrium in an external field, see (§38, [17])).

The solutions of Equation (7) depend on the boundary conditions, of course. Simple solutions are obtained by imposing rotational symmetry, namely the already mentioned *isothermal spheres* [6,20]. Actually, each solution belongs to a family of solutions related by the scale covariance [21]

$$\phi_\lambda(\mathbf{x}) = \phi(\lambda\mathbf{x}) - T \log \lambda^2. \quad (8)$$

Thus, the scale-symmetric singular isothermal sphere can be considered as the limit of regular isothermal spheres in the same family of solutions. (Let us recall that a singular isothermal sphere with $\rho(r) \propto r^{-2}$ is also a solution in General Relativity, although the equation for thermodynamic equilibrium is more complicated than Equation (7) (p. 320, [50]).) One can also obtain the symmetric solutions corresponding to lower dimension; that is to say, if we take Equation (7) in \mathbb{R}^3 , one obtains the one- or two-dimensional solutions by making ϕ depend only on one or two variables. These solutions represent a two-dimensional sheet or a one-dimensional filament, the early structures in the adhesion model. More complex solutions of Equation (7) can be obtained as functions that asymptotically are combinations of the preceding solutions. Naturally, there are also totally asymmetric solutions, some of which look like those combinations of simple solutions (some numerical solutions appear in [55]). Notice that the formulation of a partial differential equation such as Equation (7) assumes some regularity of the function, but we are mainly interested in singular solutions. We can interpret Equation (7) in a coarse-grained sense, that is to say, as an equation for the coarse-grained variable ϕ_r (or ρ_r), and eventually take the limit $r \rightarrow 0$. However, we can obtain directly singular solutions if we reformulate Equation (7) as an integral equation (e.g., in “the Hammerstein description”) [20].

Equation (7), in two dimensions, arises in the differential geometry of surfaces, where it is called Liouville’s equation. In this context, it is the equation that rules the conformal factor of a

metric on some surface [of course, the temperature T is not present, and anyway it is always scalable away in Equation (7)]. The higher dimensional generalization of Liouville's equation also rules the conformal factor of a metric and is then connected with string theory, in which the dilaton is a dynamical field in its own right. A deeper connection of these relativistic equations, which appear in theories of quantum gravity, with the Newtonian Equation (7) may exist, but we are only concerned here with the interpretation and consequences of Equation (7) in the theory of structure formation. Nevertheless, we can take advantage of the body of knowledge developed from the two-dimensional Liouville's equation and Liouville's field theory, in particular, the theory of random multiplicative cascades [22–24,27,28]. This theory arose in relation to the *lognormal model* of turbulence [25]. The lognormal model certainly has broader scope and often arises in connection with nonlinear processes.

Examples of the application of the lognormal probability distribution function in astrophysics abound, and there are often connections between them. Hubble found that the galaxy counts in cells on the sky can be fitted by a lognormal distribution [56]. A more relevant example is Zinnecker's model of star formation by hierarchical cloud fragmentation (a random multiplicative cascade model) [57]. In cosmology, a lognormal model has been proposed as a plausible approximation to the large-scale mass distribution [58]. Even though some of these models are explicitly constructed in terms of random multiplicative cascades, the form in which a scale invariant distribution arises is by no means evident. In fact, a lognormal probability distribution function $P(\rho_r)$ of the coarse-grained density ρ_r does not imply by itself any scale symmetry, even if it holds for all r . It is just the dependence of $P(\rho_r)$ on r that gives rise to scale symmetry. To be precise, in general, it is necessary that Equation (1) holds. In the lognormal model, Equation (1) is fulfilled for a particular form of $\tau(q)$.

Thus, we are led to the study of random self-similar multiplicative cascades. They are a generalization of the non-random self-similar multifractals briefly reviewed in the preceding section (Section 2). The study of random cascades is the task of the next section.

4. Random Cascades

Here, we are not interested in regular solutions of Equation (7) but in singular solutions, in particular, in solutions such that $\rho_r(x)$ is either very small or very large. As support for the existence of such solutions, we can argue that $\rho = A \exp[-\phi/T]$ is always positive but experiences great fluctuations, being close to zero at many points but very large at other points, in accord with the fluctuations of ϕ . We are also interested in solutions that fulfill the principle of statistical homogeneity and isotropy of the mass distribution. The way to obtain such solutions is by means of the theory of random multiplicative cascades.

The construction of deterministic self-similar multifractals introduced in Section 2 is based on the concept of iterated function system. A function system consists of a set of contracting similarities (with some separation condition). In addition, it is defined a set of mass ratios that defines how the initial mass is split into the sets that result from the application of the function system. There are randomized versions of this construction that are not suitable for our problem. For example, one can randomize the distribution of mass on the sets that result from the function system, by changing the order at random without altering the mass ratios [59]. This construction has mathematical interest but is not very relevant for us, because it cannot produce either nonlacunarity or statistical homogeneity and isotropy. We instead focus on the random multiplicative cascades based on Obukhov–Kolmogorov's 1962 model of turbulence [25,26].

Obukhov and Kolmogorov intended to describe spatial fluctuations of the energy dissipation rate in turbulence and were inspired by Kolmogorov's lognormal law of the size distribution in pulverization of mineral ore. This process can be essentially described as a multiplicative random cascade. In fact, similar models had been introduced earlier in economics, under the name of law of *proportionate effect* [60]. All these models are by nature only statistical. The connection with fractal geometry is due to Mandelbrot, in the late 1960s (the history is exposed in the reprint book [27]).

He observed that a spatial random cascade process of the type used in turbulence, when continued indefinitely, leads to an energy dissipation generally concentrated on a set of non-integer (fractal) Hausdorff dimension. This property was manifested in some especially simple cascade models, e.g., the β -model, in which the concentration set is monofractal [26]. However, monofractality implies a singular probability $P(\rho_r)$, because a nontrivial monofractal has to be lacunar. Therefore, $P(\rho_r)$ is unlike a lognormal probability distribution and is not relevant for models of the cosmic mass distribution. Actually, Mandelbrot's own random multiplicative cascades generically give rise to multifractal mass distributions [27]. These are the random cascades of interest here.

A *multiplicative process* is defined as follows. Suppose that we start with some positive variable of size x_0 that at each step n can grow or shrink, according to some positive random variable W_n , so that

$$x_n = x_{n-1}W_n. \quad (9)$$

Let us think of x as a mass. The proportionate effect consists in that the random growth of x is a percentage of its current mass and is independent of its current actual mass. One may want x to be as likely to grow as to shrink, on average; that is to say, its mean value must stay constant. Usually, the random variables W_n , for $n = 1, \dots$, are assumed to be independent, and even assumed to be independent copies of the same random variable W . A further condition is that W has moments of any order. In summary, the random variable W is subject to:

$$W \geq 0, \quad \langle W \rangle = 1, \quad \langle W^q \rangle < \infty, \quad \forall q > 0. \quad (10)$$

As is easy to notice, the definition of multiplicative process is analogous to the definition of additive process, which gives rise to the central limit theorem and the Gaussian distribution. However, a multiplicative process is very different. A simple example shows this: if we take W to be zero or two with equal probability, then the product of N instances of W will be zero unless all values of W_n turn out to be two, in which case the product is very large, equal to 2^N . The mean square value and the variance of the product are also large and higher moments are even larger. In fact, the moments of the product of N instances of W are the N th power of the moments of W , given by

$$\log_2 \langle W^q \rangle = q - 1. \quad (11)$$

This simple example is actually a particular case of the β -model, with $\beta = 1/2$ (§8.6.3, [26]).

The β -model is somewhat singular, because it lets W be null with a non-vanishing probability. We can easily change this feature by taking W to be $2p$ or $2(1-p)$ with equal probability, where $0 < p < 1$ ($p = 0$ or $p = 1$ give the case already considered). Now, we have

$$\log_2 \langle W^q \rangle = \log_2 \frac{(2p)^q + (2(1-p))^q}{2} = \log_2 [p^q + (1-p)^q] + q - 1. \quad (12)$$

Therefore, the q -moment of the product of N instances of W is given by the expression

$$\langle \left(\prod_{n=1}^N W_n \right)^q \rangle = \langle W^q \rangle^N = 2^{N[-\tau(q)+q-1]}, \quad (13)$$

where $\tau(q)$ is given by Equation (4), because the binomial multiplicative process of Section 2 is closely connected with this process. The binomial multifractal is possibly the simplest example of nonlacunar self-similar multifractal, but it is deterministic. The connection with the present stochastic process is based on the identification of the average over W in the multiplicative process with a "binary" spatial average over the mass distribution in the binomial multifractal. Of course, the binomial multifractal could be randomized by using a generalization of the procedure in [59]. However, this construction would have undesirable features. First, the binary subdivision procedure

lets itself to be noticed in the final result, breaking the desired statistical homogeneity (as well as statistical isotropy, in a three-dimensional generalization). Second, the total mass is fixed at the initial stage in some given volume and we would rather have that it has fluctuations that tend to zero as the volume tends to infinity.

For the application of multiplicative processes to the description of the cosmic mass distribution, we have two options. One is to consider a fixed volume and to understand that the mass fluctuation in it grows with time, according to the natural generalization of Equation (13), where N is the number of time steps. Another is the standard generalization of the definition of multiplicative processes for the construction of self-similar multifractals, where the volume is reduced in every iteration and the average mass is reduced in the same proportion. For example, for a lattice multifractal in \mathbb{R}^n , we need to replace in Equation (10) $\langle W \rangle = 1$ with $\langle W \rangle = b^{-n}$, where $b \geq 2$ is the number of subintervals (b^{-n} is the volume reduction factor) [28]. In general, if we define

$$\tau(q) = -\log_b \langle W^q \rangle - n, \quad (14)$$

then, taking into account that $r/r_0 = b^{-N}$, after N steps, and that $\rho_r = m_r/r^3$, we recover Equation (1) in the case $n = 3$.

Let us now see how the lognormal distribution arises from Equation (9). Taking logarithms,

$$\log x_N = \log x_0 + \sum_{n=1}^N \log W_n. \quad (15)$$

Assuming that the random variables $\log W_n$ fulfill the standard conditions to apply the central limit theorem, this theorem implies that the sum of the variables converges to a normal distribution and, therefore, the quotient x_N/x_0 converges for large N to a lognormal distribution. This argument has been criticized on the basis of the theory of large deviations, which just says that the limit distribution is expressed in terms of the *Cramér function*, while the central limit theorem only applies to small deviations about its maximum, in the form of a quadratic approximation (§8.6, [26]) [27]. This quadratic approximation is insufficient to characterize a full multifractal spectrum and only provides an approximation of it, whereas the Cramér function is sufficient. The insufficiency of the central limit theorem is manifest in that there are many different quadratic approximations to the multifractal spectrum. Actually, two of them are especially important: the quadratic expansions about the two distinguished points of a multifractal, namely the point where $\alpha = f(\alpha)$, which is the point of *mass concentration*, or the point where $f(\alpha)$ is maximum, which corresponds to the *support* of the mass distribution (which is the full \mathbb{R}^3 in our case, thus $f(\alpha) = 3$) [44]. Both quadratic approximations can be employed but they give different results in general. At any rate, a lognormal multiplicative process can be obtained by just requiring that the random variables W_n are lognormal themselves, so that the multifractal spectrum has an exact parabolic shape (§8.6, [26]) [27,28]. However, this lognormal model presents some problems (§8.6.5, [26]) [27].

A surprising feature of an exactly parabolic multifractal spectrum $f(\alpha)$ is that it prolongs to $f(\alpha) < 0$, so there are negative fractal dimensions. The general nature of this anomaly has been discussed by Mandelbrot [61]. He says that “the negative $f(\alpha)$ rule the sampling variability” in a random multifractal. Any set of singularities of strength α with $f(\alpha) < 0$ is *almost surely* empty, but Mandelbrot states that those values “measure usefully the degree of emptiness of empty sets.” My option in the multifractal analysis of experimental and observational cosmic mass distributions is to discard the part of the multifractal spectrum with $f(\alpha) < 0$ [62].

One more point of concern is possibly the discrete nature of multiplicative processes, which are built on a dyadic or b -adic tree from the larger to the smaller scales and produce multifractals that have two main drawbacks: they display discrete scale invariance only and are not strictly translation invariant (or isotropic in \mathbb{R}^n , $n > 1$). In statistical (or quantum) field theory, the scale symmetry takes place at *fixed points* of the renormalization group, which is usually formulated as an iteration

of a discrete transformation but admits a continuous formulation [63]. An analogous procedure can be carried out for multiplicative processes. A simple example of the continuous formulation of multiplicative processes is provided by substituting the random walk equation that produces Brownian motion, namely

$$\frac{dx}{dt} = \zeta(t), \tag{16}$$

where $\zeta(t)$ is a Gaussian process (e.g., white noise), by the multiplicative equation

$$\frac{dx}{dt} = \zeta(t) x. \tag{17}$$

The solution of this equation is

$$x(t) = x_0 \exp \int_0^t \zeta(s) ds. \tag{18}$$

It undergoes large fluctuations, such as the discrete multiplicative processes described above, and can actually be considered as a continuous formulation of the lognormal multiplicative process. A general definition of continuous multiplicative processes employs the concept of log infinitely divisible probability distribution [64]. A particularly useful class of these continuous multiplicative processes is given by the log-Lévy generators (§6.3.7, [28]). This is an interesting topic but rather technical, so we do not dwell on it.

In summary, we have a construction of multifractals that are statistically homogeneous and isotropic (in \mathbb{R}^n , $n > 1$) and have continuous scale invariance. Furthermore, they satisfy Equation (1), for some function $\tau(q)$. In fact, we can consider the multifractal as a scale symmetric mass distribution that is obtained as the continuous parameter $t \rightarrow \infty$. In this sense, it is analogous to a statistical system at a critical point defined in terms of a fixed point of the renormalization group. However, let us insist that a multifractal is *fully* scale symmetric, whereas a critical statistical system is only required to have scale symmetric *fluctuations*. Nevertheless, the mass fluctuations in a continuous scale multifractal also tend to zero as $t \rightarrow -\infty$ (at very large scale). Therefore, we may consider the transition from the fully homogeneous and isotropic cosmic mass distribution on very large scales to the multifractal distribution on smaller yet large scales as a *crossover* analogous to the ones that take place in the theory of critical phenomena in statistical physics.

We must also notice that the scale parameter t , analogous to a renormalization group parameter, can be replaced by a real scale, such as the coarse-graining scale r of Equation (1). Naturally, $r/r_0 = e^{-t}$, as a generalization of $r/r_0 = b^{-N}$ (the continuous limit can be thought of as the limit where $b \rightarrow 1$ and $N \rightarrow \infty$, with $N \ln b$ finite). The use of coarse-grained quantities is necessary to connect with the partial differential equation (Equation (7)). This connection is explained in detail in the collective work of a group of researchers in probability theory out by Rhodes and Vargas [22–24,65,66]. Here, we just need to notice the relation of a log-correlated field, with

$$\langle \phi(x)\phi(y) \rangle = -\log |x - y| \tag{19}$$

for small $|x - y|$, to the multifractal cascades. This relation comes from the mass density being

$$\rho = A \exp[-\phi/T] \tag{20}$$

in Equation (7), so that the correlations of the density field are power laws and, in particular, Equation (1) holds. Of course, this relation cannot specify the multifractal properties, given by the function $\tau(q)$. The case of a lognormal distribution was studied by Coles and Jones [58], without making connection with Equation (7). The lognormal model has a quadratic function $\tau(q)$, whose Legendre transform gives the parabolic multifractal spectrum $f(\alpha)$ (§6.3.16, [28]).

5. Experimental and Observational Results

Fortunately, nowadays we have good knowledge of the cosmic mass distribution, based on the considerable amount of data from cosmological simulations and from observations of the galaxy distribution. These data are suitable for various statistical analyses and, in particular, for multifractal analyses. Naturally, the highest quality data come from cosmological N -body simulations. There has been a steady increase in the number of bodies N that computers are capable of handling, and state-of-the-art simulations handle billions of particles, and thus afford excellent mass resolution. Moreover, N -body simulations have a numerical experimentation capability, because one can tune several aspects of the cosmic evolution, such as the initial conditions, the content of baryons in relation to dark matter, etc.

A number of multifractal analyses of the cosmic mass distribution were made years ago [11,12,40–42]. However, the quality of the data then was not sufficient to obtain reliable results. I have carried out multifractal analyses of the mass distribution in recent several N -body simulations, which have consistently yielded the same shape of the multifractal spectrum. Probably, the most interesting results to quote are the most recent ones, from the *Bolshoi* simulation [35,67]. This simulation has very good mass resolution of the cosmic structure: it contains $N = 2048^3$ particles in a volume of $(250 \text{ Mpc})^3 / h^3$ [h is the Hubble constant normalized to $100 \text{ km}/(\text{s Mpc})$]; so that each particle represents a mass of $1.35 \cdot 10^8 h^{-1} M_{\odot}$, which is the mass of a small galaxy. Therefore, it is possible to establish convergence of several coarse-grained spectra to a limit function, as the coarse-graining length is shrunk to its minimum value available in the particle distribution. Actually, the shape of the multifractal spectrum is stable along a considerable range of scales, proving the self-similarity of the mass distribution (Figure 2). It is to be noticed that the *Bolshoi* simulation only describes dark matter particles. However, the *Mare-Nostrum* simulation describes both dark matter and baryon gas particles and gives essentially the same results [68].

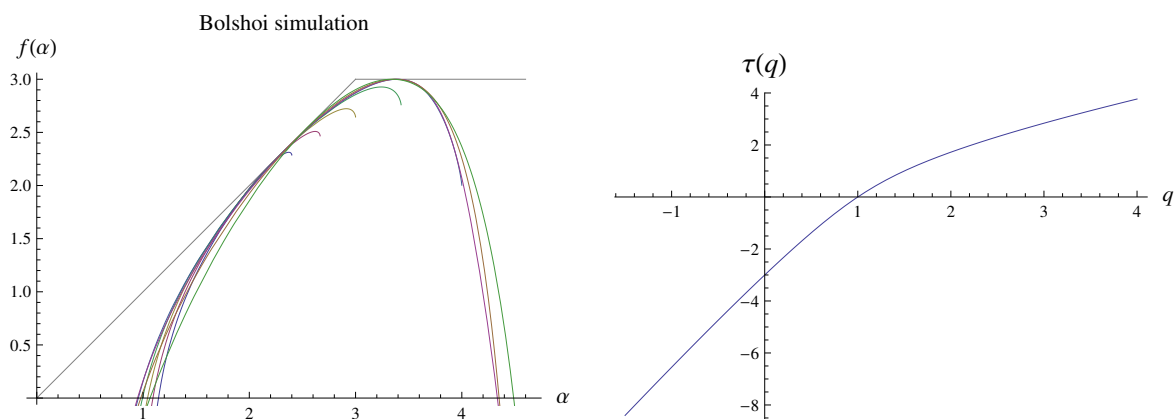


Figure 2. (Left) The multifractal spectra of the dark matter distribution in the Bolshoi simulation (coarse-graining lengths $l = 3.91, 1.95, 0.98, 0.49, 0.24, 0.122, 0.061, 0.031 \text{ Mpc}/h$). (Right) The function $\tau(q)$ (calculated at $l = 0.98 \text{ Mpc}/h$), showing that $\tau(0) = -3$, corresponding to nonlacunarity. This property is also patent in the left graph, because it shows that $\max f(\alpha) = 3$ (for sufficiently large l).

Of course, the most realistic results about the cosmic mass distribution must come from observations of the real universe. Even though there are increasingly better observations of the dark matter, the main statistical analyses of the overall mass distribution come from observations of the galaxy distribution. I have carried out a multifractal analysis of the distribution of stellar mass employing the rich Sloan Digital Sky Survey (data release 7) [62]. The stellar mass distribution is a proxy of the full baryonic matter distribution and is simply obtained from the distribution of galaxy positions by taking into account the stellar masses of galaxies, which are available for the Sloan Digital Sky Survey (data release 7).

We can assert a good concordance between the multifractal geometry of the cosmic structure in cosmological N -body simulations and galaxy surveys, to the extent that the available data allow us to test it, that is to say, in the important part of the multifractal spectrum $f(\alpha)$ up to its maximum (the part such that $q > 0$) [35,62]. The other part, such that $q < 0$, has $\alpha > 3$ and would give information about voids in the stellar mass distribution, but the resolution of the SDSS data is insufficient in this range. The common features of the multifractal spectrum found in Ref. [62] and visible in Figure 2 are: (i) a minimum singularity strength $\alpha_{\min} = 1$; (ii) a “supercluster set” of dimension $\alpha = f(\alpha) \simeq 2.5$ where the mass concentrates; and (iii) $\max f(\alpha) = 3$, giving a non-lacunar structure (without totally empty voids). As regards Point (i), it is to be remarked that $\alpha_{\min} = 1$, with $f(\alpha_{\min}) = 0$, corresponds to the edge of diverging gravitational potential. However, the multifractal spectrum $f(\alpha)$ prolongs to $f(\alpha) < 0$, giving rise to stronger singularities, which have null probability of appearing in the limit of vanishing coarse-graining length, $l \rightarrow 0$ (a set with negative dimension is *almost surely* empty). Nevertheless, these strong singularities do appear in any coarse-grained mass distribution and correspond to negative peaks of the gravitational potential ϕ , which must not be divergent in the $l \rightarrow 0$ limit. Thus, we seem to have a mass distribution in which the potential can become large (in absolute value) but is finite everywhere (recall the options brought up in Section 3).

Given that we now know the multifractal spectrum of the cosmic mass distribution with reasonable accuracy, we can look for the type of random multiplicative cascade that produces such spectrum. This is an appealing task that is left for the future.

6. Discussion

We show that there is a considerable range of scales in the universe in which scale symmetry is effectively realized, that is to say, the mass distribution is a self-similar multifractal, with identical appearance and properties at any scale. This symmetry is a consequence of the absence of any intrinsic length scale in Newtonian gravitation, which is the theory that rules the mass distribution on scales beyond the size of galaxies but small compared to the Hubble length. Indeed, it is found in the analysis of cosmological N -body simulations and in the analysis of the stellar mass distribution (with less precision) that the self-similarity extends from a fraction of Megaparsec to several Megaparsecs. On larger scales, the multifractal mass distribution shows signs of undergoing the transition to the expected homogeneity of the Friedmann–Lemaître–Robertson–Walker relativistic model of the universe, in accord with the standard cosmological principle.

We describe models that enforce scale symmetry in combination with the other relevant symmetries, namely the translational and rotational symmetries that impose homogeneity and isotropy and that must be understood in a statistical sense, related to Mandelbrot’s conditional cosmological principle. We show that those models are given by the theory of continuous random multiplicative cascades. We show how random multiplicative cascades can be constructed, made continuous, and produce the type of multifractal mass distribution that we need.

Of course, the use of continuous random multiplicative cascades could be regarded as somewhat ad hoc, as they seem unrelated to the gravitational physics. However, we have explained the close connection of those models with the partial differential equation that arises from an approximate model of gravitational physics, namely the Poisson–Boltzmann–Emden equation that follows from the assumption of thermodynamic equilibrium. While this assumption may not be fully realized in the cosmological evolution of structure formation, it is a reasonable approach to the states of virial equilibrium, say, to the regime of strong and stable clustering.

We also mention that the early stage of structure formation is approximately described by the adhesion model, which predicts a self-similar cosmic web somewhat different from the result of a continuous random multiplicative cascade related to the Poisson–Boltzmann–Emden equation, as can be perceived in Figure 1. It seems that both types of structures should be combined, being the web morphology appropriate on the larger scales and the full self-similar multifractal structure, related to the Poisson–Boltzmann–Emden equation, appropriate on smaller scales. This combination

should take into account that the matter sheets and the corresponding voids present no problem in Newtonian gravity, whereas the matter filaments and point-like singularities do and they should be replaced by weaker singularities, of power law type, precisely such as the ones that are found as simple solutions of the Poisson–Boltzmann–Emden equation; namely, radial or axial singular isothermal distributions. The combined structure achieves a mass distribution without singularities of the gravitational potential.

Since the Poisson–Boltzmann–Emden equation describes halo-like structures (Section 3), we can expect that a coarse-grained formulation of the above proposed combination of singular solutions of the Poisson–Boltzmann–Emden equation with a larger-scale web structure would be equivalent to the fractal distribution of halos that can be deduced from a coarse-grained multifractal [43,44]. Moreover, such a distribution of halos should lie in web sheets or filaments, as is expected [54,67].

Finally, we can just mention here that the formalism described in this paper can possibly be applied in a very different range of scales and with a different theory of gravity, namely the very small scales that constitute the realm of quantum gravity. The potential of scale symmetry and, furthermore, of conformal symmetry in theories of quantum gravity is well established and, in fact, string theory is essentially based on the conformal symmetry. Anyway, this very interesting connection lies beyond the scope of the present work.

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