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Fixed Point Theorems Applied in Uncertain Fractional Differential Equation with Jump

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Received: 7 April 2020; Accepted: 1 May 2020; Published: 6 May 2020

Abstract: No previous study has involved uncertain fractional differential equation (FDE, for short) with jump. In this paper, we propose the uncertain FDEs with jump, which is driven by both an uncertain V-jump process and an uncertain canonical process. First of all, for the one-dimensional case, we give two types of uncertain FDEs with jump that are symmetric in terms of form. The next, for the multidimensional case, when the coefficients of the equations satisfy Lipschitz condition and linear growth condition, we establish an existence and uniqueness theorems of uncertain FDEs with jump of Riemann-Liouville type by Banach fixed point theorem. A symmetric proof in terms of form is suitable to the Caputo type. When the coefficients do not satisfy the Lipschitz condition and linear growth condition, we just prove an existence theorem of the Caputo type equation by Schauder fixed point theorem. In the end, we present an application about uncertain interest rate model.

Keywords: uncertain fractional differential equations; V-jump process; existence and uniqueness; banach fixed point theorem; schauder fixed point theorem

1. Introduction

Wiener process is a type of stationary-independent increment stochastic process with normal random increments designed by Wiener in 1923 [1]. Then stochastic differential equation (SDE, for short) was proposed by Itô in 1951 as a vital tool to model stochastic dynamic systems [2]. Following that, many areas such as noted European option pricing model [3] by SDEs and famous stochastic epidemic dynamic model hidden in the observed data [4] were developed. As all we know, The SDEs based on probability theory need a large of available sample data. However, when we lack of data or the size of sample data applied in practice are less in many situations, we need to invite some domain experts to evaluate the belief degree that each event happens.

Human uncertainty with respect to belief degrees [5] can play an important role in addressing the issue of indeterminate phenomenon. For describing the evolution of uncertain phenomenon, the uncertain differential equation (UDE, for short) was first proposed by Liu [6]. Following that, Liu [7] also proposed the concept of stability of UDEs. Later, Chen and Liu [8] proved an existence and uniqueness theorem for an UDE and Yao et al. [9] proved some stability theorems. Besides, a large and growing body of literature [10–14] about stability theorems for UDEs have been investigated. Further, Yao and Chen [15] first proposed Euler's method combined with 99-method to obtain the numerical solution of the UDEs. With the perfect of theory and maturity of numerical method of the UDEs, The UDEs have been successfully applied to many area such as optimal control theory [16,17], differential game theory [18,19], wave equation [20–22] and finance theory [23]. To understand developing process of the UDEs comprehensively, the readers can refer the book [24].

V-jumps uncertain processes proposed by Deng et al. [25] are often used to describe the evolution of uncertain phenomenon with jumps, in which the uncertain process may be caused a sudden change by emergency, such as economics crisis, outbreaks of infectious diseases, earthquake, war, etc. Here, It is needed to see that the cadlag functions [26] (right-continuous with left limits) are vital to deal with point process and related applications. The definition of V-jump uncertain process is as follows

Definition 1. *An uncertain process V_k with respect to time k is said to be a V-jump process with parameters θ_1 and θ_2 $(0 < \theta_1 < \theta_2 < 1)$ for $k \geq 0$ if*
(i) $V_0 = 0$,
(ii) V_k has stationary and independent increments,
(iii) For any given $k > 0$, every increment $V_{r+k} - V_r$ is a $\mathcal{Z}$ jump uncertain variable $\xi \sim \mathcal{Z}(\theta_1, \theta_2, k)$ for $\forall r > 0$, whose uncertainty distribution is

$$\Phi(x) = \begin{cases} 0 & if\ x < 0, \\ \dfrac{2\theta_1}{k}x & if\ 0 \leq x < \dfrac{k}{2}, \\ \theta_2 + \dfrac{2(1-\theta_2)}{k}(x - \dfrac{k}{2}) & if\ \dfrac{k}{2} \leq x < k, \\ 1 & if\ x \geq k. \end{cases}$$

Deng et al. [27] proved an existence and uniqueness of solution to UDE with V-jump under Lipschitz condition and linear growth condition on the coefficients. The uncertain differential equation with V-jump is expressed as follows

$$dZ_k = p_1(Z_k, k)dk + p_2(Z_k, k)dC_k + p_3(Z_k, k)dV_k,$$

where C_k is an uncertain canonical process with respect to time k, V_k is an uncertain V-jump process with respect to time k, and p_1, p_2 and p_3 are some given functions.

Uncertain differential equations with V-jumps are widely applied to uncertain optimal control with V-jumps. Some related references can be seen in [28–33]. For the phenomena of complex systems, fractional differential equations (FDEs) [34] are very suitable for characterizing materials and processes with memory and genetic properties. When considering the research of uncertain complex systems, we are eagerly looking forward to having a usable mathematical tool and basic principles to model these complex systems. To better describe the uncertain complex phenomena, Zhu [35] proposed two types of uncertain fractional differential equations in the one-dimensional case, which is the Riemann-Liouville type and Caputo type, respectively. In the same year, Zhu [36] proved the existence and uniqueness of two types of uncertain fractional differential equations in the multidimensional case. The expressions of these two types of equations are as follows

$$D^p Z_k = f(k, Z_k) + g(k, Z_k)\frac{\mathrm{d}C_k}{\mathrm{d}k}$$

and

$${}^cD^p Z_k = f(k, Z_k) + g(k, Z_k)\frac{\mathrm{d}C_k}{\mathrm{d}k}$$

where D^pZ_k and ${}^cD^pZ_k$ denote the Riemann-Liouville type and Caputo type fractional derivative of the function Z_k, respectively. C_k is an uncertain canonical process with respect to time k, f, g are given functions.

Based on the above uncertain FDEs, Lu et al. [37] further analyzed the solution of the uncertain linear FDE. Lu et al. [38] proposed the numerical methods for uncertain FDEs and compared some principles [39] for FDEs with the Caputo derivatives. Jin et al. [40] simulated the extreme values for solution to uncertain FDE and applied it to American stock model. To model discrete fractional calculus, Lu et al. [41] proposed uncertain fractional forward difference equations for Riemann-Liouville type. Furthermore, Lu et al. [42] investigated finite-time stability of uncertain FDEs. However, the uncertain FDEs with jump has not been studied so far. Inspired by Zhu [35,36] and Deng et al. [25,27], for describing the state of the uncertain fractional differential system with jumps more accurately, we propose uncertain FDEs with jump, which is very significant for the characterization of uncertain complex systems when meeting a sudden change by emergency.

The remainder of the paper is organized as follows. In Section 2, we recall some concepts of fractional order derivatives. Section 3 first gives two types of uncertain FDEs with jump in the one-dimensional case, then analyzes the multidimensional case, gives existence and uniqueness theorem of uncertain FDEs with jump by fixed point theorem, finally discuss an application about uncertain interest rate model. In Section 4, we give a brief conclusion.

2. Fractional Order Derivatives

We first recall two classes of fractional order derivatives in the one-dimensional case.

Definition 2. *[43] The fractional primitive of order $p > 0$ of a function $\phi : [u, v] \to \mathbb{R}$ is defined by*

$$I^p_{u+}\phi(k) = \frac{1}{\Gamma(p)}\frac{\mathrm{d}}{\mathrm{d}k}\int_u^k (k-r)^{p-1}\phi(r)\mathrm{d}r,$$

where Γ is the gamma function satisfying

$$\Gamma(\varrho) = \int_0^\infty k^{\varrho-1}e^{-k}\mathrm{d}k, \quad \varrho > 0$$

Remark 1. *[43] The properties of the gamma function are as follows:*

$$\Gamma(\varrho+1) = \varrho\Gamma(\varrho),\ \varrho > 0;\ \Gamma(1) = 1;\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Besides, the beta function satisfying

$$B(p,q) = \int_0^1 \vartheta^{p-1}(1-\vartheta)^{q-1}d\vartheta, p > 0, q > 0,$$

and $B(p,q) = B(q,p)$. The relation between them is

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, p > 0, q > 0.$$

Definition 3. *[43] For a function ϕ given on interval $[u, v]$, the pth Riemann-Liouville fractional order derivative of ϕ is defined by*

$$D^p_{u+}\phi(k) = \frac{1}{\Gamma(m-p)}\frac{\mathrm{d}^m}{\mathrm{d}k^m}\int_u^k (k-r)^{m-p-1}\phi(r)\mathrm{d}r,$$

where $m - 1 < p \le m$.

Define $D^0_{u+} = I^0_{u+} = I$*, where* I *is identity operator, it holds that*

$$D^p_{u+} I^p_{u+} = I, I^p_{u+} D^p_{u+} \neq I, p \geq 0.$$

For a power function $(k-u)^\eta$*, it holds that*

$$D^p_{u+}(k-u)^\eta = \frac{\Gamma(\eta+1)}{(\eta+1-p)}(k-u)^{\eta-p}, p > 0, \eta > -1, k > u.$$

Definition 4. *[43] Let* $\phi : [u,v] \to \mathbb{R}$ *at least be a* m *order differentiable function. The pth Caputo fractional derivative of* ϕ *is defined by*

$${}^cD^p_{u+}\phi(k) = \frac{1}{\Gamma(m-p)} \int_u^k (k-r)^{m-p-1} \phi^{(m)}(r) \mathrm{d}r,$$

where $m - 1 < p \leq m$*, and* $\phi^{(m)}(r)$ *is the m-derivative of* ϕ*.*

Remark 2. *[43] For* $m-1 < p \leq m$ *and* $k > 0$*, it holds that*

$$D^p_{u+}\phi(k) = {}^cD^p_{u+}\phi(k) + \sum_{l=0}^{m-1} \frac{(k-u)^{l-p}}{\Gamma(l-p+1)} \phi^{(l)}(u).$$

Remark 3. *[43] For convenience, we use* I^p*,* D^p *and* ${}^cD^p$ *denote by* I^p_{0+}*,*D^p_{0+} *and* ${}^cD^p_{0+}$*, respectively. We next recall two classes of fractional order derivatives in the multidimensional case.*

(a) The pth Riemann-Liouville fractional order derivative of the function $\phi : [0,T] \to \mathbb{R}^n$ *is defined by*

$$D^p\phi(k) = \frac{1}{\Gamma(1-p)} \frac{\mathrm{d}}{\mathrm{d}k} \int_0^k (k-r)^{-p} \phi(r) \mathrm{d}r, \quad k > 0.$$

(b) The pth Caputo fractional order derivative of the function $\phi : [0,T] \to \mathbb{R}^n$ *is defined by*

$${}^cD^p\phi(k) = \frac{1}{\Gamma(1-p)} \int_0^k (k-r)^{-p} \phi'(r) \mathrm{d}r, \quad k > 0$$

where $\phi'(r)$ *is the first-order derivative of* $\phi(r)$*.*

Meanwhile, they have the following relationship

$$D^p\phi(k) = {}^cD^p\phi(k) + \frac{k^{-p}}{\Gamma(1-p)}\phi(0).$$

3. Main Results

3.1. Two Types of Uncertain FDEs with Jump in the One-Dimensional Case

Definition 5. *Let* C_k *be a canonical process and* V_k *be a V-jump process. Suppose that* $f, g, h : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *are three functions. Then*

$$D^p Z_k = f(k, Z_k) + g(k, Z_k)\frac{\mathrm{d}C_k}{\mathrm{d}k} + h(k, Z_k)\frac{\mathrm{d}V_k}{\mathrm{d}k} \tag{1}$$

is called an uncertain FDE with jump of the Riemann-Liouville type. A solution of (1) *with the initial condition*

$$\lim_{k \to 0+} k^{1-p} Z_k = z_0$$

is an uncertain process Z_k such that

$$
\begin{aligned}
Z_k =& k^{p-1}z_0 + I^p f(k,Z_k) + I^p\left(g(k,Z_k)\frac{dC_k}{dk}\right) + I^p\left(h(k,Z_k)\frac{dV_k}{dk}\right)\\
=& k^{p-1}z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} f(r,Z_r)\mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} g(r,Z_r)\mathrm{d}C_r\\
&+ \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} h(r,Z_r)\mathrm{d}V_r
\end{aligned} \tag{2}
$$

holds almost surely.

Definition 6. *Let C_k be a canonical process and V_k be a V-jump process. Suppose that $f, g, h : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are three functions. Then*

$$
{}^cD^pZ_k = f(k,Z_k) + g(k,Z_k)\frac{\mathrm{d}C_k}{\mathrm{d}k} + h(k,Z_k)\frac{\mathrm{d}V_k}{\mathrm{d}k} \tag{3}
$$

is called an uncertain FDE of the Caputo type. A solution of (3) is an uncertain process Z_k such that

$$
\begin{aligned}
Z_k =& Z_0 + I^p f(k,Z_k) + I^p\left(g(k,Z_k)\frac{dC_k}{dk}\right) + I^p\left(h(k,Z_k)\frac{dV_k}{dk}\right)\\
=& Z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} f(r,Z_r)\mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} g(r,Z_r)\mathrm{d}C_r\\
&+ \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} h(r,Z_r)\mathrm{d}V_r
\end{aligned} \tag{4}
$$

holds almost surely.

We will use the following classical Mittag-Leffler function [43]

$$
E_{p,q}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(pj+q)}, p > 0, q > 0
$$

Theorem 1. *Let C_k and V_k be two integrable uncertain processes.*

(i) The uncertain FDE with jump

$$
D^pZ_k = \mu_k + \nu_k\frac{\mathrm{d}C_k}{\mathrm{d}k} + \sigma_k\frac{\mathrm{d}V_k}{\mathrm{d}k}, k > 0
$$

with the initial condition

$$
\lim_{k\to 0+} k^{1-p}Z_k = z_0
$$

has a solution

$$
\begin{aligned}
Z_k =& k^{p-1}z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\mu_r\mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\nu_r\mathrm{d}C_r\\
&+ \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\sigma_r\mathrm{d}V_r
\end{aligned}
$$

(ii) The uncertain FDE with jump

$$
{}^cD^pZ_k = \mu_k + \nu_k\frac{\mathrm{d}C_k}{\mathrm{d}k} + \sigma_k\frac{\mathrm{d}V_k}{\mathrm{d}k}, k > 0
$$

has a solution

$$Z_k = Z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\mu_r \mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\nu_r \mathrm{d}C_r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\sigma_r \mathrm{d}V_r$$

Proof. We can obtain the conclusions from (2) and (4), respectively. □

Theorem 2. *Let a, b and e be three numbers, and* $\mu, \nu, \sigma > -1$.

(i) The uncertain FDE with jump

$$D^p Z_k = ak^\mu + bk^\nu \frac{\mathrm{d}C_k}{\mathrm{d}k} + ek^\sigma \frac{\mathrm{d}V_k}{\mathrm{d}k}, k > 0$$

with the initial condition

$$\lim_{k \to 0+} k^{1-p} Z_k = z_0$$

has a solution

$$Z_k = k^{p-1}z_0 + \frac{a\Gamma(\mu+1)}{\Gamma(p+\mu+1)}k^{p+\mu} + \frac{b}{\Gamma(p)}G_k + \frac{e}{\Gamma(p)}H_k$$

where $G_k = \int_0^k (k-r)^{p-1} r^\nu \mathrm{d}C_r$ *is a normal uncertain variable*

$$G_k \sim \mathcal{N}\left(0, \frac{\Gamma(p)\Gamma(\nu+1)}{\Gamma(p+\nu+1)}k^{p+\nu}\right)$$

where $H_k = \int_0^k (k-r)^{p-1} r^\nu \mathrm{d}V_r$ *is a* $\mathcal{Z}$ *jump uncertain variable*

$$H_k \sim \mathcal{Z}\left(\theta_1, \theta_2, \frac{\Gamma(p)\Gamma(\sigma+1)}{\Gamma(p+\sigma+1)}k^{p+\sigma}\right)$$

(ii) The uncertain FDE with jump

$$^cD^p Z_k = ak^\mu + bk^\nu \frac{\mathrm{d}C_k}{\mathrm{d}k} + ek^\sigma \frac{\mathrm{d}V_k}{\mathrm{d}k}, k > 0$$

has a solution

$$Z_k = Z_0 + \frac{a\Gamma(\mu+1)}{\Gamma(p+\mu+1)}k^{p+\mu} + \frac{b}{\Gamma(p)}G_k + \frac{e}{\Gamma(p)}H_k$$

where $G_k = \int_0^k (k-r)^{p-1} r^\nu \mathrm{d}C_r$ *is a normal uncertain variable*

$$G_k \sim \mathcal{N}\left(0, \frac{\Gamma(p)\Gamma(\nu+1)}{\Gamma(p+\nu+1)}k^{p+\nu}\right)$$

where $H_k = \int_0^k (k-r)^{p-1} r^\sigma \mathrm{d}V_r$ *is a* $\mathcal{Z}$ *jump uncertain variable*

$$H_k \sim \mathcal{Z}\left(\theta_1, \theta_2, \frac{\Gamma(p)\Gamma(\sigma+1)}{\Gamma(p+\sigma+1)}k^{p+\sigma}\right)$$

Proof. (i) It follows from Theorem (1) that

$$\begin{aligned} Z_k =& k^{p-1}z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} a r^{\mu} \mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} b r^{\nu} \mathrm{d}C_r \\ &+ \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} e r^{\sigma} \mathrm{d}V_r, (\text{let } r = \tau k, 0 \le \tau \le 1) \\ =& k^{p-1}z_0 + \frac{a}{\Gamma(p)}\int_0^1 k^{p-1}(1-\tau)^{p-1}\tau^{\mu}k^{\mu}k\mathrm{d}\tau + \frac{b}{\Gamma(p)}\int_0^k (k-r)^{p-1} r^{\nu} \mathrm{d}C_r \\ &+ \frac{e}{\Gamma(p)}\int_0^k (k-r)^{p-1} r^{\sigma} \mathrm{d}V_r, \\ =& k^{p-1}z_0 + \frac{a\Gamma(\mu+1)}{\Gamma(p+\mu+1)}k^{p+\mu} + \frac{b}{\Gamma(p)}G_k + \frac{e}{\Gamma(p)}H_k \end{aligned}$$

where $G_k = \int_0^k (k-r)^{p-1} r^{\nu} \mathrm{d}C_r$ is a normal uncertain variable, by Theorem 6.4 in [5], we have

$$G_k \sim \mathcal{N}\left(0, \int_0^k (k-r)^{p-1} r^{\nu} \mathrm{d}r\right)$$

where

$$\int_0^k (k-r)^{p-1} r^{\nu} \mathrm{d}r = \frac{\Gamma(p)\Gamma(\nu+1)}{\Gamma(p+\nu+1)}k^{p+\nu}$$

where $H_k = \int_0^k (k-r)^{p-1} r^{\sigma} \mathrm{d}V_r$ is a $\mathcal{Z}$ jump uncertain variable, by Lemma A3 in Appendix A, we have

$$H_k \sim \mathcal{Z}\left(\theta_1, \theta_2, \int_0^k (k-r)^{p-1} r^{\sigma} \mathrm{d}r\right),$$

where

$$\int_0^k (k-r)^{p-1} r^{\sigma} \mathrm{d}r = \frac{\Gamma(p)\Gamma(\sigma+1)}{\Gamma(p+\sigma+1)}k^{p+\sigma}$$

(ii) The proof is similar to that of (i). □

Theorem 3. *Let a be a real number and μ_k, ν_k, σ_k two functions on $[0, T]$. Then*

$$D^p Z_k = aZ_k + \mu_k + \nu_k \frac{\mathrm{d}C_k}{\mathrm{d}k} + \sigma_k \frac{\mathrm{d}V_k}{\mathrm{d}k}, k \in (0, T] \tag{5}$$

with the initial condition

$$\lim_{k \to 0+} k^{1-p} Z_k = z_0$$

has a solution

$$\begin{aligned} Z_k =& z_0\Gamma(p)k^{p-1}E_{p,p}(ak^p) + \int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\mu_r \mathrm{d}r \\ &+ \int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\nu_r \mathrm{d}C_r \\ &+ \int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\sigma_r \mathrm{d}V_r \end{aligned} \tag{6}$$

Proof. It is obvious that

$$\lim_{k \to 0+} k^{1-p} Z_k = \lim_{k \to 0+} z_0\Gamma(p)E_{p,p}(ak^p) + \lim_{k \to 0+} k^{1-p}\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\mu_r \mathrm{d}r$$

$$
\begin{aligned}
&+\lim_{k\to 0+} k^{1-p}\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\nu_r \mathrm{d}C_r \\
&+\lim_{k\to 0+} k^{1-p}\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\sigma_r \mathrm{d}V_r \\
=&z_0.
\end{aligned} \tag{7}
$$

For Z_k provided by (6), we have

$$
\begin{aligned}
&\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} a Z_r dr \\
=&\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} a z_0 \Gamma(p) r^{p-1} E_{p,p}(ar^p) dr \\
&+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} a \int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\mu_s \mathrm{d}s\mathrm{d}r \\
&+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} a \int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\nu_s \mathrm{d}C_s\mathrm{d}r \\
&+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} a \int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\sigma_s \mathrm{d}V_s\mathrm{d}r \\
=&az_0\int_0^k (k-r)^{p-1} r^{p-1} E_{p,p}(ar^p) dr \\
&+\frac{a}{\Gamma(p)}\int_0^k (k-r)^{p-1} \int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\mu_s \mathrm{d}s\mathrm{d}r \\
&+\frac{a}{\Gamma(p)}\int_0^k (k-r)^{p-1} \int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\nu_s \mathrm{d}C_s\mathrm{d}r \\
&+\frac{a}{\Gamma(p)}\int_0^k (k-r)^{p-1} \int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\sigma_s \mathrm{d}V_s\mathrm{d}r
\end{aligned} \tag{8}
$$

It follows from Theorem 3 in Ref [35] and the Mittag-Leffler function that

$$
az_0\int_0^k (k-r)^{p-1} r^{p-1} E_{p,p}(ar^p) dr = z_0\Gamma(p)k^{p-1}E_{p,p}(ak^p) - z_0 k^{p-1}. \tag{9}
$$

In addition, we have

$$
\begin{aligned}
&\tfrac{a}{\Gamma(p)}\int_0^k (k-r)^{p-1}\int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\sigma_s \mathrm{d}V_s\mathrm{d}r \\
=&\tfrac{a}{\Gamma(p)}\int_0^k \left(\int_s^k (k-r)^{p-1}(r-s)^{p-1}E_{p,p}(a(r-s)^p)\mathrm{d}r\right)\sigma_s \mathrm{d}V_s
\end{aligned} \tag{10}
$$

We let $r = s+\tau(k-s), 0\le\tau\le 1$ in (10), then

$$
\begin{aligned}
&\frac{a}{\Gamma(p)}\int_0^k (k-r)^{p-1}\int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\sigma_s \mathrm{d}V_s\mathrm{d}r \\
=&\frac{a}{\Gamma(p)}\int_0^k \left(\int_0^1 (k-s)^{p-1}(1-\tau)^{p-1}\tau^{p-1}(k-s)^{p-1}E_{p,p}(a\tau^p(k-s)^p)\mathrm{d}\tau\right)\sigma_s \mathrm{d}V_s \\
=&\frac{a}{\Gamma(p)}\int_0^k (k-s)^{2p-1}\left(\int_0^1 (1-\tau)^{p-1}\tau^{p-1}\sum_{j=0}^{\infty}\frac{a^j\tau^{pj}(k-s)^{pj}}{\Gamma(p(k+1))}d\tau\right)\sigma_s \mathrm{d}V_s \\
=&\frac{a}{\Gamma(p)}\int_0^k (k-s)^{2p-1}\sum_{j=0}^{\infty}\frac{a^j(k-s)^{pj}}{\Gamma(p(k+1))}\left(\int_0^1 (1-\tau)^{p-1}\tau^{p(j+1)-1}d\tau\right)\sigma_s \mathrm{d}V_s \\
=&a\int_0^k (k-s)^{2p-1}\sum_{j=0}^{\infty}\frac{a^j(k-s)^{pj}}{\Gamma(p(j+2))}\sigma_s \mathrm{d}V_s
\end{aligned} \tag{11}
$$

$$=\int_0^k (k-s)^{p-1}\sum_{j=1}^{\infty}\frac{a^j(k-s)^{pj}}{\Gamma(p(j+1))}\sigma_s\mathrm{d}V_s$$
$$=\int_0^k (k-s)^{p-1}(E_{p,p}(a(k-s)^p)-\frac{1}{\Gamma(p)})\sigma_s\mathrm{d}V_s$$
$$=\int_0^k (k-s)^{p-1}E_{p,p}(a(k-s)^p\sigma_s\mathrm{d}V_s-\frac{1}{\Gamma(p)}\int_0^k (k-s)^{p-1}\sigma_s\mathrm{d}V_s$$

Similar to (11), we can get

$$\begin{aligned}&\tfrac{a}{\Gamma(p)}\textstyle\int_0^k (k-r)^{p-1}\int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\mu_s\mathrm{d}s\mathrm{d}r\\ =\ &\textstyle\int_0^k (k-s)^{p-1}E_{p,p}(a(k-s)^p\mu_s\mathrm{d}s-\tfrac{1}{\Gamma(p)}\int_0^k (k-s)^{p-1}\mu_s\mathrm{d}s\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\tfrac{a}{\Gamma(p)}\textstyle\int_0^k (k-r)^{p-1}\int_0^r (r-s)^{p-1}E_{p,p}(a(r-s)^p)\nu_s\mathrm{d}C_s\mathrm{d}r\\ =\ &\textstyle\int_0^k (k-s)^{p-1}E_{p,p}(a(k-s)^p\nu_s\mathrm{d}C_s-\tfrac{1}{\Gamma(p)}\int_0^k (k-s)^{p-1}\nu_s\mathrm{d}C_s\end{aligned} \tag{13}$$

Substituting (9), (11)–(13) into (8) yields

$$\begin{aligned}&\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}aZ_rdr\\
=&z_0\Gamma(p)k^{p-1}E_{p,p}(ak^p)-z_0k^{p-1}\\
&+\int_0^k (k-s)^{p-1}E_{p,p}(a(k-s)^p\mu_s\mathrm{d}s-\frac{1}{\Gamma(p)}\int_0^k (k-s)^{p-1}\mu_s\mathrm{d}s\\
&+\int_0^k (k-s)^{p-1}E_{p,p}(a(k-s)^p\nu_s\mathrm{d}C_s-\frac{1}{\Gamma(p)}\int_0^k (k-s)^{p-1}\nu_s\mathrm{d}C_s\\
&+\int_0^k (k-s)^{p-1}E_{p,p}(a(k-s)^p\sigma_s\mathrm{d}V_s-\frac{1}{\Gamma(p)}\int_0^k (k-s)^{p-1}\sigma_s\mathrm{d}V_s\end{aligned}$$

Hence,

$$\begin{aligned}&z_0k^{p-1}+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}aZ_rdr+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\mu_r\mathrm{d}r\\
&+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\nu_r\mathrm{d}C_r+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}\sigma_r\mathrm{d}V_r\\
=&z_0\Gamma(p)k^{p-1}E_{p,p}(ak^p)+\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\mu_r\mathrm{d}r\\
&+\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\nu_r\mathrm{d}C_r+\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\sigma_r\mathrm{d}V_r\\
=&Z_k.\end{aligned}$$

Thus, (6) is a solution of (5) by Definition 5. □

Theorem 4. *Let a be a real number and μ_k,ν_k,σ_k two functions on $[0,T]$. Then*

$$^cD^pZ_k=aZ_k+\mu_k+\nu_k\frac{\mathrm{d}C_k}{\mathrm{d}k}+\sigma_k\frac{\mathrm{d}V_k}{\mathrm{d}k},k\in(0,T] \tag{14}$$

has a solution

$$Z_k=E_{p,1}(ak^p)Z_0+\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\mu_r\mathrm{d}r$$

$$+\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\nu_r \mathrm{d}C_r \qquad (15)$$
$$+\int_0^k (k-r)^{p-1}E_{p,p}(a(k-r)^p)\sigma_r \mathrm{d}V_r$$

Proof. The proof of Theorem 4 is similar to that of Theorem 3, we omit here. □

Remark 4. *In this part, we introduce the Riemann-Liouville type and the Caputo type of uncertain FDE with jump in the one-dimensional case. Now we state those concepts in a multidimensional case. In the next part, we will always assume $p \in (0,1]$. Let $C_k = (C_{1k}, C_{2k}, \cdots, C_{lk})^T$ be an l-dimensional canonical process and $V_k = (V_{1k}, V_{2k}, \cdots, V_{lk})^T$ be an l-dimensional V-jump process.*

3.2. Existence and Uniqueness of Uncertain FDEs with Jump in the Multidimensional Case

Definition 7. *Let C_k be a canonical process and V_k be a V-jump process. Suppose that $f : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, and $g, h : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are three functions. Then,*

$$D^p Z_k = f(k, Z_k) + g(k, Z_k)\frac{\mathrm{d}C_k}{\mathrm{d}k} + h(k, Z_k)\frac{\mathrm{d}V_k}{\mathrm{d}k} \qquad (16)$$

is called an uncertain FDE with jump of the Riemann-Liouville type. A solution of (16) with the initial condition

$$\lim_{k\to 0+} k^{1-p} Z_k = z_0$$

is an uncertain process Z_k such that

$$Z_k = k^{p-1}z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} f(r, Z_r)\mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} g(r, Z_r)\mathrm{d}C_r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} h(r, Z_r)\mathrm{d}V_r \qquad (17)$$

holds almost surely.

Definition 8. *Let C_k be a canonical process and V_k be a V-jump process. Suppose that $f : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, and $g, h : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are three functions. Then,*

$${}^cD^p Z_k = f(k, Z_k) + g(k, Z_k)\frac{\mathrm{d}C_k}{\mathrm{d}k} + h(k, Z_k)\frac{\mathrm{d}V_k}{\mathrm{d}k} \qquad (18)$$

is called an uncertain FDE of the Caputo type. A solution of (18) is an uncertain process Z_k such that

$$Z_k = Z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} f(r, Z_r)\mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} g(r, Z_r)\mathrm{d}C_r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} h(r, Z_r)\mathrm{d}V_r \qquad (19)$$

holds almost surely.

For simplicity, we use $|\cdot|$ to denote a norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times l}$. Let $C_{[u,v]}$ denote the space of continuous $\mathbb{R}^n$-valued functions on $[u, v]$, which is a Banach space with the norm

$$\|Z_k\| = \max_{k\in[u,v]} |Z_k|, \quad \text{for } Z_k \in C_{[u,v]}.$$

Give three functions $f(k,z) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $g(k,z) : [0, T] \times R^n \to \mathbb{R}^{n\times l}$ and $h(k,z) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$. Now we introduce the following mapping Φ on $C_{[0,T]}$: for $Z_k \in C_{[0,T]}$,

$$\Phi(Z_k) = k^{p-1}z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} f(r, Z_r)\mathrm{d}r + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} g(r, Z_r)\mathrm{d}C_r$$

$$+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}h(r,Z_r)\mathrm{d}V_r \tag{20}$$

where z_0 is a given initial state.

Lemma 1. *For uncertain process $Z_k \in C_{[0,T]}$, the mapping Ψ defined by*

$$\begin{aligned}\Psi(Z_k) = &\left(k^{p-1}-\tilde{u}\right)z_0 + Z_a + \frac{1}{\Gamma(p)}\int_a^k (k-r)^{p-1}f(r,Z_r)\mathrm{d}r\\ &+\frac{1}{\Gamma(p)}\int_u^k (k-r)^{p-1}g(r,Z_r)\mathrm{d}C_r,\\ &+\frac{1}{\Gamma(p)}\int_u^k (k-r)^{p-1}h(r,Z_r)\mathrm{d}V_r, \quad k>a\geq 0\end{aligned} \tag{21}$$

is sample-continuous, where $\tilde{u} = u^{p-1}$ if $u > 0$, or $\tilde{u} = 1$ if $u = 0$, and f, g and h satisfy the linear growth condition

$$|f(k,z)| + |g(k,z)| + |h(k,z)| \leq L(1+|z|), \quad \forall z \in \mathbb{R}^n, \quad k \in [0,+\infty)$$

where L is a positive constant.

Proof. Actually, for $\gamma \in \Gamma$ and $k > s > u$, it holds that

$$\begin{aligned}&|\Psi(Z_k(\gamma)) - \Psi(Z_s(\gamma))|\\ =&\left|\left(k^{p-1}-s^{p-1}\right)z_0 + \frac{1}{\Gamma(p)}\int_s^k (k-r)^{p-1}f(r,Z_r(\gamma))\mathrm{d}r\right.\\ &+\frac{1}{\Gamma(p)}\int_s^k (k-r)^{p-1}g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma)\\ &+\frac{1}{\Gamma(p)}\int_s^k (k-r)^{p-1}h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma)\\ &+\frac{1}{\Gamma(p)}\int_u^s \left[(k-r)^{p-1}-(s-r)^{p-1}\right]f(r,Z_r(\gamma))\mathrm{d}r\\ &+\frac{1}{\Gamma(p)}\int_u^s \left[(k-r)^{p-1}-(s-r)^{p-1}\right]g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma)\\ &\left.+\frac{1}{\Gamma(p)}\int_u^s \left[(k-r)^{p-1}-(s-r)^{p-1}\right]h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma)\right|\\ \leq&\left(s^{p-1}-k^{p-1}\right)|z_0| + \frac{1}{\Gamma(p)}\int_s^k (k-r)^{p-1}|f(r,Z_r(\gamma))|\mathrm{d}r\\ &+\frac{1}{\Gamma(p)}\left|\int_s^k (k-r)^{p-1}g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma)\right|\\ &+\frac{1}{\Gamma(p)}\left|\int_s^k (k-r)^{p-1}h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma)\right|\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(p)}\int_u^s\left[(k-r)^{p-1}-(s-r)^{p-1}\right]|f(r,Z_r(\gamma))|\mathrm{d}r\\
&+\frac{1}{\Gamma(p)}\left|\int_u^s\left[(k-r)^{p-1}-(s-r)^{p-1}\right]g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma)\right|\\
&+\frac{1}{\Gamma(p)}\left|\int_u^s\left[(k-r)^{p-1}-(s-r)^{p-1}\right]h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma)\right|\\
\le&\left(s^{p-1}-k^{p-1}\right)|z_0|+\frac{1}{\Gamma(p)}\int_s^k(k-r)^{p-1}|f(r,Z_r(\gamma))|\mathrm{d}r\\
&+\frac{K_\gamma}{\Gamma(p)}\int_s^k(k-r)^{p-1}|g(r,Z_r(\gamma))|\mathrm{d}r+\frac{1}{\Gamma(p)}\int_s^k(k-r)^{p-1}|h(r,Z_r(\gamma))|\mathrm{d}r\\
&+\frac{1}{\Gamma(p)}\int_u^s\left[(k-r)^{p-1}-(s-r)^{p-1}\right]|f(r,Z_r(\gamma))|\mathrm{d}r\\
&+\frac{K_\gamma}{\Gamma(p)}\int_u^s\left[(k-r)^{p-1}-(s-r)^{p-1}\right]|g(r,Z_r(\gamma))|\,\mathrm{d}r\quad\text{(by Lemma A1 in Appendix A)}\\
&+\frac{1}{\Gamma(p)}\int_u^s\left[(k-r)^{p-1}-(s-r)^{p-1}\right]|h(r,Z_r(\gamma))|\,\mathrm{d}r\quad\text{(by Lemma A2 in Appendix A)}\\
\le&\left(s^{p-1}-k^{p-1}\right)|z_0|+\frac{L}{\Gamma(p+1)}(1+\|Z_k(\gamma)\|)(2+K_\gamma)\left[(k-u)^p-(s-u)^p\right]
\end{aligned}
$$

by the linear growth condition. So, $|\Psi(Z_k(\gamma))-\Psi(Z_s(\gamma))|\to 0$ as $|k-s|\to 0$. In other words, $\Psi(Z_k)$ is sample-continuous. □

Theorem 5. *(Existence and uniqueness) The uncertain FDE (16) or (18) has a unique solution Z_k in $[0,+\infty)$ if the coefficients $f(k,z)$, $g(k,z)$ and $h(k,z)$ satisfy the Lipschitz condition*

$$|f(k,z)-f(k,\hat{z})|+|g(k,z)-g(k,\hat{z})|+|h(k,z)-h(k,\hat{z})|\le L|z-\hat{z}|,\quad\forall z,\hat{z}\in\mathbb{R}^n,\quad k\in[0,+\infty)$$

and the linear growth condition

$$|f(k,z)|+|g(k,z)|+|h(k,z)|\le L(1+|z|),\quad\forall z\in\mathbb{R}^n,\quad k\in[0,+\infty)$$

where L is a positive constant. Furthermore, Z_k is sample-continuous.

Proof. We here only give the proof for the uncertain FDE with jump (16). A symmetric proof in terms of form is suitable to the uncertain FDE with jump (18). Let $T>0$ be an arbitrarily given number, and let Φ be a mapping defined by (20) on $C_{[0,T]}$.

Give $\gamma\in\Gamma$. For $\lambda\in[0,T)$, assume $d>0$ such that $\lambda+d\le T$. Define a mapping Ψ on $C_{[\lambda,\lambda+d]}$: for $Z_k\in C_{[\lambda,\lambda+d]}, k\in[\lambda,\lambda+d]$,

$$
\begin{aligned}
\psi(Z_k)=&\left(k^{p-1}-\tilde{\lambda}\right)z_0+Z_\lambda+\frac{1}{\Gamma(p)}\int_\lambda^k(k-r)^{p-1}f(r,Z_r)\mathrm{d}r\\
&+\frac{1}{\Gamma(p)}\int_\lambda^k(k-r)^{p-1}g(r,Z_r)\mathrm{d}C_r+\frac{1}{\Gamma(p)}\int_\lambda^k(k-r)^{p-1}h(r,Z_r)\mathrm{d}V_r
\end{aligned}
$$

where $\tilde{\lambda} = \lambda^{p-1}$ if $\lambda > 0$, or $\tilde{\lambda} = 1$ if $\lambda = 0$. For $Z_k(\gamma) \in C_{[\lambda,\lambda+d]}$, It follows from Lemma 1 that $\Psi(Z_k(\gamma)) \in C_{[\lambda,\lambda+d]}$.

Let $Z_k(\gamma)$,$\hat{Z}_k(\gamma) \in C_{[\lambda,\lambda+d]}$. $\forall k \in [\lambda, \lambda + d]$, it holds that

$$\left\|\psi(Z_k(\gamma)) - \psi(\hat{Z}_k(\gamma))\right\| = \max_{k\in[\lambda,\lambda+d]} \left|\psi(Z_k(\gamma)) - \psi(\hat{Z}_k(\gamma))\right|$$

$$\le \max_{k\in[\lambda,\lambda+d]} \left|\frac{1}{\Gamma(p)} \int_\lambda^k (k-r)^{p-1} \left[f(r, Z_r(\gamma)) - f(r, \hat{Z}_r(\gamma))\right] \mathrm{d}r\right.$$

$$+ \frac{1}{\Gamma(p)} \int_\lambda^k (k-r)^{p-1} \left[g(r, Z_r(\gamma)) - g(r, \hat{Z}_r(\gamma))\right] \mathrm{d}C_r(\gamma)$$

$$\left. + \frac{1}{\Gamma(p)} \int_\lambda^k (k-r)^{p-1} \left[h(r, Z_r(\gamma)) - h(s, \hat{Z}_r(\gamma))\right] \mathrm{d}V_r(\gamma)\right|$$

$$\le \max_{k\in[\lambda,\lambda+d]} \left\{\frac{1}{\Gamma(p)} \int_\lambda^k (k-r)^{p-1} | f(r, Z_r(\gamma)) - f(r, \hat{Z}_r(\gamma))|\mathrm{d}r\right.$$

$$+ \frac{K_\gamma}{\Gamma(p)} \int_\lambda^k (k-r)^{p-1} |g(r, Z_r(\gamma)) - g(r, \hat{Z}_r(\gamma))|\mathrm{d}r$$

$$\left. + \frac{1}{\Gamma(p)} \int_\lambda^k (k-r)^{p-1} |h(r, Z_r(\gamma)) - h(r, Z_r(\gamma))|\mathrm{d}r\right\}$$

$$\le \frac{(2+K_\gamma)L}{\Gamma(p)} \max_{k\in[\lambda,\lambda+d]} \int_\lambda^k (k-r)^{p-1} |Z_r(\gamma) - \hat{Z}_r(\gamma)|\mathrm{d}r \quad \text{(by Lipschitz condition)}$$

$$\le \frac{(2+K_\gamma)Ld^p}{\Gamma(p+1)} \|Z_k(\gamma) - \hat{Z}_k(\gamma)\|.$$

Let $\kappa(\gamma) = (2 + K_\gamma)Ld^p/\Gamma(p + 1)$. We take a suitable $d = d(\gamma) > 0$ such that $\kappa(\gamma) \in (0, 1)$. In other words, Ψ is a contraction mapping on $C_{[\lambda,\lambda+d]}$. Thus, by the classical Banach fixed point theorem, we can obtain a unique fixed point $Z_k(\gamma)$ of Ψ in $C_{[\lambda,\lambda+d]}$. And then, $Z_k(\gamma) = \lim_{j\to\infty} \psi(Z_{k,j}(\gamma))$ where

$$Z_{k,j}(\gamma) = \psi(Z_{k,j-1}(\gamma)),\ j = 1, 2, \cdots \tag{22}$$

for any given $Z_{k,0}(\gamma) = Z_k \in C_{[\lambda,\lambda+d]}$.

Suppose that $[0, d], [d, 2d], \cdots, [jd, T]$ are the subsets of $[0, T]$ with $jd < T \le (j + 1)d$. The above proof means that the mapping Ψ has a unique fixed point $Z_k^{(i+1)}(\gamma)$ with $Z_{id}^{(i+1)}(\gamma) = Z_{id}^{(i)}(\gamma)$ on the interval $[id, (i + 1)d]$ for $i = 0, 1, 2, \cdots, j$, where we set $(j + 1)d = T$. Define $Z_k(\gamma)$ on the interval $[0, T]$ by setting

$$Z_k(\gamma) = \begin{cases} Z_k^{(1)}(\gamma), & k \in [0, d], \\ Z_k^{(2)}(\gamma), & k \in [d, 2d], \\ \cdots & \\ Z_k^{(j+1)}(\gamma), & k \in [jd, T]. \end{cases}$$

It holds that $Z_k(\gamma)$ is the unique fixed point of Φ defined by (20) in $C_{[0,T]}$. Besides, $Z_k(\gamma) = \lim_{j\to\infty}\Phi(Z_{k,j}(\gamma))$ where

$$Z_{k,j}(\gamma) = \Phi(Z_{k,j-1}(\gamma)),\ j = 1,2,\cdots \tag{23}$$

for any given $Z_{k,0}(\gamma) = z_k \in C_{[0,T]}$. Because $Z_{k,j}$ are uncertain vectors for $j = 1,2,\cdots$, It is obvious that Z_k is an uncertain vector by Theorem 3 in [36]. By the arbitrariness of $T > 0$, it holds that Z_k is the unique solution of uncertain FDE with jump (16). Furthermore, owing to $Z_k(\gamma)$ is in $C_{[0,T]}$, so Z_k is sample-continuous. The proof is completed.

If the coefficients f, g and h do not satisfy the Lipschitz condition and linear growth condition, we will give the existence theorem just for continuous f, g and h as follows. □

Theorem 6. *(Existence) Let f(k,z),g(k,z) and h(k,z) be continuous in*

$$G = [0,T] \times \{z \in \mathbb{R}^n : |z - z_0| \le c\}.$$

Then, uncertain FDE of the Caputo type (18) has a solution Z_k in $k \in [0,T]$ with the crisp initial condition $Z_0 = z_0 \in \mathbb{R}^n$.

Proof. For any $\gamma \in \Gamma$, let $d > 0$ be a positive number such that

$$\frac{H\,(2+K_\gamma)}{\Gamma(p+1)}d^p = c \tag{24}$$

where K_γ is the Lipschitz constant of the canonical process C_k, and $H = max_{(k,z)} \in G|f(k,z)| \vee |g(k,z)| \vee |h(k,z)|$. Denote

$$Q = \left\{Z_k(\gamma) \in D_{[0,t]}\ : Z_k \text{ is an uncertain vector and } \|Z_k(\gamma) - z_0\| \le \frac{H(2+K_\gamma)}{\Gamma(p+1)}t^p\right\} \tag{25}$$

where $t = \min\{T, d\}$.

It holds that Q is a closed convex set. Define a mapping Φ on Q by

$$\begin{aligned}\Phi(Z_k(\gamma)) =& z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} f(r, Z_r(\gamma))\mathrm{d}r\\ &+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} g(r, Z_r(\gamma))\mathrm{d}C_r(\gamma),\\ &+\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1} h(r, Z_r(\gamma))\mathrm{d}V_r(\gamma), \quad 0 \le k \le t.\end{aligned} \tag{26}$$

For $Z_k(\gamma) \in Q$, we have

$$\begin{aligned}\|\Phi(Z_k(\gamma)) - z_0\| =& \max_{0\le k\le t} |\Phi(Z_k(\gamma)) - z_0|\\ \le& \max_{0\le k\le t}\left\{\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}|f(r, Z_r(\gamma))|\mathrm{d}r\right.\\ +& \frac{K_\gamma}{\Gamma(p)}\int_0^k (k-r)^{p-1}|g(r, Z_r(\gamma))|\mathrm{d}r\\ +& \left.\frac{1}{\Gamma(p)}\int_0^k (k-r)^{p-1}|h(r, Z_r(\gamma))|\mathrm{d}r\right\}\\ \le& \max_{0\le k\le t}\frac{H(2+K_\gamma)}{\Gamma(p)}\int_0^k (k-r)^{p-1}\mathrm{d}r\end{aligned} \tag{27}$$

$$\leq \frac{H(2+K_\gamma)}{\Gamma(p+1)}h^p.$$

This means $\Phi(Z_k(\gamma)) \in Q$, and the mapping Φ is bounded uniformly in $Z_k(\gamma) \in Q$. Besides, for $0 \leq k_1 < k_2 \leq t$, it holds that

$$\begin{aligned}
|\Phi(Z_{k_1}(\gamma)) - \Phi(Z_{k_2}(\gamma))| = \Bigg| &\frac{1}{\Gamma(p)}\int_{k_1}^{k_2}(k_2-r)^{p-1}f(r,Z_r(\gamma))\mathrm{d}r \\
&+ \frac{1}{\Gamma(p)}\int_{k_1}^{k_2}(k_2-r)^{p-1}g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma) \\
&+ \frac{1}{\Gamma(p)}\int_{k_1}^{k_2}(k_2-r)^{p-1}h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma) \\
&+ \frac{1}{\Gamma(p)}\int_0^{k_1}\left[(k_2-r)^{p-1}-(k_1-r)^{p-1}\right]f(r,Z_r(\gamma))\mathrm{d}r \\
&+ \frac{1}{\Gamma(p)}\int_0^{k_1}\left[(k_2-r)^{p-1}-(k_1-r)^{p-1}\right]g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma) \\
&+ \frac{1}{\Gamma(p)}\int_0^{k_1}\left[(k_2-r)^{p-1}-(k_1-r)^{p-1}\right]h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma), \Bigg| \\
&\leq \frac{H(2+K_\gamma)}{\Gamma(p+1)}\left(k_2^p - k_1^p\right)
\end{aligned}$$

we can get a conclusion that Φ is equicontinuous for $Z_k(\gamma) \in Q$ in $[0,t]$. We know that Φ is a compact mapping on Q by the Ascoli-Arzela theorem.

Let $Z_{k,i}(\gamma)$ converge to $Z_k(\gamma) \in Q$ as $i \to \infty$. In other words, $Z_{k,i}(\gamma)$ converges to $Z_k(\gamma)$ uniformly in $k \in [0,t]$. Hence,

$$\begin{aligned}
\Phi(Z_{k,i}(\gamma)) = &\frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}f(r,Z_{r,i}(\gamma))\mathrm{d}r \\
&+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}g(r,Z_{r,i}(\gamma))\mathrm{d}C_r(\gamma) \\
&+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}h(r,Z_{r,i}(\gamma))\mathrm{d}V_r(\gamma) \\
&\to \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}f(r,Z_r(\gamma))\mathrm{d}r \\
&+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma) \\
&+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma) \\
&= \Phi(Z_k(\gamma))
\end{aligned}$$

uniformly in $k \in [0,t]$. It is easy to see that that Φ is continuous on Q.

We know that Φ has a fixed point $Z_k(\gamma)$ on Q by Schauder fixed point theorem. Thus,

$$\begin{aligned}
Z_k(\gamma) = z_0 &+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}f(r,Z_r(\gamma))\mathrm{d}r \\
&+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}g(r,Z_r(\gamma))\mathrm{d}C_r(\gamma) \\
&+ \frac{1}{\Gamma(p)}\int_0^k(k-r)^{p-1}h(r,Z_r(\gamma))\mathrm{d}V_r(\gamma)
\end{aligned} \tag{28}$$

(10) for $k \in [0, t]$. Extending this way, there exists $Z_k(\gamma)$ satisfying (28) in $k \in [0, T]$. In other words, Z_k is a solution of (18). Hence, the proof is completed. □

3.3. Application

In this subsection, we will discuss an application of the present study. We give an uncertain interest rate model that the short interest rate Z_k satisfies the following uncertain FDE with jump

$$ {}^{c}D^{p}Z_k = (m - \mu Z_k) + \nu\frac{\mathrm{d}C_k}{\mathrm{d}k} + \sigma\frac{\mathrm{d}V_k}{\mathrm{d}k}, k > 0 \tag{29} $$

where m, μ, ν, and σ are positive numbers. The above model is the FDE with jump form of the model in Ref [43]. Then, the price of a zero-coupon bond is

$$ \mathcal{P} = E\left[\exp\left(-\int_0^d Z_k dk\right)\right], $$

where E is the uncertain expected value [5], d is a maturity date.

Theorem 7. *Let Z_0 be the crisp initial state z_0. Then*

$$ \mathcal{P} = E\left[\exp\left(-\delta_1 z_0 - m\delta_2 - \nu\xi_1 - \sigma\xi_2\right)\right] $$

where $\delta_1 = dE_{p,2}(-\mu d^p)$, $\delta_2 = d^{p+1}E_{p,p+2}(-\mu d^p)$, $\xi_1 = \int_0^d \int_r^d (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)dk\mathrm{d}C_r$ is a normal uncertain variable

$$ \xi_1 \sim \mathcal{N}\left(0, d^{p+1}E_{p,p+2}(-\mu d^p)\right) $$

$\xi_2 = \int_0^d \int_r^d (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)dk\mathrm{d}V_r$ is a $\mathcal{Z}$ jump uncertain variable

$$ \xi_2 \sim \mathcal{Z}\left(\theta_1, \theta_2, d^{p+1}E_{p,p+2}(-\mu d^p)\right) $$

Proof. By (15), we can easily obtain the following solution of (29)

$$ \begin{aligned} Z_k =& E_{p,1}(-ak^p)z_0 + m\int_0^k (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)\mathrm{d}r \\ &+ \nu\int_0^k (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)\mathrm{d}C_r \\ &+ \sigma\int_0^k (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)\mathrm{d}V_r \end{aligned} \tag{30} $$

It holds that

$$ \begin{aligned} \int_0^d Z_k dk =& \int_0^d E_{p,1}(-ak^p)z_0 dk + m\int_0^d \int_0^k (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)\mathrm{d}r dk \\ &+ \nu\int_0^d \int_0^k (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)\mathrm{d}C_r dk \\ &+ \sigma\int_0^d \int_0^k (k-r)^{p-1}E_{p,p}(-\mu(k-r)^p)\mathrm{d}V_r dk \end{aligned} \tag{31} $$

It follows from Theorem 5 in Ref [35] that

$$ \int_0^d E_{p,1}(-ak^p)dk = dE_{p,2}(-\mu d^p), $$

$$\int_0^d \int_0^k (k-r)^{p-1} E_{p,p}(-\mu(k-r)^p)\mathrm{d}r dk = d^{p+1} E_{p,p+2}(-\mu d^p),$$

and

$$\begin{aligned}\int_0^d \int_0^k (k-r)^{p-1} E_{p,p}(-\mu(k-r)^p)\mathrm{d}C_r dk = &\int_0^d \int_r^d (k-r)^{p-1} E_{p,p}(-\mu(k-r)^p) dk\mathrm{d}C_r \\ &\triangleq \xi_1 \sim \mathcal{N}(0, \lambda_1)\end{aligned}$$

where

$$\lambda_1 = \int_0^d \int_r^d (k-r)^{p-1} E_{p,p}(-\mu(k-r)^p) dk\mathrm{d}r = d^{p+1} E_{p,p+2}(-\mu d^p)$$

Similarly, by Lemma A3 in Appendix A, we can obtain that

$$\begin{aligned}\int_0^d \int_0^k (k-r)^{p-1} E_{p,p}(-\mu(k-r)^p)\mathrm{d}V_r dk = &\int_0^d \int_r^d (k-r)^{p-1} E_{p,p}(-\mu(k-r)^p) dk\mathrm{d}V_r \\ &\triangleq \xi_2 \sim \mathcal{Z}(\theta_1, \theta_2, \lambda_2)\end{aligned}$$

where $\lambda_2 = \lambda_1$. Hence, we can obtain that

$$\begin{aligned}\mathcal{P} =&E\left[\exp\left(-\int_0^d Z_k dk\right)\right] \\ =&E\left[\exp\left(-dE_{p,2}(-\mu d^p)z_0 - md^{p+1}E_{p,p+2}(-\mu d^p) - \nu\xi_1 - \sigma\xi_2\right)\right] \\ =&E\left[\exp\left(-\delta_1 z_0 - m\delta_2 - \nu\xi_1 - \sigma\xi_2\right)\right]\end{aligned}$$

where $\delta_1 = dE_{p,2}(-\mu d^p)$, $\delta_2 = d^{p+1}E_{p,p+2}(-\mu d^p)$, the proof is completed. □

4. Conclusions

The main goal of the current study is to prove existence and uniqueness for uncertain FDEs with jump. In addition, we give an application about uncertain interest rate model. Of course, the readers can further study more complex models by uncertain FDEs with jump, such as uncertain stock model and uncertain optimal control model. One source of weakness in our study is the lack of numerical methods, these will be the focus of our future research.

Author Contributions: All authors contributed equally and significantly in writing this article: writing-original draft preparation, Z.J., review, X.L. and C.L., investigation, Z.J., funding acquisition, X.L. and C.L. All authors read and approved the final manuscript.

Funding: National Natural Science Foundation of China (61374183, 51535005, 71561001), the Major Projects of North Minzu University (ZDZX201805).

Acknowledgments: This work was supported by the National Natural Science Foundation of China (61374183, 51535005, 71561001), the Major Projects of North Minzu University (ZDZX201805).

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Lemma A1. *[17] Suppose that C_k is an l-dimensional canonical Liu process, and Z_k is an integrable $n \times l$-dimensional uncertain process on $[u, v]$ with respect to k. Then, the inequality*

$$\left\| \int_a^b Z_k(\gamma) dC_k(\gamma) \right\|_\infty \leq K_\gamma \int_a^b \|Z_k(\gamma)\|_\infty dk$$

holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $C_k(\gamma)$ with the norm $\| \cdot \|_\infty$.

Lemma A2. *Suppose that V_k is an l-dimensional uncertain V-jump process, and Z_k is an integrable $n \times l$-dimensional uncertain process on $[u, v]$ with respect to k. Then, for any sample γ, the inequality*

$$\left\| \int_a^b Z_k(\gamma) dV_k(\gamma) \right\|_\infty \leq \int_a^b \|Z_k(\gamma)\|_\infty dk$$

holds,

Proof. For the multidimensional case, it is easy to obtain the result similar to Theorem 3.2 in [27]. □

Lemma A3. *Let V_k be a V-jump process and let f_k be an integrable and determinstic function with respect to time k. Then the uncertain integral*

$$\int_0^s f_k dV_k$$

is a $\mathcal{Z}$ jump uncertain variables at each time s, i.e.,

$$\int_0^s f_k dV_k \sim \mathcal{Z}(\theta_1, \theta_2, \int_0^s |f_k| dk).$$

Proof. Because the V-jump process has stationary and independent increments and every increment is a V-jump uncertain variable, for any partition of closed interval [0,s] with $0 = k_1 < k_2 < \cdots < k_{m+1} = s$, it follows from Theorem 3.1 in [25] that

$$\sum_{i=1}^{m} f(k_i)(V_{k_{i+1}} - V_{k_i}) \sim \mathcal{Z}\left(\theta_1, \theta_2, \sum_{i=1}^{m} |f(k_i)|(k_{i+1} - k_i)\right).$$

In other words, the sum is also a V-jump uncertain variable . Because f is an integrable function, it holds that

$$\sum_{i=1}^{m} |f(k_i)|(k_{i+1} - k_i) \to \int_0^s |f_k| dk$$

as the mesh $\Delta \to 0$, where the mesh is written as $\Delta = \max_{1 \leq i \leq m} |k_{i+1} - k_i|$. Thus we obtain

$$\int_0^s f_k dV_k = \lim_{\Delta \to 0} \sum_{i=1}^{m} f(k_i)(V_{k_{i+1}} - V_{k_i}) \sim \mathcal{Z}(\theta_1, \theta_2, \int_0^s |f_k| dk)$$

□

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