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Cohomology of Lie Superalgebras

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Abstract: In this paper we compute the Betti numbers for complex nilpotent Lie superalgebras of dimension ≤ 5.

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1. Introduction

The study of the cohomology theory of algebras is a very interesting topic both in mathematics and physics. Among its applications, the obtainment of low degree cohomology allows to parametrize outer derivations, central extensions and deformations of a given algebra.

One of the simpler applications is given by the central extension of the group $\mathbb{R}^{2n}$ of translations on both position and momentum, which results in the Heisenberg group. Related to this, one obtains the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ as the central extension of the abelian Lie algebra $\mathfrak{a}_{2n}$. Moreover, the Bargmann group, which is a symmetry group of the Schrödinger equation, is the central extension of the Galilei group. In addition, Kac-Moody and Virasoro algebras are central extensions of polynomials loop-algebras and the Witt algebra, respectively. For more examples regarding central extensions and Physics, see for instance [1].

Recently, the deformations of the static kinematical Lie algebra which are themselves kinematical have been studied in [2] and then this classification has been used to provide a classification of kinematical superpaces in [3]. Moreover, an interesting and related problem is the one of deformation theory of the universal central extension of the static kinematical Lie algebra, which uses cohomology theory twice.

Contrary to the classical case, where the cohomology of finite dimensional Lie algebras vanishes for degrees greater than the dimension of the algebra, the cohomology of a finite dimensional Lie superalgebra does not vanish in general. This makes the problem of computing the cohomology of a given Lie superalgebra a very difficult task. There are not many results concerning the explicit computation of all Betti numbers of algebras. In the case of Lie algebras we can mention, for instance, [4–11]. For Lie superalgebras there are several results regarding low dimensional degree cohomology for particular Lie superalgebras (see for instance [12–15]), and very few containing the hole picture (see for instance [16–21]).

In this work, we compute all the dimensions of the trivial cohomology for nilpotent Lie superalgebras of dimension ≤ 5.

2. Preliminaries

For general results about Lie superalgebras and its applications see for instance [22–26]. A $\mathbb{Z}_2$-graded vector space (or vector superspace) is a direct sum of vector spaces $V = V_0 \oplus V_1$. Elements in V_0 (resp. V_1) are called even (resp. odd). Non-zero elements of $V_0 \cup V_1$ are called homogeneous and the degree of an homogeneous element $v \in V_i$ is $|v| = i$.

A Lie superalgebra is a vector superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a bilinear map $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

1. $[\![\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\!] \subset \mathfrak{g}_{\alpha+\beta}$, for $\alpha, \beta \in \mathbb{Z}_2$, ($\mathbb{Z}_2$-grading)
2. $[\![x, y]\!] = -(-1)^{|x||y|}[\![y, x]\!]$, (super skew-symmetry)
3. $(-1)^{|x||z|}[\![x, [\![y, z]\!]]\!] + (-1)^{|y||x|}[\![y, [\![z, x]\!]]\!] + (-1)^{|z||y|}[\![z, [\![x, y]\!]]\!] = 0$, (super Jacobi-identity)

for homogeneous $x, y, z \in \mathfrak{g}$.

A linear map between Lie superalgebras $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie superalgebra morphism if ϕ is even (i.e., $\phi(\mathfrak{g}_i) \subset \mathfrak{h}_i$, for $i \in \mathbb{Z}_2$), and $\phi([\![x, y]\!]_\mathfrak{g}) = [\![\phi(x), \phi(y)]\!]_\mathfrak{h}$, for $x, y \in \mathfrak{g}$.

The lower central series $\{\mathfrak{g}^i\}$ of the Lie superalgebra $\mathfrak{g}$, is the series of ideals defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{i+1} = [\![\mathfrak{g}, \mathfrak{g}^i]\!] \quad \text{for } i \geq 0.$$

We say that $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^n = 0$ for some n.

The superspace of q-dimensional cochains of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with coefficients in the $\mathfrak{g}$-module $A = A_0 \oplus A_1$ is given by

$$C^q(\mathfrak{g}, A) = \bigoplus_{q_0+q_1=q} \mathrm{Hom}(\Lambda^{q_0}\mathfrak{g}_0 \otimes S^{q_1}\mathfrak{g}_1, A).$$

$C^q(\mathfrak{g}, A)$ is $\mathbb{Z}_2$-graded by $C^q(\mathfrak{g}, A) = C^q_0(\mathfrak{g}, A) \oplus C^q_1(\mathfrak{g}, A)$ with

$$C^q_p(\mathfrak{g}, A) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r\equiv p \mod 2}} \mathrm{Hom}(\Lambda^{q_0}\mathfrak{g}_0 \otimes S^{q_1}\mathfrak{g}_1, A_r), \quad p \in \mathbb{Z}_2.$$

The differential $d : C^q(\mathfrak{g}, A) \to C^{q+1}(\mathfrak{g}, A)$ is defined by

$$\begin{aligned}
(d\varphi)(x_1, \dots, x_{q_0}, y_1, \dots, y_{q_1}) = &\sum_{1\leq i<j\leq q_0} (-1)^{i+j+1}\varphi([\![x_i, x_j]\!], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{q_0}, y_1, \dots, y_{q_1}) \\
&+ \sum_{i=1}^{q_0}\sum_{j=1}^{q_1}(-1)^{i-1}\varphi(x_1, \dots, \widehat{x}_i, \dots, x_{q_0}, [\![x_i, y_j]\!], y_1, \dots, \widehat{y}_j, \dots, y_{q_1}) \\
&+ \sum_{1\leq i<j\leq q_1} \varphi([\![y_i, y_j]\!], x_1, \dots, x_{q_0}, y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_{q_1}) \\
&+ \sum_{i=1}^{q_0}(-1)^i x_i \cdot \varphi(x_1, \dots, \widehat{x}_i, \dots, x_{q_0}, y_1, \dots, y_{q_1}) \\
&+ (-1)^{q_0-1}\sum_{i=1}^{q_1} y_i \cdot \varphi(x_1, \dots, x_{q_0}, y_1, \dots, \widehat{y}_i, \dots, y_{q_1}),
\end{aligned}$$

where $\varphi \in C^q(\mathfrak{g}, A)$, $x_i \in \mathfrak{g}_0$ and $y_i \in \mathfrak{g}_1$. Since $d \cdot d = 0$ and $d(C^q_p(\mathfrak{g}, A)) \subset C^{q+1}_p(\mathfrak{g}, A)$ for $q = 0, 1, 2, \dots$ and $p \in \mathbb{Z}_2$, we obtain the cohomology groups

$$H^q_p(\mathfrak{g}, A) = \frac{Z^q_p(\mathfrak{g}, A)}{B^q_p(\mathfrak{g}, A)},$$

where the elements of $Z_0^q(\mathfrak{g}, A)$ (resp. $Z_1^q(\mathfrak{g}, A)$) are called even q-cocycles (resp. odd q-cocycles). Analogously, the elements of $B_0^q(\mathfrak{g}, A)$ (resp. $B_1^q(\mathfrak{g}, A)$) are even q-coboundaries (resp. odd q-coboundaries).

In low degrees, the following results are known:

1. $H_0^0(\mathfrak{g}, A) = \mathrm{Ann}_{A_0}\,\mathfrak{g} = \{a \in A_0 |\ xa = 0 \text{ for all } x \in \mathfrak{g}\}$.
2. $H_0^1(\mathfrak{g}, A) = \mathrm{Der}_0(\mathfrak{g}, A) / \mathrm{IDer}_0(\mathfrak{g}, A)$, where Der_0 are the even derivations from $\mathfrak{g}$ to A and IDer_0 are the even inner derivations from $\mathfrak{g}$ to A.
3. $H_0^2(\mathfrak{g}, A)$ parameterizes the equivalence classes of extensions of $\mathfrak{g}$ by A.
4. $H_0^2(\mathfrak{g}, \mathfrak{g})$ parameterizes the set of infinitesimal deformations of $\mathfrak{g}$.

3. The Betti Numbers for Nilpotent Lie Superalgebras

Here we tabulate all Betti numbers of indecomposable nilpotent Lie superalgebras of dimension ≤ 5 (see Table 1). The classification of these superalgebras is listed in Appendix A. We set the following notation: $b_i = \dim H^i(\mathfrak{g}, \mathbb{C}) = \dim H_0^i(\mathfrak{g}, \mathbb{C}) | \dim H_1^i(\mathfrak{g}, \mathbb{C})$.

Theorem 1. *The Betti numbers for indecomposable nilpotent Lie superalgebras of dimension ≤ 5 are:*

Table 1. Betti numbers.

Lie Superalgebra	b_1	b_2	b_3	b_4	$b_k,\ k \geq 5$ odd	$b_k,\ k \geq 5$ even
$(1\|1)_1$	0|1	0|0	0|0	0|0	0|0	0|0
$(1\|2)_2$	0|2	2|0	0|2	2|0	0|2	2|0
$(1\|2)_3$	1|1	1|1	1|1	1|1	1|1	1|1
$(3\|1)_3$	2|1	1|2	0|1	0|0	0|0	0|0
$(2\|2)_1$	0|2	1|0	0|0	0|0	0|0	0|0
$(2\|2)_4$	0|2	1|1	1|1	1|1	1|1	1|1
$(2\|2)_6$	1|1	1|1	1|1	1|1	1|1	1|1
$(1\|3)_1$	1|1	2|1	2|2	3|2	$\frac{k+1}{2} \| \frac{k+1}{2}$	$\frac{k}{2}+1 \| \frac{k}{2}$
$(1\|3)_5$	0|3	5|0	0|7	9|0	$0\|2k+1$	$2k+1\|0$
$(4\|1)_4$	2|1	2|2	1|2	0|1	0|0	0|0
$(4\|1)_6$	2|1	2|2	1|2	1|1	0|1	1|0
$(3\|2)_5$	0|2	0|2	0|2	0|0	0|0	0|0
$(3\|2)_{10}$	2|2	3|4	4|4	5|4	$2k-4\|2k-4$	$2k-4\|2k-4$
$(3\|2)_{11}$	2|1	3|3	4|5	5|4	4|5	5|4
$(3\|2)_{12}$	2|2	2|2	3|4	4|3	3|4	4|3
$(3\|2)_{13}$	1|1	1|1	1|3	3|1	1|3	3|1
$(2\|3)_5$	0|3	4|0	0|4	4|0	0|4	4|0
$(2\|3)_6$	0|3	4|0	0|4	4|0	0|4	4|0
$(2\|3)_8$	0|3	4|0	0|5	6|0	$0\|k+2$	$k+2\|0$
$(2\|3)_9$	0|3	4|0	0|5	6|0	$0\|k+2$	$k+2\|0$
$(2\|3)_{10}$	0|3	4|0	0|5	6|0	$0\|k+2$	$k+2\|0$
$(2\|3)_{11}$	0|3	4|0	0|5	6|0	$0\|k+2$	$k+2\|0$
$(2\|3)_{13}$	1|2	2|2	2|2	2|2	2|2	2|2
$(2\|3)_{14}$	1|2	3|2	3|4	5|6	$k\|k+1$	$k+1\|k$
$(2\|3)_{16}$	1|2	2|2	3|2	2|4	$k\|2$	$2\|k$
$(2\|3)_{17}$	2|2	4|3	4|5	6|5	$k+1\|k+2$	$k+2\|k+1$
$(2\|3)_{18}$	0|2	2|0	0|2	2|0	0|2	2|0
$(2\|3)_{19}$	2|1	2|3	4|3	4|5	$k+1\|k$	$k\|k+1$
$(2\|3)_{21}$	1|1	1|3	2|1	3|3	$k-1\|3$	$3\|k-1$
$(2\|3)_{22}$	1|1	1|1	2|1	1|3	$k-1\|1$	$1\|k-1$
$(2\|3)_{23}$	2|1	2|1	1|2	2|1	1|2	2|1
$(1\|4)_4$	0|4	9|0	0|16	25|0	$0\|(k+1)^2$	$(k+1)^2\|0$
$(1\|4)_7$	1|1	2|1	2|2	4|2	$2(k-3)\|2(k-2)$	$2(k-2)\|2(k-3)$
$(1\|4)_8$	1|2	4|2	3|6	9|4	$k \left\| \frac{(k+1)(k+3)}{4} \right.$	$\left. \frac{(k+2)^2}{4} \right\| k$

Proof. We will consider two examples in detail. The remaining follow analogously.

First, consider the Lie superalgebra $(1|3)_1$. An easy computation shows that:

$$\begin{aligned} d\left((f^1)^\alpha (f^2)^\beta (f^3)^\gamma\right) &= (\beta+1)\, e^1 (f^1)^{\alpha-1} (f^2)^{\beta+1} (f^3)^\gamma \\ &\quad + (\gamma+1)\, e^1 (f^1)^\alpha (f^2)^{\beta-1} (f^3)^{\gamma+1}. \end{aligned} \tag{1}$$

Notice that if we fix $k = \alpha+\beta+\gamma$, then the number of elements of the form $(f^1)^\alpha (f^2)^\beta (f^3)^\gamma$ is the same as of elements of the form $e^1 (f^1)^\alpha (f^2)^\beta (f^3)^\gamma$. Therefore, if we compute the number of k-cocycles of the first form, we obtain the number of $(k+1)$-cocycles of the second form which are not coboundaries.

We will identify every vector of the form $e^1 (f^1)^\alpha (f^2)^\beta (f^3)^\gamma$ as the first or second term of a particular vector, given by Equation (1). This is univocally determined. For instance, $e^1 (f^1)^\alpha (f^2)^{\beta-1} (f^3)^{\gamma+1}$ appears as part of the image of two vectors (in one as the second term, and in the other as the first term):

$$\begin{aligned} d\left((f^1)^\alpha (f^2)^\beta (f^3)^\gamma\right) &= (\beta+1)\, e^1 (f^1)^{\alpha-1} (f^2)^{\beta+1} (f^3)^\gamma + (\gamma+1)\, \underline{e^1 (f^1)^\alpha (f^2)^{\beta-1} (f^3)^{\gamma+1}}, \\ d\left((f^1)^{\alpha+1} (f^2)^{\beta-2} (f^3)^{\gamma+1}\right) &= (\beta-1)\, \underline{e^1 (f^1)^\alpha (f^2)^{\beta-1} (f^3)^{\gamma+1}} + (\gamma+2)\, e^1 (f^1)^{\alpha+1} (f^2)^{\beta-3} (f^3)^{\gamma+2}. \end{aligned} \tag{2}$$

We want to count the number of cocycles which are linear combinations of N vectors of the form $(f^1)^\alpha (f^2)^\beta (f^3)^\gamma$.

- If $N = 1$ then by Equation (1), the only possibility is to have $\alpha = \beta = 0$ and therefore the only vector is $(f^3)^\gamma$.
- If $N = 2$, by Equation (3) we must cancel the repeated terms, and the other two terms $e^1 (f^1)^{\alpha-1} (f^2)^{\beta+1} (f^3)^\gamma$ and $e^1 (f^1)^{\alpha+1} (f^2)^{\beta-3} (f^3)^{\gamma+2}$ must be zero. This is only possible when $\alpha = 0$ and $\beta = 2$. Therefore the resulting cocycle is

$$\varphi = (f^2)^2 (f^3)^\gamma - (\gamma+1)(f^1)(f^3)^{\gamma+1}.$$

  If $k \geq 2$ then there is only one k-cocycle which is the linear combination of two vectors of the form $(f^1)^\alpha (f^2)^\beta (f^3)^\gamma$.
- If $N = 3$ the situation is similar. We have

$$\begin{aligned} d\left((f^1)^\alpha (f^2)^\beta (f^3)^\gamma\right) &= (\beta+1)\, e^1 (f^1)^{\alpha-1} (f^2)^{\beta+1} (f^3)^\gamma + (\gamma+1)\, \underline{e^1 (f^1)^\alpha (f^2)^{\beta-1} (f^3)^{\gamma+1}}, \\ d\left((f^1)^{\alpha+1} (f^2)^{\beta-2} (f^3)^{\gamma+1}\right) &= (\beta-1)\, \underline{e^1 (f^1)^\alpha (f^2)^{\beta-1} (f^3)^{\gamma+1}} + (\gamma+2)\, \underline{\underline{e^1 (f^1)^{\alpha+1} (f^2)^{\beta-3} (f^3)^{\gamma+2}}}, \\ d\left((f^1)^{\alpha+2} (f^2)^{\beta-4} (f^3)^{\gamma+2}\right) &= (\beta-3)\, \underline{\underline{e^1 (f^1)^{\alpha+1} (f^2)^{\beta-3} (f^3)^{\gamma+2}}} + (\gamma+2)\, e^1 (f^1)^{\alpha+2} (f^2)^{\beta-5} (f^3)^{\gamma+3}. \end{aligned} \tag{3}$$

  Then the extreme terms are cancelled only by taking $\alpha = 0$ and $\beta = 4$. Then the cocyle is

$$\varphi = 3(f^2)^4 (f^3)^\gamma - (\gamma+1)(f^1)(f^2)^2 (f^3)^{\gamma+1} + (\gamma+1)(\gamma+2)(f^1)^2 (f^3)^{\gamma+2}.$$

  If $k \geq 4$ then there is only one k-cocycle which is the linear combination of three vectors of the form $(f^1)^\alpha (f^2)^\beta (f^3)^\gamma$.

In general, if $k \geq 2(N-1)$ then there is exactly one k-cocycle which is a linear combination of N vectors of the form $(f^1)^\alpha (f^2)^\beta (f^3)^\gamma$. Therefore, the number of k-cocycles of the form $(f^1)^\alpha (f^2)^\beta (f^3)^\gamma$ is $\left\lfloor \dfrac{k+2}{2} \right\rfloor$.

With this we obtain that the number of distinct classes in $H^k((1|3)_1, \mathbb{C})$ is $\left\lfloor \dfrac{k+2}{2} \right\rfloor + \left\lfloor \dfrac{k+1}{2} \right\rfloor$, and then

$$b_k = \begin{cases} \frac{k+1}{2} \left| \frac{k+1}{2} \right. & \text{if } k \text{ is odd,} \\ \\ \frac{k+2}{2} \left| \frac{k}{2} \right. & \text{if } k \text{ is even.} \end{cases}$$

Now consider the Lie superalgebra $(2|3)_{14}$. In this case, we obtain the following:

$$\begin{aligned} d(e^1e^2(f^1)^\alpha(f^2)^\beta(f^3)^\gamma) &= -\,(\beta+1)(\gamma+1)\, e^1(f^1)^\alpha(f^2)^{\beta+1}(f^3)^{\gamma+1}, \\ d(e^1(f^1)^\alpha(f^2)^\beta(f^3)^\gamma) &= 0, \\ d(e^2(f^1)^\alpha(f^2)^\beta(f^3)^\gamma) &= (\gamma+1)\, e^1e^2(f^1)^{\alpha-1}(f^2)^\beta(f^3)^{\gamma+1} + (\beta+1)(\gamma+1)\, (f^1)^\alpha(f^2)^{\beta+1}(f^3)^{\gamma+1}, \\ d((f^1)^\alpha(f^2)^\beta(f^3)^\gamma) &= (\,\gamma+1)\, e^1(f^1)^{\alpha-1}(f^2)^\beta(f^3)^{\gamma+1}. \end{aligned}$$

From this we deduce that the cocycles are:

(1) $v_1 = e^1(f^1)^\alpha(f^2)^\beta(f^3)^\gamma$: If $\gamma \geq 1$ then $d\left(\frac{1}{\gamma}\,(f^1)^{\alpha+1}(f^2)^\beta(f^3)^{\gamma-1}\right) = v_1$. Therefore v_1 is also a coboundary for $\gamma \geq 1$.
(2) $v_2 = (f^2)^\beta(f^3)^\gamma$: If $\beta,\gamma \geq 1$ then $d\left(\frac{1}{\beta\gamma}\, e^2(f^2)^{\beta-1}(f^3)^{\gamma-1}\right) = v_2$. Therefore v_2 is also a coboundary for $\beta,\gamma \geq 1$.
(3) $v_3 = e^1e^2(f^1)^\alpha(f^2)^\beta(f^3)^\gamma + (\beta+1)\ (f^1)^{\alpha+1}(f^2)^{\beta+1}(f^3)^\gamma$: If $\gamma \geq 1$ then $d\left(\frac{1}{\gamma}\, e^2(f^1)^{\alpha+1}(f^2)^\beta(f^3)^{\gamma-1}\right) = v_3$. Therefore v_3 is also a coboundary for $\gamma \geq 1$.

With all this, we obtain that the representatives for the cohomology $H^k((2|3)_{14}, \mathbb{C})$ are:

(1) $e^1(f^1)^\alpha(f^2)^\beta$ with $\alpha+\beta = k-1$. This gives us k vectors.
(2a) $(f^2)^k$.
(2b) $(f^3)^k$.
(3) $e^1e^2(f^1)^\alpha(f^2)^\beta + (\beta+1)\ (f^1)^{\alpha+1}(f^2)^{\beta+1}$ with $\alpha+\beta = k-2$. This gives us $k-1$ vectors.

Finally we obtain

$$b_k = \begin{cases} k\,|k+1 & \text{if } k \text{ is odd,} \\ \\ k+1\,|k & \text{if } k \text{ is even.} \end{cases}$$

□

Remark 1. *The previous proof shows the difficulties of computing cohomology of Lie superalgebras in comparison with the obtainment of cohomology of Lie algebras of the same dimension. In fact, the Lie superalgebra* $(1|3)_1$ *can be seen as the 4-dimensional model filiform Lie algebra L, by "forgetting" the* $\mathbb{Z}_2$*-grading. The filiform algebra L has Lie product:*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4,$$

and Betti numbers:

$$b_1 = b_2 = b_3 = 2, \quad b_4 = 1.$$

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Appendix A. Nilpotent Lie Superalgebras of Dimension ≤ 5

There are several classifications of Lie superalgebras of dimension ≤ 5. Among them, we mention [27–29] (the latter corrects the previous two). The classification of non-trivial indecomposable nilpotent Lie superalgebras of dimension ≤ 5, is then obtained from [29]. For every dimension $(m|n)$ we consider a basis $\{e_1, \dots, e_m | f_1, \dots, f_n\}$ and write the brackets in terms of this basis. We will not consider Lie superalgebras of dimension $(m|0)$ since they are just Lie algebras.

Theorem A1 ([29]). *Indecomposable nilpotent Lie superalgebras of dimension* ≤ 5 *are up to isomorphism:*

Table A1. Indecomposable nilpotent Lie superalgebras.

Notation [29]	Product			
	Dimension $(1\|1)$			
$(1\|1)_1$:	$\llbracket f_1, f_1 \rrbracket = e_1$.			
	Dimension $(1\|2)$			
$(1\|2)_2$:	$\llbracket f_1, f_2 \rrbracket = e_1$			
$(1\|2)_3$:	$\llbracket e_1, f_2 \rrbracket = f_1$			
	Dimension $(3\|1)$			
$(3\|1)_3$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket f_1, f_1 \rrbracket = e_3$		
	Dimension $(2\|2)$			
$(2\|2)_1$:	$\llbracket f_1, f_1 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = e_2$		
$(2\|2)_4$:	$\llbracket f_1, f_2 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = e_2$		
$(2\|2)_6$:	$\llbracket e_2, f_2 \rrbracket = f_1$,	$\llbracket f_2, f_2 \rrbracket = e_1$		
	Dimension $(1\|3)$			
$(1\|3)_1$:	$\llbracket e_1, f_2 \rrbracket = f_1$,	$\llbracket e_1, f_3 \rrbracket = f_2$		
$(1\|3)_5$:	$\llbracket f_1, f_1 \rrbracket = e_1$,	$\llbracket f_2, f_3 \rrbracket = e_1$		
	Dimension $(4\|1)$			
$(4\|1)_4$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket f_1, f_1 \rrbracket = e_4$,		
$(4\|1)_6$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket e_1, e_3 \rrbracket = e_4$,	$\llbracket f_1, f_1 \rrbracket = e_4$,	
	Dimension $(1\|4)$			
$(1\|4)_4$:	$\llbracket f_1, f_1 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = e_1$,	$\llbracket f_3, f_3 \rrbracket = e_1$,	$\llbracket f_4, f_4 \rrbracket = e_1$
$(1\|4)_7$:	$\llbracket e_1, f_2 \rrbracket = f_1$,	$\llbracket e_1, f_3 \rrbracket = f_2$,	$\llbracket e_1, f_4 \rrbracket = f_3$.	
$(1\|4)_8$:	$\llbracket e_1, f_2 \rrbracket = f_1$,	$\llbracket e_1, f_4 \rrbracket = f_3$.		
	Dimension $(3\|2)$			
$(3\|2)_5$:	$\llbracket f_1, f_1 \rrbracket = e_2$	$\llbracket f_1, f_2 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = e_3$.	
$(3\|2)_{10}$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket f_1, f_1 \rrbracket = e_3$,	$\llbracket f_2, f_2 \rrbracket = e_3$.	
$(3\|2)_{11}$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket e_1, f_2 \rrbracket = f_1$,		
$(3\|2)_{12}$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket e_1, f_2 \rrbracket = f_1$,	$\llbracket f_2, f_2 \rrbracket = e_3$,	
$(3\|2)_{13}$:	$\llbracket e_1, e_2 \rrbracket = e_3$,	$\llbracket e_1, f_2 \rrbracket = f_1$,	$\llbracket f_1, f_2 \rrbracket = e_3$,	$\llbracket f_2, f_2 \rrbracket = 2e_2$
	Dimension $(2\|3)$			
$(2\|3)_5$:	$\llbracket f_1, f_1] = e_1$,	$\llbracket f_2, f_2 \rrbracket = e_2$	$\llbracket f_3, f_3 \rrbracket = e_1$.	
$(2\|3)_6$:	$\llbracket f_1, f_1 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = e_2$	$\llbracket f_3, f_3 \rrbracket = e_1 + e_2$.	
$(2\|3)_8$:	$\llbracket f_1, f_2 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = 2e_2$,	$\llbracket f_2, f_3 \rrbracket = e_2$.	
$(2\|3)_9$:	$\llbracket f_1, f_2 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = 2e_2$,	$\llbracket f_3, f_3 \rrbracket = e_1$.	
$(2\|3)_{10}$:	$\llbracket f_1, f_2 \rrbracket = e_1$,	$\llbracket f_2, f_2 \rrbracket = 2e_2$,	$\llbracket f_3, f_3 \rrbracket = e_1 + e_2$.	
$(2\|3)_{11}$:	$\llbracket f_1, f_2] = e_1$,	$\llbracket f_2, f_2] = 2e_2$,	$\llbracket f_2, f_3] = e_2$.	$\llbracket f_3, f_3 \rrbracket = e_1$.
$(2\|3)_{13}$:	$\llbracket e_1, f_3 \rrbracket = f_1$,	$\llbracket f_2, f_2 \rrbracket = e_2$.		
$(2\|3)_{14}$:	$\llbracket e_1, f_3 \rrbracket = f_1$,	$\llbracket f_2, f_3 \rrbracket = e_2$.		
$(2\|3)_{16}$:	$\llbracket e_1, f_3 \rrbracket = f_1$,	$\llbracket f_2, f_2 \rrbracket = e_2$,	$\llbracket f_3, f_3 \rrbracket = e_2$.	
$(2\|3)_{17}$:	$\llbracket e_1, f_3 \rrbracket = f_1$,	$\llbracket e_2, f_2 \rrbracket = f_1$.		

Table A1. *Cont.*

Notation [29]	Product			
	Dimension (2|3)			
$(2\|3)_{18}$:	$[\![e_1, f_3]\!] = f_1,$	$[\![e_2, f_2]\!] = f_1,$	$[\![f_2, f_2]\!] = 2e_1,$	$[\![f_2, f_3]\!] = -e_2.$
$(2\|3)_{19}$:	$[\![e_1, f_3]\!] = f_1,$	$[\![e_2, f_3]\!] = f_2.$		
$(2\|3)_{21}$:	$[\![e_1, f_2]\!] = f_1,$	$[\![e_1, f_3]\!] = f_2,$	$[\![f_3, f_3]\!] = e_2.$	
$(2\|3)_{22}$:	$[\![e_1, f_2]\!] = f_1,$	$[\![e_1, f_3]\!] = f_2,$	$[\![f_1, f_3]\!] = -e_2,$	$[\![f_2, f_2]\!] = e_2.$
$(2\|3)_{23}$:	$[\![e_1, f_2]\!] = f_1$	$[\![e_1, f_3]\!] = f_2,$	$[\![e_2, f_3]\!] = f_1.$	

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