Variational Principles for Two Kinds of Coupled Nonlinear Equations in Shallow Water

Xiao-Qun Cao [1,2,*] , **Ya-Nan Guo** [1,2] , **Shi-Cheng Hou** [1,2] , **Cheng-Zhuo Zhang** [1] **and Ke-Cheng Peng** [1]

[1] College of Meteorology and Oceanography, National University of Defense Technology, Changsha 410073, China; guoyanan@nudt.edu.cn (Y.-N.G.); houshicheng17@nudt.edu.cn (S.-C.H.); zhangchengzhuo18@nudt.edu.cn (C.-Z.Z.); pengkch15@lzu.edu.cn (K.-C.P.)

[2] College of Computer, National University of Defense Technology, Changsha 410073, China

* Correspondence: caoxiaoqun@nudt.edu.cn

Abstract: It is a very important but difficult task to seek explicit variational formulations for nonlinear and complex models because variational principles are theoretical bases for many methods to solve or analyze the nonlinear problem. By designing skillfully the trial-Lagrange functional, different groups of variational principles are successfully constructed for two kinds of coupled nonlinear equations in shallow water, i.e., the Broer-Kaup equations and the (2+1)-dimensional dispersive long-wave equations, respectively. Both of them contain many kinds of soliton solutions, which are always symmetric or anti-symmetric in space. Subsequently, the obtained variational principles are proved to be correct by minimizing the functionals with the calculus of variations. The established variational principles are firstly discovered, which can help to study the symmetries and find conserved quantities for the equations considered, and might find lots of applications in numerical simulation.

Keywords: variational principle; calculus of variations; Broer-Kaup equations; (2+1)-dimensional dispersive long-wave equations

1. Introduction

Partial differential equations (PDEs) are usually used to model different phenomena in nonlinear sciences, ranging from physics to mechanics, biology, chemistry, meteorology, ocean, and so on [1–3]. Additionally, numerous mathematical techniques have been developed to explore the approximate or exact solutions [3–25]. Variational-based methods, such as Ritz technique [13], variational iteration method [14–18], and variational approximation method [19–25] et al., have been and continue to be useful and effective tools for nonlinear analysis. For example, the soliton solutions and their dynamics of lots of nonlinear equations were accurately captured by the variational approximation method [19–25], which always substitutes some ansatzes into the obtained Lagrange functional, and find the variational parameters by solving the corresponding Euler-Lagrange equations. When contrasted with other numerical or analytical methods, variational-based methods show a lot of advantages. Firstly, they can be used in investigating practical problems from a global perspective and provide physical insight into the nature of the solutions. Secondly, they can help to study the symmetries and conserved quantities for the discussed nonlinear problems. Last but not least, the obtained solutions are the best among all possible trial-functions and require much less strong local differentiability of variables than the methods that directly solve PDEs. Because variational principles are the theoretical bases for many kinds of variational methods, it is very important but difficult to seek explicit variational formulations for nonlinear and complex models usually expressed by the nonlinear PDEs. It is an inverse problem to directly find variational principles from a set of known equations by calculus rules

of variations. Recently, many scientists have made many attempts and great success for constructing different kinds of variational principles in various fields such as fluid dynamics, meteorology, ocean, mathematical biology, solid state physics, optics, and plasma physics, and so forth [19–36]. In this paper, different groups of variational principles are established for the Broer-Kaup equations [37,38] and the (2+1)-dimensional dispersive long-wave equations [39–41], respectively, which contains various solitary waves. Additionally, the solitary waves are always symmetric or anti-symmetric in space. Although the Broer-Kaup equations and (2+1)-dimensional dispersive long-wave equations considered in this paper have been extensively studied for a long time by some scientists [37–41], but, up to now, variational principle for them has not been dealt with. Therefore, finding variational principles for them is of great value and maybe helpful for harbor and coastal designs.

2. Variational Principles for the Broer-Kaup Equations

The nonlinear coupled Broer-Kaup equations are widely applied in shallow water wave modeling, which are also important and famous in physical fluids, statistical physics, nonlinear optical fiber communication, plasma physics, and so on [37,38,42–44]. The equation studied in this part is given, as following

$$\begin{cases} u_t + uu_x + v_x = 0 \\ v_t + u_x + u_x v + uv_x + u_{xxx}/4 = 0 \end{cases} \tag{1}$$

where $u = u(x,t)$ and $v = v(x,t)$ denotes the wave horizontal velocity and wave height, respectively, and both of them are dimensionless variables [37,38]. In the above equation, uu_x, $u_x v$ and uv_x are nonlinear terms, while u_{xxx} represents the dispersive term. If we introduce the variable transformations:

$$\begin{cases} u = -w \\ v = \eta - 1 - w_x/2 \end{cases} \tag{2}$$

Equation (1) becomes

$$\begin{cases} w_t - \frac{1}{2}(w^2 + 2\eta - w_x)_x = 0 \\ \eta_t - (w\eta + \frac{1}{2}\eta_x)_x = 0 \end{cases} \tag{3}$$

which is the famous integrable Broer-Kaup equations. Using the Darboux transformation method and homogeneous balance method, respectively, many kinds of soliton solutions are obtained [37,38]. However, up to now, variational principles for both Equations (1) and (3) are not discovered. Equation (1) can also be expressed in the conservative forms:

$$\begin{cases} u_t + (\frac{u^2}{2} + v)_x = 0 \\ v_t + (u + uv + u_{xx}/4)_x = 0 \end{cases} \tag{4}$$

It is obvious that finding Lagrangian representations for the above given equations is a nontrivial problem. Additionally, it is necessary to replace original wave variables by their representation as derivatives of potential fields. According to the first equation in (4), a potential function Φ can be introduced, as following

$$\begin{aligned} \Phi_x &= u \\ \Phi_t &= -(v + u^2/2) \end{aligned} \tag{5}$$

so that the first equation in Equation (4) is automatically satisfied. Similarly, in the light of the second equation in Equation (4), another potential function Π can also be introduced, being defined as:

$$\begin{aligned} \Pi_x &= v \\ \Pi_t &= -(u + uv + u_{xx}/4) \end{aligned} \tag{6}$$

and the second equation in Equation (4) is automatically satisfied. We will construct different groups of variational principles (VPs), according to the field Equations (5) and (6), respectively.

For establishing the variational principles, whose Euler-Lagrange equations will be equivalent to the Broer-Kaup equations, we can firstly set a trial-functional in the following form:

$$J = \iint L\,dxdt \tag{7}$$

where L is the trial-Lagrange functional. In view of Equations (4) and (5), we design that the L can be written as following

$$L = v\Phi_t + (u + uv + u_{xx}/4)\Phi_x + F(u,v) \tag{8}$$

which is not a standard Lagrangian. Additionally, F is an unknown functional of velocity u and wave height v and their derivatives. There are many alternative methods for constructing the trial-functional, see References [19–32]. The considerable merit of the above trial-Lagrange functional (8) is whose stationary condition with respect to Φ leads to the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Phi_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_t}\right) = 0 \tag{9}$$

In view of Equations (8) and (9) is identical to the second one in the Broer-Kaup Equation (1). Subsequently, by calculating the stationary conditions of Equation (8) with respect to u and v, respectively, we obtain:

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) + \frac{\delta F}{\delta u} = 0 \tag{10}$$

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial v_t}\right) + \frac{\delta F}{\delta v} = 0 \tag{11}$$

where $\frac{\delta F}{\delta u}$ and $\frac{\delta F}{\delta v}$ is called the Frechet's variational derivative [19–30] of F. By using Equations (8), (10) and (11) can be rewritten, as follows:

$$\Phi_x + v\Phi_x + \frac{\Phi_{xxx}}{4} + \frac{\delta F}{\delta u} = 0 \tag{12}$$

$$\Phi_t + u\Phi_x + \frac{\delta F}{\delta v} = 0 \tag{13}$$

We hope to find such an F, so that Equations (12) and (13) turn out to be the field Equation (5). Accordingly, after substituting Equation (5) into Equations (12) and (13), we get:

$$\frac{\delta F}{\delta u} = -u - uv - \frac{u_{xx}}{4} \tag{14}$$

$$\frac{\delta F}{\delta v} = v - \frac{u^2}{2}. \tag{15}$$

From the above Equations (14) and (15), we can identify F successfully, as follows

$$F = \frac{v^2 - u^2 - u^2v}{2} + \frac{u_x^2}{8} \tag{16}$$

or

$$F = \frac{v^2 - u^2 - u^2v}{2} - \frac{uu_{xx}}{8}. \tag{17}$$

After substituting the expressions (16) or (17) into (8), respectively, two different trial-Lagrange functionals can be generated. Furthermore, by substituting the trial-Lagrange functionals into the expression (8), we finally get the first group of variational principles for the Broer-Kaup equations, which read

$$J(u,v,\Phi) = \iint [v\Phi_t + (u + uv + u_{xx}/4)\Phi_x + (v^2 - u^2 - u^2v)/2 + u_x^2/8]dxdt \tag{18}$$

and

$$J(u, v, \Phi) = \iint [v\Phi_t + (u + uv + u_{xx}/4)\Phi_x + (v^2 - u^2 - u^2v)/2 - uu_{xx}/8]dxdt. \tag{19}$$

The established variational principles provide conservation laws in an energy forms and may find lots of applications in numerical simulation of Equation (1). In the following, we will prove the obtained variational principles correct. By making anyone of the above functionals, Equations (18) and (19), stationary with respect to three independent functions u, v, and Φ severally, we can obtain three different Euler-Lagrange equations:

$$\delta\Phi: \quad -v_t - (u + uv + u_{xx}/4)_x = 0 \tag{20}$$

$$\delta u: \quad (1 + v + \frac{1}{4}\frac{\partial^2}{\partial x^2})(\Phi_x - u) = 0 \tag{21}$$

$$\delta v: \quad \Phi_t + u\Phi_x + v - \frac{u^2}{2} = 0 \tag{22}$$

in which $\delta\Phi$, δu, and δv is the first-order variation for Φ, u, and v. Obviously, the Equation (20) is equivalent to the second one in Equation (1) or Equation (4). Because the Equation (21) must be established in all definition domains, we conclude that $\Phi_x = u$, which is identical to the first equation in (5). Substituting $\Phi_x = u$ into Equation (22) leads to $\Phi_t = -(v + \frac{u^2}{2})$, which is identical to the second equation in (5). Hence, successfully, we proved the obtained variational principles (18) and (19) correct.

In the following, we will establish the second groups of variational principles for the Broer-Kaup equations. By using the potential function Π defined in Equation (6), the trial-Lagrange functional can be skillfully designed in the following form

$$L = u\Pi_t + (v + \frac{u^2}{2})\Pi_x + F(u, v) \tag{23}$$

which is not a standard Lagrangian. Additionally F is an unknown functional of velocity u and wave height v and their derivatives. Obviously, the stationary condition of L with respect to Π results in:

$$\frac{\partial L}{\partial \Pi} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial \Pi_x}) - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \Pi_t}) = 0 \tag{24}$$

In view of (23), (24) is equivalent to the first equation in (1) or (6). Furthermore, by calculating the stationary condition of L with respect to u and v, respectively, we get:

$$\Pi_t + u\Pi_x + \frac{\delta F}{\delta u} = 0 \tag{25}$$

$$\Pi_x + \frac{\delta F}{\delta v} = 0 \tag{26}$$

According to Equations (6), (25) and (26) become:

$$\frac{\delta F}{\delta u} = u + \frac{u_{xx}}{4} \tag{27}$$

$$\frac{\delta F}{\delta v} = -v \tag{28}$$

From the above Equations (27) and (28), we can identify F easily, as follows:

$$F = \frac{u^2 - v^2}{2} - \frac{u_x^2}{8}) \tag{29}$$

or

$$F = \frac{u^2 - v^2}{2} + \frac{uu_{xx}}{8}. \tag{30}$$

Substituting (29) or (30) into (23), two new trial-Lagrange functionals are produced. Eventually, we can obtain the second group of variational principles for the specific Broer-Kaup equations, denoted as

$$J(u, v, \Pi) = \iint [u\Pi_t + (v + \frac{u^2}{2})\Pi_x + \frac{u^2 - v^2}{2} - \frac{u_x^2}{8}]dxdt \tag{31}$$

and

$$J(u, v, \Pi) = \iint [u\Pi_t + (v + \frac{u^2}{2})\Pi_x + + \frac{u^2 - v^2}{2} + \frac{uu_x}{8}]dxdt \tag{32}$$

Proof. According to calculus rules of variations, we can obtain a set of Euler-Lagrange equations by making anyone of the above functionals, Equations (31) and (32), stationary with respect to three independent functions Π, u, and v.

$$\delta\Pi: \quad -u_t - (v + \frac{u^2}{2})_x = 0 \tag{33}$$

$$\delta u: \quad \Pi_t + u\Pi_x + u + u_{xx}/4 = 0 \tag{34}$$

$$\delta v: \quad \Pi_x - v = 0 \tag{35}$$

in which $\delta\Pi$, δu, and δv is the first-order variation of Π, u, and v, respectively. Obviously, equation (33) is equivalent to the first equation in Equation (1) or Equation (4). From Equation (35), we have $\Pi_x = v$, which is identical to the first equation in (6). Substituting $\Pi_x = v$ into Equation (34) yields $\Pi_t = -(uv + u + u_{xx}/4)$, which is equivalent to the second equation in (6). Successfully, we prove the second group of variational principles (31)-(32) correct. The above variational principles can help us to study the symmetries and conserved quantities for the Broer-Kaup equations, and can be applied to the corresponding numerical simulations. □

3. Variation Principles for the (2+1)-Dimensional Dispersive Long-Wave Equations

The dispersive long-wave equations can describe many kinds of solitary waves in shallow water, which have completely different formulations [39–41,45–48]. In this part, we will establish variational principles for the integrable equations, given as following:

$$\begin{cases} u_t - u_{xx} - 2u_x v - 2uv_x = 0 \\ v_{ty} + v_{xxy} - 2u_{xx} - 2v_x v_y - 2vv_{xy} = 0 \end{cases} \tag{36}$$

where $u = u(x, y, t)$ and $v = v(x, y, t)$ are the two-dimensional field of wave velocity and height, respectively, and both of them are dimensionless variables [39–41]. Some solitary wave solutions are obtained for Equation (36) by a simple transformation and homogeneous balance method [40]. Additionally, the exact solutions are also derived through combining the extended F/G-expansion method and variable separation method, such as the bright and dark dromion solution and the periodic solitary wave solutions [41]. However, up to now, variational principles for (36) are not established. Because finding Lagrangian formulations for Equation (36) is a nontrivial problem, it is necessary to replace original variables by their representation as derivatives of potential fields. Equation (36) can be rewritten in the following forms:

$$\begin{cases} u_t - (u_x + 2uv)_x = 0 \\ [v_t + (v_x - v^2)_x]_y - 2u_{xx} = 0 \end{cases} \tag{37}$$

According to the special structures of the second equation in (37), two potential functions Φ and Ψ can be introduced skillfully, being defined as

$$\begin{aligned}
\Phi_x &= v \\
\Phi_t &= v^2 - v_x + 2\Psi_x \\
\Psi_y &= u
\end{aligned} \tag{38}$$

so that the second equation in (37) is automatically satisfied.

At first, the trial-functional can be given as following:

$$J(u, v, \Phi, \Psi) = \iiint L\,dx\,dy\,dt \tag{39}$$

where L is the trial-Lagrange functional to be determined. In view of Equations (37) and (38), we design that the L can be written as

$$L = u\Phi_t + u\Phi_{xx} - 2uv\Phi_x + F \tag{40}$$

which is not a standard Lagrangian. Additionally, F is an unknown functional free of Φ, that is, only dependent on u, v, Ψ, and their derivatives. Obviously, the stationary condition of L with respect to Φ leads to:

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Phi_x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial \Phi_{xx}}\right) = 0 \tag{41}$$

After substituting Equation (40) into Equation (41), we get

$$-u_t + u_{xx} + (2uv)_x = 0$$

which is completely identical to the first equation in Equation (37).

Subsequently, by calculating the stationary condition of L with respect to u and v, respectively, we obtain:

$$\frac{\partial L}{\partial u} + \frac{\delta F}{\delta u} = 0 \tag{42}$$

$$\frac{\partial L}{\partial v} + \frac{\delta F}{\delta v} = 0 \tag{43}$$

and $\frac{\delta F}{\delta u}$, $\frac{\delta F}{\delta v}$ is called the Frechet's variational derivative. By using Equations (40), (42) and (43) can be rewritten, as follows:

$$\Phi_t + \Phi_{xx} - 2v\Phi_x + \frac{\delta F}{\delta u} = 0 \tag{44}$$

$$-2u\Phi_x + \frac{\delta F}{\delta v} = 0 \tag{45}$$

Our aim is to find such an F, so that Equations (44) and (45) are identical to the field equations of (38). Substituting the first and second equation of (38) into (44) and (45) yields:

$$\frac{\delta F}{\delta u} = v^2 - 2\Psi_x \tag{46}$$

$$\frac{\delta F}{\delta v} = 2uv \tag{47}$$

From the Equations (46) and (47), F can be determined, as follows

$$F = uv^2 - 2u\Psi_x + F_1 \tag{48}$$

Substituting (48) into (40), the trial-Lagrange functional is updated as:

$$L = u\Phi_t + u\Phi_{xx} - 2uv\Phi_x + uv^2 - 2u\Psi_x + F_1 \tag{49}$$

and F_1 is a functional to be determined, only of Ψ and/or its derivatives. Furthermore, by calculating the stationary condition of the above new trial-Lagrangian L with respect to Ψ, we obtain:

$$2u_x + \frac{\delta F_1}{\delta \Psi} = 0 \tag{50}$$

We search for such an F_1, so that Equation (50) becomes the last field equation of (38). Substituting the last one of Equation (38) into (50) yields:

$$\frac{\delta F_1}{\delta \Psi} = -2\Psi_{xy} \tag{51}$$

From the above Equation (51), F_1 is identified as two different forms:

$$F_1 = \Psi_x \Psi_y \tag{52}$$

or

$$F_1 = -\Psi \Psi_{xy}. \tag{53}$$

Substituting the expressions (52), (53) into (49) leads to two new trial-Lagrange functionals. Furthermore, we obtain the variational formulations for the (2+1)-dimensional dispersive long-wave equations (36), which read:

$$J(u, v, \Phi, \Psi) = \iiint [u\Phi_t + u\Phi_{xx} - 2uv\Phi_x + uv^2 - 2u\Psi_x + \Psi_x\Psi_y]dxdydt \tag{54}$$

and

$$J(u, v, \Phi, \Psi) = \iiint [u\Phi_t + u\Phi_{xx} - 2uv\Phi_x + uv^2 - 2u\Psi_x - \Psi\Psi_{xy}]dxdydt. \tag{55}$$

Proof. According to calculus rules of variations, we obtain the following Euler-Lagrange equations by minimizing anyone of the above functionals, Equations (54) and (55) with respect to four independent functions Φ, Ψ, u, and v.

$$\delta\Phi: \quad -u_t + u_{xx} + (2uv)_x = 0 \tag{56}$$

$$\delta\Psi: \quad 2(u - \Psi_y)_x = 0 \tag{57}$$

$$\delta u: \quad \Phi_t + \Phi_{xx} - 2v\Phi_x + v^2 - 2\Psi_x = 0 \tag{58}$$

$$\delta v: \quad -2u\Phi_x + 2uv = 0 \tag{59}$$

in which $\delta\Phi$, $\delta\Psi$, δu, and δv is the first-order variation of Φ, Ψ, u, and v, respectively. Evidently, Equation (56) is equivalent to the first one in Equation (36) or Equation (37). From the Equation (57), we have $\Psi_y = u$, which is identical to the third one in (38). From Equation (59), we have $\Phi_x = v$, which is identical to the first one in (38). Substituting $\Phi_x = v$ into Equation (58) yields $\Phi_t + v_x - v^2 - 2\Psi_x = 0$, which is equivalent to the second equation in (38). Successfully, we prove the variational principles (54), (55) correct. The above variational principles can be used to study possible solution structures for solitary waves by using variational approximation method [19–25], and understand the physical relations and interactions between velocity and wave height for the (2+1)-dimensional dispersive long-wave equations that are considered in this paper. □

4. Conclusions

In the second and third parts, different groups of variational principles have been successfully constructed for the Broer-Kaup equations and the (2+1)-dimensional dispersive long-wave equations, respectively, by the calculus of variations and designing skillfully trial-Lagrange functionals. Subsequently, the obtained variational principles have proved correct by minimizing the corresponding functionals. From the results of analysis, it is concluded that the variational principle for the nonlinear equations studied in this paper is not unique, but it has many different integral formulations. According to the obtained variational principles, on one hand, we can study possible solution structures for

solitary waves, and deeply understand the physical relations and interactions between horizontal velocity and wave height. On the other hand, they also provide hints for numerical algorithms, then the Equations (1) and (36) can be solved numerically by the variational-based methods. In the analytical analysis and numerical simulations, it is of great importance to choose an appropriate variational principle according to practical needs. Our work in the future will focus on the dynamics of soliton in the Broer-Kaup Equation (1) and the (2+1)-dimensional dispersive long-wave Equation (36), by the variational approximation method using the established variational principles in this paper.

It is noteworthy that all the Lagrangians determined in this paper are neither standard nor non-standard Lagrangians in the strict sense, but more close to the standard ones. As we all know, there are two major families of Lagrangians, the so-called standard and non-standard Lagrangians. The former is typically expressed as the difference between terms that can be identified as the kinetic and potential energy, while the latter are entitled non-natural and they can take different forms, such as exponential forms, logarithmic forms, and power-law function, etc. [49–60]. The non-standard Lagrangians play a significant role in many kinds of equations, such as the nonlinear second-order Riccati equations and the nonlinear differential equation with Lienard type, and the dissipative systems [49–60]. Additionally, for all of the equations, standard Lagrangians become invalid. However, the problem of the exploration and the application of non-standard Lagrangians are still open, and demand lots of deep research. In the past several years, many scientists have made many attempts and great success for studying and applications of non-standard Lagrangians in various fields [49–60]. All of the Lagrangians established in this paper are evidently not the standard Lagrangians, because no obvious identification of the kinetic and potential energy terms can be made. On the hand, from the proof process, we successfully derive all of the original nonlinear equations from the Lagrangians, without using any other constraints. Accordingly, the obtained Lagrangians are also not the non-standard Lagrangians. Hence, we can conclude that the Lagrangians constructed in this paper belong to neither standard types nor non-standard ones technically.

Author Contributions: All authors have important contributions in this paper. The details are as following. Conceptualization, X.-Q.C.; Methodology, X.-Q.C. and Y.-N.G.; Validation, K.-C.P.; Writing original draft, X.-Q.C.; Writing review & editing, Y.-N.G., S.-C.H. and C.-Z.Z. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ablowitz, M.J.; Clarkson, P.A. *Solitons, Nonlinear Evolution Equations and Inverse Scatting*; Cambridge University Press: Cambridge, UK, 1991.
2. Gu, C.H. *Soliton Theory and Its Application*; Zhejiang Science and Technology Publishing House: Hangzhou, China, 1990.
3. He, J.H.; Li, Z.B. Converting Fractional Differential Equations into Partial Differential Equations. *Therm. Sci.* **2012**, *16*, 331–334. [CrossRef]
4. García-Ros, G.; Alhama, I.; Alhama, F. Dimensionless characterization of the non-linear soil consolidation problem of Davis and Raymond. Extended models and universal curves. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 61–78. [CrossRef]
5. Lakshminarayana, P.; Vajravelu, K.; Sucharitha, G.; Sreenadh, S. Peristaltic slip flow of a Bingham fluid in an inclined porous conduit with Joule heating. *Appl. Math. Nonlinear Sci.* **2018**, *3*, 41–54. [CrossRef]
6. Wang, M.; Zhou, Y.; Li, Z. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Phys. Lett. A* **1996**, *216*, 67–75. [CrossRef]
7. Liu, S.K.; Fu, Z.T. Expansion method about the Jacobi elliptic function and its applications to nonlinear wave equations. *Acta Phys. Sin.* **2001**, *50*, 2068–2073.

8. Ma, H.C. Exact solutions of nonlinear fractional partial differential equations by fractional sub-equation method. *Therm. Sci.* **2015**, *19*, 1239–1244. [CrossRef]

9. Li, Z.B. Exact solutions of time-fractional heat conduction equation by the fractional complex transform. *Therm. Sci.* **2012**, *16*, 335–338. [CrossRef]

10. He, J.H. Exp-function method for fractional differential equations. *Int. J. Nonlinear Sci. Numer. Simul.* **2013**, *14*, 363–366. [CrossRef]

11. He, J.H. Some asymptotic methods for strongly nonlinear equations. *Int. J. Mod. Phys. B* **2006**, *20*, 1141–1199. [CrossRef]

12. Guner, O.; Bekir, A. Exp-function method for nonlinear fractional differential equations. *Nonlinear Sci. Lett. A* **2017**, *8*, 41–49.

13. Wu, Y. Variational approach to higher-order water-wave equations. *Chaos Solitons Fractals* **2007**, *32*, 195–203. [CrossRef]

14. Gazzola, F.; Wang, Y.; Pavani, R. Variational formulation of the Melan equation. *Math. Methods Appl. Sci.* **2018**, *41*, 943–951. [CrossRef]

15. Baleanu, D. A modified fractional variational iteration method for solving nonlinear gas dynamic and coupled KdV equations involving local fractional operator. *Therm. Sci.* **2018**, *22*, S165–S175. [CrossRef]

16. Durgun, D.D. Fractional variational iteration method for time-fractional nonlinear functional partial differential equation having proportional delays. *Therm. Sci.* **2018**, *22*, S33–S46.

17. He, J.H.; Liu, F.J. Local Fractional Variational Iteration Method for Fractal Heat Transfer in Silk Cocoon Hierarchy. *Nonlinear Sci. Lett. A* **2013**, *4*, 15–20.

18. Yang, X.J.; Baleanu, D. Fractal heat conduction problem solved by local fractional variation iteration method. *Therm. Sci.* **2013**, *17*, 625–628. [CrossRef]

19. Malomed, B.A.; Weinstein, M.I. Soliton dynamics in the discrete nonlinear Schrödinger equation. *Phys. Lett. A* **1996**, *220*, 91–96. [CrossRef]

20. Malomed, B.A.; Anderson, D.; Lisak, M.; Quiroga-Teixeiro, M.L.; Stenflo, L. Dynamics of solitary waves in the Zakharov model equations. *Phys. Rev. E* **1997**, *55*, 962–968. [CrossRef]

21. Malomed, B.A. Variational methods in nonlinear fiber optics and related fields. *Prog. Opt.* **2002**, *43*, 71–193.

22. Chong, C.; Pelinovsky, D.E. Variational approximations of bifurcations of asymmetric solitons in cubic-quintic nonlinear schrödinger lattices. *Discret. Contin. Dyn. Syst.* **2011**, *4*, 1019–1031. [CrossRef]

23. Kaup, D.J. Variational solutions for the discrete nonlinear Schrödinger equation. *Math. Comput. Simul.* **2005**, *69*, 322–333. [CrossRef]

24. Chong, C.; Pelinovsky, D.E.; Schneider, G. On the validity of the variational approximation in discrete nonlinear Schrödinger equations. *Phys. D Nonlinear Phenom.* **2011**, *241*, 115–124. [CrossRef]

25. Putri, N.Z.; Asfa, A.R.; Fitri, A.; Bakri, I.; Syafwan, M. Variational approximations for intersite soliton in a cubic-quintic discrete nonlinear Schrödinger equation. *J. Phys. Conf. Ser.* **2019**, *1317*, 012015. [CrossRef]

26. He, J.H. Variational principles for some nonlinear partial differential equations with variable coefficients. *Chaos Solitons Fractals* **2004**, *19*, 847–851. [CrossRef]

27. He, J.H. A modified Li-He's variational principle for plasma. *Int. J. Numer. Methods Heat Fluid Flow* **2019**. [CrossRef]

28. He, J.H. Generalized equilibrium equations for shell derived from a generalized variational principle. *Appl. Math. Lett.* **2017**, *64*, 94–100. [CrossRef]

29. He, J.H.; Sun, C. A variational principle for a thin film equation. *J. Math. Chem.* **2019**, *57*, 2075–2081. [CrossRef]

30. He, J.H. Variational principle for the generalized KdV-burgers equation with fractal derivatives for shallow water waves. *J. Appl. Comput. Mech.* **2020**, *6*, 735–740.

31. Cao, X.Q. Variational principles for two kinds of extended Korteweg-de Vries equations. *Chin. Phys. B* **2011**, *20*, 94–102. [CrossRef]

32. Cao, X.Q. Generalized variational principles for Boussinesq equation systems. *Acta Phys. Sin.* **2011**, *60*, 105–113.

33. Wang, Y. A variational formulation for anisotropic wave traveling in a porous medium. *Fractals* **2019**, *27*, 1950047. [CrossRef]

34. Wang, K.L.; He, C.H. A remark on Wang's fractal variational principle. *Fractals* **2019**, *27*, 1950132. [CrossRef]

35. El-Kalaawy, O.H. New Variational principle-exact solutions and conservation laws for modified ion-acoustic shock waves and double layers with electron degenerate in plasma. *Phys. Plasmas* **2017**, *24*, 032308. [CrossRef]

36. El-Kalaawy, O.H. Variational principle, conservation laws and exact solutions for dust ion acoustic shock waves modeling modified Burger equation. *Comput. Math. Appl.* **2016**, *72*, 1013–1041. [CrossRef]

37. Liu, P. Darboux transformation of broer-kaup system and its soliton solutions. *Acta Math. Sci.* **2006**, *26A*, 999–1007.

38. Zhou, Z.J.; Li, Z.B. A Darboux transformation and new exact solutions for Broer-Kaup system. *Acta Phys. Sin.* **2003**, *52*, 262–266.

39. Liu, S.K.; Liu, S.D. *Nonlinear Equations in Physics*; The Series of Advanced Physics of Peking University; Peking University Press: Beijing, China, 2000.

40. Liu, L.H. Explicit solution of (2+1) dimensional dispersive long water waves equation and broer-kaup equation. *J. Jilin Norm. Univ. (Nat. Sci. Ed.)* **2011**, *2*, 60–66.

41. Yang, J.; Yu, Y.S. Exact solutions for (2+1)-dimensional dispersive long wave equation. *Coll. Phys.* **2017**, *36*, 26–33.

42. Jiang, B.; Bi, Q.S. Peaked periodic wave solutions to the broer-kaup equation. *Commun. Theor. Phy.* **2017**, *67*, 22–26. [CrossRef]

43. Tang, D.B.; Zhao, X.Q.; Wang, L.M.; Zhang, Y.M. Some new soliton wave solutions for (2+1)-dimensional Broer-Kaup equations. *Acta Phys. Sin.* **2003**, *52*, 1827–1831.

44. Naranmandula, W.K. Multisoliton-like solutions for (2+1)-dimensional dispersive long wave equations and (2+1)-dimensional Broer-Kaup equations. *Acta Phys. Sin.* **2003**, *52*, 1565–1568.

45. Chen, C.L.; Tang, X.Y.; Lou, S.Y. Solutions of a (2+1)-dimensional dispersive long wave equations. *Phys. Rev. E* **2002**, *66*, 036605. [CrossRef] [PubMed]

46. Li, D.S.; Zhang, H.Q. Some new exact solutions of the dispersive long-wave equation in (2+1)-dimensional spaces. *Commun. Theor. Phys.* **2003**, *40*, 143–146.

47. Lin, F.Z.; Ma, S.H. New exact solutions and complex wave excitations for the (2+1)-dimensional dispersive long wave equation. *Acta Phys. Sin.* **2014**, *63*, 040508.

48. Zeng, X.; Zhang, H.Q. New soliton-like solutions to the (2+1)-dimensional dispersive long wave equations. *Acta Phys. Sin.* **2005**, *54*, 504–510.

49. Musielak, Z. Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients. *J. Phys. A Math. Theor.* **2008**, *41*, 055205. [CrossRef]

50. Musielak, Z.; Roy, D.; Swift, L. Method to derive Lagrangian and Hamiltonian for a nonlinear dynamical system with variable coefficients. *Chaos Solitons Fractals* **2008**, *38*, 894–902. [CrossRef]

51. Musielak, Z. General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems. *Chaos Solitons Fractals* **2009**, *42*, 2645–2652. [CrossRef]

52. Musielak, Z.; Davachi, N.; Rosario-Franco, M. Lagrangians, gauge transformations and Lie groups for commutative semigroup of differential equations. *arXiv* **2019**, arXiv:1902.01014.

53. Musielak, Z.; Davachi, N.; Rosario-Franco, M. Lagrangian formalism and its auxiliary conditions: Special function equations and Bateman oscillators. *arXiv* **2019**, arXiv:1902.01013.

54. Zhang, Y.; Zhou, X.-S. Noether theorem and its inverse for nonlinear dynamical systems with nonstandard Lagrangians. *Nonlinear Dyn.* **2016**, *84*, 1867–1876. [CrossRef]

55. Zhang, Y.; Wang, X.-P. Mei symmetry and invariants of quasi-fractional dynamical systems with non-standard lagrangians. *Symmetry* **2019**, *11*, 1061. [CrossRef]

56. Song, J.; Zhang, Y. Routh method of reduction for dynamical systems with nonstandard Lagrangians on time scales. *Indian J. Phys.* **2019**, *94*, 501–506. [CrossRef]

57. El-Nabulsi, A.R. Non-linear dynamics with non-standard Lagrangians. *Qual. Theory Dyn. Syst.* **2013**, *12*, 273–291. [CrossRef]

58. El-Nabulsi, A. Modified Proca equation and modified dispersion relation from a power-law Lagrangian functional. *Indian J. Phys.* **2013**, *87*, 465–470. [CrossRef]

59. Rami, E.-N.A.; Soulati, T.; Rezazadeh, H. Non-standard complex Lagrangian dynamics. *J. Adv. Res. Dyn. Control Syst.* **2013**, *5*, 50–62.

60. El-Nabulsi, R.A. Non-standard power-law Lagrangians in classical and quantum dynamics. *Appl. Math. Lett.* **2015**, *43*, 120–127. [CrossRef]