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Construction of Weights for Positive Integral Operators

Ron Kerman

Department of Mathematics, Brock University, St. Catharines, ON L2S 3A1, Canada; rkerman@brocku.ca

Received: 15 March 2020; Accepted: 16 April 2020; Published: 12 June 2020

Abstract: Let (X, M, μ) be a σ-finite measure space and denote by $P(X)$ the μ-measurable functions $f\colon X \to [0, \infty]$, $f < \infty$ μ ae. Suppose $K\colon X \times X \to [0, \infty)$ is $\mu \times \mu$-measurable and define the mutually transposed operators T and T' on $P(X)$ by $(Tf)(x) = \int_X K(x, y) f(y)\, d\mu(y)$ and $(T'g)(y) = \int_X K(x, y) g(x)\, d\mu(x), f, g \in P(X), x, y \in X$. Our interest is in inequalities involving a fixed (weight) function $w \in P(X)$ and an index $p \in (1, \infty)$ such that: (*): $\int_X [w(x)(Tf)(x)]^p d\mu(x) \lesssim C \int_X [w(y) f(y)]^p d\mu(y)$. The constant $C > 1$ is to be independent of $f \in P(X)$. We wish to construct all w for which (*) holds. Considerations concerning Schur's Lemma ensure that every such w is within constant multiples of expressions of the form $\phi_1^{1/p-1}\phi_2^{1/p}$, where $\phi_1, \phi_2 \in P(X)$ satisfy $T\phi_1 \leq C_1\phi_1$ and $T'\phi_2 \leq C_2\phi_2$. Our fundamental result shows that the ϕ_1 and ϕ_2 above are within constant multiples of (**): $\psi_1 + \sum_{j=1}^{\infty} E^{-j} T^{(j)} \psi_1$ and $\psi_2 + \sum_{j=1}^{\infty} E^{-j} T'^{(j)} \psi_2$respectively; here $\psi_1, \psi_2 \in P(X)$, $E > 1$ and $T^{(j)}, T'^{(j)}$ are the jth iterates of T and T'. This result is explored in the context of Poisson, Bessel and Gauss–Weierstrass means and of Hardy averaging operators. All but the Hardy averaging operators are defined through symmetric kernels $K(x, y) = K(y, x)$, so that $T' = T$. This means that only the first series in (**) needs to be studied.

Keywords: weights; positive integral operators; convolution operators

MSC: 2000 Primary 47B34; Secondary 27D10

1. Introduction

Consider a σ-finite measure space (X, M, μ) and a positive integral operator T defined through a nonnegative kernel $K = K(x, y)$ which is $\mu \times \mu$ measurable on $X \times X$; that is, T is given on the class, $P(X)$, of μ-measurable functions $f : X \to [0, \infty]$, $f < \infty$ μ ae, by

$$(Tf)(x) = \int_X K(x, y) f(y)\, d\mu(y), \; x \in X.$$

The transpose, T', of T at $g \in P(X)$ is

$$(T'g)(y) = \int_X K(x, y) g(x)\, d\mu(x), \; y \in X;$$

it satisfies

$$\int_X gTf d\mu = \int_X fT'g\, d\mu, \; f, g \in P(X).$$

Our focus will be on inequalities of the form

$$\int_X [uTf]^p\, d\mu \leq B^p \int_X [vf]^p\, d\mu, \tag{1}$$

with the index p fixed in $(1, \infty)$ and $B > 0$ independent of $f \in P(X)$; here, $u, v \in P(X)$, $0 \le u, v < \infty$, μ *ae*, are so-called weights.

The equivalence need only be proved in one direction. Suppose, then, (1) holds and $g \in P(x)$ satisfies $\int_X [u^{-1}g]^p \, d\mu < \infty$. Then

$$\left[\int_X [v^{-1}T'g]^{p'} \, d\mu\right]^{\frac{1}{p'}} = \sup \int_X fv^{-1}T'g \, d\mu,$$

the supremum being take over $f \in P(X)$ with $\int_X f^p \, d\mu \le 1$. But, Fubini's Theorem ensures

$$\begin{aligned}
\int_X fv^{-1}T'g \, d\mu &= \int_X gT(fv^{-1}) \, d\mu \\
&= \int_X (u^{-1}g)uT(fv^{-1}) \, d\mu \\
&\le \left[\int_X [u^{-1}g]^{p'} \, d\mu\right]^{\frac{1}{p'}} \left[B^p \int_X [vfv^{-1}]^p \, d\mu\right]^{\frac{1}{p}} \\
&\le \left[B^{p'} \int_X [u^{-1}g]^{p'} \, d\mu\right]^{\frac{1}{p'}}.
\end{aligned}$$

Further, (1) holds if and only if the dual inequality

$$\int_X [v^{-1}T'g]^{p'} \, d\mu \le B^{p'} \int_X [u^{-1}g]^{p'} \, d\mu, \; p' = \frac{p}{p-1}, \tag{2}$$

does.

Inequality (1) has been studied for various operators T in such papers as [1–9].

In this paper, we are interested in constructing weights u and v for which (1) holds. We restrict attention the case $u = v = w$; the general case will be investigated in the future. Our approach is based on the observation that, implicit in a proof of the converse of Schur's lemma, given in [10], is a method for constructing w. An interesting application of Schur's lemma itself to weighted norm inequalities is given in Christ [11].

In Section 2, we prove a number of general results the first of which is the following one.

Theorem 1. *Let (X, M, μ) be a σ-finite measure space with $u, v \in P(X)$, $0 \le u, v < \infty$, μ ae. Suppose that T is a positive integral operator on $P(X)$ with transpose T'. Then, for fixed p, $1 < p < \infty$, one has (1), with $C > 1$ independent of $f \in P(X)$, if and only if them exists a function $\phi \in P(X)$ and a constant $C > 1$ for which*

$$T(v^{-1}\phi^{p'}) \le Cu^{-1}\phi^{p'} \text{ and } T'(u\phi^p) \le Cv\phi^p. \tag{3}$$

In this case, B_0, the smallest B possible in (1) and Co, the smallest possible C so that (3) holds for some ϕ, satisfy

$$B_0 \le C_0 = \max\left[B_1^p, B_1^{p'}\right],$$

where $B_1 = B_0^{1/p} + B_0^{1/p'}$.

Theorem 1 has the following consequence.

Corollary 1. *Under the condition of Theorem 1, (1) holds for $u = v = w$ if and only if $w = \phi_1^{-1/p'}\phi_2^{1/p}$, where ϕ_1, ϕ_2 are functions in $P(X)$ satisfying*

$$T\phi_1 \le C\phi_1 \text{ and } T'\phi_2 \le C\phi_2, \tag{4}$$

for some $C > 1$.

Though it is often possible to work with the inequalities (4) directly (see Remark 1) it is important to have a general method to construct the functions ϕ_1 and ϕ_2. This method is given in our principal result.

Theorem 2. *Suppose* X, μ *and* T *are as in Theorem 1. Let* $\phi \in P(X)$*. Then,* ϕ *satisfies an inequality of the form*

$$T\phi \leq C_1\phi,\ C_1 > 0 \text{ constant}, \tag{5}$$

if and only if there is a constant $C > 1$ *such that*

$$C^{-1}\phi \leq \psi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\psi \leq C\phi, \tag{6}$$

where $\psi \in P(X)$, $C_2 > 1$ *is constant and* $T^{(j)} = T \circ T \cdots \circ T$, j *times.*

The kernels of operator of the form

$$\sum_{j=1}^{\infty} C^{-j} T^{(j)} \text{ and } \sum_{j=1}^{\infty} C^{-j} T'^{(j)}$$

will be called the weight generating kernels of T. In Sections 3–6 these kernels will be calculated for particular T. All but the Hardy operators considered in Section 6 operate on the class $P(R^n)$ of nonnegative, Lebesgue-measurable functions on R^n.

The operators last referred to are, in fact, convolution operators

$$(T_k f)(x) = (k * f)(x) = \int_{R^n} k(x-y)f(y)\,dy,\ x \in R^n,$$

with even integrable kernels k, $\int_{R^n} k(y)\,dy = 1$. In particular, the kernel $k(x-y)$ is symmetric, so $T_k' = T_k$, whence only the first series in (**) need be considered.

Further, the convolution kernels are part of an approximate identity $\{k_t\}_{t>0}$ on

$$L^P(R^n) = \left\{ f \text{ Leb. meas: } \left[\int_{R_n} |f|^p\right]^{1/p} < \infty \right\},$$

see [12]. Thus, it becomes of interest to characterize the weights w for which $\{k_t\}_{l>0}$ is an approximate identity on

$$L^p(w) = L^p(R^n, w) = \left\{ f \text{ Leb. meas: } \|f\|_{p,w} = \left[\int_{R^n} |wf|^p\right]^{1/p} < \infty \right\};$$

that is $k_t * f \in L^p(w)$ and

$$\lim_{t \to 0+} \|k_t * f - f\|_{p,w} = 0$$

for all $f \in L^p(w)$. It is a consequence of the Banach-Steinhaus Theorem that this will be so if and only if

$$\sup_{0<t<a} \|k_t\| < \infty$$

for some fixed $a > 0$, where $\|k_t\|$ denotes the operator norm of T_{k_t} on $L^p(w)$. We remark here that the operators in Sections 3–5 are bounded on $L^p(w)$ and, indeed, form part of an approximate identity on $L^p(w)$, if w satisfies the A_p condition, namely,

$$\sup \left[\frac{1}{|Q|}\int_Q w^p\right]\left[\frac{1}{|Q|}\int_Q w^{-p'}\right]^{1/p'} < \infty,\ p' = \frac{p}{p-1}, \tag{7}$$

the supremum being taken over all cubes Q in R^n whose sides are parallel to the coordinate axes with $\infty > |Q| =$ Lebesgue measure of Q. See ([13], p. 62) and [14].

Finally, all the convolution operators are part of a convolution semigroup $(k_t)_{t>0}$; that is $k_t(x) = t^{-n}k\left(\frac{x}{t}\right)$ and $k_{t_1} * k_{t_2} = k_{t_1+t_2}$, $t_1, t_2 > 0$. The approximate identity result can thus be interpreted as the continuity of the semigroup.

We conclude the introduction with some remarks on terminology and notation. The fact that T is bounded on $L^p(w)$ if and only if T' is bounded on $L^{p'}(w^{-1})$ is called the principle of duality or, simply, duality. Two functions $f, g \in P(X)$ are said to be equivalent if a constant $C > 1$ exists for which

$$C^{-1}g \leq f \leq Cg. \tag{8}$$

We indicate this by $f \approx g$, with the understanding that C is independent of all parameters appearing, (except dimension) unless otherwise stated. If only one of the inequalities in (8) holds, we use the notation $f \succeq g$ or $f \preceq g$, as appropriate. Lastly, a convolution operator and its kernel are frequently denoted by the same symbol.

2. General Results

In this section we give the proofs of the results stated in the Introduction, together with some remarks.

Proof of Theorem 1. The conditions (3) are, respectively, equivalent to

$$T' : L^1(u^{-1}\phi^{p'}) \to L^1(v^{-1}\phi^{p'})$$
$$\text{i.e., } T : L^\infty(v\phi^{-p'}) \to L^\infty(u\phi^{-p'})$$

and

$$T : L^1(v\phi^p) \to L^1(u\phi^p).$$

It will suffice to deal with the first condition in (3). So, Fubini's Theorem yields

$$\int_X v^{-1}\phi^{p'} T' f\, d\mu \leq C\int_X u^{-1}\phi^{p'} f\, d\mu$$

equivalent to

$$\int_X f\, T(v^{-1}\phi^{p'})\, d\mu \leq C\int_X f u^{-1}\phi^{p'}\, d\mu, \quad f \in P(X),$$

and hence to

$$T(v^{-1}\phi^{p'}) \leq Cu^{-1}\phi^{p'},$$

since f is arbitrary.

According to the main result of [15], then,

$$T : L^p\left((v\phi^p)^{1/p}(v\phi^{-p'})^{1/p'}\right) \to L^p\left((u\phi^p)^{1/p}(u\phi^{-p'})^{1/p'}\right)$$

i.e., $T : L^p(v) \to L^p(u)$, with norm $\leq C$, so that (1) holds with $B \leq C$.

Suppose now (1) holds. Following [10], choose $g \in P(X)$ with

$$\int_X g^{pp'}\, d\mu = 1.$$

Let $T_1 g = \left[uT(v^{-1}g^{p'})\right]^{1/p'}$ and $T_2 g = \left[v^{-1}T'(ug^p)\right]^{1/p}$. Set

$$S = T_1 + T_2,\ A = B_0 + \varepsilon \text{ and } \phi = \sum_{j=0}^{\infty} A^{-j} S^{(j)} g.$$

As in [10], conclude $T_1\phi \leq A\phi$ and $T_2\phi \leq A\phi$, so that (2) is satisfied for $C_0 \leq \left[B_1^p, B_1^{p'}\right]$, where $B_1 = B_0^{1/p} + B_0^{1/p'}$. □

Proof of Corollary 1. Given (1), one has (2) and Theorem 1 then implies (3), with T replaced by T', namely for $u = v = w$,

$$T(w^{-1}\phi^{p'}) \leq Cw^{-1}\phi^{p'} \text{ and } T(w\phi^p) \leq Cw\phi^p,$$

whence the inequalities (4) are satisfied by $\phi_1 = w\phi^p$ and $\phi_2 = w^{-1}\psi^{p'}$. Conversely, given (4), and taking $u = v = w = \phi_1^{1/p-1}\phi_2^{1/p}$, one readily obtains (3), with $\psi = (\psi_1\psi_2)^{1/pp'}$. □

Proof of Theorem 2. Clearly, if (6) holds,

$$T\phi \leq C\left[T\psi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j+1)}\psi\right] = CC_2 \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\psi \leq C^2 C_2 \phi.$$

Suppose $\phi \in P(X)$ satisfies (5). Then,

$$T^{(j)}\phi \leq C_1^j \phi_1,\ j = 1, 2, \ldots.$$

It only remains to observe that

$$\left(1 + \frac{C_1}{\varepsilon}\right)^{-1} \phi \leq \phi + \sum_{j=1}^{\infty} (C_1 + \varepsilon)^{-j} T^{(j)}\phi \leq \phi + \sum_{j=1}^{\infty} \left(\frac{C_1}{C_1 + \varepsilon}\right)^j \phi \leq \left(1 + \frac{C_1}{\varepsilon}\right)\phi,$$

for any $\varepsilon > 0$. □

Remark 1. *The class of functions ϕ determined by the weight-generating operators $\sum_{j=1}^{\infty} C^{-j} T^{(j)}$ effectively remains the same as C increases. Thus, suppose $0 < C_1 < C_2$, $\psi \in P(X)$ and $\phi = \psi + \sum_{j=1}^{\infty} C_1^{-j} T^{(j)}\psi$. Then, ϕ is equivalent to $\psi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\psi$, since*

$$\begin{aligned}
\phi \leq \phi + \sum_{j=1}^{\infty} C_2^{-j} T^{(j)}\phi &= \sum_{j=0}^{\infty} C_2^{-j} \sum_{k=0}^{\infty} C_1^{-k} T^{(j+k)}\psi = \sum_{l=0}^{\infty} \left(\sum_{j+h=l} C_1^{-k} C_2^{-j}\right) T^{(l)}\psi \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{C_1}{C_2}\right)^j C_1^{-l} T^{(l)}\psi = \frac{C_2}{C_2 - C_1} \sum_{l=0}^{\infty} C_1^{-l} T^{(l)}\psi \\
&= \frac{C_2}{C_2 - C_1}\phi.
\end{aligned}$$

This means that in dealing with weight-generating operators we need only consider $C > 1$.

We conclude this section with the following observations on approximate identities in weighted Lebesgue spaces.

Remark 2. *Suppose $\{k_t\}_{t>0}$ is an approximate identity in $L^p(R^n)$, $1 < p < \infty$. If the inequalities (4) involving ϕ_1 and ϕ_2 can be shown to hold for T_{kt}, $t \in (0, a]$ for some $a > 0$, with $C > 1$ independent of such t, then $\{k_t\}_{t>0}$ will also be an approximate identity in $L^p(w) = L^p(R^n, w)$, $w = \phi_1^{-1/p'} \phi_2^{1/p}$.*

Example 1. *Let $k = k(|x|)$ be any bounded, nonnegative radial function on R^n which is a decreasing function of $|x|$ and suppose $\int_{R^n} k(x)\,dx = 1$. It is well-known, see ([13], p. 63), that $k_t(x) = t^{-n}k(x/t)$, $x \in R^n$, is an approximate identity in $L^p(R^n)$, $1 < p < \infty$.*

The weight $w(x) = 1 + |x|^{-n/p}(1 + \log^+(1/|x|))^{-1}$, for fixed p, $1 < p < \infty$, has the interesting properly that $T_{k_t} : L^p(w) \to L^p(w)$ for all $t > 0$, yet $\{k_t\}_{t>0}$ is never an approximate identity in $L^p(w)$.

To obtain the boundedness assertion take $\phi_1(x) = 1$ and $\phi_2(x) = 1 + |x|^{-n}\,(1 + \log^+(1/|x|))^{-p}$ in Corollary 1.

Arguments similar to those in [6] show that if $\{k_t\}_{t>0}$ is an approximate identity in $L^p(w)$, then w must satisfy the A_p condition for all cubes Q will sides parallel to the coordinate axes and $|Q| \leq a$ for some $a > 0$. However, the weight w does not have this property.

3. The Poisson Integral Operators

We recall that for $t > 0$ and $y \in R^n$, the Poisson kernel, P_t, is defined by

$$P_t(y) = c_n t(t^2 + |y|^2)^{-(n+1)/2}, \; c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2).$$

Theorem 3. *The weight-generating kernels for P_t, $t > 0$, are equivalent to $P \equiv P_0$. Indeed, given $\psi \in P(R^n)$, with $P_\psi < \infty$ a.e.,*

$$C_t^{-1}P_\psi \leq \sum_{j=1}^{\infty} C^{-j}P_{jt}\psi \leq C_t' P_\psi, \tag{9}$$

where $C > 1$, $C_t = C\max[t^{-1}, t^n]$ and $C_t' = C_t \sum_{j=1}^{\infty} C^{-j}\max[jt, (jt)^{-n}]$.

Proof. Observe that by the semigroup property $P_t^{(j)} = P_{jt}$, $j = 1, 2, \ldots$.

Also,

$$\min[t, t^{-n}]P \leq P_t \leq \max[t, t^{-n}]P.$$

Now, suppose

$$\psi + \sum_{j=1}^{\infty} C^{-j}P_{jt}\psi \text{ is in } P(R^n),$$

with $C > 1$. Then,

$$\begin{aligned} P_\psi &\leq C_t P_t\psi + \sum_{j=1}^{\infty} C^{-j}P_{(j+1)t}\psi \leq C_t \sum_{j=1}^{\infty} C^{-j}P_{jt}\psi \leq C_t \sum_{j=1}^{\infty} C^{-j}\max[jt, (j)^{-n}]P_\psi \\ &\leq C_t' P_\psi. \end{aligned}$$

□

As stated in Section 1, $w \in A_p$ is sufficient for $\{P_t\}_{t>0}$ to be an approximate identify in $L^P(w)$. Moreover, $w \in A_p$ is also necessary for this in the periodic case. See [6,8,16]. It is perhaps surprising then that the class of approximate identity weights is much larger than A_p, as is seen in the next result.

Proposition 1. *Let $w_\alpha(x) = [1+|x|]^\alpha$, $\alpha \in R$. Then, for any $t > 0$, P_t is bounded on $L^p(w_\alpha)$ if any only if $-\frac{n}{p} - 1 < \alpha < \frac{n}{p'} + 1$. Moreover, on that range of α one has*

$$\lim_{t \to 0+} \|P_t * f - f\|_{p,\omega_\alpha} = 0, \tag{10}$$

for all $f \in L^p(\omega_\alpha)$. The set of α for which $w_\alpha \in A_p$, however, is

$$-\frac{n}{p} < \alpha < \frac{n}{p'}.$$

Proof. We omit the easy proof of the assertion concerning the α for which $w_\alpha \in A_p$.

To obtain the "if" part of the other assertion we will show

$$P_t * w_\beta \leq C w_\beta,\ t > 0, \tag{11}$$

if and only if $-n-1 \leq \beta < 1$, with $C > 1$ independent of both s and t, if $t \in (0,1)$. Corollary 1 and Remark 2, then yield (10) when $-\frac{n}{p} - 1 < \alpha < \frac{n}{p'} + 1$.

Consider, then, fixed $x \in R^n$ and $0 < t < 1$. We have

$$\begin{aligned}(P_t * w_\beta)(x) &= \left(\int_{|y| \leq \frac{|x|}{2}} + \int_{\frac{|x|}{2} < |y| < 2|x|} + \int_{|y| \geq 2|x|} \right) P_t(y) w_\beta(s-t)\, dy \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Now,

$$I_1 \leq w_\beta(x) \int_{|y| < \frac{|x|}{2}} P_t(y)\, dy \leq C w_\beta(x),$$

for all $\beta \in R$.

Again,

$$I_2 \geq c P_t(x) \int_{|x-y| \leq 1} (1+|x-y|)^\beta\, dy \geq c P_t(x) \geq c|x|^{-n-1},$$

so we require $\beta > n-1$, if (11) is to hold.

Moreover, for $x \in R^n$ and $0 < t < 1$,

$$\begin{aligned} I_2 &\approx P_t(x) \left[|x|^n \chi_{|x| \leq 1} + |x|^{\beta+n} \chi_{|x|>1} \right] \\ &\approx \left(\frac{|x|}{t} \right)^n \chi_{|x| \leq 1} + \frac{t}{|x|} \chi_{t \leq |x| \leq 1} + \frac{t}{|x|} |x|^\beta \chi_{|x| \geq 1} \\ &\leq C w_\beta(x). \end{aligned}$$

Next, for $|x| \gg 1$

$$I_3 = \int_{|y|>2|x|} P_t(y) w_\beta(y)\, dy \preceq t \int_{|y|>2|x|} |y|^{-n-1+\beta} dy$$

which requires $\beta < 1$ to have $I_3 < \infty$. In that case

$$I_3 \preceq \int_{r > 2|x|} r^{-n-1+\beta} r^{n-1}\, dr \preceq |x|^{\beta-1} \preceq w_\beta(x).$$

That P_t is not bounded on $L^p(w_\alpha)$ when $\alpha \leq -\frac{n}{p} - 1$ can be seen by noting that, for appropriate $\varepsilon > 0$, the function $f(x) = |x|[\log(1+|x|)]^{-(1+\varepsilon)/p}$ is in $L^p(w_\alpha)$, while $P_t f \equiv \infty$. The range $\alpha \geq n/p + 1$ is then ruled out by duality. □

4. The Bessel Potential Operators

The Bessel kernel, $G_\alpha, \alpha > 0$, can be defined explicitly by

$$G_\alpha(y) = C_\alpha |y|^{(\alpha-n)/2} K_{(n-\alpha)/2}(|y|),\ y \in R^n,$$

where K_r is the modified Bessel function of the third kind and

$$C_\alpha^{-1} = \pi^{n/2} 2^{(n+\alpha-2)} \Gamma(\alpha/2).$$

It is, however, more readily recognized by its Fourier transformation

$$\hat{G}_\alpha(z) = (2\pi)^{-n/2}[1 + |z|^2]^{-\alpha/2}.$$

Using the latter formula one picks out the special cases G_{n-1} and G_{n+1} which, except for constant multiplies, are, respectively, $|y|^{-1}e^{-|y|}$ and the Picard kernel $e^{-|y|}$.

The semigroup properly $G_\alpha * G_\beta = G_{\alpha+\beta}$ holds and so the jth convolution iterate has kernel $G_{j\alpha}$. Also, $\int_{R^n} G_\alpha(y)\, dy = 1$.

We use the integral representation

$$G_\alpha(y) = g_{\alpha,n}(|y|) = (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1} \int_0^\infty e^{-|y|^2 t/4} e^{-1/t} t^{(n-2)/2} \frac{dt}{t} \tag{12}$$

to show in Lemma 1 below that known estimates [17], are in fact, sharp.

Lemma 1. *Suppose $n, \alpha > 0, n \in Z_+$. Set $m = n - \alpha$ and define r^{-m_+} to be r^{-m}, $\log_+\left(\frac{2}{r}\right)$ or 1, according as $m > 0, m = 0$ or $m < 0$. Then, a constant $C > 1$ exists, depending on n, such that*

$$\begin{aligned} C^{-1} r^{-m+} \le g_{\alpha,n}(r) \le C r^{-m+},\ 0 < r < 1, \\ C^{-1} r^{-(m+1)/2} e^{-r} \le g_{\alpha,n}(r) \le C r^{-(m+1)/2} e^{-r},\ r \ge 1. \end{aligned} \tag{13}$$

Proof. As in [17], p. 296

$$g_{\alpha,n}(r) = C_\alpha e^{-r} (\alpha/r)^{m/2} \int_1^\infty e^{-\frac{r}{2}(x+\frac{1}{x}-2)} \left[x^{m/2} + x^{-m/2}\right] \frac{dx}{x}$$

with $C_\alpha = (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1}$. Clearly,

$$g_{\alpha,n}(r) \approx r^{-m/n} e^{-r} \int_1^\infty e^{-\frac{r}{2}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2} x^{|m|/2} \frac{dx}{x}. \tag{14}$$

Let $y = \sqrt{x} - 1/\sqrt{x}$, so that $x = \dfrac{2 + y^2 + \sqrt{(1+y^2)^2 - 4}}{2}$ which is essentially 1, when $0 < y < 2$ and y^2 when $y > 2$. The integral in (14) is thus equivalent to

$$\int_0^{\sqrt{2}} e^{-\frac{r}{2}y^2} dy + \int_{\sqrt{2}}^\infty e^{-\frac{r}{2}y^2} y^{|m|} \frac{dy}{y}. \tag{15}$$

Next, let $y = \sqrt{2z/t}$ to get (15) equivalent to

$$r^{-1/2} \int_0^r e^{-2} \frac{dz}{\sqrt{z}} + \int_r^\infty e^{-z} |z|^{|m|/2} \frac{dz}{z}. \tag{16}$$

Using L'Hospital's Rule and the asymptotic formula for the incomplete gamma function we find that the expression (16) is effectively $r^{-|m|/2}$ in $(0,0)$ and $r^{-1/2}$ in $(1,\infty)$. This completes the proof when $m \neq 0$. The case $m = 0$ is left to the reader. □

Remark 3. *For $p \in (1,\infty)$, let $W_{\alpha,p}$ denote the class of weights w for which G_α is bounded on $L^p(w)$. Then $W_{\alpha,p}$ increases with α and $W_{\alpha,p} = W_{p,p}$, whenever $\alpha, \beta > n$. These facts follow from the semigroup property, the estimates (13) and the inequality $G_{\alpha_t} \leq CG^{1-t}_{\alpha_1}, G^t_{\alpha_2}$ which holds for $\alpha_t = (1-t)_{\alpha_1} + t\alpha_2$, provided $0 < \alpha_1 < \alpha_2$, $0 < t < 1$ and either $\alpha_2 < n$ or $\alpha_1 > n$. However,no two classes $W_{\alpha,p}$ are identical, as is shown in the following proposition.*

Proposition 2. *Fix $p \in (1,\infty)$ and α, $\beta \in (0,n)$, with $\alpha < \beta/p$. Then, there is a weight $w \in W_{\beta,p} - W_{\alpha,p}$.*

Proof. Let $\phi_\gamma(x) = 1 + \sum_{k=1}^{\infty} |x - 4^{-k}|^{-\gamma} \chi_{E_k}(x)$, where

$$E_k = \left\{ x \in R^n : |x - 4^{-k}| \leq \tfrac{1}{2} 4^k \right\}.$$

One readily shows $G_\beta \phi_\gamma \leq C\phi_\gamma$, if $0 < \gamma < \beta$. Hence, taking $w_\gamma = \phi_\gamma^{1/p}$, we have $w_\gamma \in W_{\beta,p}$.
For $0 < \delta < n$, $L^p(w_\gamma)$ contains the function

$$f(x) = \sum_{k=1}^{\infty} |x - x_k|^{-\delta/p} \chi_{F_k}$$

where

$$x_k = \tfrac{1}{2} \left[\tfrac{3}{2} \cdot 4^{k+1} + \tfrac{1}{2} \cdot 4^k \right] = 7/4k + 2$$

and

$$F_k = \left\{ x \in R^n : |x - x_k| < \tfrac{1}{2} \cdot 4k + 1 \right\}.$$

We seek conditions on r and δ so that $w_\gamma \notin W_{\alpha,p}$.
Now, $G_\alpha f = 4^{k[\delta/p - \bar{\alpha}]}$ on E_k, so

$$\|G_\alpha f\|^p_{p,w_\gamma} \geq \sum_{k=1}^{\infty} 4^{k[\delta - \alpha p + \gamma - n]} = \infty,$$

if $\delta - \alpha p + \gamma - n \geq 0$. By taking γ sufficiently close to β and δ sufficiently closed to n, this condition can be met. □

Theorem 4. *Suppose n, α, m and m_+ are as in Lemma 1. Fix $C > 1$ and set $k = [1 - C^{-2/\gamma}]^{1/2}$. Then, the weight-generating kernel for G_α corresponding to C is equivalent to*

$$|y|^{m_+}, |y| \leq 1,$$

and

$$\left[|y|^{-(m+1)/2} + |y|^{(1-n)/2} \right] e^{-k|y|}, \ |y| \geq 1.$$

In particular, for $\alpha \in (0,2]$, the kernel is equivalent to $G_\alpha(ky) + G_2(ky)$.

Proof. In view of (12), the kernel is given by

$$(4\pi)^{-n/2} \int_0^\infty e^{-(r^2/4)t} e^{-1/t} t^{\frac{n}{2}-1} S(t)\, dt,$$

where $r = |y|$ and
$$S(t) = \sum_{j=1}^{\infty} \frac{[C^{-1}t^{-\alpha/2}]^j}{\Gamma(j\alpha/2)},$$

When $C^{-1}t^{-\alpha/2} \leq 1$, that is, $t \geq C^{-2/\alpha} \equiv c$, the sum $S(t)$ is, effectively, $t^{-\alpha/2}$, as is seen from the inequalities
$$\frac{C^{-1}t^{-\alpha/2}}{\Gamma(\alpha/2)} \leq S(t) \leq \frac{C^{-1}t^{-\alpha/2}}{\Gamma(\alpha/2)} \left[1 + \sum_{j=1}^{\infty} \frac{1}{\Gamma(j\alpha/2)}\right].$$

Here, we have used $\Gamma(x+y) \geq \Gamma(x)\Gamma(y)$ when $x, y > 0$.

For $t \leq c$, the asymptotic expression
$$\sum_{j=1}^{\infty} \frac{t^j}{\Gamma(\ell j)} = t^{1/l} e^{t^{1/l}} [1 + 0(t^{-1})], \text{ as } t \to \infty,$$

given in [8], yields
$$S(t) \approx t^{-1} e^{a/t},\ t \leq c.$$

Thus, the kernel is, essentially,
$$\int_0^c e^{-(r^2/4)t} e^{(c-1)/t} t^{(n/2)-2}\, dt + \int_c^{\infty} e^{-(r^2/4)t} e^{-1/t} t^{(n-\alpha)/2} \frac{dt}{t}. \tag{17}$$

Now, the first term in (17) is bounded on $0 \leq r \leq 1$, while the second term is equivalent to G_α for all $r \geq 0$. It only remains to show the first integral, I, satisfies $I \approx r^{(1-n)/2} e^{-kr}$ for $r \geq 1$. To this end set $s = rt/2$ in I to obtain
$$I \approx r^{(2-n)/2} e^{-kr} \int_0^{cr/2} e^{-r\left[\sqrt{s} - k/\sqrt{s}\right]^2/2} \cdot S^{\frac{n}{2}-2}\, ds$$

Next, let $y = \sqrt{s} - k/\sqrt{s}$ so that
$$I \approx r^{(2-n)/2} e^{-kr} \int_{-\infty}^{\beta(r)} e^{-ry^2/2} [y + f(y)]^{n-3} [1 + yf(y)^{-1}]\, dy,$$

where $\beta(r) = \sqrt{cr/2} - k\sqrt{2/cr}$ and $f(y) = \sqrt{y^2 + 4l} = \sqrt{s} + \frac{k}{\sqrt{s}}$.

Finally, take $z = \sqrt{r/2}y$ to get
$$\begin{aligned} I \approx r^{(1-n)/2} e^{-kr} \int_{-\infty}^{\gamma(r)} e^{-z^2} &\left[\sqrt{2/r}z + f\left(\sqrt{2/r}z\right)\right]^{n-3} \\ &\left[1 + \sqrt{2/r}zf\left(\sqrt{2/r}z\right)^{-1}\right] dz, \end{aligned}$$

with $\gamma(r) = \sqrt{c}r/2 - k/\sqrt{c}$. We have now just to observe that when $z \in \mathbb{R}$ and $r \geq 1$
$$0 \leq 1 + \sqrt{2/r}zf\left(\sqrt{2/r}z\right)^{-1} < 2$$

while $\sqrt{2/r}z + f\left(\sqrt{2/r}z\right)$ lies between $2k^{1/2}$ and $\sqrt{2z^2 + 4k}$. □

Typical of G_α weights are the exponential functions $e^{\beta x}$, $-1 < \beta < 1$.

Proposition 3. *Suppose $\alpha \in (0, 1/2)$ and $p \in (1, \infty)$. Set $w_\beta(f) = e^{\beta|x|}$, $x \in R^n$. Then, G_α is bounded on $L^p(w_\beta)$ if and only if $-1 < \beta < 1$. Moreover, on this range of β, one has*
$$\lim_{\alpha \to 0+} \|G_\alpha * f - f\|_{p, w_\beta} = 0$$

for all $f \in L^p(w_\beta)$.

Proof. Fix $\beta \in (-1,1)$. We show $C > 1$ exists, independent of $\alpha \in (0, 1/2)$, such that

$$(G_\alpha w_\beta)(x) \leq C w_\beta(x),\ x \in R^n.$$

The "if" part then follows by Remark 2.

Using the simple inequalities $|x+y| \leq |x| + |y|$ when $\beta > 0$ and $|x-y| \geq |x| - |y|$ when $\beta < 0$ we obtain

$$(G_\alpha w_\beta)(x) \leq w_\beta(x) \int_{R^n} e^{|\beta|\,|y|} G_\alpha(y)\, dy.$$

But, the proof of Lemma 1 shows

$$\begin{aligned} \int_{R^n} e^{|\beta|\,|y|} G_\alpha(y)\, dy &\leq \int_{|y|\leq 1} e^{|\beta|\,|y|} |y|^{\alpha-n}\, dy + \int_{|y|>1} e^{[|\beta|-1]|y|} |y|^{-\frac{n}{2}-\frac{1}{4}}\, dy \\ &\approx 1, \end{aligned}$$

when $\alpha \in (0,1)$.

To prove the "only if" part, only the care $\beta = -1$ needs to be considered. We observed that $f(x) = \frac{e^{|x|}}{1+|x|^{n+1}}$ is in $L^p(w_{-1})$ and that G_α bounded on $L^p(w_{-1})$ implies the same of $G_{j\alpha}, j = 2, 3, \ldots$. However, for $j \geq \dfrac{n+3}{\alpha}$, $G_{j\alpha} f \equiv \infty$. □

Example 2. *Consider the Bessel potential* $G_2(y)$ *so that the weight-generating kernels are equivalent to* $G_2(ky)$, $0 < k < 1$. *These are especially simple when the dimension,* n, *is 1 or 3. In the first case* $G_2(y)$ *is essentially equal to the Picard kernel,* $e^{-|y|}$, *and in the second case to* $|y|^{-1}e^{-|y|}$.

According to Corollary 1, then, T_{G_2} *is bounded on* $L^p(e^{k/p'|y|})$ *and* $L^p(e^{-k/p|y|})$ *when* $n = 1$; *on* $L^p\left(|y|^{1/p'} e^{k/p'|y|}\right)$ *and* $L^p\left(|y|^{1/p} e^{-k/p|y|}\right)$ *when* $n = 3$.

5. The Gauss–Weierstrass Operators

In this section, we briefly treat the Gauss–Weierstrass kernels, $\{W_t\}_{t>0}$, defined by

$$W_t(y) = (4\pi t)^{-n/2} \exp(-|y|^2/4t),\ y \in R^n.$$

The iterates of W_t satisfy $W_t^{(h)} = W_{ht}, h = 1, 2, \ldots$.

Proposition 4. *Fix* $p \in (1,\infty)$ *and set* $w_\beta(x) = e^{\beta|x|}$. *Then,* W_t *is bounded on* $L^p(w_\beta)$ *for all* $\beta \in (-\infty,\infty)$. *Moreover, one has*

$$\lim_{t\to 0_+} \|W_t * f - f\|_{p,w_\beta} = 0, \tag{18}$$

for every $f \in L^p(w_\beta)$.

Proof. Only $\beta \geq 0$ need by considered, the result for $\beta < 0$ follows by duality.

It will suffice to show that for each $\beta \geq 0$,

$$(W_t * e^{\beta|\cdot|})(x) \leq C e^{\beta|x|},$$

with $C > 1$ independent of $x \in R^n$ and $t \in (0,1)$.

Now,

$$\int_{R^n} W_t(y) e^{\beta|x-y|}\, dy \leq \int_{R^n} W_t(y) e^{\beta[|x|+|y|]}\, dy = e^{\beta|x|} \int_{R^n} W_t(y) e^{\beta|y|}\, dy,$$

from which the boundedness assertion follows. Again $W_t(y)$ is an increasing function of t for fixed y with $|y| \geq \sqrt{2nt}$ so,

$$\begin{aligned}\int_{R^n} W_t(y)e^{\beta|y|}\,dy &= \left(\int_{|y|<\sqrt{2nt}} + \int_{|y|>\sqrt{2nt}}\right) W_t(y)e^{\beta|y|}\,dy \\ &\leq e^{\beta\sqrt{2nt}} \int_{|y|<\sqrt{2nt}} W_t(y)\,dy + \int_{|y|>\sqrt{2nt}} W_1(y)e^{\beta|y|}\,dy \\ &\leq e^{\beta\sqrt{2n}} + (4\pi)^{-n/2} \int_{R_n} \exp(-|y|^2/4)e^{\beta|y|}\,dy\end{aligned}$$

when $t \in (0,1)$, thereby yielding (18). □

Theorem 5. *Fix $C > 1$. Then, the weight-generating kernel for W_l corresponding to C is equivalent to*

$$t^{-\frac{n}{4}-\frac{1}{2}}|y|^{1-n/2}\exp(-t^{-1/2}k|y|),\ k = \sqrt{\log K},\ \textit{for some } K > 1,$$

with the constants of equivalence independent of $t \in (0,a)$, $|y| > 4ka^{1/2}$, where $0 < a < 1$.

Proof. The desired kernel is

$$\sum_{j=1}^{\infty} C^{-j}(4\pi tj)^{-n/2}\exp(-r^2/4jt) \tag{19}$$

where $r = |y|$.

Let $f(r,t,u) = C^{-u}(4\pi tu)^{-n/2}\exp(-r^2/4ut)$, $u > 0$, and let $\alpha = t^{-1/2}kr$. Denote by I_1, I_2 and I_3 the intervals $(0, \alpha/4k^2), (\alpha/4k^2, 2\alpha/k^2)$ and $(2\alpha/k^2, \infty)$, respectively. It is easily shown that when $r > 1$ and $t \in (0,1)$, the function f, as a function of u, increases on I_1, decreases on I_3 and satisfies $K^{-1}f(r,t,u) \leq f(r,t,u+s) \leq Kf(r,t,u)$ for some $K > 1$ and all $u \in I_2$, $s \in (0,1)$. Thus, the study of the sum in (19) amounts to looking at the integrals

$$J_i = \int_{I_i} f(r,t,u)\,du,\ i = 1,2,3.$$

Indeed, $C^{-u} = e^{-k^2u}$, therefore,

$$\begin{aligned}C^{-1}(J_1+J_2+J_3) &= C^{-1}\left(\int_0^{[\alpha/4h^2]+1} + \int_{[\alpha/4k^2]+1}^{[2\alpha/k^2]} + \int_{[2\alpha/k^2]}^{\infty}\right) f(r,t,u)\,du \\ &\leq \sum_{j=1}^{\infty} f(r,t,j) \\ &= \left(\sum_{j=1}^{[\alpha/4k^2]} + \sum_{j=[\alpha/4k^2]+1}^{[2\alpha/k^2]} + \sum_{j=[2\alpha/k^2]+1}^{\infty}\right) f(r,t,u)\,du \\ &\leq C(J_1+J_2+J_3).\end{aligned}$$

We have

$$\begin{aligned}J_1 &\leq t^{-n/2}\left(\frac{\alpha}{4k^2}\right)^{-n/2}\exp\left(-k^2\frac{\alpha}{4k^2}\right)\exp\left(-|y|^2\Big/\frac{4\alpha t}{4k^2}\right)\frac{\alpha}{4k^2} \\ &\leq t^{-\frac{n}{4}-\frac{1}{2}}|y|^{1-\frac{n}{2}}\exp\left(-\frac{5}{4}t^{-1/2}k|y|\right)\end{aligned}$$

Again,

$$J_3 \le t^{-n/2}\left(\frac{2\alpha}{k^2}\right)^{-n/2}\exp\left(-r^2\Big/\frac{4\alpha}{4k^2}t\right)\exp\left(-k^2\frac{\alpha}{4k^2}\right)$$
$$\le t^{-n/4}|y|^{-n/2}\exp\left(-\frac{5}{4}t^{-1/2}k|y|\right) \le J_1.$$

Finally, in J_2 take $u = \alpha v/2k^2$ to get

$$J_2 \le t^{-n/4}|y|^{-n/2}\int_{1/2}^{4}\exp\left(-\frac{\alpha}{2}\left[v+\frac{1}{v}\right]\right)v^{-n/2}\,dv$$
$$\le t^{-\frac{n}{4}-\frac{1}{2}}|y|^{1-\frac{n}{2}}\exp\left(-t^{-1/2}k|y|\right).$$

Altogether, then,

$$\int_0^\infty f(|y|,t,u)\,du \le t^{-\frac{n}{4}-\frac{1}{2}}|y|^{1-\frac{n}{2}}\exp\left(-t^{-1/2}k|y|\right).$$

□

Remark 4. *The weight-generating kernels are similar to those of G_2 on R^1 and R^3 (see Example 2), whence the exponential weights of Proposition 4 are in some sense typical. This illustrates a general theorem of Lofstrom, [18], which asserts that no translation-invariant operator is bounded on $L^p(w)$, when w is a rapidly varying weight such as $w(\alpha) = \exp(|x|^\alpha), \alpha > 1$.*

6. The Hardy Averaging Operators

In this section we consider Lebesgue-measurable functions defined on the set

$$R^n_+ = \{y \in R^n : y_i > 0, i = 1,\ldots,n\},$$

where, as usual, we write $y = (y_1,\ldots,y_n)$. Given $x \in R^n_+$, we define the sets

$$E_n(x) = \{y \in R^n_+ : 0 < y_i < x_i, i = 1,\ldots,n\}$$

and

$$F_n(x) = \{y \in R^n_+ : 0 < x_i < y_i, i = 1,\ldots,n\}.$$

Finally, we denote the product $x_1^{-1}\ldots x_n^{-1}$ by x^{-1} or $\frac{1}{x}$ and the product $(\log\frac{x_1}{y_1})\ldots(\log\frac{x_n}{y_n})$ by $\log\frac{x}{y}$; here, $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ belong to R^n_+.

The Hardy averaging operators, P_n and Q_n, are defined at $f \in P(R^n_+)$, $x \in R^n_+$, by

$$(P_n f)(x) = x^{-1}\int_{E_n(x)} f(y)\,dy$$

and

$$(Q_n f)(x) = \int_{F_n(x)} f(y)\,\frac{dy}{y}.$$

These operators, which are the transposes of one another, are generalizations to n-dimensions of the well-known ones, considered in [5] for example. A simple induction argument leads to the following formulas for the iterates of P_n and Q_n :

$$\left(P_n^{(j)} f\right)(x) = \frac{x^{-1}}{\Gamma(j)^n}\int_{F_n(s)} f(y)[\log x/y]^{j-1}\frac{dy}{y},$$

and
$$\left(Q_n^{(j)} f\right)(x) = \frac{1}{\Gamma(j)^n} \int_{F_n(s)} f(y)[\log y/x]^{j-1} \frac{dy}{y},$$
in which $x \in R_+^n$ and $j = 0, 1, \ldots$.

From Theorem 1 of [19], we obtain the representations of the weight-generating kernels of P_n and Q_n described below.

Theorem 6. *For $C > 1$ and set $\alpha = nC^{-1/n}$. Then, the weight-generating kernels for P_n and Q_n corresponding to C are equivalent, respectively, to*
$$x^{-1}\left[1 + (\log x/y)^{1/2(n-1)} \exp[\alpha(\log x/y)^{1/n}]\right] \chi_{E_n(x)}(y) \tag{20}$$
and
$$y^{-1}\left[1 + (\log y/x)^{1/2(n-1)} \exp[\alpha(\log y/x)^{1/n}]\right] \chi_{F_n(x)}(y). \tag{21}$$

Proposition 5. *Let $w_\beta(x) = [1 + |x|]^\beta$, $\beta \in R$. Then P_n is bounded on $L^p(w_\beta)$ if and only if $\beta < 1/p'$; by duality, Q_n is bounded on $L^p(w_\beta)$ of and only if $\beta > -1/p$.*

Proof. For simplicity, we consider $n = 2$ only.

Take $\psi = w_\gamma$ and fix $\alpha \in (0, 2)$. Denote by g the weight-generating kernel (20) applied to ψ. The change of variable $y_1 = x_1 z_1$, $y_2 = x_2 z_2$ in the integral giving $g(x)$ yields
$$g(x) = \int_0^1 \int_0^1 \left[1 + \sqrt{x_1^2 z_1^2 + x_2^2 z_2^2}\right]^\gamma$$
$$\left[1 + (\log 1/z_1 \log 1/z_2)^{-1/4} \times \exp\left[\alpha(\log 1/z_1 \log 1/z_2)^{1/2}\right]\right] dz_1\, dz_2$$

Hence, when $r > -1$, we find
$$g(x) \approx \begin{cases} 1, & 0 < x_1,\ x_2 \le 1 \\ x_2^\gamma, & 0 < x_1 \le 1,\ x_2 > 1 \\ x_1^\gamma, & x_1 > 1,\ 0 < x_2 \le 1 \\ \max\left[x_1^\gamma, x_2^\gamma\right], & x_1,\ x_2 \ge 1; \end{cases}$$
that is, $g(x) \approx w_\gamma(x)$, provided $r > -1$. This proves the "if" part, since $\beta = -\gamma/p' < 1/p'$.

To see that we must have $\gamma < 1/p'$, note that $h = \chi_{E_2}(\dot{x})$, $\dot{x} = (1,1)$, is in $L^p(w_\gamma)$ and
$$(P_2 h)(x) = \begin{cases} 1, & 0 < x_1,\ x_2 \le 1 \\ x_2^{-1}, & 0 < x_1 \le 1,\ x_2 > 1 \\ x_1^{-1}, & x_1 > 1,\ 0 < x_2 \le 1 \\ x_1^{-1} x_2^{-1}, & x_1,\ x_2 \ge 1 \end{cases}$$
so
$$\int_{R_+^2} [w_\beta P_2 h]^p = \infty, \text{ if } \beta \ge 1/p'.$$

□

Theorem 7. *Denote by G_1 and G_2 the positive integral operators on $P(R_+^n)$ with kernels (20) and (21), respectively. Suppose $\psi_i \in P(R_+^n)$ is such that $G_i \psi_i < \infty$ ae on R_+^n, $i = 1, 2$. Take $\phi_i = \psi_i + G_i \psi_i$, $i = 1, 2$ and set $w = \phi_1^{-\frac{1}{p'}} \phi_2^{\frac{1}{p}}$. Then,*
$$P_n \colon L^p(R_+^n) \to L^p(R_+^n). \tag{22}$$

Moreover, any weight w satisfying (22) is equivalent to one in the above form.

Proof. This result is a consequence of Corollary 1 and Theorem 2. $\square$

Remark 5. *When $n = 1$, the functions x^β, $\beta > -1$, are eigenfunctions of the operator P corresponding to the eigenvalue $(\beta+1)^{-1}$. As a result, if $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for all x and if $a_k > 0$, then there exists $\psi \in P(R_+)$ for which $\psi + \sum_{j=1}^{\infty} C^{-j} P^{(j)} \psi \approx \phi$, $C > 1$; namely. $\psi(x) = b_0 + \sum_{k=1}^{\infty} b_k x^k$, where $b_k = a_k \left(1 + \sum_{j=1}^{\infty} \frac{c^{-j}}{(k+1)} j\right)_k^{-1}$, $k = 0, 1, \ldots$.*

For example, $\phi_1(x) = e^{\beta p' e^x}$, $\beta > 0$, is an entire function with $\phi^{(k)}(0) > 0$, $k = 0, 1, \ldots$. Combining this $\phi_1(x)$ with $\phi_2(x) = x^{\gamma p}$ we obtain the P-weight $x^\gamma e^{-\beta e^x}$, $\gamma < 0 < \beta$. Interpolation with change of measure shows one can, in fact, take all $\gamma < 1/p'$.

Similar results are obtained when $\phi(x_1, \ldots, x_n)$ is everywhere on R^n the sum of a power series in $x_1, \ldots, x_n$ with nonnegative coefficients. To take a specific example, consider a power series in one variable, $\sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, which converges for all $x \in R$. Then, $\phi(x_1, \ldots, x_n) = \sum_{k=0}^{\infty} a_k (x_1 \ldots x_n)^k$ leads to the P_n-weights $w(x_1, \ldots, x_n) = x_1^{\gamma_1} \ldots x_n^{\gamma_n}\, \phi(x_1, \ldots, x_n)^{1/p'}$, where $\gamma_i < 1/p'$, $i = 1, \ldots, n$.

Criteria for the boundedness of Hardy operators between weighted Lebesgue spaces with possibly different weights are given in [5] for the case $n = 1$ and in [7] for the case $n = 2$.

Added in Proof: While this work was in press the author came across the paper [20]. In it Bloom proves our Theorem 1 using complex interpolation rather than interpolation with change of measure. A (typical) application of his result to the Hardy operators substitutes them in the necessary and sufficient conditions, thereby giving a criterion *for their two* weighted boundedness. This is in contrast to our Theorem 6, in which the explicit form of a *single weight* is given.

Funding: This research received no external funding.

Acknowledgments: The author is grateful to son Ely and Vít Musil for technical aid.

Conflicts of Interest: The author declares no conflict of interest.

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