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Comment

Comments on the Paper “Lie Symmetry Analysis, Explicit Solutions, and Conservation Laws of a Spatially Two-Dimensional Burgers–Huxley Equation”

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Abstract: This comment is devoted to the paper “Lie Symmetry Analysis, Explicit Solutions, and Conservation Laws of a Spatially Two-Dimensional Burgers–Huxley Equation” (Symmery, **2020**, vol.12, 170), in which several results are either incorrect, or incomplete, or misleading.

Keywords: Burgers–Fitzhugh–Nagumo equation; nonlinear evolution equation; Lie symmetry; exact solution

MSC: 35K57; 35K58; 35B06; 35Cxx

1. Introduction

The recently published paper [1] is devoted to Lie symmetry analysis, exact solutions, and conservation laws of the equation

$$u_t = u_{xx} + u_{yy} + uu_x + uu_y + u(u-\delta)(1-u). \quad (1)$$

The equation is called the Burgers–Huxley equation, although it is a generalization of the famous the Fitzhugh–Nagumo (FN) equation

$$u_t = u_{xx} + u(u-\delta)(1-u),\ 0 < \delta < 1. \quad (2)$$

This equation is a simplification of the classical model describing nerve impulse propagation [2,3] and was extensively studied in the 1990s by symmetry-based methods using the terminology the FN equation or Nagumo equation [4–6] (see more details and references in Chapter 3 of the book [7]). In [1], all these papers from the 1990s are not mentioned.

In the particular case $\delta = 0$, the FN equation becomes the known Huxley equation [5,6]

$$u_t = u_{xx} + u^2(1-u) \quad (3)$$

There are some other important particular cases, including the case $\delta < 0$ (see for details Chapter 3 in [7]). Thus, Equation (1) should be called as a two-dimensional (in space) generalization of the the FN equation or the Burgers–Fitzhugh–Nagumo equation. Terminology used for this equation in [1] was wrongly introduced much later.

In the following section, the results derived in [1] are analyzed and it is presented the rigorous Lie symmetry analysis to the the Burgers–Fitzhugh–Nagumo (BFN) Equation (1). In the last section, some discussion concerning the problem of finding exact solutions for Equation (1) is presented.

2. Lie Symmetries, Optimal Subalgebras and Exact Solutions

In [1], three pages are devoted to finding Lie symmetries of the BFN Equation (1). It can be easily noted that this equation contains only a unique parameter δ. Nowadays, there are many computer algebra packages, which allow to calculate Lie symmetries of such kind PDEs for a few seconds. I used MAPLE and the result is the same, namely, the 3-dimensional Lie algebra with the basic operators

$$X_1 = \frac{\partial}{\partial t} \equiv \partial_t, \quad X_2 = \frac{\partial}{\partial x} \equiv \partial_x, \quad X_3 = \frac{\partial}{\partial y} \equiv \partial_y. \tag{4}$$

Remark 1. *According to the general Lie group theory, this Lie algebra generates the three-parameter Lie group of the time and space translations. Thus, if $u(t,x,y,)$ is a solution of Equation (1), then $u(t+t_0, x+x_0, y+y_0)$ is also a solution (t_0, x_0, y_0 are arbitrary constants).*

Having the Lie algebra (4), the authors claim that they found an "optimal system of one and two dimensional subalgebras" (see P.11 in [1]). However, there are no two-dimensional subalgebras in the paper. Moreover, the optimal system of one-dimensional subalgebras presented on P.5 [1] is incorrect. In fact, all such systems for Lie algebras of low dimensionality were derived many years ago and are presented in the seminal work [8]. Obviously the 3-dimensional algebra (4) is Abelian and can be easily identified in Table I [8]. Therefore, a correct optimal system of one-dimensional subalgebras consist of 3 (and not 5) algebras

$$X_3, \quad X_2 + bX_3, \quad X_1 + aX_2 + bX_3, \quad a \in \mathbb{R}, b \in \mathbb{R}. \tag{5}$$

The optimal system of two-dimensional subalgebras consist of the following algebras,

$$\langle X_2, X_3 \rangle, \quad \langle X_1 + aX_2, X_3 \rangle \quad \langle X_1 + aX_3, X_2 + bX_3 \rangle. \tag{6}$$

Now I present the reductions of the BFN Equation (1) using the correct set of subalgebras listed above with the relevant conclusions. The detailed calculations of new variables are omitted here because it is a standard routine.

Using the first one-dimensional subalgebra X_3, one immediately realizes that the (1+2)-dimensional Equation (1) reduces to the same equation but in one space dimensionality

$$v_t = v_{xx} + vv_x + v(v-\delta)(1-v), \tag{7}$$

where $u(t,x,y) = v(t,x)$. It means that any solution of the (1+1)-dimensional BFN Equation (7) is automatically the solution of the BFN Equation (1). Actually, it is a trivial result.

Using the second one-dimensional subalgebra $X_2 + bX_3$, one obtains the ansatz

$$u(t,x,y) = v(t,z), \quad z = y - bx, \tag{8}$$

therefore the reduced equation is

$$v_t = (1+b^2)v_{zz} + (1-b)vv_z + v(v-\delta)(1-v). \tag{9}$$

The latter can be simplified to the form

$$v_t = v_{z^*z^*} + \lambda vv_{z^*} + v(v-\delta)(1-v) \tag{10}$$

by the transformation $z = \sqrt{1+b^2}z^*$, $\lambda = (1-b)/\sqrt{1+b^2}$. Therefore, we again arrive at the (1+1)-dimensional BFN equation. However, now we can generalize each known solution of Equation (7) to that of the (1+2)-dimensional Equation (1) by the transformation

$$x \to z^* = \frac{y-bx}{\sqrt{1+b^2}}.$$

For example, let us take the known exact solutions in notations v and z^* (see (4.72)–(4.73) [7])

$$v = u_0 + \frac{\sum_{i=1}^{3} \alpha_i C_i \exp\left[\alpha_i \frac{\lambda+\kappa}{4}\left(z^* + (\lambda u_0 + \frac{3\kappa-\lambda}{4}\alpha_i)t\right)\right]}{\sum_{i=1}^{3} C_i \exp\left[\alpha_i \frac{\lambda+\kappa}{4}\left(z^* + (\lambda u_0 + \frac{3\kappa-\lambda}{4}\alpha_i)t\right)\right]}, \tag{11}$$

and

$$v = u_0 + \frac{\sum_{i=1}^{3} \alpha_i C_i \exp\left[\alpha_i \frac{\lambda-\kappa}{4}\left(z^* + (\lambda u_0 - \frac{3\kappa+\lambda}{4}\alpha_i)t\right)\right]}{\sum_{i=1}^{3} C_i \exp\left[\alpha_i \frac{\lambda-\kappa}{4}\left(z^* + (\lambda u_0 - \frac{3\kappa+\lambda}{4}\alpha_i)t\right)\right]} \tag{12}$$

where

$$u_0 = \frac{1}{3}(\delta+1),\ \alpha_1 = -\frac{1}{3}(\delta+1),\ \alpha_2 = \frac{1}{3}(2\delta-1),\ \alpha_3 = \frac{1}{3}(2-\delta),\ \kappa = \sqrt{\lambda^2+8}$$

(hereafter C_i are arbitrary constants). Thus, formulaes (11) and (12) with $z^* = \frac{y-bx}{\sqrt{1+b^2}}$ and $\lambda = (1-b)/\sqrt{1+b^2}$ present the four-parameter families of the exact solutions of the BFN Equation (1).

It should be noted that the family of solutions (12) of the (1+1)-dimensional BFN equation was found for the first time in [9] using a generalized conditional symmetry, and both families of solutions (11) and (12) were firstly constructed in [10] using Q-conditional symmetries. Such kinds of solutions are called two-shock waves. In the case when, say $C_3 = 0$, the standard traveling front is obtained, which can be rewritten in terms of the function tanh.

Remark 2. *In the case $\lambda = 0$, formulaes (11) and (12) produce the known exact solutions of the FN equation, which were identified the first time in [11] and later rediscovered and generalized in [12]. In these works, such exact solutions are called the interaction solutions of traveling waves.*

Using the last one-dimensional subalgebra $X_1 + aX_2 + bX_3$ from formula (7), one obtains the ansatz

$$u(t,x,y,) = v(z_1,z_2), \quad z_1 = x - at,\ z_2 = y - bt \tag{13}$$

leading to the reduced equation

$$v_{z_1z_1} + v_{z_2z_2} + (v+a)v_{z_1} + (v+b)v_{z_2} + v(v-\delta)(1-v) = 0. \tag{14}$$

This equation coincides with one (17) in [1] and posses the trivial Lie symmetries

$$Z_1 = \partial_{z_1}, \quad Z_2 = \partial_{z_2}, \tag{15}$$

which produce two different ansätze

$$v(z_1,z_2) = W(\omega) \quad \omega = z_1 = x - at \tag{16}$$

and

$$v(z_1,z_2) = W(\omega) \quad \omega = dz_1 + z_2, d \in \mathbb{R}. \tag{17}$$

Note that ansatz (16) was lost while only a particular case of Equation (17) was studied in [1]. Using ansätze Equations (16) and (17) the reduced equations

$$W'' + (W + a)W' + W(W - \delta)(1 - W) = 0 \quad (18)$$

and

$$(1 + d^2)W'' + ((1 + d)W + da + b)W' + W(W - \delta)(1 - W) = 0, \quad (19)$$

are obtained, respectively (hereafter the upper primes in ODEs mean the differentiation with regard to the relevant variable). Obviously, the second-order ODE (19) contains Equation (18) as a particular case, therefore one may analyze only ODE (19). In [1], only two particular cases of ODE (19) were under study (see (15) and (21) therein) and formal solutions in the form of power series were constructed. Such solutions are useless because they are not global, moreover their convergence radius was not found in [1] (only existence of such radius was proved in a questionable way).

Here, we show that ODE (19) is reducible to well-known ODEs and different exact solutions can be derived depending on parameters. First of all, ODE (19) can be rewritten in the form

$$W'' + (\lambda W + \lambda_0)W' + W(W - \delta)(1 - W) = 0 \quad (20)$$

by the transformation $W(\omega) = W(\omega_*)$, $\omega = \sqrt{1 + d^2}\omega^*$, $\lambda = (1 + d)/\sqrt{1 + d^2}$, $\lambda_0 = (da + b)/\sqrt{1 + d^2}$. It should be noted that this ODE naturally arises when exact solutions of the (1+1)-dimensional BFN Equation (7) are constructed (see equation (4.40) in [7]) and one is not integrable for arbitrary coefficients. However, its particular solutions can be constructed in explicit forms (see examples below).

In the general case, ODE (20) is reducible to the Abel-type equation of the second kind

$$FF' + (\lambda W + \lambda_0)F + W(W - \delta)(1 - W) = 0 \quad (21)$$

by the non-local substitution $W' = F(W)$. It is well-known that such equations are integrable only in very exceptional cases. To the best of my knowledge, the special case for Equation (21) occurs only if $\lambda = \lambda_0 = 0$ and then the general solution can be easily derived, therefore one arrives at the solution of Equation (20)

$$\int \frac{dW}{\sqrt{2W^4 - \frac{8}{3}(b + 1)W^3 + 4bW^2 + C_1}} = \pm 2(\omega_* + C_0). \quad (22)$$

Thus, taking into account the formulaes (13) and (17), the exact solutions of the BFN Equation (1)

$$\int \frac{du}{\sqrt{2u^4 - \frac{8}{3}(\delta + 1)u^3 + 4\delta u^2 + C_1}} = \pm\sqrt{2}(2at - x + y + C_0). \quad (23)$$

is obtained. The above integral leads to elliptic functions; however, there are several special cases (e.g., $C_1 = 0$), when elementary functions arise. The detailed analysis lays beyond scope of this paper.

In the case $|\lambda| + |\lambda_0| \neq 0$, the particular solutions of ODE (21) can be only found. For example, let me consider the ad hoc ansatz

$$F = e_2 W^2 + e_1 W + e_0. \quad (24)$$

Simple calculations show that

$$e_2 = \frac{1}{4}(-\lambda \pm \sqrt{\lambda^2 + 8}), \quad (25)$$

$$e_1 = -\lambda_0 \; or \; e_0 = 0. \quad (26)$$

Let me examine the case $e_1 = -\lambda_0$ then immediately $e_0 = \frac{2\delta}{\lambda \pm \sqrt{\lambda^2+8}}$ and the restriction $\lambda_0 = \frac{2(\delta+1)}{\lambda \pm \sqrt{\lambda^2+8}}$ are obtained. As a result, the particular exact solution of ODE (21)

$$F = D_0(W^2 - (\delta+1)W + \delta), \quad D_0 = \frac{1}{4}(-\lambda \pm \sqrt{\lambda^2+8}) \tag{27}$$

is constructed. Notably, this solution is a generalization of that presented in the handbook [13] (see Equation 13.3.1.6 with $k = 2$). Thus, one arrives at the equation

$$\int \frac{dW}{D_0(W^2 - (\delta+1)W + \delta)} = \omega_* + C_0 \tag{28}$$

for finding the function W. Because the above integral can be expressed in terms of elementary functions, the different exact solutions can be constructed depending on δ. The simplest case occurs for $\delta = 1$, therefore $W = 1 - \frac{1}{D_0(\omega_* + C_0)}$.

Assuming $\delta \neq 1$, Equation (28) produces the solutions

$$W = \frac{\delta+1}{2} + \frac{1-\delta}{2} \coth\left(\frac{1-\delta}{2} D_0(\omega_* + C_0)\right) \tag{29}$$

and

$$W = \frac{\delta+1}{2} + \frac{1-\delta}{2} \tanh\left(\frac{1-\delta}{2} D_0(\omega_* + C_0)\right). \tag{30}$$

Thus, taking into account the formulaes (13), (17) and the notations after Equation (20), the following exact solutions of the BFN Equation (1) are constructed,

$$u(t,x,y) = 1 - \frac{\sqrt{d^2+1}}{D_0(dx + y - \lambda_0\sqrt{d^2+1}\,t)}, \quad \delta = 1; \tag{31}$$

$$u(t,x,y) = \frac{\delta+1}{2} + \frac{1-\delta}{2} \coth\left(\frac{D_0(1-\delta)}{2\sqrt{d^2+1}}(dx + y - \lambda_0\sqrt{d^2+1}\,t)\right), \quad \delta \neq 1 \tag{32}$$

$$u(t,x,y) = \frac{\delta+1}{2} + \frac{1-\delta}{2} \tanh\left(\frac{D_0(1-\delta)}{2\sqrt{d^2+1}}(dx + y - \lambda_0\sqrt{d^2+1}\,t)\right), \quad \delta \neq 1, \tag{33}$$

where D_0 and λ_0 are defined above, the parameter d satisfies the equality $\lambda = (1+d)/\sqrt{1+d^2}$ and λ is an arbitrary constant. C_0 is skipped because this constant reflects the obvious fact that Equation (1) is invariant with regard to the time translations (see Remark 1).

From applicability point of view, the most interesting solution is Equation (33) because it is the traveling front, which connects two steady-states points $u = 1$ and $u = \delta$ of the BFN Equation (1).

Remark 3. *The case $e_1 = 0$ (see Equation (26)) can be examined in a similar way; however, the exact solutions obtained have the same structure as solutions (32) and (33).*

Now I turn to the two-dimensional subalgebras (6). Because these subalgebras are two-dimensional, one obtains the reduced equations in the form of ODEs. Obviously, the first subalgebra $\langle X_2, X_3 \rangle$ leads to the ansatz $u(t,x,y) = U(t)$, therefore the reduced equation is the integrable first-order ODE

$$U' = U(U - \delta)(1 - U).$$

However, each solution of this ODE generates the space-independent solution of the BFN Equation (1) and such solutions are not interesting from the applicability point of view.

The second subalgebra $\langle X_1 + aX_2, X_3 \rangle$ generates to the ansatz $u(t,x,y) = W(\omega), \quad \omega = x - at$. Therefore, the reduced equation is Equation (18).

Finally, the last subalgebra from formula (6) leads the ansatz $u(t,x,y) = W(\omega), \quad \omega = at - bx + y$, therefore the reduced equation is

$$(1+b^2)W'' + ((1-b)W - a)W' + W(W-\delta)(1-W) = 0. \tag{34}$$

Now one sees that ODE (34) coincides with Equation (19) (up to notations). Thus, it is proved the statement: *no essentially new solutions of the BFN Equation (1) can be derived using the two-dimensional subalgebras (6)*.

3. Discussion

In this work, a rigorous Lie symmetry analysis of the Burgers–Fitzhugh–Nagumo Equation (1) is presented. It means that the known algorithm (see, e.g., Section 1.3 in [7]) was realized. At the first step, the 3-dimensional Lie algebra of symmetries was found using the computer algebra package MAPLE. At the second step, all inequivalent subalgebras (the optimal systems of subalgebras) were identified using the results from paper [8]. At the third step, the full list of reductions of the equation in question to equations of lower dimensionality was derived. Finally, all the reduced equations were analyzed with the aim to find their exact solutions and to transform the latters into solutions of the BFN Equation (1). It should be stressed that the above steps were not properly realized in [1], therefore several results are either wrong (optimal systems of subalgebras), incomplete (some reductions are lost), or misleading (there are no explicit solutions although this is announced in the paper title).

In conclusion, I would like to point out the following. If the Lie algebra of invariance of a PDE has low dimensionality r (say, $r = 2, 3, 4$), then another way exists at the second step of the algorithm. One simply takes a general linear combination of the Lie symmetries and solves the corresponding first-order equation. In the case of the Lie algebra (4), one obtains the equation

$$au_t + bu_x + cu_y = 0.$$

This equation leads exactly to three inequivalent solutions, depending on the parameters a, b and c. Using these solutions, the relevant ansätze can be easily constructed, which reduce the BFN Equation (1) to the (1+1)-dimensional equations presented in Section 2. As a result, the same exact solutions are obtainable. However, this task becomes a nontrivial problem if the maximal algebra of invariance of the PDE in question has the high dimensionality (or one is infinite-dimensional). Nonlinear multidimensional PDE arising in real world applications are often invariant under such algebras. For example, the (1+2)-dimensional Burgers equation

$$u_t = u_{xx} + u_{yy} + uu_x + uu_y$$

is invariant with regard to the 5-dimensional Li algebra [14] and the nonlinear equation with Burgers term

$$u_t = (uu_x)_x + (uu_y)_y + 8uu_x + 8u^2$$

admits even the 6-dimensional Li algebra [15] with very unusual structure, which does not occurs in the one-dimensional space case [7] (see Table 2.5 therein). Therefore, it would be more convenient to have optimal systems of subalgebras in such cases (see, e.g., a highly non-trivial example of constructing and application of the optimal systems of subalgebras in [16]). On the other hand, it is a sophisticated algebraic problem in the general case (i.e., r is arbitrary), which is still not solved (see more details in the recent monograph [17] and the references cited therein).

The problem of construction of exact solutions for the BFN Equation (1) is still open. It is shown here that Lie symmetries do not lead to exact solutions with essentially new structures. In fact, all the solutions presented above are straightforward generalizations of the corresponding solutions of the (1+1)-dimensional BFN Equation (2). Therefore, one may claim that all essentially new solutions of Equation (1) are non-Lie solutions, therefore other methods should be applied, for example conditional symmetries [18], generalized conditional symmetries [9,19], etc.

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