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Multiple Techniques for Studying Asymptotic Properties of a Class of Differential Equations with Variable Coefficients

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Abstract: This manuscript is concerned with the oscillatory properties of 4th-order differential equations with variable coefficients. The main aim of this paper is the combination of the following three techniques used: the comparison method, Riccati technique and integral averaging technique. Two examples are given for applying the criteria.

Keywords: delay differential equations; oscillation; fourth-order

1. Introduction

Differential equations of fourth-order have applications in dynamical systems, optimization, and in the mathematical modeling of engineering problems [1]. The p-Laplace equations have some significant applications in elasticity theory and continuum mechanics, see, for example, [2,3]. Symmetry plays an important role in determining the right way to study these equations [4]. The main aim of this paper is the combination of the following three techniques used:

(a) The comparison method.
(b) Riccati technique.
(c) Integral averaging technique.

We consider the following fourth-order delay differential equations with p-Laplacian like operators

$$\left(a(\zeta)\left|u'''(\zeta)\right|^{p-2}u'''(\zeta)\right)' + q(\zeta)\, g(u(\eta(\zeta))) = 0, \tag{1}$$

where $\zeta \geq \zeta_0$. Throughout this work, we suppose that:

K1: $p > 1$ is a real number.
K2: $a \in C^1([\zeta_0, \infty), \mathbb{R})$, $a(\zeta) > 0$, $a'(\zeta) \geq 0$ and under the condition

$$\int_{\zeta_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} \mathrm{d}s = \infty, \tag{2}$$

K3: $q \in C([\zeta_0, \infty), \mathbb{R})$, $q(\zeta) > 0$,

K4: $\eta \in C([\zeta_0, \infty), \mathbb{R})$, $\eta(\zeta) \leq \zeta$, $\lim_{\zeta\to\infty} \eta(\zeta) = \infty$,

K5: $g \in C(\mathbb{R}, \mathbb{R})$ such that $g(u) \geq m|u|^{p-2} u > 0$, for $u \neq 0$ and m is a constant.

Definition 1. *The function $u \in C^3[\zeta_u, \infty)$, $\zeta_u \geq \zeta_0$ is called a solution of (1), if $a(\zeta)|u'''(\zeta)|^{p-2} u'''(\zeta) \in C^1[\zeta_u, \infty)$, and $u(\zeta)$ satisfies (1) on $[\zeta_u, \infty)$. Moreover, the equation (1) is oscillatory if all its solutions oscillate.*

In the last few decades, there have been a constant interest to investigate the asymptotic property for oscillations of differential equation, see [5–25]. Furthermore, there are some results that study the oscillatory behavior of 4th-order equations with p-Laplacian, we refer the reader to [26,27].

Now the following results are presented.

Grace and Lalli [28], Karpuz et al. [29] and Zafer [30] studied the even-order equation

$$u^{(\gamma)}(\zeta) + q(\zeta) u(\eta(\zeta)) = 0,$$

they used the Riccati substitution to find several oscillation criteria and established the following results, respectively:

$$\int_{\zeta_0}^{\infty} \left(\delta(s) q(s) - \frac{(\gamma-1)! (\delta'(s))^2}{2^{3-2\gamma} \eta^{\gamma-2}(s) \eta'(s) \delta(s)} \right) ds = \infty, \tag{3}$$

where $\delta \in C^1([\zeta_0, \infty), (0, \infty))$.

$$\liminf_{\zeta\to\infty} \int_{\eta(\zeta)}^{\zeta} q(s) \eta^{\gamma-2}(s)\, ds > \frac{(\gamma-1) 2^{(\gamma-1)(\gamma-2)}}{\mathrm{e}} \tag{4}$$

and

$$\liminf_{\zeta\to\infty} \int_{\eta(\zeta)}^{\zeta} q(s) \eta^{\gamma-2}(s)\, ds > \frac{(\gamma-1)!}{\mathrm{e}}. \tag{5}$$

Zhang et al. [31,32] studied the even-order equation

$$\left(a(\zeta) \left(u^{(\gamma-1)}(\zeta) \right)^{\beta} \right)' + q(\zeta) u^{\beta}(\eta(\zeta)) = 0, \tag{6}$$

where β is a quotient of odd positive integers. They proved that it is oscillatory, if

$$\liminf_{\zeta\to\infty} \int_{\zeta}^{\eta(\zeta)} \frac{q(s)}{a(\eta(s))} \left(\eta^{\gamma-2}(s) \right)^{\beta} ds > \frac{((\gamma-1)!)^{\beta}}{e}, \tag{7}$$

where $\gamma \geq 2$ is even and they used the compare with first order equations. If there exists a function $\delta \in C^1([\zeta_0, \infty), (0, \infty))$ for all constants $M > 0$ such that

$$\liminf_{\zeta\to\infty} \int_{\zeta_0}^{\infty} \delta(s) \left(q(s) - \frac{a(s) \left(\theta M \eta^{\gamma-2}(s) \eta'(s) \right)^{1-p}}{p^p} \left(\frac{\delta'(s)}{\delta(s)} - \frac{a(s)}{r(s)} \right)^p \right) ds = \infty,, \tag{8}$$

for some constant $\theta \in (0, 1)$.

Our aim in this work is to complement results in [28–32]. Two examples are given for applying the criteria.

2. Some Auxiliary Lemmas

Lemma 1. *[13] Fixing $V > 0$ and $U \geq 0$, we have that*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^{\beta}}.$$

Lemma 2. *[14] For $i = 0, 1, ..., \gamma$, let $u^{(i)}(\zeta) > 0$, and $u^{(\gamma+1)}(\zeta) < 0$, then*

$$\frac{u(\zeta)}{\zeta^{\gamma}/\gamma!} \geq \frac{u'(\zeta)}{\zeta^{\gamma-1}/(\gamma-1)!}.$$

Lemma 3. *[16] Suppose that u is an eventually positive solution of (1). Then, we distinguish the following situations:*

$$\begin{array}{ll} (\mathbf{S}_1) & u(\zeta) > 0,\ u'(\zeta) > 0, u''(\zeta) > 0,\ u'''(\zeta) > 0,\ u^{(4)}(\zeta) < 0, \\ (\mathbf{S}_2) & u(\zeta) > 0,\ u'(\zeta) > 0, u''(\zeta) < 0,\ u'''(\zeta) > 0,\ u^{(4)}(\zeta) < 0, \end{array}$$

for $\zeta \geq \zeta_1$, where $\zeta_1 \geq \zeta_0$ is sufficiently large.

3. Main Results

Let the differential equation

$$\left[a(\zeta)(u'(\zeta))^{\beta}\right]' + q(\zeta) u^{\beta}(g(\zeta)) = 0, \quad \zeta \geq \zeta_0, \tag{9}$$

where $a,\ q \in C([\zeta_0, \infty), \mathbb{R}^+)$, is nonoscillatory if and only if $\zeta \geq \zeta_0$, and a function $\varsigma \in C^1([\zeta, \infty), \mathbb{R})$, satisfying the inequality

$$\varsigma'(\zeta) + \gamma a^{-1/\beta}(\zeta)(\varsigma(\zeta))^{(1+\beta)/\beta} + q(\zeta) \leq 0, \quad \text{on } [\zeta, \infty).$$

Definition 2. *Let*

$$D = \{(\zeta, s) \in \mathbb{R}^2 : \zeta \geq s \geq \zeta_0\} \text{ and } D_0 = \{(\zeta, s) \in \mathbb{R}^2 : \zeta > s \geq \zeta_0\}.$$

A kernel function $H_i \in C(D, \mathbb{R})$ is said to belong to the function class $\Im$, written by $H \in \Im$, if, for $i = 1, 2$,

(i) $H_i(\zeta, s) = 0$ *for* $\zeta \geq \zeta_0$, $H_i(\zeta, s) > 0$, $(\zeta, s) \in D_0$;
(ii) $H_i(\zeta, s)$ *has a continuous and nonpositive partial derivative $\partial H_i/\partial s$ on D_0 and there exist functions $\delta, \vartheta \in C^1([\zeta_0, \infty), (0, \infty))$ and $h_i \in C(D_0, \mathbb{R})$ such that*

$$\frac{\partial}{\partial s} H_1(\zeta, s) + \frac{\delta'(s)}{\delta(s)} H_1(\zeta, s) = h_1(\zeta, s) H_1^{\beta/(\beta+1)}(\zeta, s) \tag{10}$$

and

$$\frac{\partial}{\partial s} H_2(\zeta, s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(\zeta, s) = h_2(\zeta, s) \sqrt{H_2(\zeta, s)}. \tag{11}$$

Theorem 1. *Let (2) holds. If the equations*

$$\left(\frac{2a^{\frac{1}{p-1}}(\zeta)}{(\theta\zeta^2)^{p-1}}(u'(\zeta))^{p-1}\right)' + kq(\zeta)\left(\frac{\eta^3(\zeta)}{\zeta^3}\right)^{p-1} u^{p-1}(\zeta) = 0 \tag{12}$$

and

$$u''(\zeta)+u(\zeta)\int_{\zeta}^{\infty}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty}q(s)\left(\frac{\eta(\zeta)}{\zeta}\right)^{p-1}\mathrm{d}s\right)^{1/p-1}\mathrm{d}\varsigma=0 \tag{13}$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Assume, for the sake of contradiction, that u is a positive solution of (1). Then, we let $u(\zeta)>0$ and $u(\eta(\zeta))>0$. By Lemma 3, we have $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let case $(\mathbf{S}_1)$ holds. Using [25], [Lemma 2.2.3], we find

$$u'(\zeta)\geq\frac{\theta}{2}\zeta^{2}u'''(\zeta), \tag{14}$$

for every $\theta\in(0,1)$.

From Lemma 2, we get

$$\frac{u'(\zeta)}{u(\zeta)}\leq\frac{3}{\zeta}.$$

Integrating from $\eta(\zeta)$ to ζ, we find

$$\frac{u(\eta(\zeta))}{u(\zeta)}\geq\frac{\eta^{3}(\zeta)}{\zeta^{3}}. \tag{15}$$

Defining

$$\varphi(\zeta):=\delta(\zeta)\left(\frac{a(\zeta)(u'''(\zeta))^{p-1}}{u^{p-1}(\zeta)}\right),\varphi(\zeta)>0, \tag{16}$$

where $\delta\in C^{1}([\zeta_0,\infty),(0,\infty))$ and

$$\begin{aligned}\varphi'(\zeta) &= \delta'(\zeta)\frac{a(\zeta)(u'''(\zeta))^{p-1}}{u^{p-1}(\zeta)}+\delta(\zeta)\frac{\left(a(u''')^{p-1}\right)'(\zeta)}{u^{p-1}(\zeta)}\\ &\quad-(p-1)\delta(\zeta)\frac{u^{p-2}(\zeta)u'(\zeta)a(\zeta)(u'''(\zeta))^{p-1}}{u^{2(p-1)}(\zeta)}.\end{aligned}$$

Combining (14) and (16), we obtain

$$\begin{aligned}\varphi'(\zeta) &\leq \frac{\delta'_{+}(\zeta)}{\delta(\zeta)}\varphi(\zeta)+\delta(\zeta)\frac{\left(a(\zeta)(u'''(\zeta))^{p-1}\right)'}{u^{p-1}(\zeta)}\\ &\quad-(p-1)\delta(\zeta)\frac{\theta}{2}\zeta^{2}\frac{a(\zeta)(u'''(\zeta))^{p}}{u^{p}(\zeta)}\\ &\leq \frac{\delta'(\zeta)}{\delta(\zeta)}\varphi(\zeta)+\delta(\zeta)\frac{\left(a(\zeta)(u'''(\zeta))^{\beta}\right)'}{u^{\beta}(\zeta)}\\ &\quad-\frac{(p-1)\theta\zeta^{2}}{2(\delta(\zeta)a(\zeta))^{\frac{1}{p-1}}}\varphi^{\frac{p}{p-1}}(\zeta).\end{aligned} \tag{17}$$

From (1) and (17), we find

$$\varphi'(\zeta)\leq\frac{\delta'(\zeta)}{\delta(\zeta)}\varphi(\zeta)-m\delta(\zeta)\frac{q(\zeta)u^{p-1}(\eta(\zeta))}{u^{p-1}(\zeta)}-\frac{(p-1)\theta\zeta^{2}}{2(\delta(\zeta)a(\zeta))^{\frac{1}{p-1}}}\varphi^{\frac{p}{p-1}}(\zeta).$$

From (15), we have

$$\varphi'(\zeta) \leq \frac{\delta'(\zeta)}{\delta(\zeta)}\varphi(\zeta) - m\delta(\zeta)q(\zeta)\left(\frac{\eta^3(\zeta)}{\zeta^3}\right)^{p-1} - \frac{(p-1)\theta\zeta^2}{2(\delta(\zeta)a(\zeta))^{\frac{1}{p-1}}}\varphi^{\frac{p}{p-1}}(\zeta). \tag{18}$$

Let $\delta(\zeta) = m = 1$ in (18), we have

$$\varphi'(\zeta) + \frac{(p-1)\theta\zeta^2}{2a^{\frac{1}{p-1}}(\zeta)}\varphi^{\frac{p}{p-1}}(\zeta) + q(\zeta)\left(\frac{\eta^3(\zeta)}{\zeta^3}\right)^{p-1} \leq 0.$$

Hence, the equation (12) is nonoscillatory, which is a contradiction.
Let case $(\mathbf{S}_2)$ holds. By Lemma 2, we find

$$\frac{u'(\zeta)}{u(\zeta)} \leq \frac{1}{\zeta}.$$

Integrating again from $\eta(\zeta)$ to ζ, we find

$$\frac{u(\eta(\zeta))}{u(\zeta)} \geq \frac{\eta(\zeta)}{\zeta}. \tag{19}$$

Defining

$$\psi(\zeta) := \vartheta(\zeta)\frac{u'(\zeta)}{u(\zeta)} > 0,$$

where $\vartheta \in C^1([\zeta_0, \infty), (0, \infty))$ and

$$\psi'(\zeta) = \frac{\vartheta'(\zeta)}{\vartheta(\zeta)}\psi(\zeta) + \vartheta(\zeta)\frac{u''(\zeta)}{u(\zeta)} - \frac{1}{\vartheta(\zeta)}\psi(\zeta)^2. \tag{20}$$

Integrating (1) from ζ to x and using $u'(\zeta) > 0$, we have

$$a(x)(u'''(x))^{p-1} - a(\zeta)(u'''(\zeta))^{p-1} = -\int_\zeta^x q(s)g(u(\eta(s)))ds.$$

From (19), we get

$$a(x)(u'''(x))^{p-1} - a(\zeta)(u'''(\zeta))^{p-1} \leq -ky^{p-1}(\zeta)\int_\zeta^x q(s)\left(\frac{\eta(s)}{s}\right)^{p-1} ds.$$

Letting $x \to \infty$, we have

$$a(\zeta)(u'''(\zeta))^{p-1} \geq ky^{p-1}(\zeta)\int_\zeta^\infty q(s)\left(\frac{\eta(s)}{s}\right)^{p-1} \mathrm{d}s$$

and so

$$u'''(\zeta) \geq u(\zeta)\left(\frac{m}{a(\zeta)}\int_\zeta^\infty q(s)\left(\frac{\eta(s)}{s}\right)^{p-1}\mathrm{d}s\right)^{1/(p-1)}.$$

Integrating again from ζ to ∞, we get

$$u''(\zeta) + u(\zeta)\int_\zeta^\infty\left(\frac{m}{a(\varsigma)}\int_\varsigma^\infty q(s)\left(\frac{\eta(s)}{s}\right)^{p-1}\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma \leq 0. \tag{21}$$

Combining (20) and (21), we find

$$\psi'(\zeta) \le \frac{\vartheta'(\zeta)}{\vartheta(\zeta)}\psi(\zeta) - \vartheta(\zeta)\int_{\zeta}^{\infty}\left(\frac{m}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\left(\frac{\eta(s)}{s}\right)^{p-1}\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma - \frac{1}{\vartheta(\zeta)}\psi(\zeta)^2. \tag{22}$$

If $\vartheta(\zeta) = m = 1$ in (22), we get

$$\psi'(\zeta) + \psi^2(\zeta) + \int_{\zeta}^{\infty}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\left(\frac{\eta(s)}{s}\right)^{p-1}\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma \le 0.$$

Thus, the Equation (13) is nonoscillatory, which is a contradiction. The proof of the theorem is complete. □

Next, we obtain the following Hille and Nehari type oscillation criteria for (1) with $p = 2$.

Theorem 2. *Let $p = 2,\ m = 1$. Assume that*

$$\int_{\zeta_0}^{\infty}\frac{\theta\zeta^2}{2a(\zeta)}\mathrm{d}\zeta = \infty$$

and

$$\liminf_{\zeta\to\infty}\left(\int_{\zeta_0}^{\zeta}\frac{\theta s^2}{2a(s)}\mathrm{d}s\right)\int_{\zeta}^{\infty} q(s)\left(\frac{\eta^3(s)}{s^3}\right)\mathrm{d}s > \frac{1}{4}, \tag{23}$$

for some constant $\theta \in (0,1)$,

$$\liminf_{\zeta\to\infty}\zeta\int_{\zeta_0}^{\zeta}\int_{v}^{\infty}\left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\left(\frac{\eta(s)}{s}\right)\mathrm{d}s\right)\mathrm{d}\varsigma\mathrm{d}v > \frac{1}{4}, \tag{24}$$

then all solutions of (1) is oscillatory.

In this theorem, we use the integral averaging technique:

Theorem 3. *Let (2) holds. If there exist positive functions $\delta, \vartheta \in C^1([\zeta_0,\infty),\mathbb{R})$ such that*

$$\limsup_{\zeta\to\infty}\frac{1}{H_1(\zeta,\zeta_1)}\int_{\zeta_1}^{\zeta}\left(H_1(\zeta,s)\,m\delta(s)\,q(s)\left(\frac{\eta^3(s)}{s^3}\right)^{p-1} - \pi(s)\right)\mathrm{d}s = \infty \tag{25}$$

and

$$\limsup_{\zeta\to\infty}\frac{1}{H_2(\zeta,\zeta_1)}\int_{\zeta_1}^{\zeta}\left(H_2(\zeta,s)\,\vartheta(s)\,\varpi(s) - \frac{\vartheta(s)\,h_2^2(\zeta,s)}{4}\right)\mathrm{d}s = \infty, \tag{26}$$

where

$$\pi(s) = \frac{h_1^p(\zeta,s)\,H_1^{p-1}(\zeta,s)}{p^p}\frac{2^{p-1}\delta(s)\,a(s)}{(\theta s^2)^{p-1}},$$

for all $\theta \in (0,1)$, and

$$\varpi(s) = \left(\frac{1}{a(\varsigma)}\int_{\varsigma}^{\infty} q(s)\left(\frac{\eta(s)}{s}\right)^{p-1}\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}\varsigma,$$

then (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 1. Assume that ($\mathbf{S}_1$) holds. From Theorem 1, we get that (18) holds. Multiplying (18) by $H_1(\zeta, s)$ and integrating the resulting inequality from ζ_1 to ζ, we find that

$$\begin{aligned}\int_{\zeta_1}^{\zeta} H_1(\zeta,s)\, m\delta(s)\, q(s) \left(\frac{\eta^3(s)}{s^3}\right)^{p-1} ds \leq\ & \varphi(\zeta_1) H_1(\zeta,\zeta_1) + \int_{\zeta_1}^{\zeta} \left(\frac{\partial}{\partial s} H_1(\zeta,s) + \frac{\delta'(s)}{\delta(s)} H_1(\zeta,s)\right) \varphi(s)\, ds \\ & - \int_{\zeta_1}^{\zeta} \frac{(p-1)\,\theta s^2}{2(\delta(s)\, a(s))^{\frac{1}{p-1}}} H_1(\zeta,s)\, \varphi^{\frac{p}{p-1}}(s)\, ds.\end{aligned}$$

From (10), we get

$$\begin{aligned}\int_{\zeta_1}^{\zeta} H_1(\zeta,s)\, m\delta(s)\, q(s) \left(\frac{\eta^3(s)}{s^3}\right)^{p-1} ds \leq\ & \varphi(\zeta_1) H_1(\zeta,\zeta_1) + \int_{\zeta_1}^{\zeta} h_1(\zeta,s)\, H_1^{(p-1)/p}(\zeta,s)\, \varphi(s)\, ds \\ & - \int_{\zeta_1}^{\zeta} \frac{(p-1)\,\theta s^2}{2(\delta(s)\, a(s))^{\frac{1}{p-1}}} H_1(\zeta,s)\, \varphi^{\frac{p}{p-1}}(s)\, ds. \qquad (27)\end{aligned}$$

Using Lemma 1 with $V = (p-1)\,\theta s^2 / \left(2(\delta(s)\, a(s))^{\frac{1}{p-1}}\right) H_1(\zeta,s)$, $U = h_1(\zeta,s)\, H_1^{(p-1)/p}(\zeta,s)$ and $u = \varphi(s)$, we get

$$\begin{aligned}& h_1(\zeta,s)\, H_1^{(p-1)/p}(\zeta,s)\, \varphi(s) - \frac{(p-1)\,\theta s^2}{2(\delta(s)\, a(s))^{\frac{1}{p-1}}} H_1(\zeta,s)\, \varphi^{\frac{p}{p-1}}(s) \\ \leq\ & \frac{h_1^p(\zeta,s)\, H_1^{p-1}(\zeta,s)}{p^p} \frac{2^{p-1}\delta(s)\, a(s)}{(\theta s^2)^{p-1}},\end{aligned}$$

which, with (27) gives

$$\frac{1}{H_1(\zeta,\zeta_1)} \int_{\zeta_1}^{\zeta} \left(H_1(\zeta,s)\, m\delta(s)\, q(s) \left(\frac{\eta^3(s)}{s^3}\right)^{p-1} - \pi(s) \right) ds \leq \varphi(\zeta_1).$$

This contradicts (25).

Assume that ($\mathbf{S}_2$) holds. From Theorem 1, (22) holds. Multiplying (22) by $H_2(\zeta, s)$ and integrating the resulting inequality from ζ_1 to ζ, we get

$$\begin{aligned}\int_{\zeta_1}^{\zeta} H_2(\zeta,s)\, \vartheta(s)\, \varpi(s)\, ds \leq\ & \psi(\zeta_1) H_2(\zeta,\zeta_1) \\ & + \int_{\zeta_1}^{\zeta} \left(\frac{\partial}{\partial s} H_2(\zeta,s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(\zeta,s)\right) \psi(s)\, ds \\ & - \int_{\zeta_1}^{\zeta} \frac{1}{\vartheta(s)} H_2(\zeta,s)\, \psi^2(s)\, ds.\end{aligned}$$

Thus, from (11), we get

$$\begin{aligned}\int_{\zeta_1}^{\zeta} H_2(\zeta,s)\, \vartheta(s)\, \varpi(s)\, ds \leq\ & \psi(\zeta_1) H_2(\zeta,\zeta_1) + \int_{\zeta_1}^{\zeta} h_2(\zeta,s) \sqrt{H_2(\zeta,s)}\, \psi(s)\, ds \\ & - \int_{\zeta_1}^{\zeta} \frac{1}{\vartheta(s)} H_2(\zeta,s)\, \psi^2(s)\, ds \\ \leq\ & \psi(\zeta_1) H_2(\zeta,\zeta_1) + \int_{\zeta_1}^{\zeta} \frac{\vartheta(s)\, h_2^2(\zeta,s)}{4} ds\end{aligned}$$

and so

$$\frac{1}{H_2(\zeta,\zeta_1)}\int_{\zeta_1}^{\zeta}\left(H_2(\zeta,s)\vartheta(s)\varpi(s)-\frac{\vartheta(s)h_2^2(\zeta,s)}{4}\right)\mathrm{d}s\le\psi(\zeta_1),$$

which contradicts (26). The proof of the theorem is complete. □

Example 1. *Consider the equation*

$$u^{(4)}(\zeta)+\frac{q_0}{\zeta^4}u\left(\frac{9\zeta}{10}\right)=0,\ \zeta\ge 1,\ q_0>0. \tag{28}$$

Let $p=2,\ a(\zeta)=1,\ q(\zeta)=q_0/\zeta^4$ *and* $\eta(\zeta)=9\zeta/10$. *If we set* $m=1,\ H_1(\zeta,s)=(\zeta-s)^2$ *and* $\delta(s)=s^3$, *then* $h_1(\zeta,s)=(\zeta-s)\left(5-3\zeta s^{-1}\right)$, *and conditions (23) becomes*

$$\begin{aligned}
&\limsup_{\zeta\to\infty}\frac{1}{H_1(\zeta,\zeta_1)}\int_{\zeta_1}^{\zeta}\left(H_1(\zeta,s)m\delta(s)q(s)\left(\frac{\eta^3(s)}{s^3}\right)^{p-1}-\pi(s)\right)\mathrm{d}s\\
=\ &\limsup_{\zeta\to\infty}\frac{1}{(\zeta-1)^2}\int_{\zeta_1}^{\zeta}\left(\frac{729q_0\zeta^2s^{-1}}{1000}+\frac{729q_0s}{1000}-\frac{729q_0\zeta}{500}-\frac{s\left(25+9\zeta^2s^{-2}-30\zeta s^{-1}\right)}{2\theta}\right)\mathrm{d}s\\
=\ &\infty,
\end{aligned}$$

if $q_0>500/(81\theta)$ *for some* $\theta\in(0,1)$, *letting* $\theta=81/82$, *then* $q_0>6.25$.

Also, set $H_2(\zeta,s)=(\zeta-s)^2$ *and* $\vartheta(s)=s$, *then* $h_2(\zeta,s)=(\zeta-s)\left(3-\zeta s^{-1}\right)$, $\varpi(s)=3q_0/\left(20\zeta^2\right)$ *and conditions (24) becomes*

$$\begin{aligned}
&\limsup_{\zeta\to\infty}\frac{1}{H_2(\zeta,\zeta_1)}\int_{\zeta_1}^{\zeta}\left(H_2(\zeta,s)\vartheta(s)\varpi(s)-\frac{\vartheta(s)h_2^2(\zeta,s)}{4}\right)\mathrm{d}s\\
=\ &\limsup_{\zeta\to\infty}\frac{1}{(\zeta-1)^2}\int_{\zeta_1}^{\zeta}\left(\frac{3q_0\zeta^2s^{-1}}{20}+\frac{3q_0s}{20}-\frac{3q_0\zeta}{10}-\frac{s\left(9-6\zeta s^{-1}+\zeta^2s^{-2}\right)}{4}\right)\mathrm{d}s\\
=\ &\infty,
\end{aligned}$$

if $q_0>5/3$, *From Theorem 3, all solutions of (28) are oscillatory, if* $q_0>6.25$.

Remark 1. *By comparing our results with previous results*

1. By applying condition (3) in [28], we get

$$q_0>1728,$$

2. By applying condition (4) in [29], we get

$$q_0>919.6,$$

3. By applying condition (5) in [30], we get

$$q_0>28.73,$$

4. By applying condition (7) in [31], we get

$$q_0>28.73,$$

5. The condition (8) in [32] cannot be applied to Equation (28) due to the arbitrariness in the choice of θ. *Therefore, our result complement results [28–32].*

Example 2. *Let the equation*

$$u^{(4)}(\zeta) + \frac{q_0}{\zeta^4} u\left(\frac{1}{2}\zeta\right) = 0, \ \zeta \geq 1, \ q_0 > 0. \tag{29}$$

Let $a(\zeta) = 1$, $q(\zeta) = q_0/\zeta^4$ *and* $\eta(\zeta) = \zeta/2$. *If we set* $m = 1$, *then condition (23) becomes*

$$\begin{aligned} \liminf_{\zeta\to\infty} \left(\int_{\zeta_0}^{\zeta} \frac{\theta s^2}{2a(s)} \mathrm{d}s \right) \int_{\zeta}^{\infty} q(s) \left(\frac{\eta^3(s)}{s^3} \right) \mathrm{d}s &= \liminf_{\zeta\to\infty} \left(\frac{\zeta^3}{3} \right) \int_{\zeta}^{\infty} \frac{q_0}{8s^4} ds \\ &= \frac{q_0}{72} > \frac{1}{4} \end{aligned}$$

and condition (24) becomes

$$\begin{aligned} \liminf_{\zeta\to\infty} \zeta \int_{\zeta_0}^{\zeta} \int_{v}^{\infty} \left(\frac{1}{a(\varsigma)} \int_{\varsigma}^{\infty} q(s) \left(\frac{\eta(s)}{s} \right) \mathrm{d}s \right) \mathrm{d}\varsigma \mathrm{d}v &= \liminf_{\zeta\to\infty} \zeta \left(\frac{q_0}{12\zeta} \right) \\ &= \frac{q_0}{12} > \frac{1}{4}. \end{aligned}$$

Hence, by Theorem 2, all solution equation (29) is oscillatory if $q_0 > 18$.

Remark 2. *We point out that continuing this line of work, we can have oscillatory results for a fourth order equation of the type:*

$$\left(a(\zeta) \left| u'''(\zeta) \right|^{p-2} u'''(\zeta) \right)' + \sum_{i=1}^{m} q_i(\zeta) \left| u(\eta_i(\zeta)) \right|^{p-2} u(\eta_i(\zeta)) = 0, \ where \ \zeta \geq \zeta_0, \ m \geq 1,$$

under the condition

$$\int_{\zeta_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} \mathrm{d}s < \infty.$$

4. Conclusions

In this article, we studied some oscillation conditions for 4th-order differential equations by the comparison method, Riccati technique and integral averaging technique.

Further, in the future work we study Equation (1) under the condition $\int_{\zeta_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} \mathrm{d}s < \infty$.

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