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Symmetries in Phase Portrait

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Abstract: We construct polynomial dynamical systems $x' = P(x)$ with symmetries present in the local phase portrait. This point of view on symmetry yields the approaches to ODEs construction being amenable to classical methods of the Spectral Analysis.

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MSC: 37J35; 37K10

1. Introduction

The standard meaning embedded in the concept of symmetry sounds like a similarity or exact correspondence between different things. For sketching a (crude) phase portrait of a polynomial regular autonomous system $x' = P(x)$ we usually start by determining the behavior of the trajectories near the stationary (equilibrium) points $x^* \in \mathrm{Null}(P)$, where $\mathrm{Null}(P) = \{x^* : P(x^*) = 0\}$ is a zero set of P. The complex systems created are better if its symmetric constituents are regularly assembling.

In other words, we consider dynamical systems $x' = P(x)$ as locally equivalent in a neighborhood of two equilibrium points x_1^*, x_2^* if the phase portrait is "locally qualitatively similar", namely, if one portrait can be obtained from another by a diffeomorphism $\varphi(x)$, $\varphi : P(U_{x_1^*}) \to P(U_{x_2^*})$, it often turns out to be polynomial automorphism, cf. [1].

In this case, it is appropriate to determine the geometric symmetry in the totality of equilibrium points as the existence of discrete transformation in neighborhoods of a different $x^* \in \mathrm{Null}(P)$. It should be pointed out that symmetries in ODEs theory are often formulated in terms of infinitesimal rather than finite discrete transformations.

The fact that the given system of ODEs $x' = P(x)$ has a discrete symmetry imposes strict limits on the similarity in the set of its equilibrium points $x^* \in \mathrm{Null}(P)$. Namely, denote by $\mathrm{Spec}(P)$ the set of (complex) eigenvalues of the Jacobian matrix $J_P(x^*)$; as x^* varies in $Null(P)$.

One of the several equivalent formulations of a similarity in phase portrait is

Definition 1. *Phase portrait of $x' = P(x)$ will be called similar at equilibrium points $x_1^*, x_2^* \in Null(P)$ iff* $\mathrm{Spec}(J_P(x_1^*)) = \mathrm{Spec}(J_P(x_2^*))$.

Let $\mathrm{Null}(P)$ become stratified by the above-stated similarity relation. Imagine $\mathrm{Null}(P)$ as union of it subsets $\mathrm{Null}(P)_0, \dots, \mathrm{Null}(P)_N$. Each strata $\mathrm{Null}(P)_i$ contains only similar points in the sense of Definition 2. The phase portrait called having high symmetry iff number N of a strata is minimal.

The question arises whether the exact symmetry on the totality of the equilibrium points $\mathrm{Null}(P)$ of some ODEs can be implemented to symmetry ODE construction (see [2,3]). Without loss of generality, suppose in the sequel, $P(0) = 0$ and $0 \in \mathrm{Null}(P)$. The above-defined similarity form always is point group symmetries acting on $\mathrm{Null}(P)$ (an isometry) that keeps 0 fixed.

2. Equilibrium Points Set Stratification

Evidently, $\text{Null}(P)$ can be stratified by some symmetry relation g. Namely, the totality $\text{Null}(P)$ can be stratified on strata $\text{Null}(P) = \{\text{Null}(P)_0 \cup \text{Null}(P)_1 \cup \ldots \cup \text{Null}(P)_N\}$ if there exists group g_k for $k \in \{0, 1, \ldots, N\}$ (some of them possible trivial) that acts transitively on a corresponding strata $\text{Null}(P)_k$, $k = 0, 1, \ldots, N$.

Example 1. *The following system of non-linear differential equations in three variables may be considered as a variant of the Einstein General Relativity equation (particularly discovered by E. Kasner (1925)):*

$$x' = ayz - x^2 + x, \quad y' = axz - y^2 + y, \quad z' = axy - z^2 + z \tag{1}$$

There are eight equilibrium points of (1) for $a \neq 1$, namely $\text{Null}(P) = \{x_0^*, x_1^*, \ldots, x_7^*\}$:

$$x_0^* = [0, 0, 0],\ x_1^* = [0, 0, 1],\ x_2^* = [0, 1, 0],\ x_3^* = [1, 0, 0], \tag{2}$$

$$x_4^* = \left[\frac{a+1}{a^2+a+1}, \frac{a+1}{a^2+a+1}, \frac{-a}{a^2+a+1}\right],\ x_5^* = \left[\frac{a+1}{a^2+a+1}, \frac{-a}{a^2+a+1}, \frac{a+1}{a^2+a+1}\right], \tag{3}$$

$$x_6^* = \left[\frac{-a}{a^2+a+1}, \frac{a+1}{a^2+a+1}, \frac{a+1}{a^2+a+1}\right],\ x_7^* = \left[\frac{1}{1-a}, \frac{1}{1-a}, \frac{1}{1-a}\right], \tag{4}$$

The set of an equilibrium points $\text{Null}(P)$ for Equation (1) is stratified into four components

$$\text{Null}(P) = \{x_0^*\} \cup \{x_1^*, x_2^*, x_3^*\} \cup \{x_4^*, x_5^*, x_6^*\} \cup \{x_7^*\}.$$

All these strata are invariant under the dihedral group D_3 action, which still has the rotation axis, but no mirror planes as it is shown schematically on the Figure 1.

The Jacobian matrices in each of these four groups of equilibrium points are similar and have the following eigenvalues correspondingly:

$$[1.1.1], \quad \left[-1, \frac{1+2a}{a-1}, \frac{1+2a}{a-1}\right], \quad \left[-1, \frac{2a^2+3a+1}{a^2+a+1}, \frac{2a^2-a-1}{a^2+a+1}\right], \quad [-1, 1-a, 1+a]. \tag{5}$$

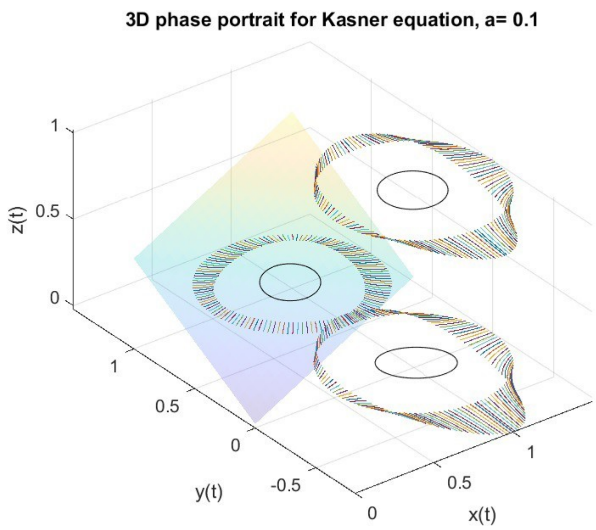

Figure 1. D_3 symmetry stratification in phase portrait of (1) near x_1^*, x_2^*, x_3^*.

Example 2. *Consider* $\text{Null}(P)$ *as configuration of four point set lie at the vertices of the equilateral triangle in* $\mathbb{R}^2$ *supplemented with center thereof. Suppose* $\text{Null}(P)$ *is an equilibrium set of some quadratic plane system*

$x' = f(x,y)$, $y' = g(x,y)$ such that the local dynamic near the vertices of the equilateral triangle $\mathrm{Null}(P)$ *are similar. What would $f(x,y)$, $g(x,y)$ be in this case?*

Let $P(x,y) = \{f(x,y), g(x,y)\}$ and $P(x^*,y^*) = 0$ at $(x^*,y^*) \in \{(0,-1), (\pm\sqrt{3}/2), 1/2\})\}$. Then

$$f(x,y) = ax^2 + 2bxy - ay^2 - bx - ay, \quad g(x,y) = Ax^2 + 2Bxy - Ay^2 - Bx - Ay. \tag{6}$$

By assumption, the Jacobian matrix $J_P(x^*,y^*)$ must have the same eigenvalues at all (x^*,y^*). That focuses of the equality: $A = b$, $B = -a$ in (6). The correct answer on the question in Example 2 is:

$$P = aP_1 + bP_2, \quad P_1(x,y) = \{x^2 - y^2 - y, -2xy + x\}, \quad P_2(x,y) = \{2xy - x, x^2 - y^2 - y\},. \tag{7}$$

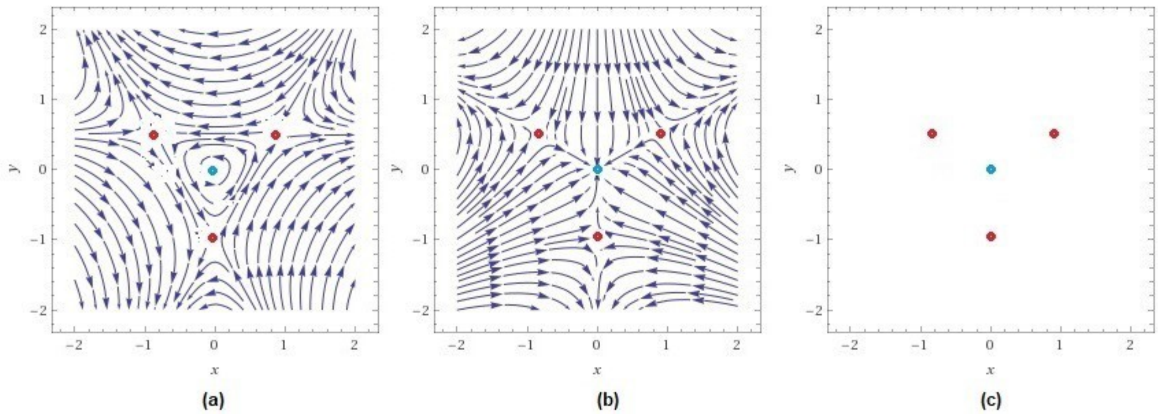

Figure 2. Phase portraits of P_1, P_2 in (**a**,**b**) follow the same set of a hyperbolic equilibrium points in (**c**).

Clearly, the dihedral group $D_3 = \mathrm{Sym}(\mathrm{Null}(P))$ carries out the point symmetry of $\mathrm{Null}(P)$ in Example 2. The matrix generators S and R of D_3 in $\mathbb{R}^2$ are of form:

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}.$$

If P is defined in (7) then $SP(x) = P(Sx)$ and $RP(x) = P(Rx)$. Therefore, D_3 is a symmetry of $x' = P(x)$.

The direct calculations show that the Jacobian matrix at the origin in the presence of symmetry has eigenvalues $[-b + ai, -b - ai]$, which means an origin is always foci or center. The eigenvalues of the Jacobian at the remaining points of an equilateral triangle all are equal to

$$[-b + \sqrt{3a^2 + 4b^2}, -b - \sqrt{3a^2 + 4b^2}].$$

Therefore, they all are cusps. It should be noted that $Null(P)$ can be decomposed into two parts, zero at the origin and zeroes at the remaining points of an equilateral triangle. The number of strata, in this case, is $N = 2$. Phase Portraits of the vector fields in Example 2 are present in Figure 2.

The similar example of a quadratic polynomial vector field with group Z_2 as symmetry on set of its equilibrium points may be present as five parametric family of vector fields:

$$P = \{cx^2 - 2dxy - ey^2 + bx + e,\ ax^2 + \frac{2d^2}{e}xy + dy^2 - \frac{bd}{e}x - d\}. \tag{8}$$

In (8) the set $Null(P)$ is composed of two points $[0,1]$ and $[0,-1]$ and one strata. The Jacobian matrix at both of these equilibrium points has eigenvalues $[0,b]$. Phase Portraits of the vector fields in (8) are present in Figure 3.

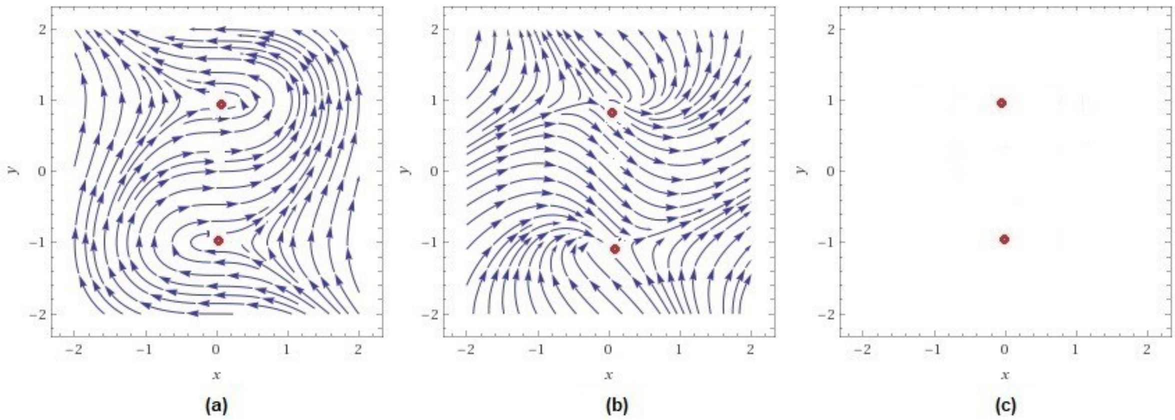

Figure 3. Dynamic (**a**,**b**) in (8) follow all (non hyperbolic) equilibrium points set in (**c**).

In the next section, we focused mainly on the polynomial ODEs with the isolated equilibrium points. Recall an equilibrium point hyperbolic if none of the eigenvalues of the Jacobian matrix at this point have a zero real part. Equilibrium points in (8) are non-hyperbolic due to the presence of zero eigenvalues.

3. Basic Facts

By the Grobman–Hartman theorem [4,5] the local behavior comparison for the given system of ODEs in the vicinity of a couple of isolated hyperbolic equilibrium points can be implemented by juxtaposition of the eigenvalues of the Jacobi matrix at each of the above-stated equilibrium points.

Definition 2. *Let* $\mathrm{Null}(P) = \{x^* \in \mathbb{C}^n, P(x^*) = 0\}$ *be a set of the equilibrium points to the polynomial ODE* $x' = P(x)$ *and let them all be hyperbolic. Denote by* $J_P(x)$ *the Jacobian matrix of* $P(x)$*. By assumption of hyperbolicity,* $J_P(x^*)$ *is not singular for all* $x^* \in \mathrm{Null}(P)$ *. In these settings, the dynamics in a small neighborhood of any two points* $x_1^*, x_2^* \in \mathrm{Null}(P)$ *is said to be "topologically equivalent" iff* $J_P(x_1^*)$ *and* $J_P(x_2^*)$ *are the similar matrices,* $J_P(x_1^*) \sim J_P(x_2^*)$*. (There exists a nondegenerate matrix C, such that* $CJ_P(x_1^*) = J_P(x_2^*)C$*.)*

The similarity should provide information on a model in which this symmetry can sketch besides having intrinsic value.

To ensure what kind of an above-described similarity takes place, we use the following version of the celebrated Euler–Jacobi formula (see [6] (p. 106) (see also Theorem 4.3 in [7])).

3.1. Euler–Jacobi Formula

Theorem 1 (Euler–Jacobi Formula)**.** *Let* $P = (P_1, \dots, P_n) : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial map and let* $\widetilde{P}$ *be the polynomial map, whose components are the highest homogeneous terms of the components of P. Suppose that any complex root* $a \in \mathbb{C}^n$*,* $P(a) = 0$ *is simple and, in addition, suppose* $\widetilde{P}(x) = \{0\}$ *only for* $x = 0$*. Then, for any polynomial* $h : \mathbb{C}^n \to \mathbb{C}$ *of degree less than the degree of the Jacobian determinant of P, i.e.,* $\deg h < -n + \sum_{i=1}^{n} \deg P_i$*, one has*

$$\sum_{P(a)=0} \frac{h(a)}{\det[J_P(a)]} = 0 \tag{9}$$

where $J_P(\cdot)$ *denotes the Jacobian matrix of P.*

3.2. Application of the Euler–Jacobi Formula

Following the Euler–Jacobi principle [6,8] it is possible to derive the adequate equivalence formula proposed for the symmetry in totality of equilibrium points to polynomial ODEs.

Corollary 1. *There are no examples of a fully symmetric set of all isolated hyperbolic equilibrium points. Suppose* $\det[J_P(a)] = d = \text{const}$ *for all complex roots a of the equation* $P(a) = 0$*. Then using* (9)

$$\sum_{P(a)=0} \frac{1}{\det[J_P(a)]} = 0, \quad \textit{but} \quad 1/d \neq 0.$$

This obvious fact deserves special mention.

Corollary 2. *Consider polynomial ODE* $x' = S(x) - x$, $x \in \mathbb{C}^n$ *where* $S(x)$ *is homogeneous of order k part,* $S(\lambda x) = \lambda^k S(x)$ *and define the characteristic polynomial of Jacobian matrix* $J_S(x)$ *as* $p_\lambda(x) = \det(\lambda I - J_S(x))$*. Suppose* $p_1(a) \neq 0$ *for all roots a of the equation* $S(a) = a$*. Then by Theorem 1 (see also [9])*

$$\sum_{S(a)=a} \frac{p_\lambda(a)}{p_1(a)} = k^n, \tag{10}$$

Theorem 2. *If dynamics at all nonzero hyperbolic isolated equilibrium point of ODE* $x' = S(x) - x$ *defined in Corollary 2 are locally topologically equivalent then the characteristic polynomial of the Jacobian matrix* $J_S(x)$ *at all nonzero equilibrium points* $x = a$ *is* $p_\lambda(a) = \lambda^n - k^n$*.*

Proof. In general there exists $k^n - 1$ nonzero isolated equilibrium points a. Suppose the spectrum at all of them are the same . Then using (10) and the fact that $p_\lambda(0) = \lambda^n$, we obtain:

$$(k^n - 1)\frac{p_\lambda(a)}{p_1(a)} + \frac{p_\lambda(0)}{p_1(0)} = k^n \quad \Rightarrow \quad \frac{p_\lambda(a)}{p_1(a)} = \frac{k^n - \lambda^n}{k^n - 1} \quad \Rightarrow \quad p_\lambda(a) = \lambda^n - k^n. \tag{11}$$

By Euler's formula $S(x) = A(x)x$, $A(x) = 1/k\, J_S(x)$. Here $J_S(x)$ is the Jacobian matrix of $S(x)$ being calculated at point x. $A(a)x$ is often called a linearization, or the linear part of $S(x)$ near $x = a$. It should be pointed out that in the presence of the described symmetry, the characteristic polynomial of the linearization at any nonzero isolated equilibrium point may be calculated in a unified form for any homogeneous part $S(x)$. Namely, $\det(\lambda I - A(a)) = \lambda^n - 1$. □

3.3. The Scope of the Euler–Jacobi Formula

To overcome the contradictions with Theorem 1 for symmetry construction, we need to extend the concept of equilibria.

Definition 3. *An equilibrium point* $x^* \in \mathbb{C}^n$ *for ODE* $x' = P(x)$ *is called critical if* $P(x^*) = 0$ *and* $\det(J_P(x^*)) = 0$*.*

Proposition 1. *A non isolated equilibrium point is always critical.*

Proof. Suppose $x = x(s)$ is a continuous differentiable set of an equilibria. Then for ODE $x' = P(x)$, $P(x(s)) = 0$,

$$\frac{d}{ds}P(x(s)) = 0, \quad \Rightarrow \quad J_P(x(s))\, x(s) = 0, \quad \Rightarrow \quad \det(J_P(x(s))) = 0;$$

It is readily visible that the formula (9) is unacceptable if there are critical points. □

The reason why the formula (9) is inapplicable for critical points is two-fold:

1. Nonisolated equilibrium points do not make sense for summation in (9).
2. Zero determinant in the denominator does not allow division.

The formula (8) referred to the possibility of constructing an ODEs with all equilibrium points to have the same spectrum. The Euler-Jacobi formula is not applicable here but the Generalized Euler–Jacobi formula discussed in the next Subsection allowed to do it.

3.4. Generalized Euler-Jacobi Formula

Let $P : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Denote by A_s (resp. A_d) the set of simple (resp. double) roots of P. The statement below is an immediate consequence of the Global Residue Theorem (see [10]), Euler-Jacobi formula for simple singularities (see [10]), and the Beźout Theorem (see, for example, [11])

Below we present a criterium for a root of a polynomial map to be multiple. Observe that in the scalar case, multiple roots rely on the divisibility of polynomials. Our criterium can be traced back to the following concept known as the Gleason B property: a domain $D \subset \mathbb{C}^n$ have the Gleason B property at a point $z = a \in D$ if for every $f \in B(D)$ such that $f(a) = 0$ there exist functions $f_1, f_2, \ldots, f_n \in B(D)$ such that $f(z) = \sum_{j=1}^{n}(z_j - a_j)f_j(z)$.

Proposition 2 (Factorization Lemma (see [11,12]))**.** *Let $P = (p_1, \ldots, p_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $P(a) = 0$. Then, a is an isolated multiple root of P if and only if there exist coordinates $z = (z_1, \ldots z_n)$ in $\mathbb{C}^n$ and natural $m \geq 2$ such that*

$$p_k(z) = (z_1 - a_1)^m q_{k1}(z_1) + \sum_{i=2}^{n}(z_i - a_i)q_{ki}(z) \quad (k = 1, \ldots, n), \tag{12}$$

where $R(z) = \{q_{ki}(z)\}_{k,i=1}^{n}$ is a polynomial matrix, the vector $q_1(z_1) := \{q_{11}(z_1), \ldots, q_{n1}(z_1)\}^t$ depends only on z_1 and is not equal to zero identically. Representation (12) of linearization of P is unique up to the Jordan blocks permutation.

Using Proposition 2 we are in a position to formulate the generalized Euler–Jacobi formula for polynomial maps with simple and double singularities. The following Theorem was proven in [11]:

Theorem 3. *Let $P = (P_1, \ldots, P_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map $(\deg(P_i) = d_i,\ i = 1, \ldots, n)$. Denote by A_s (resp. A_d) the set of simple (resp. double) roots of P and assume that*

$$\mathrm{card}(A_s \cup A_d) = \prod_{i=1}^{n} d_i$$

(each root counted according to its multiplicity). Denote by $\widetilde{P}$ the polynomial map, whose components are the highest homogeneous terms of the components of P and assume that $\widetilde{P}(x) = 0 \iff x = \{0\}$. Finally, let $r : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with $\deg(r) \leq \sum_{m=1}^{n} d_m - (n+1)$. Then,

$$\sum_{a_j \in A_s} \frac{r(a_j)}{\det(J_p(a_j))} + \sum_{a_j \in A_d} \frac{\partial}{\partial \xi_{a_j}} \left(\frac{r(z)}{\det(R(z))} \right) \bigg|_{z=a_j} = 0, \tag{13}$$

where $z = (z_1, \ldots, z_n)$, $\frac{\partial}{\partial \xi_{a_j}}(\cdot)$ stands for the directional derivative along the unit eigenvector of $J_P(a_j)$ corresponding to the zero eigenvalue, and $R(z)$ is a polynomial matrix given by (12) corresponding to the "factorization" of P near a double point a_j.

Remark 1. We refer to (8), where a symmetry with non hyperbolic critical equilibrium points for $F_1(x, y) = (1 - y^2, x^2)$. Using Theorem 3 we obtain:

$$\det R(x, y) = 4xy, \qquad \frac{\partial}{\partial_{\eta_{0,\pm 1}}} = \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x} \left(\frac{1}{\det(R(x, y))} \right) \bigg|_{0,\pm 1} = \pm 1/4. \tag{14}$$

4. Symmetry Constructions

In this Section we formulate the basic principle for the purposes of the geometric symmetry construction for homogeneous plus linear ODEs $x' = S(x) - x$ where $S(\lambda x) = \lambda^k S(x)$, $x \in \mathbb{C}^n$. First, we construct a quadratic system with high order symmetry.

Definition 4. *(Linear pencil) A linear pencil is an expression of the form*

$$L(x_1, x_2, \dots, x_n) = A_0 + \sum_{i=1}^{n} A_i x_i$$

where the A_i, $(i = 0, 1, \dots, n)$ are $n \times n$ matrices and the $x_i \in \mathbb{C}$ are variables.

Any system of quadratic ODEs $x' = P(x)$ for $P(x) := Q(x) + Ax$ where $Q(x)$ is a homogeneous quadratic map and Ax is a constant matrix may be rewritten in form

$$x' = L(x_1, x_2, \dots, x_n)x \tag{15}$$

Evidently, $x = 0$ is an equilibrium point of (15) whose type is defined by a spectrum of matrix A_0. Let $\text{Null}(P)_0 = \{x^* \neq 0 : L(x_1^*, x_2^*, \dots, x_n^*)x^* = 0\}$ be a set of non zero equilibrium points to (15). Local dynamics near the points $x^* \in \text{Null}(P)_0$ by Definition 2 are topologically equivalent iff Jacobian matrices $J_{Lx}(x)$ at points $x = x^* \in \text{Null}(P)_0$ are the similar. This means they have one spectrum to be equal.

$$\text{spec } J_{Lx}(x_1^*) = \text{spec } J_{Lx}(x_2^*), \quad \text{for all} \quad x_1^*, x_2^* \in \text{Null}(P)_0. \tag{16}$$

Due to results in [9] and Theorem 1, the characteristic polynomial of Lx at symmetric equilibrium point is a product of linear factors with distinct roots. Therefore, all matrices in Lx have to be simultaneously diagonalizable.

4.1. Matrices with the Same Spectrum

There is a natural way to think about this problem. Given a class of matrices or operators, one asks if there is a transformation, a change of basis, in which their matrix representations all have the same spectrum. The simplest scenario we consider here is when a class of matrices can be simultaneously diagonalized. Indeed the simultaneously diagonalized matrices were not required to have the one typical spectrum. The inverse statement is true.

Definition 5. *Two $n \times n$ matrices A, B are said to be simultaneously diagonalizable over field $\mathbb{F}$ if there is a nonsingular matrix $S \in GL(n, \mathbb{F})$ such that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal matrices. Clearly,*

(a) *Two matrices A and B are simultaneously diagonalizable then they commute $AB = BA$,*
(b) *Not every pair of commuting matrices are simultaneously diagonalizable.*
(c) *Two diagonalizable matrices preserves eachother's eigenspaces.*
(d) *The set of simultaneously diagonalizable matrices generate a toral Lie algebra*

4.2. Circulant Matrices

Definition 6. *By circulant matrix* $\text{Circ}(c_0, c_1, \dots, c_{n-1})$ *we call matrix of form*

$$\text{Circ}(c_0, c_1, \dots, c_{n-2}, c_{n-1}) = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \dots & c_3 & c_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n-1} & c_{n-2} & c_{n-3} & \dots & c_1 & c_0 \end{bmatrix}.$$

Circulant matrices serve an important application in various disciplines including mathematics, physics, image processing, probability and statistics, number theory, geometry, and in the numerical methods of ordinary and partial differential equations. Some of these applications are outlined in [2]. The spectral decompositions of this paper are given in explicit form which can be quickly evaluated in computer programs and provide a useful basis for theoretical investigations.

Since diagonalizing transformations are made up of eigenvectors of a matrix, then a set of matrices is simultaneously diagonalizable iff they share a full set of eigenvectors. An equivalent condition is that they each are diagonalizable, and they all mutually commute.

Secondly, given a mutually commuting set of matrices, by finding their shared eigenvectors, one finds the transformation that simultaneously diagonalizes all of them. Such a transformation of circulant matrices can be carried out by Discrete Fourier Transform (DFT)

The DFT transform matrix W can be defined as a particular case of the Vandermonde matrix.

$$W = \frac{1}{\sqrt{n}}\left[\omega^{jk}\right]_{j,k=0,\ldots,n-1} = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-2} & \omega^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-2)} & \omega^{(n-1)^2} \end{bmatrix} \quad (17)$$

where $\omega = e^{-2\pi i/n}$ is a primitive n-th root of unity.

Given an n-vector v, its DFT is another n-vector defined by $\hat{v} = Wv$. A remarkable fact is that for any given circulant matrix C, its eigenvalues are easily computed.

4.3. Diagonalization of Circulant Matrices

The circulant matrix is diagonalized by the DFT matrix W, namely

$$W\,\mathrm{Circ}(c_0, c_1, \ldots, c_{n-1}) = D(c_0, c_1, \ldots, c_{n-1})W, \quad (18)$$

where D is a diagonal matrix with eigenvalues of $\mathrm{Circ}(c)$ at the main diagonal:

$$D(c_0, c_1, \ldots, c_{n-1}) = \mathrm{diag}\left(\sum_{k=0}^{n-1} c_k, \sum_{k=0}^{n-1} \omega^k c_k, \ldots \sum_{k=0}^{n-1} \omega^{k(n-1)} c_k\right). \quad (19)$$

Therefore, any column of a DFT matrix W is in fact an eigenvector of $\mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$. Since each row of $\mathrm{Circ}(c)$ contains the same entries (just in a different order), the sum in (19) is also the same.

n independent eigenvectors characterize the circulant matrices, but the circulant matrix does not have equally distributed eigenvalues.

4.4. Circulant Convolution

There is one last step involved in order to build the system of ODEs with equally distributed eigenvalues [2].

Definition 7. *Given two n-vectors a and x, their circular convolution $y = a \odot x$ is another n-vector defined by*

$$y = a \odot x, \quad y_k = \sum_{m=0}^{n-1} a_k x_{n-k}, \quad k = 0, 1, \ldots, n-1 \quad (20)$$

where the indices in the sum are evaluated modulo n.

Comparing Definition 6 with (21), we see that multiplying a vector by a circulant matrix is equivalent to convolution operation the vector with the vector defining the circulant matrix (6). Namely, $y = Circ(a_0, a_1, \ldots, a_{n-1})\ x \equiv a \odot x$ for any n-vector column x.

With Definition 8 in hand let us define the following circulant powers:

Definition 8. *Given n-vector a, it circular n-power $a^{\odot n}$ is another n-vector defined reccurently as*

$$a^{\odot 1} = a, \quad a^{\odot 2} = a \odot a, \qquad a^{\odot 3} = a \odot a^{\odot 2}, \quad \dots \quad , a^{\odot n} = a \odot a^{\odot (n-1)}. \tag{21}$$

Theorem 4. *[2,3] There exists a diagonal matrix D such that the ODEs written in the circulant convolution notation*

$$x' = D\, x^{\odot k} - x, \tag{22}$$

have an equally distributed set of the eigenvalues at any of its non zero equilibrium points.

Proof. Determinant d_a of the circulant matrix Circ(a) has the well-known representation:

$$\det(\mathrm{Circ(a)}) = \prod_{k=0}^{n-1} \left(a_0 + a_1\omega^k + a_2\omega^{2k} + \dots + a_{n-1}\omega^{k(n-1)}\right),$$

where $\omega = exp\{2\pi i/n\}$ is a primitive n-th root of unity.

Define a diagonal matrix $D = \mathrm{diag}(1, \omega, \omega^2, \dots, \omega^{n-1})$. Using DFT transform (17) with matrix W we obtain $WD = U_1\, W$ where U_m is a shift by m unit matrix,

$$U_m = \{u_{ij}\} = \begin{cases} 1, & \text{if } j - i = m \bmod n, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

In addition, using (18) we obtain:

$$W\, D\, x^{\odot k} = U_1\, \mathrm{diag}\left(\sum_{k=0}^{n-1} x_k, \sum_{k=0}^{n-1} \omega^k x_k, \dots \sum_{k=0}^{n-1} \omega^{k(n-1)} x_k\right)^{k-1} Wx, \tag{24}$$

$$\det\left(D\, x^{\odot k} = U_1\, \mathrm{diag}\left(\sum_{k=0}^{n-1} x_k, \sum_{k=0}^{n-1} \omega^k x_k, \dots \sum_{k=0}^{n-1} \omega^{k(n-1)} x_k\right)^{k-1} Wx, \tag{25}$$

All equilibrium points $x^* \in \mathrm{Null}(P)$ of (22) fulfill the equations $D\ (x^*)^{\odot k} = x^*$. By straightforward calculations, we obtain $p_\lambda(x^*) = \det(\lambda I - U_1\, \mathrm{Circ}(x^*)^k) = \lambda^k - 1$, cf. with (11). □

Example 3. *The ODE (22) in the plane acquire an extremely simple form. Using complex number notations let us define $z = x + iy$. Then the following complex dynamics*

$$\frac{d}{dt}z = \bar{z}^k - z, \quad \bar{z} = x - iy, \quad k = 2, 3, \dots, \tag{26}$$

have a maximal possible symmetry group, in fact this is the dihedral group D_{k+1}. The phase portraits of such complex ODE for $k = 2, 3, 4$ is present in Figure A1. The non zero real equilibrium points are here equidistributed on the unit circle. They all are saddles. The eigenvalues of the Jacobian matrix are $\lambda_k = \{k - 1, -k - 1\}$.

It should be pointed out that the vector field (26) with the dihedral D_4 symmetry group is not unique. Some further examples of the cubic vector fields with the equilibrium points set $z = \{\pm 1; \pm i\}$ and the D_4 symmetry group present in Figure A2.

Example 4. *In dimension four one can rewrite (22) for $k = 2$ and $x = [x_1, x_2, x_3, x_4]$ explicitly as*

$$x_1' = {x_1}^2 + 2\,x_2 x_4 + {x_3}^2 - x_1,\ x_2' = 2\,i x_1 x_2 + 2\,i x_3 x_4 - x_2,$$
$$x_3' = -2\,x_1 x_3 - {x_2}^2 - {x_4}^2 - x_3,\ x_4' = -2\,i x_2 x_3 - 2\,i x_1 x_4 - x_4.$$

It can be verified that this ODE has exactly 15 non zero equilibrium points arranged on the complex unit ball in $\mathbb{C}^4$. At any non zero equilibrium point the Jacobian matrix has the same set of eigenvalues, namely $\lambda = \{1, -3, 1 \pm 2i\}$.

5. Conclusions

1. Our main goal is to create examples of ODEs that have a high symmetry.
2. We show the necessary conditions for homogeneous ODEs with high symmetry to have charpoly of the Jacobian matrix of form $\lambda^n - k^n$ (see Theorem 2).
3. We explicitly indicate the connection of the circulant matrices theory on possible symmetries in phase portraits. Namely, the system $x' = Dx^{\odot k} - x$ written using circulant convolution notation, has a high symmetry in its phase portrait. (See Theorem 4 and it graphical representation in Appendix A)

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Appendix A. Symmetries in the Plane Phase Portrait

Example A1. *Phase portraits of $z' = \bar{z}^k - z$ for $z = x + iy$.*

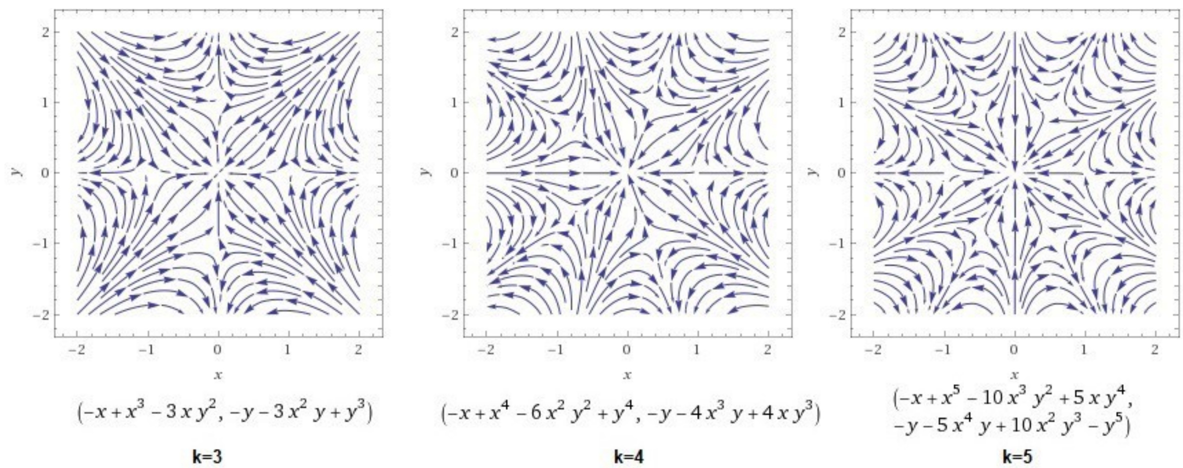

Figure A1. Polynomial of order k plane ODEs with the dihedral symmetry group D_{k+1}.

Example A2. *Different cubic plane ODEs may have D_4 symmetries. In Figure A2 the different dynamics in cases (a)-(c) are accompanied by explicit formulas of corresponding vector fields placed above it:*

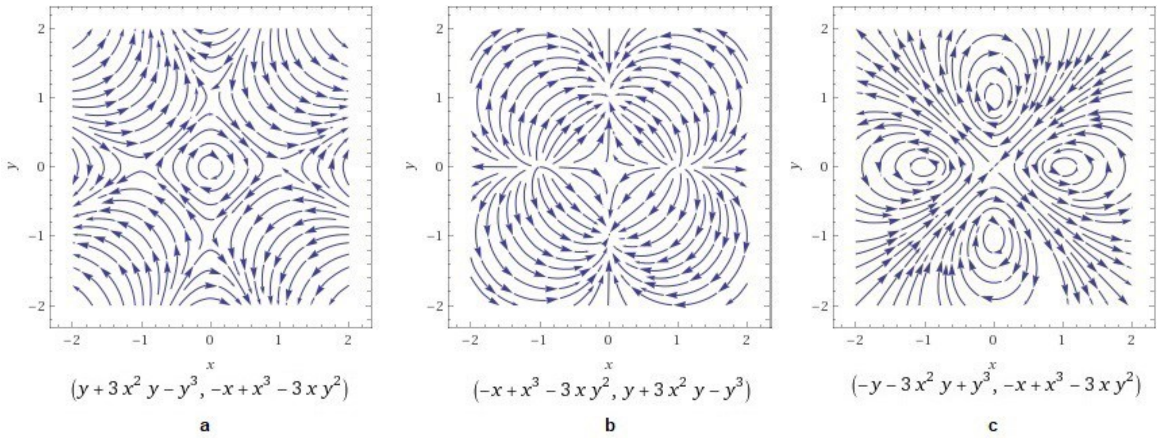

Figure A2. Phase portraits with D_4 symmetry at its equilibrium points.

Using complex number notations the phase portraits in Figure A2 are: a). $z' = z^3 - \bar{z}$, b). $z' = i(\bar{z}^3 - z)$, c). $z' = i(z^3 - \bar{z})$. All of these phase portraits are produced from Example A1, case $k = 3$.

Appendix B. Basic Definitions

- We consider an autonomous system $x' = P(x)$.
- An equilibrium (singular) point x^* of an ODE $x'(t) = P(x(t))$ is one where a solution of the differential equation remains fixed at x^* for all time.
- The solutions to the linearized system near an equilibrium point x^* are approximately closed to the solutions of the actual system provided that x^* is hyperbolic.
- A diffeomorphism is an invertible map such that both the mapping and its inverse are continuously differentiable.
- The definition of the geometrical equivalence can be generalized to cover more general cases when the state space is a complete metric or, in particular, is a Banach space.
- The phase portraits of topologically equivalent systems are often also called topologically equivalent.

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