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Binary Operations in the Unit Ball: A Differential Geometry Approach

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Abstract: Within the framework of differential geometry, we study binary operations in the open, unit ball of the Euclidean n-space $\mathbb{R}^n$, $n \in \mathbb{N}$, and discover the properties that qualify these operations to the title *addition* despite the fact that, in general, these binary operations are neither commutative nor associative. The binary operation of the Beltrami-Klein ball model of hyperbolic geometry, known as Einstein addition, and the binary operation of the Beltrami-Poincaré ball model of hyperbolic geometry, known as Möbius addition, determine corresponding metric tensors in the unit ball. For a variety of metric tensors, including these two, we show how binary operations can be recovered from metric tensors. We define corresponding scalar multiplications, which give rise to gyrovector spaces, and to norms in these spaces. We introduce a large set of binary operations that are algebraically equivalent to Einstein addition and satisfy a number of nice properties of this addition. For such operations we define sets of gyrolines and co-gyrolines. The sets of co-gyrolines are sets of geodesics of Riemannian manifolds with zero Gaussian curvatures. We also obtain a special binary operation in the ball, which is isomorphic to the Euclidean addition in the Euclidean n-space.

Keywords: differential geometry; binary operation; Einstein addition; gyrogroup; gyrovector space; geodesics

1. Introduction

Let $\mathbb{B}$ be the unit, open ball in the Euclidean n-space $\mathbb{R}^n$,

$$\mathbb{B} = \{x \in \mathbb{R}^n : \ \|x\| < 1\}, \tag{1}$$

$n \in \mathbb{N}$, where $\| \cdot \|$ is the Euclidean norm.

Einstein addition is a binary operation, $\oplus_E$, in the ball $\mathbb{B} \subset \mathbb{R}^n$ that stems from his velocity composition law in the ball $\mathbb{B} \subset \mathbb{R}^3$ of relativistically admissible velocities. Seemingly structureless, Einstein addition is neither commutative nor associative. However, Einstein addition turns out to be both *gyrocommutative* and *gyroassociative*, thus giving rise to the rich algebraic structures that became known as a (gyrocommutative) gyrogroup and a gyrovector space, the definitions of which are presented in Definitions 1–3, Section 2.

Einstein addition, $\oplus_E$, and its isomorphic copy, Möbius addition, $\oplus_M$, are studied in the literature algebraically, along with applications to the hyperbolic geometry of Lobachevsky and Bolyai. Naturally, one may expect that the rich algebraic structure of Einstein addition can find home in differential geometry, giving rise to a novel branch called *Binary Operations in the Ball*.

Accordingly, the aim of this paper is to develop a differential geometry approach to Einstein addition and, hence, to discover the resulting novel branch of differential geometry that involves binary operations in the ball. We thus begin with the study of an arbitrary binary operation in the ball that satisfies some general conditions.

A binary operation in $\mathbb{B}$ is a function f: $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$. We consider functions f of class C^2, that is, functions f having continuous second derivatives. This operation determines a metric tensor $G(x)$ in $\mathbb{B}$ given by

$$g(x) = \frac{\partial f(-x,y)}{\partial y}|_{y=x}, \qquad G(x) = g(x)^\top g(x) \in \mathbb{R}^{n\times n}, \tag{2}$$

$x \in \mathbb{B}$, where $^\top$ denotes transposition, and $\mathbb{R}^{n\times n}$ is the space of all real $n \times n$-matrices. Then, $\mathbb{B}$ is a Riemannian manifold with a metric tensor G.

We pay special attention to the following three binary operations in the ball, along with their associated scalar multiplication:

- Einstein addition $\oplus_{E}$ in the ball, presented in (136), and the scalar multiplication $\otimes_E$ that it admits, presented in (138), are recovered in Section 5 within the framework of differential geometry. The triple $(\mathbb{B}, \oplus_{E}, \otimes_E)$ is an Einstein gyrovector space that forms the algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry.
- Möbius addition $\oplus_{M}$ in the ball, presented in (153), and the scalar multiplication $\otimes_M$ that it admits, presented in (154), are recovered in Section 6 within the framework of differential geometry. The triple $(\mathbb{B}, \oplus_{M}, \otimes_M)$ is a Möbius gyrovector space that forms the algebraic setting for the Beltrami–Poincaré ball model of hyperbolic geometry.
- A novel, interesting binary operation $\oplus$ in the ball, presented in (199), and the scalar multiplication $\otimes$ that it admits, presented in (205), are discovered in Section 7 within the framework of differential geometry. Remarkably, the triple $(\mathbb{B}, \oplus, \otimes)$ is a vector space isomorphic to the Euclidean vector space $(\mathbb{R}^n, +, \cdot)$. As such, the binary operation $\oplus$ in $\mathbb{B}$ is commutative, associative and distributive. Accordingly, the vector space $(\mathbb{B}, \oplus, \otimes)$ forms the algebraic setting for a novel n-dimensional Euclidean geometry ball model.

A procedure that we present in this paper enables binary operations in the ball, like $\oplus_{E}$, $\oplus_{M}$ and $\oplus$, to be obtained from corresponding metric tensors G. Interestingly, a metric tensor G is determined by the behavior of the function f in a neighborhood of the set $D = \{(-x, x) \;:\; x \in \mathbb{B}\}$, rather than in the whole of the space $\mathbb{B} \times \mathbb{B}$. Hence, the global operations turn out to be determined by local properties of the functions f and their second derivatives in the set D.

The procedure is formulated in terms of geodesics and a parallel transport. It is applied to a wide class of metric tensors that satisfy three properties: (i) smoothness, (ii) rotation invariance, and (iii) plane invariance.

For Einstein addition $\oplus_{E}$ the operation of scalar multiplication $t \otimes_E a$ with $a \in \mathbb{B}$, $t \in \mathbb{R}$ is well defined [1]. It gives rise to a structure called a gyrovector space. For each metric tensor considered in this paper we define an operation $t \otimes a$ of scalar multiplication, which leads to corresponding gyrovector spaces.

For every $a, b \in \mathbb{B}$ the set $\{f(b, t \otimes a) \;:\; t \in \mathbb{R}\}$ is called a gyroline. The set of gyrolines defines a metric tensor G. The sets $\{f(t \otimes a, b) \;:\; t \in \mathbb{R}\}$ are called co-gyrolines. It is proved in the paper that for operations f isomorphic to Einstein addition, the co-gyrolines are gyrolines for other binary operations, f_{co}, with metric tensors that we denote by G_{co}. The Gaussian curvatures of spaces with metric tensors G_{co} are equal to zero.

Various algebraic and geometric properties of the operations $\oplus_{E}$ and $\oplus_{M}$ have been intensively studied in recent papers and monographs; see, for instance, References [1–10]).

Einstein addition and Möbius addition are neither commutative nor associative. As such, they do not form group operations. Yet, the lack of the commutative and associative laws is compensated by the *gyrocommutative* and *gyroassociative* laws that these binary operations obey. As such, these binary operations give rise to the algebraic objects known as *gyrogroups* and *gyrovector spaces*. Remarkably, gyrovector spaces form the algebraic setting for various models of analytic hyperbolic geometry, just as the standard vector spaces form the algebraic setting for analytic Euclidean geometry; see, for instance, References [4,5,9,11–13].

The special interest of our study of both Einstein addition and Möbius addition within the framework of differential geometry stems from the result that they are *gyrocommutative gyrogroup operations*. Indeed they give rise to (i) Einstein gyrogroups $(\mathbb{B}, \oplus_{\mathrm{E}})$ and Einstein gyrovector spaces $(\mathbb{B}, \oplus_{\mathrm{E}}, \otimes_E)$; and to (ii) Möbius gyrogroups $(\mathbb{B}, \oplus_{\mathrm{M}})$ and Möbius gyrovector spaces $(\mathbb{B}, \oplus_{\mathrm{M}}, \otimes_M)$. The definitions of the abstract (gyrocommutative) gyrogroup and gyrovector space are presented in Section 2.

The organization of the paper is as follows. In Section 2 we present the definitions of the abstract (gyrocommutative) gyrogroup and gyrovector space. In Section 3 we introduce metric tensors satisfying three properties: (i) smoothness; (ii) rotation invariance; and (iii) plane invariance. Then we derive the equations for geodesics and parallel transport. In Section 4 we (i) introduce binary operations defined by metric tensors G; (ii) define corresponding operations of scalar multiplication; and (iii) introduce distances and gyronorms. In Section 5 we prove that the operation f, defined in terms of the metric tensor of Einstein addition, coincides with Einstein addition. In Sections 6 and 7 similar results are obtained for Möbius addition and for a novel addition. Section 8 is devoted to properties of binary operations similar to those of Einstein addition: (i) left cancellation law; (ii) existence and properties of unitary gyrators; and (iii) a gyrocommutative law.

In the paper, we use the following notation—$\mathbb{R}$ is the set of real numbers. I is the identity matrix. A real square matrix U is called unitary [14] if $U^\top U = I$. The function sign is the signum function in $\mathbb{R}$. Components of an n-vector x are denoted by x_j, $j = 1, \ldots, n$.

2. Gyrogroups and Gyrovector Spaces

Definition 1 ((Gyrogroup), [4] Definition 2.5). *A groupoid $(S, \oplus)$ is a nonempty set S with a binary operation. A groupoid $(S, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In S there is at least one element,* 0*, called a left identity, satisfying*

(G1) $\quad 0 \oplus a = a,$

for all $a \in S$. There is an element $0 \in S$ satisfying axiom $(G1)$ such that for each $a \in S$ there is an element $\ominus a \in S$, called a left inverse of a, satisfying

(G2) $\quad \ominus a \oplus a = 0\,.$

Moreover, for any $a, b, c \in S$ there exists a unique element $\mathrm{gyr}[a, b]c \in S$ *such that the binary operation obeys the left gyroassociative law*

(G3) $\quad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c\,.$

The map $\mathrm{gyr}[a, b] : S \to S$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid $(S, \oplus)$, that is,*

(G4) $\quad \mathrm{gyr}[a, b] \in \mathrm{Aut}(S, \oplus)\,,$

and the automorphism $\mathrm{gyr}[a, b]$ *of S is called the gyroautomorphism, or the gyration, of S generated by $a, b \in S$. The operator* $\mathrm{gyr} : S \times S \to \mathrm{Aut}(S, \oplus)$ *is called the gyrator of S. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any $a, b \in S$ possesses the left reduction property*

(G5) $\quad \mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]\,,$

called the reduction axiom.

The gyrogroup axioms $(G1)$–$(G5)$ in Definition 1 split up into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.
2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups split up into gyrocommutative and non-gyrocommutative ones.

Definition 2 ((Gyrocommutative Gyrogroup), [4] Definition 2.6). *A gyrogroup $(S, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law*

(G6) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$,
for all $a, b \in S$.

The first concrete example of a gyrogroup was discovered in 1988 [15]. It became known as an Einstein gyrogroup. Einstein gyrogroups are employed, for instance, in References [5,16–21]. Möbius gyrogroups are employed as well, for instance, in References [4,22–24]. In full analogy with groups, there are topological gyrogroups, studied in References [25,26]. Gyrogroups share remarkable analogies with groups studied, for instance, in References [27–32]. Applications of gyrogroups in harmonic analysis are found in References [17,23,33]. For other interesting studies of gyrogroups see References [34–39]. Einstein gyrogroups and gyrovector spaces are extended to Einstein bi-gyrogroups and bi-gyrovector spaces in References [10,40], along with an application to relativistic quantum entanglement of multi-particle systems.

The gyrocommutative gyrogroups that we study in this paper admit scalar multiplication, turning themselves into gyrovector spaces, the formal definition of which follows.

Definition 3 ((Gyrovector space), [4] Definition 6.2)**.** *A real inner product gyrovector space* $(G, \oplus, \otimes)$ *(gyrovector space in short) is a gyrocommutative gyrogroup* $(G, \oplus)$ *that obeys the following axioms:*

(1) G is a subset of a real inner product vector space V called the carrier of G, $G \subset V$*, from which it inherits its inner product,* $\cdot$*, and norm,* $\|\cdot\|$*, which are invariant under gyroautomorphisms, that is,*

(V0) $gyr[u, v]a \cdot gyr[u, v]b = a \cdot b$ *for all* $a, b, u, v \in G$.

(2) G admits a scalar multiplication, $\otimes$*, possessing the following properties. For all real numbers* $r, r_1, r_2 \in \mathbb{R}$ *and all points* $a \in G$*:*

(V1) $1 \otimes a = a$,
(V2) $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a$,
(V3) $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$,
(V4) $\frac{|r| \otimes a}{\|r \otimes a\|} = \frac{a}{\|a\|}, \quad a \neq 0, \ r \neq 0$,
(V5) $gyr[u, v](r \otimes a) = r \otimes gyr[u, v]a$,
(V6) $gyr[r_1 \otimes v, r_2 \otimes v] = I$,
(V7) $\|r \otimes a\| = |r| \otimes \|a\|$,
(V8) $\|a \oplus b\| \leq \|a\| \oplus \|b\|$.

Like gyrogroups, also gyrovector spaces are studied in the literature. The papers [41–45] are devoted to various aspects of Möbius gyrovector spaces. Einstein and Möbius gyrovector spaces in the context of a gyrovector space approach to hyperbolic geometry are the subject of Reference [6]. Generalized gyrovector spaces are studied in References [46–48]. Interesting results about the differential geometry of some gyrovector spaces may be found in Reference [49]. Other interesting studies of gyrovector spaces are found in References [50–55].

In this paper we introduce, within the framework of differential geometry, a large number of gyrocommutative gyrogroup operations, which enjoy key properties of Einstein addition and Möbius addition. Furthermore, we present corresponding scalar multiplications that turn these gyrocommutative gyrogroups into gyrovector spaces.

3. Metric Tensors

In this section we consider metric tensors that satisfy certain conditions, and derive equations for geodesics and parallel transports for these tensors.

3.1. Parametrization of Metric Tensors

For every smooth binary operation $f \in C^1 \colon \mathbb{B} \times \mathbb{B} \to \mathbb{B}$,

$$f(a, b) = a \oplus b, \tag{3}$$

we determine a matrix function $G \in \mathbb{R}^{n\times n}$ by the equations

$$\begin{aligned} g(x) &= \frac{\partial}{\partial y}[(-x) \oplus y]|_{y=x} \\ G(x) &= g(x)^\top g(x)\,, \end{aligned} \tag{4}$$

for all $x \in \mathbb{B}$.

We consider three conditions on the tensors G. These are—(i) rotation invariance; (ii) two-dimensional plane invariance; and (iii) smoothness.

Rotation invariance. For every unitary $n \times n$-matrix U and $x \in \mathbb{B}$ we have

$$G(Ux) = U^\top G(x)U. \tag{5}$$

Plane invariance. For every $a, b \in \mathbb{B}$ and every two-dimensional plane that contains a and b, the whole geodesic $x(\cdot)$ with initial conditions $x(0) = a$ and $\dot{x}(0) = b$ belongs to this plane.

Smoothness. The function G is differentiable in $\mathbb{B}$.

The first property means invariance with respect to rotations. The second property means that every geodesic belongs to a plane that contains its initial data. In particular, if a and b belong to the same line l, then $f(a, b) \in l$. The third property is a standard assumption that allows to derive the differential equations for geodesics.

These three properties are valid if there exist differentiable functions m_0, m_1: $[0, 1) \to (0, \infty)$ such that $m_0(0) = m_1(0)$, $m_0'(s) > 0$, $m_1'(s) > 0$ for all $s \in [0, \infty)$, and

$$G(x) = \{g_{ij}(x)\}_{i,j=1}^n = m_0(\|x\|^2)(I - \frac{xx^\top}{\|x\|^2}) + m_1(\|x\|^2)\frac{xx^\top}{\|x\|^2}. \tag{6}$$

The inverse matrix $G^{-1}(x)$ of $G(x)$ in (6) exists, being given by

$$G^{-1}(x) = \{g^{js}(x)\}_{j,s=1}^n = \frac{1}{m_0(\|x\|^2)}(I - \frac{xx^\top}{\|x\|^2}) + \frac{1}{m_1(\|x\|^2)}\frac{xx^\top}{\|x\|^2}. \tag{7}$$

We say that the metric tensor $G(x)$ in (6) is parametrized by the two functions m_0 and m_1. Special attention will be paid to two parametrizations of $G(x)$.

The first parametrization of $G(x)$ is when m_0 and m_1 are given by

$$m_0(r^2) = \frac{1}{1-r^2}\,, \qquad m_1(r^2) = \frac{1}{(1-r^2)^2}\,, \qquad \forall r \in [0, 1), \tag{8}$$

which will be associated with Einstein addition. Einstein addition, in turn, is the binary operation that results from Einstein's addition law of relativistically admissible velocities in special relativity theory [1].

The second parametrization of $G(x)$ is when m_0 and m_1 are given by

$$m_0(r^2) = \frac{1}{(1-r^2)^2}\,, \qquad m_1(r^2) = \frac{1}{(1-r^2)^2}, \qquad \forall r \in [0, 1), \tag{9}$$

which will be associated with Möbius addition. Möbius addition, in turn, is a natural generalization to the ball of a well known Möbius transformation of the disc [4].

In order to construct binary operations that result from metric tensors G we will find the differential equations for geodesics and parallel transports. To this end we have to calculate the Christoffel symbols in Section 3.3.

3.2. Geodesics

Assume x: $[0,1] \to \mathbb{B}$ is a curve in $\mathbb{B}$. The length of this curve is

$$l = \int_0^1 \sqrt{\dot{x}(s)^\top G(x(s))\dot{x}(s)}ds. \tag{10}$$

Denote $G(x) = \{g_{ij}(x)\}_{i,j=1}^n$, $a = x(0)$, $b = x(1)$, and

$$F(x(s), \dot{x}(s)) = \sqrt{\dot{x}(s)^\top G(x(s))\dot{x}(s)}. \tag{11}$$

Assume $x(\cdot)$ minimizes l over all smooth curves x such that $x(0) = a$, $x(1) = b$. Use a new parametrization

$$dt = \frac{F(x(s), \frac{dx(s)}{ds})ds}{\int_0^1 F(x(u), \dot{x}(u))du}. \tag{12}$$

Then, $F(x(t), \frac{dx}{dt}(t)) \equiv const.$

According to Euler-Lagrange equations for all $k = 1, \ldots, n$ we have

$$\frac{\partial F}{\partial x_k} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_k} = 0. \tag{13}$$

This equation takes the form

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial g_{ij}(x)}{\partial x_k} \frac{dx_i(t)}{dt} \frac{dx_j(t)}{dt} - \frac{d}{dt} \sum_{j=1}^n g_{kj}(x) \frac{dx_j(t)}{dt} = 0. \tag{14}$$

Denote the entries of the matrix $G^{-1}(x)$ by $g^{kp}(x)$. Then (14) is equivalent to the following. For all $p = 1, \ldots, n$

$$\ddot{x}_p + \frac{1}{2} \sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n \{\frac{\partial g_{kj}(x)}{\partial x_i} + \frac{\partial g_{ki}(x)}{\partial x_j} - \frac{\partial g_{ij}(x)}{\partial x_k}\} g^{kp}(x)\dot{x}_i\dot{x}_j = 0. \tag{15}$$

Denote

$$\Gamma_{ij}^p(x) = \frac{1}{2} \sum_{k=1}^n \{\frac{\partial g_{kj}(x)}{\partial x_i} + \frac{\partial g_{ki}(x)}{\partial x_j} - \frac{\partial g_{ij}(x)}{\partial x_k}\} g^{kp}(x). \tag{16}$$

Then (15) and (16) yield the well known equations of geodesics [56],

$$\ddot{x}_p + \sum_{i=1}^n \sum_{j=1}^n \Gamma_{ij}^p(x)\dot{x}_i\dot{x}_j = 0. \tag{17}$$

Note that along every geodesic the function

$$F(x(t), \dot{x}(t)) = \sqrt{\sum_{i=1}^n \sum_{j=1}^n g_{ij}(x(t))\dot{x}_i(t)\dot{x}_j(t)} \tag{18}$$

is constant.

3.3. Christoffel Symbols

For the sake of brevity we omit the arguments of the functions m_j: $m_j = m_j(\|x\|^2)$, and of the derivatives of these functions,

$$m_j'(r^2) = \frac{dm_j(z)}{dz}|_{z=r^2}. \tag{19}$$

We also use the Kronecker delta δ_{ij} which is equal to one if $i = j$, and to zero otherwise. The elements $g_{ij}(x)$ of the metric tensor $G(x)$ in (6) have the form

$$g_{ij}(x) = m_0(\delta_{ij} - \frac{x_i x_j}{\|x\|^2}) + m_1 \frac{x_i x_j}{\|x\|^2}. \tag{20}$$

Then

$$\begin{aligned} \frac{\partial m_j}{\partial x_k} &= m_j' 2x_k \\ \frac{\partial}{\partial x_k}\left(\frac{x_i x_j}{\|x\|^2}\right) &= -\frac{2x_i x_j x_k}{\|x\|^4} + \frac{\delta_{ik}x_j + \delta_{jk}x_i}{\|x\|^2}. \end{aligned} \tag{21}$$

The partial derivatives of $g_{ij}(x)$ are given by

$$\begin{aligned} \frac{\partial g_{ij}(x)}{\partial x_k} &= m_0' 2x_k(\delta_{ij} - \frac{x_i x_j}{\|x\|^2}) + m_1' 2x_k \frac{x_i x_j}{\|x\|^2} \\ &+ (m_1 - m_0)(\frac{\delta_{ik}x_j + \delta_{jk}x_i}{\|x\|^2} - \frac{2x_i x_j x_k}{\|x\|^4}). \end{aligned} \tag{22}$$

Now we introduce the functions $\Gamma_{ijk}(x)$,

$$\begin{aligned} \Gamma_{ijk}(x) &= \frac{1}{2}(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}) \\ &= m_0'(\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k - \frac{x_i x_j x_k}{\|x\|^2}) \\ &+ m_1' \frac{x_i x_j x_k}{\|x\|^2} + (m_1 - m_0)(\frac{\delta_{ij}x_k}{\|x\|^2} - \frac{x_i x_j x_k}{\|x\|^2}). \end{aligned} \tag{23}$$

Therefore, Christoffel symbols $\Gamma^s_{ij}(x)$ are given by

$$\begin{aligned} \Gamma^s_{ij}(x) &= \sum_{k=1}^{n} \Gamma_{ijk}(x) g^{ks}(x) \\ &= \frac{m_1'}{m_1}\frac{x_i x_j x_s}{\|x\|^2} + \frac{m_0'}{m_0}(\delta_{is}x_j + \delta_{js}x_i - \frac{2x_i x_j x_s}{\|x\|^2}) \\ &+ \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1}(\delta_{ij}x_s - \frac{x_i x_j x_s}{\|x\|^2}). \end{aligned} \tag{24}$$

Having the Christoffel symbols $\Gamma^s_{ij}(x)$, we are in the position to calculate the equation of geodesics in Section 3.4.

3.4. Geodesics

The well known differential equation of geodesics is (17). Substituting the values of Christoffel symbols $\Gamma^s_{ij}(x)$ from (24) into (17) yields

$$\ddot{x} + \frac{m_0'}{m_0} 2(x^\top \dot{x})\dot{x} + [(\frac{m_1'}{m_1} - \frac{2m_0'}{m_0})\frac{(x^\top \dot{x})^2}{\|x\|^2} + \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1}(\|\dot{x}\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2})]x = 0. \tag{25}$$

In the special case when the initial values $x(0)$ and $\dot{x}(0)$ are parallel, that is, $x(0) = \lambda_1 a$ and $\dot{x}(0) = \lambda_2 a$ for some vector a and numbers λ_1, λ_2, the solution of (25) has the form

$$x(t) = r(t)a\,. \tag{26}$$

Here, the scalar function r, to be determined, satisfies the initial conditions $r(0) = \lambda_1$ and $\dot{r}(0) = \lambda_2$, along with the differential equation

$$\ddot{r} + \frac{m_0'}{m_0} 2\|a\|^2 r\dot{r}^2 + (\frac{m_1'}{m_1} - \frac{2m_0'}{m_0})\|a\|^2 r\dot{r}^2 = 0 \tag{27}$$

or, equivalently,

$$\ddot{r} + \frac{m_1'}{m_1}\|a\|^2 r\dot{r}^2 = 0\,. \tag{28}$$

Noting the notation $m_1 = m_1(\|x\|^2) = m_1(\|a\|^2 r(t)^2)$ that we use, (28) can be integrated, obtaining

$$\ln|\dot{r}(t)| + \frac{1}{2}\ln|m_1(\|a\|^2 r(t)^2)| = C_1, \tag{29}$$

where C_1 is an arbitrary constant.

The resulting first order Equation (29) can be written as

$$\dot{r}(t) = \frac{C}{\sqrt{m_1(\|a\|^2 r(t)^2)}}\,, \tag{30}$$

where

$$C = \dot{r}(0)\sqrt{m_1(\|x(0)\|^2)}\,. \tag{31}$$

Equation (30) is separable, giving rise to the equation

$$\sqrt{m_1(\|a\|^2 r^2)}dr = C\,dt\,. \tag{32}$$

Integrating (32) yields

$$\int_{r(0)}^{r(1)} \sqrt{m_1(\|a\|^2 s^2)}ds = C\,, \tag{33}$$

so that, finally, by (31) and (33),

$$\dot{r}(0) = \int_{r(0)}^{r(1)} \sqrt{\frac{m_1(\|a\|^2 s^2)}{m_1(\|a\|^2 r(0)^2)}}ds\,. \tag{34}$$

Let us now consider the important special case when $r(0) = 0$ and $r(1) = 1$ (this means $x(0) = 0$ and $x(1) = a$). In this special case (34) reduces to

$$\dot{r}(0) = \int_0^1 \sqrt{\frac{m_1(\|a\|^2 s^2)}{m_1(0)}}ds = \frac{1}{\|a\|}\int_0^{\|a\|} \sqrt{\frac{m_1(s^2)}{m_1(0)}}ds. \tag{35}$$

Formula (35) will prove useful in the definition of special binary operations and in the introduction of norms associated with these operations.

3.5. Parallel Transport

The well known differential equation for a parallel transport of a vector $X = \{X_s\}_{s=1}^n$ along a curve x is [56]

$$\dot{X}_s + \sum_{i=1}^n \sum_{j=1}^n \Gamma_{ij}^s \dot{x}_i X_j = 0, \qquad s = 1, \ldots, n. \tag{36}$$

Substituting the values of Christoffel symbols $\Gamma_{ij}^s(x)$ from (24) into (36) yields

$$\begin{aligned} \dot{X} + \frac{m_1'}{m_1}\frac{(x^\top \dot{x})(x^\top X)}{\|x\|^2}x + \{\frac{m_0'}{m_0}[(x^\top X)(I - \frac{xx^\top}{\|x\|^2})\dot{x} \\ +(x^\top \dot{x})(I - \frac{xx^\top}{\|x\|^2})X] + \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1}\dot{x}^\top(I - \frac{xx^\top}{\|x\|^2})X\}\; x = 0\,. \end{aligned} \tag{37}$$

If the curve x is linear, joining the origin and a point $a \in \mathbb{R}^n$, then $x(t) = at$, and (37) takes the form

$$\dot{X} + \frac{m_1'}{m_1} t a^\top X\, a + \frac{m_0'}{m_0}[-t a^\top X\, a + t\|a\|^2 X] = 0\,. \tag{38}$$

With the initial condition $X(0) = X_0$, (38) has the solution

$$X(t) = \sqrt{\frac{m_0(\|x(0)\|^2)}{m_0(\|x(t)\|^2)}}(I - \frac{aa^\top}{\|a\|^2})X_0 + \sqrt{\frac{m_1(\|x(0)\|^2)}{m_1(\|x(t)\|^2)}}\frac{aa^\top}{\|a\|^2}X_0\,. \tag{39}$$

In particular, at $t = 1$, we have

$$X(1) = \sqrt{\frac{m_0(0)}{m_0(\|a\|^2)}}(I - \frac{aa^\top}{\|a\|^2})X_0 + \sqrt{\frac{m_1(0)}{m_1(\|a\|^2)}}\frac{aa^\top}{\|a\|^2}X_0. \tag{40}$$

If the vectors X_0 and a are not parallel, then the first term in (39) describes the evolution of $X(t)$ in a direction orthogonal to a in the plane determined by the vectors X_0 and a, and the second term in (39) describes the evolution of $X(t)$ in a direction parallel to a.

If we multiply the functions m_0 and m_1 by the same positive number, then equation (25) of geodesics and (37) of parallel transport remain unchanged. Hence, without loss of generality we may assume that $m_0(0) = m_1(0) = 1$.

Assumption 1. *The functions m_0 and m_1 satisfy $m_0(0) = m_1(0) = 1$.*

4. Binary Operations

We are now in the position to define a binary operation $\oplus$ in $\mathbb{B}$ that results from the metric tensor G in (6).

4.1. Vector Addition

In this subsection we define a binary operation $\oplus$ in the ball $\mathbb{B}$. Let vectors $a, b \in \mathbb{B}$ be given. If $b = 0$, then $a \oplus b = a$. If $a = 0$, then $a \oplus b = b$. For the case $b \neq 0$, $a \neq 0$ we perform the following four steps that lead to the definition of $a \oplus b$.

Step 1. We calculate the value $\dot{r}(0)$ using Formula (35) with a replaced by b,

$$\dot{r}(0) = \frac{1}{\|b\|}\int_0^{\|b\|}\sqrt{m_1(s^2)}ds. \tag{41}$$

Step 2. We calculate the value $X(1)$ using Formula (40) with initial condition $X_0 = \dot{r}(0)\, b$,

$$X(1) = \frac{1}{\sqrt{m_0(\|a\|^2)}}(I - \frac{aa^\top}{\|a\|^2})X_0 + \frac{1}{\sqrt{m_1(\|a\|^2)}}\frac{aa^\top}{\|a\|^2}X_0. \tag{42}$$

Step 3. We solve the second order differential Equation (25),

$$\begin{aligned} &\ddot{x} + \frac{m_0'}{m_0}2(x^\top \dot{x})\dot{x} \\ &+ [(\frac{m_1'}{m_1} - \frac{2m_0'}{m_0})\frac{(x^\top \dot{x})^2}{\|x\|^2} + \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1}(\|\dot{x}\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2})]x \\ &= 0 \end{aligned} \tag{43}$$

with initial conditions $x(0) = a$, $\dot{x}(0) = X(1)$.

Step 4. A binary operation $a \oplus b$ is defined by the equation

$$a \oplus b = x(1). \tag{44}$$

We say that the binary operation $\oplus$ in $\mathbb{B}$ is generated by the metric tensor G given by (2).

4.2. Elementary Properties of the Binary Operations $\oplus$

Lemma 1. *For every vector $v \in \mathbb{B}$*

$$0 \oplus v = v, \tag{45}$$

$$(-v) \oplus v = 0. \tag{46}$$

Proof. The first equation stems from the definition of $\oplus$. Assume $a = -v$, $b = v$, $v \neq 0$. Applying steps 1-4, we have

$$X_1 = \frac{1}{\sqrt{m_1(\|v\|^2)}}v. \tag{47}$$

Consider a solution x of Equation (43) with initial conditions $x(0) = -v$, $\dot{x}(0) = X_1$. Since the vectors $x(0)$, $\dot{x}(0)$ are parallel to v, the solution x has the form $x(t) = g(t)v$, where g is a scalar function to be determined. Equation (43) shows that g satisfies the conditions

$$\begin{aligned} \ddot{g} + \frac{m_1'}{m_1}(\dot{g})^2 g\|v\|^2 &= 0, \\ g(0) &= -1, \\ \dot{g}(0) &= \frac{1}{\sqrt{m_1(\|v\|^2)}}\int_0^1 \sqrt{m_1(s^2\|v\|^2)}ds. \end{aligned} \tag{48}$$

The solution g of this initial value problem satisfies the equation

$$\int_{-1}^{g(t)} \sqrt{m_1(s^2\|v\|^2)}ds = t\int_0^1 \sqrt{m_1(s^2\|v\|^2)}ds. \tag{49}$$

Obviously, $g(1) = 0$. Hence, $(-a) \oplus a = x(1) = 0$. □

4.3. Metric Tensors Associated with Binary Operations

In this section we show that the metric tensor G that generates the operation $\oplus$ can be recovered from the operation $\oplus$.

Theorem 1. *Let G be given by (6) and let $\oplus$ be the binary operation (44) generated by G. Furthermore, for every $x_0 \in \mathbb{B}$ let $g(x_0)$ be the $n \times n$-matrix such that*

$$(-x_0) \oplus (x_0 + \Delta x) = g(x_0)\Delta x + o(\|\Delta x\|)\,. \tag{50}$$

Then

$$g(x_0)^\top g(x_0) = G(x_0) = m_0(\|x_0\|^2)\left[I - \frac{x_0 x_0^\top}{\|x_0\|^2}\right] + m_1(\|x_0\|^2)\frac{x_0 x_0^\top}{\|x_0\|^2}\,. \tag{51}$$

Proof. The matrix

$$g(x_0) = \frac{\partial[(-x_0) \oplus y]}{\partial y}|_{y=x_0} \tag{52}$$

exists since the functions m_0 and m_1 are smooth. It is sufficient to prove that

$$g(x_0)\Delta x = \sqrt{m_0(\|x_0\|^2)}\Delta x \tag{53}$$

when $x_0^\top \Delta x = 0$, and

$$g(x_0)\Delta x = \sqrt{m_1(\|x_0\|^2)}\Delta x \tag{54}$$

when $x_0 \| \Delta x$.

Let's assume that the vectors x_0 and Δx are parallel, that is, for some number δ we have $\Delta x = \delta x_0$. We use the procedure described in Section 4.1 with $a = -x_0$ and $b = (1+\delta)x_0$. Then, following (40),

$$X(1) = \int_0^{(1+\delta)\|x_0\|} \sqrt{\frac{m_1(s^2)}{m_1(\|x_0\|^2)}}ds\frac{x_0}{\|x_0\|}\,. \tag{55}$$

The vectors $x(t)$ and $\dot{x}(t)$ are parallel. Hence (25) takes the form

$$\ddot{x} + \frac{m_1'}{m_1}\|\dot{x}\|^2 x = 0 \tag{56}$$

with initial conditions $x(0) = -x_0$, $\dot{x}(0) = X(1)$. The unique solution of this initial value problem has the form $x(t) = q(t)x_0$, where the scalar function q satisfies the second order differential equation

$$\ddot{q} + \dot{q}\frac{1}{2}\frac{d}{dt}\left[\ln\; m_1(\|x_0\|^2 q^2)\right] = 0 \tag{57}$$

and the initial conditions

$$\begin{aligned} q(0) &= -1 \\ \dot{q}(0) &= \int_0^{1+\delta} \sqrt{\frac{m_1(\|x_0\|^2 s^2)}{m_1(\|x_0\|^2)}}ds\,. \end{aligned} \tag{58}$$

Integrating Equation (57) yields

$$\dot{q}(t) = \dot{q}(0)\sqrt{\frac{m_1(\|x_0\|^2)}{m_1(\|x_0\|^2 q(t)^2)}}. \tag{59}$$

Equation (59) is separable. Integrating it over the interval $t \in [0,1]$ yields

$$\int_{-1}^{q(1)} \sqrt{\frac{m_1(\|x_0\|^2 q^2)}{m_1(\|x_0\|^2)}}dq = \dot{q}(0) = \int_0^{1+\delta} \sqrt{\frac{m_1(\|x_0\|^2 s^2)}{m_1(\|x_0\|^2)}}ds. \tag{60}$$

Therefore

$$\int_{-q(1)}^{1} \sqrt{m_1(\|x_0\|^2q^2)}dq = \int_0^{1+\delta} \sqrt{m_1(\|x_0\|^2s^2)}ds \tag{61}$$

and

$$\int_{-q(1)}^{0} \sqrt{m_1(\|x_0\|^2q^2)}dq = \int_1^{1+\delta} \sqrt{m_1(\|x_0\|^2s^2)}ds. \tag{62}$$

Since $m_1(0) = 1$, we get

$$q(1) = \delta\sqrt{m_1(\|x_0\|^2)} + o(\delta)\,. \tag{63}$$

Thus, for $\Delta x = \delta x_0$, and $x(1) = (-x_0) \oplus (x_0 + \Delta x) = q(1)x_0$ we have

$$x(1) = \sqrt{m_1(\|x_0\|^2)}\Delta x + o(\|\Delta x\|)\,, \tag{64}$$

so that (53) is proved.

We now assume that $x_0^\top \Delta x = 0$. Then $\|x_0 + \Delta x\| - \|x_0\| = o(\|\Delta x\|)$, and

$$\begin{aligned} X(1) &= \frac{\Delta x}{\|x_0\|}\frac{1}{\sqrt{m_0(\|x_0\|^2)}}\int_0^{\|x_0\|}\sqrt{m_1(s^2)}ds \\ &+ \frac{x_0}{\|x_0\|}\int_0^{\|x_0\|}\sqrt{\frac{m_1(s^2)}{m_1(\|x_0\|^2)}}ds + o(\|\Delta x\|). \end{aligned} \tag{65}$$

Let us consider the solution x of (25) with initial conditions $x(0) = -x_0$ and $\dot{x}(0) = X(1)$. For every $t \geq 0$ the vector $x(t)$ belongs to a two-dimensional plane that contains x_0 and $X(1)$. We introduce an orthonormal basis with the first unit vector $e_1 = \frac{x_0}{\|x_0\|}$, and the second unit vector $e_2 = \frac{\Delta x}{\|\Delta x\|}$, and use the notation $x_\|(t) = e_1^\top x(t)$ and $x_\perp(t) = e_2^\top x(t)$.

Then $x(t) = x_\|(t)e_1 + x_\perp(t)e_2$ for all $t \geq 0$. Owing to (25), the functions $x_\|$ and $x_\perp$ satisfy the same second order differential equation

$$\ddot{y} + h_1(x,\dot{x})\dot{y} + h_2(x,\dot{x})y = 0 \tag{66}$$

with initial conditions

$$\begin{aligned} x_\|(0) &= -\|x_0\|, \\ \dot{x}_\|(0) &= \int_0^{\|x_0\|}\sqrt{\frac{m_1(s^2)}{m_1(\|x_0\|^2)}}ds + o(\|\Delta x\|), \end{aligned} \tag{67}$$

$$\begin{aligned} x_\perp(0) &= 0, \\ \dot{x}_\perp(0) &= \frac{\|\Delta x\|}{\|x_0\|}\frac{1}{\sqrt{m_0(\|x_0\|^2)}}\int_0^{\|x_0\|}\sqrt{m_1(s^2)}ds + o(\|\Delta x\|), \end{aligned} \tag{68}$$

where

$$\begin{aligned} h_1(x,\dot{x}) &= \frac{m_0'(z)}{m_0(z)}|_{z=\|x\|^2}2(x^\top\dot{x}), \\ h_2(x,\dot{x}) &= [(\frac{m_1'(z)}{m_1(z)} - \frac{2m_0'(z)}{m_0(z)})\frac{(x^\top\dot{x})^2}{\|x\|^2} \\ &+ \frac{m_1(z) - m_0(z) - z\,m_0'(z)}{z\,m_1(z)}(\|\dot{x}\|^2 - \frac{(x^\top\dot{x})^2}{\|x\|^2})]|_{z=\|x\|^2}. \end{aligned} \tag{69}$$

Since the functions $h_1(x(\cdot), \dot{x}(\cdot))$ and $h_2(x(\cdot), \dot{x}(\cdot))$ are bounded on $[0,1]$ and $x_\perp(0) = 0$, $\dot{x}_\perp(0) = O(\|\Delta x\|)$, we have

$$x_\perp(t) = O(\|\Delta x\|), \quad \dot{x}_\perp(t) = O(\|\Delta x\|) \tag{70}$$

uniformly for $t \in [0,1]$. Denote by x^0 a solution of Equation (25) with initial data

$$x^0(0) = -x_0, \quad \dot{x}^0(0) = \frac{x_0}{\|x_0\|} \int_0^{\|x_0\|} \sqrt{\frac{m_1(s^2)}{m_1(\|x_0\|^2)}} ds. \tag{71}$$

Then $x^0_\perp(t) = 0$, $x^0_\|(t) = -\|x(t)\|$ for all $t \in [0,1]$, $x^0_\|(\cdot)$ is increasing, and

$$x^0(1) = 0, \quad \dot{x}^0(1) = \frac{x_0}{\|x_0\|} \int_0^{\|x_0\|} \sqrt{m_1(s^2)} ds. \tag{72}$$

Denote by $\hat{t}$ the value $t \in [0,1]$ such that $\|x^0(t)\| = \|\Delta x\|^{2/3}$. Then $1 - \hat{t} = O(\|\Delta x\|^{2/3})$. On the interval $[0, \hat{t}]$ we have

$$|\frac{(x^\top(t)\dot{x}(t))^2}{\|x(t)\|^2} - \dot{x}^2_\|(t)| = |\frac{(x_\|(t)\dot{x}_\|(t) + x_\perp(t)\dot{x}_\perp(t))^2}{x^2_\|(t) + x^2_\perp(t)} - \dot{x}^2_\|(t)| = O(\|\Delta x\|^{4/3}). \tag{73}$$

Since $\|x\|^2 = x^2_\| + x^2_\perp$, $x^\top \dot{x} = x_\| \dot{x}_\| + x_\perp \dot{x}_\perp$, and $\|\dot{x}\|^2 = \dot{x}^2_\| + \dot{x}^2_\perp$, Equations (69) and (70) imply

$$|h_j(x(t), \dot{x}(t)) - h_j(x_\|(t)e_1, \dot{x}_\|(t)e_1)| = O(\|\Delta x\|^{4/3}), \qquad j = 1, 2. \tag{74}$$

On the other hand for every differentiable scalar function w we have

$$\begin{aligned} h_1(w(t)e_1, \dot{w}(t)e_1) &= \frac{m_0'(z)}{m_0(z)}|_{z=w(t)^2} 2(w(t)\dot{w}(t)), \\ h_2(w(t)e_1, \dot{w}(t)e_1) &= [(\frac{m_1'(z)}{m_1(z)} - \frac{2m_0'(z)}{m_0(z)})|_{z=w(t)^2} \dot{w}(t)^2. \end{aligned} \tag{75}$$

Therefore, there exist constants C_1 and C_2 that are independent of bounded Δx, and such that

$$\begin{aligned} |h_j(x_\|(t)e_1, \dot{x}_\|(y)e_1) - h_j(x^0(t), \dot{x}^0(t))| \le \\ \le C_1\|x_\|(t)e_1 - x^0(t)\| + C_2\|\dot{x}_\|(t)e_1 - \dot{x}^0(t)\|, \quad j = 1, 2. \end{aligned} \tag{76}$$

Since x and x^0 are solutions of Equation (66), there exist constants C_3, C_4 and C_5 independent on bounded Δx and such that

$$\|\ddot{x}_\| e_1 - \ddot{x}^0\| \le C_3\|\dot{x}_\| e_1 - \dot{x}^0\| + C_4\|x_\| e_1 - x^0\| + C_5\|\Delta x\|^{4/3}. \tag{77}$$

Taking into account that $x_\|(0)e_1 = x^0 = -x_0$, $\|\dot{x}_\|(0)e_1 - \dot{x}^0(0)\| = o(\|\Delta x\|)$, we get

$$\|x_\|(\hat{t})e_1 - x^0(\hat{t})\| = o(\|\Delta x\|), \quad \|\dot{x}_\|(\hat{t})e_1 - \dot{x}^0(\hat{t})\| = o(\|\Delta x\|). \tag{78}$$

Since $1 - \hat{t} = O(\|\Delta x\|^{2/3})$, for $t \in [\hat{t}, 1]$ we have

$$\|\dot{x}(t) - \dot{x}^0(t)\| = o(\|\Delta x\|) + O(t - \hat{t}) = O(\|\Delta x\|^{2/3}), \tag{79}$$

$$\|x(t) - x^0(t)\| = o(\|\Delta x\|) + O(t - \hat{t})^2 = o(\|\Delta x\|). \tag{80}$$

In particular,

$$x_{\|}(1) = o(\|\Delta x\|), \quad \dot{x}_{\|}(1) = \int_0^{\|x_0\|} \sqrt{m_1(s^2)}ds + O(\|\Delta x\|^{2/3}). \tag{81}$$

In order to prove (54) we consider the Wronskian

$$W(t) = det \begin{pmatrix} x_{\|}(t) & x_{\perp}(t) \\ \dot{x}_{\|}(t) & \dot{x}_{\perp}(t) \end{pmatrix}. \tag{82}$$

It is well known that

$$W(1) = W(0)e^{-\int_0^1 \frac{m_0'}{m_0} 2(x^\top \dot{x})dt}. \tag{83}$$

First, we calculate the exponent. Since $m_1(0) = 1$ and $\|x(1)\| = O(\|\Delta x\|)$, we have

$$\begin{gathered} \int_0^1 \frac{m_0'}{m_0} 2(x^\top \dot{x})dt = \int_0^1 \frac{d}{dt}(\ln(m_0(\|x(t)\|^2))dt \\ = \ln\left[\frac{m_0(\|x(1)\|^2)}{m_0(\|x(0)\|^2)}\right] = \ln\left[\frac{1}{m_0(\|x_0\|^2)}\right] + O(\|\Delta x\|). \end{gathered} \tag{84}$$

Hence,

$$e^{-\int_0^1 \frac{m_0'}{m_0} 2(x^\top \dot{x})dt} = m_0(\|x_0\|^2) + O(\|\Delta x\|). \tag{85}$$

Then, we calculate the values of the Wronskian:

$$W(0) = x_{\|}(0)\dot{x}_{\perp}(0) - \dot{x}_{\|}(0)x_{\perp}(0) = -\|\Delta x\| \frac{1}{\sqrt{m_0(\|x_0\|^2)}} \int_0^{\|x_0\|} \sqrt{m_1(s^2)}ds, \tag{86}$$

$$W(1) = x_{\|}(1)\dot{x}_{\perp}(1) - \dot{x}_{\|}(1)x_{\perp}(1) = -x_{\perp}(1) \int_0^{\|x_0\|} \sqrt{m_1(s^2)}ds + o(\|\Delta x\|). \tag{87}$$

Thus

$$x_{\perp}(1) = \sqrt{m_0(\|x_0\|^2)}\|\Delta x\| + o(\|\Delta x\|). \tag{88}$$

Taking into account that $x_{\|}(1) = o(\|\Delta x\|)$ and $\Delta x || e_2$, we get

$$(-x_0) \oplus (x_0 + \Delta x) = x(1) = x_{\|}(1)e_1 + x_{\perp}(1)e_2 = \sqrt{m_0(\|x_0\|^2)}\Delta x + o(\|\Delta x\|), \tag{89}$$

and

$$g(x_0)\Delta x = \sqrt{m_0(\|x_0\|^2)}\Delta x. \tag{90}$$

This completes the proof of the theorem. □

Theorem 1 shows that there is a one-to-one correspondence between metric tensors G in the form (6), and binary operations $\oplus$ that G generates, as defined in Section 4.1.

4.4. Multiplication of Vectors by Numbers

In this subsection we define a function $\otimes$: $\mathbb{R} \times \mathbb{B} \to \mathbb{B}$ (multiplication of vectors by numbers) that is compatible with the binary operation $\oplus$ in the sense that

$$\begin{gathered} (t_1 \otimes a) \oplus (t_2 \otimes a) = (t_1 + t_2) \otimes a \\ (t_1 \otimes (t_2 \otimes a)) = (t_1 t_2) \otimes a \end{gathered} \tag{91}$$

for all $t_1, t_2 \in \mathbb{R}$, $a \in \mathbb{B}$. Notice that $t_1 + t_2$ and $t_1 t_2$ in (91) are the common addition and multiplication of t_1 and t_2 in $\mathbb{R}$. The first identity in (91) is called the *scalar distributive law*, and the second identity in (91) is called the *scalar associative law*.

Let $a, b \in \mathbb{B}$ be two nonzero parallel vectors (that is, belong to the same line passing through the origin). We calculate $a \oplus b$ using the four steps described in Section 4.1.

Step 1. We define

$$\dot{r}(0) = \frac{1}{\|b\|} \int_0^{\|b\|} \sqrt{m_1(s^2)} ds \,. \tag{92}$$

Step 2. We compute $X(1)$ using (40) with $X_0 = \dot{r}(0)b$, noting that

$$(I - \frac{aa^\top}{\|a\|^2})b = 0 \qquad \text{and} \qquad a^\top b = \text{sign}(a^\top b)\|a\|\|b\| \,. \tag{93}$$

Then

$$X(1) = \text{sign}(a^\top b) \frac{a}{\|a\|} \int_0^{\|b\|} \sqrt{\frac{m_1(s^2)}{m_1(\|a\|^2)}} ds \,. \tag{94}$$

Step 3. We integrate Equation (25) with initial data $x(0) = a$ and $\dot{x}(0) = X(1)$. We parametrize the solution x as follows: $x(t) = p(t)a$ with scalar function p. Then $p(0) = 1$, $\dot{p}(0)a = X(1)$, $x(1) = p(1)a = a \oplus b$. Equation (34) implies

$$\dot{p}(0) = \int_{p(0)}^{p(1)} \sqrt{\frac{m_1(\|a\|^2 s^2)}{m_1(\|a\|^2 p(0)^2)}} ds \,. \tag{95}$$

Therefore

$$\frac{\text{sign}(a^\top b)}{\|a\|} \int_0^{\|b\|} \sqrt{\frac{m_1(s^2)}{m_1(\|a\|^2)}} ds = \int_{p(0)}^{p(1)} \sqrt{\frac{m_1(\|a\|^2 s^2)}{m_1(\|a\|^2 p(0)^2)}} ds \,. \tag{96}$$

Following obvious simplifications we get

$$\int_0^{p(1)\|a\|} \sqrt{m_1(s^2)} ds = \int_0^{\|a\|} \sqrt{m_1(s^2)} ds + \text{sign}(a^\top b) \int_0^{\|b\|} \sqrt{m_1(s^2)} ds \,. \tag{97}$$

Step 4. We have $a \oplus b = p(1)a$.

Therefore $p(1)\|a\| = \text{sign}((a \oplus b)^\top a)\|a \oplus b\|$. We define a function h as follows. For every number p we set

$$h(p) = \int_0^p \sqrt{m_1(s^2)} ds \,. \tag{98}$$

Then, $h(p) = \text{sign}(p)h(|p|)$ and

$$\text{sign}((a \oplus b)^\top a) h(\|a \oplus b\|) = h(\|a\|) + \text{sign}(a^\top b) h(\|b\|) \,. \tag{99}$$

The function h is monotonically increasing. Therefore, h is invertible, the inverse of which is denoted by h^{-1}.

Assumption 2. *We assume that*

$$\int_0^1 \sqrt{m_1(s^2)} ds = \infty. \tag{100}$$

Under this assumption the function h is a bijection $(-1, 1) \to (-\infty, \infty)$. In particular, $h(0) = 0$, and for every $t \in \mathbb{R}$ and $p \in (-1, 1)$ there exists $h^{-1}(t\, h(p))$.

Now for every number t we define

$$t \otimes a = \frac{a}{\|a\|} h^{-1}(t\, h(\|a\|))\,. \tag{101}$$

Then $h(\|t \otimes a\|) = |t| h(\|a\|)$ and $(-t) \otimes a = -(t \otimes a)$. We assume $t_1 > 0$, $t_2 > 0$. Then

$$\begin{aligned} \text{sign}(a^\top (t_1 \otimes a)) &= \text{sign}(a^\top (t_2 \otimes a)) \\ &= \text{sign}(a^\top ((t_1 + t_2) \otimes a) \\ &= 1. \end{aligned} \tag{102}$$

Now we have

$$\begin{aligned} (t_1 \otimes a) \oplus (t_2 \otimes a) &= \frac{a}{\|a\|} h^{-1}(h(\|(t_1 \otimes a) \oplus (t_2 \otimes a)\|)) \\ &= \frac{a}{\|a\|} h^{-1}(h(\|t_1 \otimes a\|) + h(\|t_2 \otimes a\|)) \\ &= \frac{a}{\|a\|} h^{-1}(t_1 h(\|a\|) + t_2 h(\|a\|)) \\ &= \frac{a}{\|a\|} h^{-1}((t_1 + t_2) h(\|a\|)) \\ &= (t_1 + t_2) \otimes a \end{aligned} \tag{103}$$

and

$$\begin{aligned} (t_1 \otimes (t_2 \otimes a)) &= \frac{a}{\|a\|} h^{-1}(t_1 h(\|t_2 \otimes a\|)) \\ &= \frac{a}{\|a\|} h^{-1}(t_1 t_2 h(\|a\|)) \\ &= (t_1 t_2) \otimes a\,. \end{aligned} \tag{104}$$

Owing to the property $(-t) \otimes a = -(t \otimes a)$, the cases with t_1 and t_2 of arbitrary signs are considered similarly. Accordingly, the operation of multiplication by a number is well defined.

From the definition of scalar product (101) it is easy to see that

$$1 \otimes a = a \tag{105}$$

and for all $a \in \mathbb{B} \backslash \{0\}$, $r \neq 0$

$$\frac{|r| \otimes a}{\|r \otimes a\|} = \frac{a}{\|a\|}. \tag{106}$$

4.5. Distances and Norms

We introduce the standard definition of distance between points a and b of the unit ball $\mathbb{B}$.

Definition 4. *The distance between points $a, b \in \mathbb{B}$ is the minimal length of a curve connecting a and b,*

$$dist(a, b) = \min\{\int_0^1 \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} g_{ij}(x(t)) \dot{x}_i(t) \dot{x}_j(t)\, dt}\}\,, \tag{107}$$

where the minimum is taken over all smooth functions x: $[0, 1] \to \mathbb{B}$ with boundary conditions $x(0) = a$ and $x(1) = b$.

Obviously, $dist(a,b) \geq 0$ for all $a,b \in \mathbb{B}$, and $dist(a,b) = 0$ iff $a = b$ since $G(x) > 0$ for all $x \in \mathbb{B}$. Besides, we have the triangle inequality: for all $a,b,c \in \mathbb{B}$

$$dist(a,b) + dist(b,c) \geq dist(a,c). \tag{108}$$

The value $dist(0,a)$ is called the norm of a, denoted by $\|a\|_{\oplus}$.

Lemma 2. *For every $a,b \in \mathbb{B}$ there exists $z \in \mathbb{B}$ such that $a \oplus z = b$.*

Proof. Fix arbitrary points $a,b \in \mathbb{B}$. Since $\mathbb{B}$ is convex, there exist curves in $\mathbb{B}$ connecting a and b. The minimum of the lengths of such curves does exist since $G(x) \to \infty$ as $\|x\| \to 1$. Let this minimum be attained at a curve q: $[0,1] \to \mathbb{B}$. Then q is a geodesic which connects a and b: $q(0) = a$, $q(1) = b$. Consider the vector $\dot{q}(0)$. Make a parallel transport of this vector along the interval connecting a and the origin. Denote the vector at the origin by y. Consider a geodesic w: $[0,1] \to \mathbb{B}$ with initial conditions $w(0) = 0$, $\dot{w}(0) = y$. Denote $z = w(1)$. Then according to the steps 1,2,3 described in the Section 4.1 we have $b = a \oplus z$. The lemma is proved. □

Consider a geodesic x: $[0,1] \to \mathbb{B}$ with boundary conditions $x(0) = a$ and $x(1) = b$. It is known that $dist(a,b)$ is equal to the length of this geodesic, and that along the geodesic the integrand on the right-hand side of (107) is constant [56]. Therefore

$$dist(a,b) = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(x(0))\dot{x}_i(0)\dot{x}_j(0)}\,. \tag{109}$$

Let us consider a geodesic y: $[0,1] \to \mathbb{B}$ with boundary points $y(0) = 0$ and $y(1) = z$. The value of $\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(y(t))\dot{y}_i(t)\dot{y}_j(t)}$ is constant over $t \in [0,1]$. Hence,

$$\|z\|_{\oplus} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(y(0))\dot{y}_i(0)\dot{y}_j(0)}\,. \tag{110}$$

But the vector $\dot{x}(0)$ is a parallel transport of the vector $\dot{y}(0)$ along the curve $\{as \ : \ s \in [0,1]\}$. Therefore,

$$\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(x(0))\dot{x}_i(0)\dot{x}_j(0)} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(y(0))\dot{y}_i(0)\dot{y}_j(0)}\,. \tag{111}$$

Finally,

$$dist(a,b) = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(y(0))\dot{y}_i(0)\dot{y}_j(0)} = \|z\|_{\oplus}\,. \tag{112}$$

In Section 3.4 we found that $y(t) = r(t)z$, and

$$\dot{r}(0) = \frac{1}{\|z\|}\int_0^{\|z\|}\sqrt{m_1(s^2)}ds\,. \tag{113}$$

In view of the equation

$$\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} g_{ij}(y(0))\dot{y}_i(0)\dot{y}_j(0)} = \dot{r}(0)\|z\| \tag{114}$$

we have

$$\|z\|_{\oplus} = dist(a,b) = \int_0^{\|z\|} \sqrt{m_1(s^2)}ds = h(\|z\|)\,. \tag{115}$$

Interestingly, the norm $\|\cdot\|_{\oplus}$ does not depend on the function m_0. In particular, if we have two spaces with metric tensors that have the same function m_1, then the distances between points (and hence the norms) in these two spaces coincide, and the operations of multiplication by a number also coincide. We'll see this result in several examples, including the spaces with Einstein and Möbius additions.

Now let us consider a binary operation $\oplus$ applied to numbers. That is, given functions m_0, m_1, we consider the tensor G given by (6), and its resulting binary operation $\oplus$ introduced in Section 4.1. Also, let's consider a tensor with the same functions m_0 and m_1 in the one dimensional space. The corresponding operation between numbers is denoted by the same symbol, $\oplus$.

For arbitrary numbers p, q the value of $p \oplus q$ is defined by the four-steps procedure presented in Section 4.1. If $q = 0$, then $p \oplus q = p$. Assume $q \neq 0$.

Step 1. We calculate

$$\dot{r}(0) = \frac{1}{|q|}\int_0^{|q|} \sqrt{m_1(s^2)}ds. \tag{116}$$

Step 2. Since $1 - \frac{qq^\top}{|q|^2} = 0$, we have

$$X_0 = \int_0^q \sqrt{m_1(s^2)}ds, \qquad X_1 = \frac{1}{\sqrt{m_1(q^2)}}\int_0^q \sqrt{m_1(s^2)}ds. \tag{117}$$

Step 3. If x is a scalar then Equation (25) for geodesics takes the form

$$\ddot{x} + \frac{m_1'}{m_1}(\dot{x})^2 x = 0. \tag{118}$$

A solution x of (118) with the initial values $x(0) = p$, $\dot{x}(0) = X_1$ satisfies the condition

$$\int_p^{x(t)} \sqrt{m_1(s^2)}ds = t\int_0^q \sqrt{m_1(s^2)}ds. \tag{119}$$

for all $t > 0$.

Step 4. Since $p \oplus q = x(1)$, we have

$$\int_0^{p\oplus q} \sqrt{m_1(s^2)}ds = \int_0^p \sqrt{m_1(s^2)}ds + \int_0^q \sqrt{m_1(s^2)}ds. \tag{120}$$

Thus, taking into account the definition of the function h, we get

$$h(p \oplus q) = h(p) + h(q). \tag{121}$$

Similarly, the operation $\otimes$ for numbers is defined by (101). In particular, for arbitrary numbers r and p we have

$$r \otimes p = h^{-1}(r\,h(p)). \tag{122}$$

Thus, for every number r and vector $a \in \mathbb{B}$

$$\|r \otimes a\| = |h^{-1}(r\,h(\|a\|))| = |r| \otimes \|a\|. \tag{123}$$

Let $a, b \in \mathbb{B}\backslash\{0\}$. The triangle inequality implies

$$dist(a,0) + dist(0,-b) \geq dist(a,-b) \tag{124}$$

and equality is attained only if the vectors a and b belong to the same ray, that is, if there exists a positive number λ such that $a = \lambda b$.

In terms of the function h, inequality (124) yields

$$h(\|a\|) + h(\|b\|) \geq h(\|a \oplus b\|)\,. \tag{125}$$

Employing (121), we get

$$\|a\| \oplus \|b\| = h^{-1}[h(\|a\|) + h(\|b\|)] \geq h^{-1}[h(\|a \oplus b\|)] = \|a \oplus b\|\,. \tag{126}$$

We have thus proved the following theorem.

Theorem 2. *For all vectors $a, b \in \mathbb{B}$ we have the gyrotriangle inequality*

$$\|a \oplus b\| \leq \|a\| \oplus \|b\|, \tag{127}$$

and equality is attained iff the vectors a, b lie on the same ray, that is, for some nonnegative number λ we have $a = \lambda b$ or $b = \lambda a$.

Inequality (127) may be considered as a triangle inequality in the spaces of vectors and numbers with the same addition operation $\oplus$.

It is shown below that for Einstein addition and all additions isomorphic to Einstein addition (for example, Möbius addition) a solution of the equation $a \oplus x = b$ is given by $x = (-a) \oplus b$. Hence, we have the following result.

Theorem 3. *If a solution x of the equation $a \oplus x = b$ is given by $x = (-a) \oplus b$, then*

$$dist(a, b) = h(\|(-a) \oplus b\|). \tag{128}$$

5. Spaces with Einstein Addition

In this section, we consider a special binary operation in the Beltrami-Klein ball model of hyperbolic geometry that turns out to be Einstein addition [1,5].

We consider the metric tensor G in (6)–(8) parametrized by the functions

$$m_0(r^2) = \frac{1}{1-r^2} \qquad \text{and} \qquad m_1(r^2) = \frac{1}{(1-r^2)^2}\,, \tag{129}$$

so that

$$G(x) = \frac{1}{1-\|x\|^2}\left[I - \frac{xx^\top}{\|x\|^2}\right] + \frac{1}{(1-\|x\|^2)^2}\frac{xx^\top}{\|x\|^2}\,. \tag{130}$$

5.1. Einstein Addition

We follow the four steps described in Section 4:

Step 1. We evaluate

$$\dot{r}(0) = \frac{1}{\|b\|}\int_0^{\|b\|}\frac{ds}{1-s^2} = \frac{\operatorname{atanh}(\|b\|)}{\|b\|}\,. \tag{131}$$

Step 2. We find a solution of Equation (40) with $X_0 = \frac{\operatorname{atanh}(\|b\|)}{\|b\|}b$,

$$X(1) = \frac{\operatorname{atanh}(\|b\|)}{\|b\|}[\sqrt{1-\|a\|^2}(I - \frac{aa^\top}{\|a\|^2}) + (1-\|a\|^2)\frac{aa^\top}{\|a\|^2}]b\,. \tag{132}$$

Step 3. We solve Equation (25) with initial conditions $x(0) = a$, $\dot{x}(0) = X(1)$. Noticing that

$$\frac{m_1'(r^2)}{m_1(r^2)} - \frac{2m_0'(r^2)}{m_0(r^2)} = 0$$
$$m_1(r^2) - m_0(r^2) - r^2 m_0'(r^2) = 0\,, \tag{133}$$

Equation (25) takes the form

$$\ddot{x}(t) + \frac{d(\ln(m_0(\|x(t)\|^2))}{dt}\dot{x}(t) = 0\,. \tag{134}$$

The solution to the initial value problem is given by

$$x(t) = a + \frac{\tanh(t\ \text{atanh}(\|b\|))}{\text{atanh}(\|b\|)\left[1 + \frac{a^\top b}{\|b\|}\tanh(t\ \text{atanh}(\|b\|))\right]}X(1) \tag{135}$$

Step 4. Finally, we obtain the addition

$$\begin{aligned} a \oplus_E b &= a + \frac{\sqrt{1-\|a\|^2}}{1+a^\top b}[b - \frac{a^\top b}{\sqrt{1-\|a\|^2}+1}a] \\ &= \frac{1}{1+a^\top b}\{[1 + \frac{a^\top b}{\|a\|^2}(1-\sqrt{1-\|a\|^2})]a + \sqrt{1-\|a\|^2}b\}\,, \end{aligned} \tag{136}$$

for all $a, b \in \mathbb{B}$, which is recognized as Einstein addition.

The binary operation described in Section 4 enables Einstein addition (136) to be recovered from the metric tensor (130). Thus, the operation $a \oplus_E b$ is determined by its local properties in a neighborhood of the set $\{(a,b)\ :\ a, b \in \mathbb{B},\ a = -b\}$.

The binary operation $\oplus_E$ in (136) turns out to be the well known Einstein addition in the ball, studied in special relativity theory, where the speed of light in empty space is normalized to $c = 1$ [1,4,5].

5.2. Einstein Multiplication by a Number

To define an operation of multiplication by a number we evaluate the function h in (98) with the function m_1 in (129), obtaining,

$$h(p) = \int_0^p \frac{1}{1-s^2}ds = \text{atanh}(p)\,. \tag{137}$$

Therefore, by (101),

$$t \otimes_E a = \text{atanh}(t\ \tanh(\|a\|))\frac{a}{\|a\|}\,. \tag{138}$$

5.3. Derivation of the Metric Tensor Associated with Einstein Addition

We have derived Einstein addition from the metric tensor (130). According to Theorem 1 we can make also the reverse procedure, that is, we can recover the metric tensor (130) from Einstein addition.

In this subsection we demonstrate it directly. Let us consider the Einstein difference in the following chain of equations.

$$
\begin{aligned}
&(-x) \oplus_E (x+\Delta x) \\
&= -x + \frac{1}{1-x^\top(x+\Delta x)}[\sqrt{1-\|x\|^2}(x+\Delta x - \frac{x^\top(x+\Delta x)}{\|x\|^2}x) \\
&+ \frac{x^\top(x+\Delta x)}{\|x\|^2}(1-\|x\|^2)x] \\
&= \frac{1}{1-x^\top(x+\Delta x)}[\sqrt{1-\|x\|^2}\Delta x + (1-\sqrt{1-\|x\|^2})\frac{x^\top \Delta x}{\|x\|^2}x] \\
&= g(x)\Delta x + o(\Delta x)\,,
\end{aligned}
\tag{139}
$$

where the $n \times n$-matrix $g(x)$ is given by

$$
g(x) = \frac{1}{1-\|x\|^2}[\sqrt{1-\|x\|^2}(I - \frac{xx^\top}{\|x\|^2}) + \frac{xx^\top}{\|x\|^2}]\,. \tag{140}
$$

Then, the metric tensor $G(x)$ is given by the equation

$$
\begin{aligned}
G(x) &= g(x)^\top g(x) \\
&= \frac{1}{1-\|x\|^2}(I - \frac{xx^\top}{\|x\|^2}) + \frac{1}{(1-\|x\|^2)^2}\frac{xx^\top}{\|x\|^2}\,,
\end{aligned}
\tag{141}
$$

thus recovering (130).

6. Spaces with Möbius Addition

Contrasting (129), let's assume that

$$
m_0(r^2) = m_1(r^2) = \frac{1}{(1-r^2)^2}\,, \tag{142}
$$

so that the metric tensor $G(x)$ in (6) and (9) takes the form

$$
G(x) = \frac{1}{(1-\|x\|^2)^2}I\,. \tag{143}
$$

6.1. Möbius Addition

Again, we follow the four steps described in Section 4.

Step 1. Similar to the case of Einstein addition, we evaluate

$$
\dot{r}(0) = \frac{1}{\|b\|}\int_0^{\|b\|}\frac{1}{1-s^2}ds = \frac{\operatorname{atanh}(\|b\|)}{\|b\|}\,. \tag{144}
$$

Step 2. With

$$
X_0 = \frac{\operatorname{atanh}(\|b\|)}{\|b\|}\,, \tag{145}
$$

we find a solution $X(1)$ to Equation (40),

$$
X(1) = (1-\|a\|^2)\frac{\operatorname{atanh}(\|b\|)}{\|b\|}b\,. \tag{146}
$$

Step 3. We solve the initial value problem

$$\begin{aligned} \ddot{x} + \frac{2}{1-\|x\|^2}[2(x^\top \dot{x})\dot{x} - \|\dot{x}\|^2 x] = 0 \\ x(0) = a \\ \dot{x}(0) = X(1)\,, \end{aligned} \tag{147}$$

seeking a solution of the form

$$x(t) = P + R_1 \cos\varphi(t) + R_2 \sin\varphi(t)\,, \tag{148}$$

with constant vectors P, R_1, R_2.

The initial value problem (147) has a solution (148) with the following parameters and a function φ, as may be checked directly,

$$\begin{aligned} R_1 &= -\frac{(1-\|a\|^2)\|b\|^2}{2(\|a\|^2\|b\|^2 - (a^\top b)^2)}a + \frac{(1-\|a\|^2)a^\top b}{2(\|a\|^2\|b\|^2 - (a^\top b)^2)}b \\ R_2 &= \frac{1-\|a\|^2}{2\sqrt{\|a\|^2\|b\|^2 - (a^\top b)^2}}b \\ P &= a - R_1 \\ \varphi(t) &= 2\arctan\left[\frac{\sqrt{\|a\|^2\|b\|^2 - (a^\top b)^2}\tanh(\operatorname{atanh}(\|b\|)t)}{\|b\| + a^\top b \tanh(\operatorname{atanh}(\|b\|)t)}\right]. \end{aligned} \tag{149}$$

Therefore,

$$\begin{aligned} x(1) &= P + R_1 \cos\varphi(1) + R_2 \sin\varphi(1) \\ &= \frac{(1+2a^\top b + \|b\|^2)a + (1-\|a\|^2)b}{1+2a^\top b + \|a\|^2\|b\|^2}\,. \end{aligned} \tag{150}$$

We notice that

$$\|R_1\| = \|R_2\| = \|P - x(t)\|\,, \qquad \forall t \geq 0, \tag{151}$$

and

$$\|P\|^2 = \|R_1\|^2 + 1\,. \tag{152}$$

Therefore, each geodesic is a circular arc centered at P with radius $\|R_1\|$, which intersects the unit circle orthogonally, and it tends to one of these two intersections as $t \to \infty$ and as $t \to -\infty$.

Step 4. Finally, we get

$$a \oplus_M b = \frac{(1+2a^\top b + \|b\|^2)a + (1-\|a\|^2)b}{1+2a^\top b + \|a\|^2\|b\|^2}\,, \tag{153}$$

for all $a, b \in \mathbb{B}$, which is the well known formula for Möbius addition in the ball [1,4].

Möbius addition is recovered here from the metric tensor (143). Thus, the operation $a \oplus_M b$ is determined by its local properties in a neighborhood of the set $\{(a,b) \; : \; a,b \in \mathbb{B},\; a = -b\}$.

6.2. Möbius Multiplication by a Number

Since the function m_1 is the same for both Einstein and Möbius metric tensors, the multiplication by a scalar is the same as in (138) in Section 5.2,

$$t \otimes_M a = \operatorname{atanh}(t \ \tanh(\|a\|)) \frac{a}{\|a\|} \,. \tag{154}$$

6.3. Derivation of the Metric Tensor Associated with Möbius Addition

As for Einstein addition, to recover the underlying metric tensor from its associated Möbius addition it is possible to use Theorem 1. In this subsection we calculate directly the metric tensor G that determines the binary operation $\oplus_M$. Let us consider the Möbius difference

$$\begin{aligned} &(-x) \oplus_M (x + \Delta x) \\ &\quad = \frac{1 - 2x^\top (x + \Delta x) + \|x\|^2 \|x + \Delta x\|^2)(-x) + (1 - \|x\|^2)(x + \Delta x)}{1 - 2x^\top (x + \Delta x) + \|x\|^2 \|x + \Delta x\|^2} \\ &\quad = \frac{(1 - \|x\|^2)\Delta x - \|\Delta x\|^2 x}{(1 - \|x\|^2)^2 - 2x^\top (1 - \|x\|^2) + \|x\|^2 \|\Delta x\|^2} \\ &\quad = g_M(x)\Delta x + o(\Delta x)\,, \end{aligned} \tag{155}$$

where

$$g_M(x) = \frac{1}{1 - \|x\|^2} I \,. \tag{156}$$

Then, the metric tensor $G(x)$ is given by the equation

$$G(x) = g_M(x)^\top g_M(x) = \frac{1}{(1 - \|x\|^2)^2} I \,, \tag{157}$$

thus recovering (143).

7. A Space with an Operation Isomorphic to Euclidean Addition

Assume

$$m_0(r^2) = \frac{1}{1 - r^2}, \qquad m_1(r^2) = \frac{1}{(1 - r^2)^3}. \tag{158}$$

Then the metric tensor (6) takes the form

$$G(x) = \frac{1}{1 - \|x\|^2} \left[I - \frac{xx^\top}{\|x\|^2} \right] + \frac{1}{(1 - \|x\|^2)^3} \frac{xx^\top}{\|x\|^2}. \tag{159}$$

7.1. Binary Operation

We follow the four steps described in Section 4.

Step 1. We calculate

$$\dot{r}(0) = \frac{1}{\|b\|} \int_0^{\|b\|} \frac{1}{\sqrt{(1 - s^2)^3}} ds = \frac{1}{\sqrt{1 - \|b\|^2}}. \tag{160}$$

Step 2. We find a solution of Equation (40) with $X_0 = \frac{b}{\sqrt{1 - \|b\|^2}}$,

$$X(1) = \sqrt{\frac{1 - \|a\|^2}{1 - \|b\|^2}} \left(b - (a^\top b) a \right) \,. \tag{161}$$

Step 3. We solve Equation (25) with initial conditions $x(0) = a$, $\dot{x}(0) = X(1)$. Noticing that

$$\frac{m_1'(r^2)}{m_1(r^2)} - \frac{2m_0'(r^2)}{m_0(r^2)} = \frac{1}{1-r^2}, \qquad \frac{m_1(r^2) - m_0(r^2) - r^2 m_0'(r^2)}{r^2 m_1(r^2)} = 1\,, \tag{162}$$

we see that Equation (25) takes the form

$$\ddot{x} + \frac{2x^\top \dot{x}}{1-\|x\|^2}\dot{x} + \left[\|\dot{x}\|^2 + \frac{(x^\top \dot{x})^2}{1-\|x\|^2}\right] x = 0\,. \tag{163}$$

Assume vectors a and b are not parallel. The solution of the initial value problem lies entirely in the plane containing the vectors a and b. Denote by

$$b_\perp = b - \frac{a^\top b}{\|a\|^2} a \tag{164}$$

the component of vector b orthogonal to a. Denote $r(t) = \|x(t)\|$. Then the vector function x may be presented in the polar coordinates as follows:

$$x = r\cos\varphi \frac{a}{\|a\|} + r\sin\varphi \frac{b_\perp}{\|b_\perp\|}, \tag{165}$$

where φ is a scalar function, the angle of x in the plane with unit vectors $\frac{a}{\|a\|}$ and $\frac{b_\perp}{\|b_\perp\|}$. Then $r(0) = \|a\|$, $\varphi(0) = 0$, $\dot{x}(0) = \dot{r}(0)\frac{a}{\|a\|} + r(0)\dot{\varphi}(0)\frac{b_\perp}{\|b_\perp\|}$, and

$$\|b_\perp\| r(0)\dot{\varphi}(0) = b_\perp^\top \dot{x}(0) = b_\perp^\top X(1) = \sqrt{\frac{1-\|a\|^2}{1-\|b\|^2}} b_\perp^\top b > 0. \tag{166}$$

Therefore $\dot{\varphi}(0) > 0$. Moreover,

$$x^\top \dot{x} = r\dot{r}, \qquad \|\dot{x}\|^2 = \dot{r}^2 + r^2\dot{\varphi}^2, \qquad \|\dot{x}\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2} = r^2\dot{\varphi}^2, \tag{167}$$

and Equation (163) is equivalent to the following two scalar equations:

$$\ddot{r} + \frac{3}{1-r^2}\dot{r}^2 r + r(r^2-1)\dot{\varphi}^2 = 0, \tag{168}$$

and

$$r\ddot{\varphi} + 2\dot{\varphi}\frac{\dot{r}}{1-r^2} = 0. \tag{169}$$

Equation (169) can be written as

$$\frac{d}{dt}\left[\frac{r^2\dot{\varphi}}{1-r^2}\right] = 0. \tag{170}$$

Hence, there exists a number C such that

$$\frac{r^2\dot{\varphi}}{1-r^2} = C. \tag{171}$$

Since $\dot{\varphi}(0) > 0$, we have $C > 0$, and $\dot{\varphi}(t) > 0$ for all $t \in [0,1]$.

Substituting (171) into (168) yields

$$\ddot{r} + \frac{3r\dot{r}}{1-r^2} - \frac{(1-r^2)^3C^2}{r^3} = 0. \tag{172}$$

It follows that if $\dot{r}(0) \geq 0$, then $\dot{r}(t) > 0$ for all $t > 0$. If $\dot{r}(0) < 0$, then there exists a number $\hat{t} > 0$ such that $\dot{r}(t) < 0$ for all $t \in [0, \hat{t})$, $\dot{r}(\hat{t}) = 0$, and $\dot{r}(t) > 0$ for all $t > \hat{t}$. Notice that

$$r(0)\dot{r}(0) = x(0)^\top \dot{x}(0) = a^\top X(1) = a^\top b \frac{(1-\|a\|^2)^{3/2}}{(1-\|b\|^2)^{1/2}}. \tag{173}$$

Therefore, $\operatorname{sign} \dot{r}(0) = \operatorname{sign} a^\top b$. Below we consider the case $\dot{r}(0) > 0$. The case $\dot{r}(0) \leq 0$ may be studied in a very similar manner and leads to the same formula for $a \oplus b$.

Let $z = r^2$. Then

$$\dot{z} = 2r\dot{r}, \qquad \ddot{z} = 2r\ddot{r} + 2\dot{r}^2, \tag{174}$$

and Equation (172) acquires the form

$$\frac{\ddot{z}}{2} - \frac{\dot{z}^2}{4z} + \frac{3\dot{z}^2}{4(1-z)} - \frac{(1-z)^3C^2}{z} = 0. \tag{175}$$

Since

$$\frac{d}{dt}\left[\frac{\dot{z}^2}{z(1-z)^3} + \frac{4C^2(1-z)}{z}\right] = \frac{4\dot{z}}{z(1-z)^3}\left[\frac{\ddot{z}}{2} - \frac{\dot{z}^2}{4z} + \frac{3\dot{z}^2}{4(1-z)} - \frac{(1-z)^3C^2}{z}\right] = 0, \tag{176}$$

there exists a constant C_1 such that

$$\frac{\dot{z}^2}{z(1-z)^3} + \frac{4C^2(1-z)}{z} = C_1. \tag{177}$$

Equation (177) is autonomous, and therefore separable,

$$\frac{dz}{\sqrt{(1-z)^3(C_1 z - 4C^2(1-z))}} = dt. \tag{178}$$

Let's calculate C and C_1 in terms of a and b. Let

$$p = \int_0^{\|b\|} \frac{ds}{\sqrt{(1-s^2)^3}} = \frac{\|b\|}{\sqrt{1-\|b\|^2}}, \tag{179}$$

and let $\alpha \in (0, \pi/2)$ be the angle between a and b:

$$\cos\alpha = \frac{a^\top b}{\|a\|\ \|b\|}. \tag{180}$$

Then

$$\begin{aligned} x(0) &= a \\ \dot{x}(0) &= X_1 = p\sqrt{1-\|a\|^2}\left(\frac{b}{\|b\|} - \frac{a^\top b}{\|b\|}a\right). \end{aligned} \tag{181}$$

We have $r(0) = \|x(0)\| = \|a\|$,

$$2x(0)^\top \dot{x}(0) = \frac{d(\|z\|)^2}{dt} = 2r(0)\dot{r}(0)\,. \tag{182}$$

Therefore,

$$\dot{r}(0) = p\sqrt{(1-\|a\|^2)^3}\frac{a^\top b}{\|a\|\ \|b\|} = p\sqrt{(1-\|a\|^2)^3}\cos\alpha. \tag{183}$$

For every t we have

$$\dot{\varphi}^2 = \frac{1}{\|x\|^2}\left(\|\dot{x}\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2}\right). \tag{184}$$

Therefore

$$\dot{\varphi}(0) = p\frac{\sqrt{1-\|a\|^2}}{\|a\|}\sin\alpha. \tag{185}$$

From (171) we get

$$C = \frac{r(0)^2\dot{\varphi}(0)}{1-r(0)^2} = p\frac{\|a\|\sin\alpha}{\sqrt{1-\|a\|^2}}. \tag{186}$$

Now we calculate the constant C_1 using (177),

$$C_1 = \frac{4r(0)^2\dot{r}(0)^2}{r(0)^2(1-r(0)^2)^3} + \frac{4C^2(1-r(0)^2)}{r(0)^2} = 4p^2. \tag{187}$$

We integrate Equation (178) over $t \in [0,1]$ using these values of C, C_1 and the equation $z = r^2$, obtaining

$$\int_0^1 \frac{\dot{r}\, dt}{\sqrt{(1-r^2)^3\left(1-\frac{\|a\|^2(1-r^2)}{r^2(1-\|a\|^2)}\sin^2\alpha\right)}} = p. \tag{188}$$

Changing the variable of integration on the left-hand side, and using the initial condition $r(0) = \|a\|$, we have

$$\int_{\|a\|}^{r(1)} \frac{ds}{\sqrt{(1-s^2)^3\left(1-\frac{\|a\|^2(1-s^2)}{s^2(1-\|a\|^2)}\sin^2\alpha\right)}} = \int_0^{\|b\|}\frac{ds}{\sqrt{(1-s^2)^3}}. \tag{189}$$

Both integrals can be evaluated in terms of elementary functions,

$$\sqrt{\frac{r(1)^2}{1-r(1)^2} - \frac{\|a\|^2\sin^2\alpha}{1-\|a\|^2}} - \sqrt{\frac{\|a\|^2\cos^2\alpha}{1-\|a\|^2}} = \frac{\|b\|}{\sqrt{1-\|b\|^2}}. \tag{190}$$

In order to calculate $\varphi(1)$, we solve Equation (171) for $\dot{\varphi}$,

$$\dot{\varphi} = C\frac{1-r^2}{r^2}. \tag{191}$$

Now we rewrite Equation (178) in terms of r,

$$\dot{r} = p\sqrt{(1-r^2)^3(1-\frac{\|a\|^2(1-r^2)}{r^2(1-\|a\|^2)}\sin^2\alpha}. \tag{192}$$

Then, we multiply and divide the right-hand side of (190) by $\frac{dr}{dt}$, use Equation (192) and integrate (190) over $[0,1]$ taking into account that $\varphi(0) = 0$, obtaining

$$\begin{aligned}\varphi(1) &= \int_0^1 \frac{\sin\alpha\ \dot{r}\ dt}{\sqrt{r^2(1-r^2)^2\left[\frac{r^2}{1-r^2} - \frac{\|a\|^2}{1-\|a\|^2}\sin^2\alpha\right)}}\sqrt{\frac{\|a\|^2}{1-\|a\|^2}} \\ &= \int_{\|a\|}^{r(1)} \frac{\sin\alpha\ dr}{\sqrt{r^2(1-r^2)^2\left[\frac{r^2}{1-r^2} - \frac{\|a\|^2}{1-\|a\|^2}\sin^2\alpha\right)}}\sqrt{\frac{\|a\|^2}{1-\|a\|^2}} \\ &= \alpha - \arctan\sqrt{\frac{\|a\|^2\sin^2\alpha(1-r(1)^2)}{(1-\|a\|^2)r(1)^2 - (1-r(1)^2)\frac{\|a\|^2}{1-\|a\|^2}\sin^2\alpha}}.\end{aligned} \tag{193}$$

Noticing that

$$\frac{r(1)^2}{1-r(1)^2} = \|\frac{a}{\sqrt{1-\|a\|^2}} + \frac{b}{\sqrt{1-\|b\|^2}}\|^2 \tag{194}$$

and

$$\frac{r(1)^2}{1-r(1)^2} - \frac{\|a\|^2}{1-\|a\|^2}\sin^2\alpha = \left(\sqrt{\frac{\|a\|^2\cos^2\alpha}{1-\|a\|^2}} + \sqrt{\frac{\|b\|^2}{1-\|b\|^2}}\right)^2, \tag{195}$$

we have

$$\begin{aligned}\cos\varphi(1) &= \sqrt{\frac{r(1)^2}{1-r(1)^2}}\left(\sqrt{\frac{\|a\|^2}{1-\|a\|^2}} + \sqrt{\frac{\|b\|^2\cos^2\alpha}{1-\|b\|^2}}\right) \\ \sin\varphi(1) &= \sqrt{\frac{r(1)^2}{1-r(1)^2}}\sqrt{\frac{\|b\|^2\sin^2\alpha}{1-\|b\|^2}}.\end{aligned} \tag{196}$$

Taking into account (165), (195) and (196), we finally get

$$x(1) = \frac{\frac{a}{\sqrt{1-\|a\|^2}} + \frac{b}{\sqrt{1-\|b\|^2}}}{\sqrt{1+\|\frac{a}{\sqrt{1-\|a\|^2}} + \frac{b}{\sqrt{1-\|b\|^2}}\|^2}} = \frac{\gamma_a a + \gamma_b b}{\sqrt{1+\|\gamma_a a + \gamma_b b\|^2}}, \tag{197}$$

where

$$\gamma_a = \frac{1}{\sqrt{1-\|a\|^2}}. \tag{198}$$

Step 4. We set

$$a \oplus b = x(1) = \frac{\gamma_a a + \gamma_b b}{\sqrt{1+\|\gamma_a a + \gamma_b b\|^2}}. \tag{199}$$

To see that the novel operation $\oplus$ in the ball of $\mathbb{R}^n$ is isomorphic to the Euclidean vector addition in $\mathbb{R}^n$, let us consider the mapping $f\colon \mathbb{B} \to \mathbb{R}^n$ given by

$$f(x) = \gamma_x x\,. \tag{200}$$

Then

$$1 - \|a \oplus b\|^2 = \frac{1}{1+\|\gamma_a a + \gamma_b b\|^2} \tag{201}$$

and

$$f(a \oplus b) = \gamma_a a + \gamma_b b = f(a) + f(b). \tag{202}$$

Equation (202) implies that the novel operation $\oplus$ in the ball is isomorphic to the Euclidean operation, $+$, in $\mathbb{R}^n$ and hence it is commutative and associative.

7.2. Multiplication by Numbers

To define the operation $\otimes$ of multiplication by numbers satisfying properties (91) in the Section 4.4 we have to introduce the function h: $[0, 1) \to \mathbb{R}$, (98),

$$h(p) = \int_0^p \sqrt{\frac{1}{(1-s^2)^3}} ds = \frac{p}{\sqrt{1-p^2}}. \tag{203}$$

Obviously, h is an increasing bijection. Therefore the inverse function h^{-1} exists. It is easy to see that

$$h^{-1}(q) = \frac{q}{\sqrt{1+q^2}}. \tag{204}$$

The operation $\otimes$ is given by Formula (101), which in our case has the form

$$t \otimes a = h^{-1}(t\, h(\|a\|)) \frac{a}{\|a\|} = \frac{t}{\sqrt{1+(t^2-1)\|a\|^2}} \frac{a}{\|a\|}. \tag{205}$$

Combining (202) and (203), (203) and (204) we get

$$f(t \otimes a) = \frac{t \otimes a}{\sqrt{1-\|t \otimes a\|^2}} = h(\|t \ \otimes a\|) \frac{t \otimes a}{\| \otimes a\|} = th(\|a\|) \frac{a}{\|a\|} = t\, f(a). \tag{206}$$

Equations (202) and (206) imply the distributive property of the operations $\oplus$ and $\otimes$: for any numbers t_1, t_2, and vectors $a, b \in \mathbb{B}$ we have

$$f((t_1 \otimes a) \oplus (t_2 \otimes b)) = t_1 f(a) + t_2 f(b). \tag{207}$$

Thus, the operation $\oplus$ is isomorphic to Euclidean addition.

7.3. Derivation of the Metric Tensor Associated with the Binary Operation

To recover the underlying metric tensor from the operation $\oplus$ we can use Theorem 1. Noticing that

$$\begin{aligned}
(-x) \oplus (x + \Delta x) &= \frac{-\frac{x}{\sqrt{1-\|x\|^2}} + \frac{x+\Delta x}{\sqrt{1-\|x+\Delta x\|^2}}}{\sqrt{1+\| - \frac{x}{\sqrt{1-\|x\|^2}} + \frac{x+\Delta x}{\sqrt{1-\|x+\Delta x\|^2}}\|^2}} \\
&= \frac{\Delta x}{\sqrt{1-\|x\|^2}} + x\left(\frac{1}{\sqrt{1-\|x+\Delta x\|^2}} - \frac{1}{\sqrt{1-\|x\|^2}}\right) + o(\|\Delta x\|) \\
&= \left(\frac{1}{\sqrt{1-\|x\|^2}} I + \frac{xx^\top}{\sqrt{(1-\|x\|^2)^3}}\right) \Delta x + o(\|\Delta x\|)\,,
\end{aligned} \tag{208}$$

we have

$$g(x) = \frac{1}{\sqrt{1-\|x\|^2}} I + \frac{xx^\top}{\sqrt{(1-\|x\|^2)^3}}$$
$$= \frac{1}{\sqrt{1-\|x\|^2}} \left(I - \frac{xx^\top}{\|x\|^2} \right) + \frac{1}{\sqrt{(1-\|x\|^2)^3}} \frac{xx^\top}{\|x\|^2} .. \tag{209}$$

The metric tensor G associated with the operation $\oplus$ is given in (51),

$$G(x) = g(x)^\top g(x) = \frac{1}{1-\|x\|^2} \left(I - \frac{xx^\top}{\|x\|^2} \right) + \frac{1}{(1-\|x\|^2)^3} \frac{xx^\top}{\|x\|^2}. \tag{210}$$

It has the form (6) parametrized by the functions

$$m_0(r^2) = \frac{1}{1-r^2}$$
$$m_1(r^2) = \frac{1}{(1-r^2)^3}, \tag{211}$$

thus recovering (158)–(159).

8. Properties of Einstein Addition

Seemingly structureless, Einstein addition is neither commutative nor associative. The Einstein groupoid $(\mathbb{B}, \oplus_E)$ formed by the ball $\mathbb{B}$ equipped with Einstein addition $\oplus_E$ is a gyrocommutative gyrogroup. A gyrogroup is a group-like object that possesses a rich structure [1,5], formally defined in Definitions 1 and 2, Section 2. Einstein addition is a gyrocommutative gyrogroup operation, several properties of which are presented below.

1. Left Cancellation Law:

$$a \oplus_E ((-a) \oplus_E b) = b, \qquad \forall a, b \in \mathbb{B} . \tag{212}$$

2. Existence of Gyrations: for every $a, b \in \mathbb{B}$ there exists a unitary matrix denoted by $gyr[a, b]$ such that for all $c \in \mathbb{B}$ we have the following gyroassociative law:

$$a \oplus_E (b \oplus_E c) = (a \oplus_E b) \oplus_E (gyr[a, b]c), \qquad \forall a, b, c \in \mathbb{B}. \tag{213}$$

3. Gyrocommutative Law:

$$a \oplus_E b = gyr[a, b](b \oplus_E a), \qquad \forall a, b \in \mathbb{B} , \tag{214}$$

such that

$$\|a \oplus_E b\| = \|b \oplus_E a\| . \tag{215}$$

4. Left Reduction Property:

$$gyr[a \oplus_E b, b] = gyr[a, b], \qquad \forall a, b \in \mathbb{B} . \tag{216}$$

The validity of these properties was proved by using computer algebra (specifically, the software MATHEMATICA for symbolic manipulation). Below we prove these properties by hand using the definition of Einstein addition (136), and provide the matrix representation of gyrations $gyr[a, b]$.

8.1. Left Cancellation Law

By Einstein addition (136), we have

$$(-a) \oplus_E b = -a + \frac{1}{1-a^\top b}[\sqrt{1-\|a\|^2}(b - \frac{a^\top b}{\|a\|^2}a) + \frac{a^\top b}{\|a\|^2}(1-\|a\|^2)a]\,. \tag{217}$$

We notice that

$$a^\top[(-a) \oplus_E b] = -\|a\|^2 + \frac{1-\|a\|^2}{1-a^\top b}a^\top b = \frac{a^\top b - \|a\|^2}{1-a^\top b} \tag{218}$$

and

$$1 + a^\top[(-a) \oplus_E b] = \frac{1-\|a\|^2}{1-a^\top b}\,. \tag{219}$$

Therefore,

$$\begin{aligned} a \oplus_E ((-a) \oplus_E b) &= a + \frac{\sqrt{1-\|a\|^2}}{1-\|a\|^2}(1-a^\top b)\{-a \\ &+ \frac{1}{1-a^\top b}[\sqrt{1-\|a\|^2}(b - \frac{a^\top b}{\|a\|^2}a) \\ &+ \frac{a^\top b}{\|a\|^2}(1-\|a\|^2)a + \frac{(a^\top b - \|a\|^2)(\sqrt{1-\|a\|^2}-1)}{\|a\|^2}a]\} \\ &= a + \frac{1}{\sqrt{1-\|a\|^2}}[-a + a^\top b a + \sqrt{1-\|a\|^2}b - \sqrt{1-\|a\|^2}\frac{a^\top b}{\|a\|^2}a \\ &+ \frac{a^\top b}{\|a\|^2}(1-\|a\|^2)a + (a^\top b - \|a\|^2)(\sqrt{1-\|a\|^2}-1)\frac{a}{\|a\|^2}] = b\,, \end{aligned} \tag{220}$$

thus proving the left cancellation law.

The property established in this subsection may be reformulated as follows. For every vectors $a, b \in \mathbb{B}$ there exists a unique vector $x \in \mathbb{B}$ such that

$$a \oplus_E x = b \tag{221}$$

and, moreover,

$$x = (-a) \oplus_E b\,. \tag{222}$$

Eliminating x between (221) and (222) yields the *left cancellation law* (212).

8.2. Existence of Gyrations

Evaluating the squared norm $\|a \oplus_E b\|^2$ yields an elegant result,

$$\begin{aligned} \|a \oplus_E b\|^2 &= \|\frac{\sqrt{1-\|a\|^2}}{1+a^\top b}(I - \frac{aa^\top}{\|a\|^2})b + \frac{\|a\|^2 + a^\top b}{\|a\|^2(1+a^\top b)}a\|^2 \\ &= \frac{1-\|a\|^2}{(1+a^\top b)^2}(\|b\|^2 - \frac{(a^\top b)^2}{\|a\|^2}) + \frac{(\|a\|^2 + a^\top b)^2}{(1+a^\top b)^2\|a\|^2} \\ &= 1 - \frac{(1-\|a\|^2)(1-\|b\|^2)}{(1+a^\top b)^2}\,. \end{aligned} \tag{223}$$

The function on the extreme right-hand side of (223) is symmetric with respect to a, b. Hence, $\|a \oplus_E b\| = \|b \oplus_E a\|$. Owing to this symmetry, there exists a unitary matrix which is the identity matrix in the subspace orthogonal to both a and b, and which maps $b \oplus_E a$ to $a \oplus_E b$. This matrix is said to be the *gyration* generated by a and b, and is denoted by $gyr[a,b]$. Gyrations play an important role in the geometric theory of spaces with the binary operation $a \oplus_E b$. Hence, we introduce them along with their properties in Section 8.3.

8.3. Definition of Gyrations

For every three vectors $a, b, c \in \mathbb{B}$ we consider a vector $d \in \mathbb{B}$ given by

$$d = -(a \oplus_E b) \oplus_E (a \oplus_E (b \oplus_E c)). \tag{224}$$

First we check that d depends on c linearly, that is, d is equal to some matrix depending on a, b only, times c. This matrix is destined to be called the *gyration matrix* generated by a and b.

Owing to the left cancellation law, Equation (224) is equivalent to

$$(a \oplus_E b) \oplus_E d = a \oplus_E (b \oplus_E c)\,. \tag{225}$$

Using (136), we get,

$$\begin{aligned}(a \oplus_E b) \oplus_E d = {} & \frac{1}{1+(a \oplus_E b)^\top d}\{(1+\frac{(a \oplus_E b)^\top d}{\|a \oplus_E b\|^2})a \oplus_E b \\ & + \sqrt{1-\|a \oplus_E b\|^2}[d - \frac{(a \oplus_E b)^\top d}{\|a \oplus_E b\|^2}(a \oplus_E b)]\}\end{aligned} \tag{226}$$

and

$$\begin{aligned}a \oplus_E (b \oplus_E c) = {} & \frac{1}{1+a^\top(b \oplus_E c)}\{[1+\frac{a^\top(b \oplus_E c)}{\|a\|^2}(1-\sqrt{1-\|a\|^2})]a \\ & + \sqrt{1-\|a\|^2}(b \oplus_E c)\}\,.\end{aligned} \tag{227}$$

Furthermore, we need identity (228) of the following Lemma.

Lemma 3. *For every vectors $a, b, c \in \mathbb{B}$ we have*

$$(1+b^\top c)(1+a^\top(b \oplus_E c)) = (1+a^\top b)(1+(b \oplus_E a)^\top c). \tag{228}$$

Proof. By means of (136) we have,

$$(1+b^\top c)(1+a^\top(b \oplus_E c)) = 1+b^\top c + (1+\frac{b^\top c}{\|b\|^2})a^\top b + \sqrt{1-\|b\|^2}(a^\top c - \frac{b^\top c}{\|b\|^2}a^\top b) \tag{229}$$

and

$$(1+a^\top b)(1+(b \oplus_E a)^\top c) = 1+a^\top b + (1+\frac{b^\top a}{\|b\|^2})b^\top c + \sqrt{1-\|b\|^2}(a^\top c - \frac{b^\top a}{\|b\|^2}b^\top c)\,. \tag{230}$$

The right-hand sides of (229) and (230) are identical as desired, and the proof is complete. □

We solve Equation (225) for d under the assumption that

$$(a \oplus_E b)^\top d = (b \oplus_E a)^\top c. \tag{231}$$

This assumption will be verified later, below (237).

Then, taking into account Lemma 3, we see that (225) is equivalent to

$$\begin{aligned}&[1+\frac{(b\oplus_E a)^\top c}{\|a\oplus_E b\|^2}(1-\sqrt{1-\|a\oplus_E b\|^2})](a\oplus_E b)+\sqrt{1-\|a\oplus_E b\|^2}d\\&=\frac{1+b^\top c}{1+a^\top b}\{[1+\frac{a^\top(b\oplus_E c)}{\|a\|^2}(1-\sqrt{1-\|a\|^2})]a+\sqrt{1-\|a\|^2}(b\oplus_E c)\}\,.\end{aligned}\tag{232}$$

Equation (232) allows us to determine d,

$$\begin{aligned}d=\frac{1}{\sqrt{1-\|a\oplus_E b\|^2}}&\left[\frac{1+b^\top c}{1+a^\top b}\{[1+\frac{a^\top(b\oplus_E c)}{\|a\|^2}(1-\sqrt{1-\|a\|^2})]a\right.\\&+\sqrt{1-\|a\|^2}(b\oplus_E c)\}\\&\left.-[1+\frac{(b\oplus_E a)^\top c}{\|a\oplus_E b\|^2}(1-\sqrt{1-\|a\oplus_E b\|^2})](a\oplus_E b)\right].\end{aligned}\tag{233}$$

To prove that (233) is a solution of (225) for the unknown d, we need only to show that (231) holds. By means of (136), we have

$$\begin{aligned}d=&\frac{1}{\sqrt{1-\|a\oplus_E b\|^2}(1+a^\top b)}\Big[a+b^\top ca\\&+(1-\sqrt{1-\|a\|^2})[(1+\frac{b^\top c}{\|b\|^2})a^\top b+\sqrt{1-\|b\|^2}(a^\top c-\frac{b^\top c}{\|b\|^2}a^\top b)]a\\&+\sqrt{1-\|a\|^2}[(1+\frac{b^\top c}{\|b\|^2})b+\sqrt{1-\|b\|^2}(c-\frac{b^\top c}{\|b\|^2}b)]\\&-(1+a^\top b)[(a\oplus_E b)-\sqrt{1-\|a\oplus_E b\|^2}\frac{(b\oplus_E a)^\top c}{\|a\oplus_E b\|^2}(a\oplus_E b)\\&+\frac{(b\oplus_E a)^\top c}{\|a\oplus_E b\|^2}(a\oplus_E b)]\Big].\end{aligned}\tag{234}$$

The terms without c in the big square brackets cancel each other, and we get the following expression for d, which is linear in c,

$$\begin{aligned}d=&\frac{(b\oplus_E a)^\top c}{\|a\oplus_E b\|^2}(a\oplus_E b)+\frac{1}{\sqrt{1-\|a\oplus_E b\|^2}(1+a^\top b)}\Big[b^\top ca\\&+(1-\sqrt{1-\|a\|^2})[\frac{b^\top c}{\|b\|^2}a^\top b+\sqrt{1-\|b\|^2}(a^\top c-\frac{b^\top c}{\|b\|^2}a^\top b)]a\\&+\sqrt{1-\|a\|^2}[\frac{b^\top c}{\|b\|^2}b+\sqrt{1-\|b\|^2}(c-\frac{b^\top c}{\|b\|^2}b)]\\&-(1+a^\top b)\frac{(a\oplus_E b)}{\|a\oplus_E b\|^2}[(1+\frac{b^\top a}{\|b\|^2})b^\top c+\sqrt{1-\|b\|^2}(a^\top c-\frac{b^\top c}{\|b\|^2}a^\top b)]\Big].\end{aligned}\tag{235}$$

Now we check condition (231). We use the following four formulas,

$$
\begin{aligned}
a^\top(a \oplus_E b) &= \frac{|a\|^2 + a^\top b}{1 + a^\top b} \\
b^\top(a \oplus_E b) &= \frac{1}{a^\top b}[(1 + \frac{a^\top b}{\|a\|^2})a^\top b + \sqrt{1 - \|a\|^2}(\|b\|^2 - \frac{(a^\top b)^2}{\|a\|^2})] \\
c^\top(a \oplus_E b) &= \frac{1}{1 + a^\top b}[(1 + \frac{a^\top b}{\|a\|^2})c^\top b + \sqrt{1 - \|a\|^2}(c^\top b - \frac{a^\top b}{\|a\|^2}b^\top c)]
\end{aligned}
\tag{236}
$$

$$
\sqrt{1 - \|a \oplus_E b\|^2}(1 + a^\top b) = \sqrt{(1 - \|a\|^2)(1 - \|b\|^2)}\,,
$$

in the representation

$$
\begin{aligned}
&(a \oplus_E b)^\top d - (b \oplus_E a)^\top c \\
&= \frac{1}{\sqrt{(1 - \|a\|^2)(1 - \|b\|^2)}}\Big[b^\top c a^\top(a \oplus_E b) \\
&+ (1 - \sqrt{1 - \|a\|^2})[\frac{b^\top c}{\|b\|^2}a^\top b + \sqrt{1 - \|b\|^2}(a^\top c - \frac{b^\top c}{\|b\|^2}a^\top b)]a^\top(a \oplus_E b) \\
&+ \sqrt{1 - \|a\|^2}[\frac{b^\top c}{\|b\|^2}b^\top(a \oplus_E b) + \sqrt{1 - \|b\|^2}(c^\top(a \oplus_E b) - \frac{b^\top c}{\|b\|^2}b^\top(a \oplus_E b))] \\
&- (1 + a^\top b)[(1 + \frac{b^\top a}{\|b\|^2})b^\top c + \sqrt{1 - \|b\|^2}(a^\top c - \frac{b^\top c}{\|b\|^2}a^\top b)]\Big]
\end{aligned}
\tag{237}
$$

to show by direct calculations that the right-hand side of (237) vanishes. Hence, Equation (231) is true. Therefore, the vector d satisfies (225). Equation (235) shows that d depends linearly on c,

$$
d = Mc\,, \tag{238}
$$

where M is a matrix depending only on a and b. This matrix is called the gyration matrix generated by a and b, and is denoted by $gyr[a, b]$.

In particular, (225) implies the Gyroassociative law (213):

$$
a \oplus_E (b \oplus_E c) = (a \oplus_E b) \oplus_E (Mc) \quad \forall a, b, c \in \mathbb{B}\,. \tag{239}
$$

8.4. Properties of the Gyration

We have already shown that the gyration operator $gyr[a, b]$, defined by the identity

$$
(a \oplus_E b) \oplus_E gyr[a, b]c = a \oplus_E (b \oplus_E c)\,, \tag{240}
$$

is linear, that is, $gyr[a, b]c = Mc$ for some matrix M. Below we determine the matrix M and prove that M is unitary.

From (235) and (238) we obtain the following representation of the matrix M.

$$
M = gyr[a, b] = I + C_{aa}aa^\top + C_{ab}ab^\top + C_{ba}ba^\top + C_{bb}bb^\top\,, \tag{241}
$$

where the scalar functions $C_{aa}, C_{ab}, C_{ba}, C_{bb}$ are given by

$$\begin{aligned} C_{aa} &= -\frac{1}{\|a\|^2}\frac{(1-\sqrt{1-\|a\|^2})(1-\sqrt{1-\|b\|^2})}{1+a^\top b+\sqrt{(1-\|a\|^2)(1-\|b\|^2)}} \\ C_{ab} &= \frac{1+\frac{2a^\top b}{(1+\sqrt{1-\|a\|^2})(1+\sqrt{1-\|b\|^2})}}{1+a^\top b+\sqrt{(1-\|a\|^2)(1-\|b\|^2)}} \\ C_{ba} &= -\frac{1}{1+a^\top b+\sqrt{(1-\|a\|^2)(1-\|b\|^2)}} \\ C_{bb} &= -\frac{1}{\|b\|^2}\frac{(1-\sqrt{1-\|a\|^2})(1-\sqrt{1-\|b\|^2})}{1+a^\top b+\sqrt{(1-\|a\|^2)(1-\|b\|^2)}}\,. \end{aligned} \tag{242}$$

Lemma 4. *The matrix M is unitary,*

$$M^\top M = I\,. \tag{243}$$

Proof. We have

$$M^\top M = I + \alpha_{aa}aa^\top + \alpha_{ab}ab^\top + \alpha_{ba}ba^\top + \alpha_{bb}bb^\top\,, \tag{244}$$

where the numbers $\alpha_{aa}, \alpha_{ab}, \alpha_{ba}, \alpha_{bb}$ are given by

$$\begin{aligned} \alpha_{aa} &= 2C_{aa} + \|a\|^2 C_{aa}^2 + 2C_{ab}C_{aa}a^\top b + C_{ab}^2\|b\|^2 \\ \alpha_{ab} &= \alpha_{ba} = C_{ab} + C_{ba} + a^\top b(C_{aa}C_{bb} + C_{ab}C_{ba}) \\ &\quad + C_{aa}C_{ba}\|a\|^2 + C_{ab}C_{bb}\|b\|^2 \\ \alpha_{bb} &= 2C_{bb} + \|b\|^2 C_{bb}^2 + 2C_{ba}C_{bb}a^\top b + C_{ba}^2\|b\|^2\,. \end{aligned} \tag{245}$$

Straightforward calculations show that all these numbers vanish. To show these results we use the notation $x = \sqrt{1-\|a\|^2}$ and $y = \sqrt{1-\|b\|^2}$, so that $\|a\|^2 = 1-x^2$ and $\|b\|^2 = 1-y^2$. Indeed, we have

$$\begin{aligned} &\alpha_{aa}(1+a^\top b+xy)\|a\|^2 = -2(1-x)(1-y)(1+a^\top b+xy) \\ &+(1-x)^2(1-y)^2 - 2a^\top b(1-x)(1-y)(1+\frac{2a^\top b(1-x)(1-y)}{\|a\|^2\|b\|^2} \\ &+(1-x^2)(1-y^2) + 4a^\top b(1-x)(1-y) + 4(a^\top b)^2\frac{(1-x)^2(1-y)^2}{\|a\|^2\|b\|^2} = 0, \end{aligned} \tag{246}$$

$$\begin{aligned} &\alpha_{bb}(1+a^\top b+xy)\|b\|^2 = -2(1-x)(1-y)(1+a^\top b+xy) \\ &+(1-x)^2(1-y)^2 + 2a^\top b(1-x)(1-y) + (1-x^2)(1-y^2) = 0 \end{aligned} \tag{247}$$

and

$$\begin{aligned} &\alpha_{ab}(1+a^\top b+xy)\|a\|^2\|b\|^2 = ((1-x^2)(1-y^2) \\ &+2a^\top b(1-x)(1-y))(1+a^\top b+xy) - (1-x^2)(1-y^2)(1+a^\top b+xy) \\ &+a^\top b[(1-x)^2(1-y)^2 - ((1-x^2)(1-y^2) + 2a^\top b(1-x)(1-y))] \\ &+(1-x)(1-y)(1-x^2)(1-y^2) - [(1-x^2)(1-y^2) \\ &+2a^\top b(1-x)(1-y)](1-x)(1-y) = 0\,. \end{aligned} \tag{248}$$

The proof of the Lemma is, thus, complete. □

Lemma 5. *If vectors a and b are parallel, that is, there exists a non zero pair of numbers* (λ_1, λ_2) *such that*

$$\lambda_1 a + \lambda_2 b = 0, \tag{249}$$

then

$$gyr[a, b] = I. \tag{250}$$

Proof. If $\lambda_2 = 0$, then $a = 0$, and the statement follows from (241). Assume $\lambda_2 \neq 0$. Set $\lambda = -\frac{\lambda_1}{\lambda_2}$. Then $b = \lambda a$, and the representation (241) amounts to

$$gyr[a, b] = I + daa^\top, \tag{251}$$

where

$$d = C_{aa} + \lambda(C_{a(\lambda a)} + C_{(\lambda a)a}) + \lambda^2 C_{(\lambda a)(\lambda a)} = 0 \tag{252}$$

□

Corollary 1. *For every numbers* r_1, r_2 *and a vector* $v \in \mathbb{B}$ *we have*

$$gyr[r_1 \otimes v, r_2 \otimes v] = I. \tag{253}$$

Proof. By the definition of the scalar multiplication in (101), the vectors $r_1 \otimes v$ and $r_2 \otimes v$ are parallel. Hence, (253) follows from Lemma 5. □

8.5. Reduction Property

In this subsection we establish property (216) of Einstein addition. Let

$$\gamma_v = \frac{1}{\sqrt{1 - \|v\|^2}} \tag{254}$$

be the well known *Lorentz gamma factor* of special relativity theory [1,5]. Then, in view of (242) we have

$$\begin{aligned} C_{aa} &= -\frac{\gamma_a^2}{\gamma_a + 1}\frac{\gamma_b - 1}{\gamma_a\gamma_b(1 + a^\top b) + 1} \\ C_{ab} &= \frac{\gamma_a\gamma_b + 2a^\top b\frac{\gamma_a^2\gamma_b^2}{(\gamma_a+1)(\gamma_b+1)}}{\gamma_a\gamma_b(1 + a^\top b) + 1} \\ C_{ba} &= -\frac{\gamma_a\gamma_b}{\gamma_a\gamma_b(1 + a^\top b) + 1} \\ C_{bb} &= -\frac{\gamma_b^2}{\gamma_b + 1}\frac{\gamma_a - 1}{\gamma_a\gamma_b(1 + a^\top b) + 1}. \end{aligned} \tag{255}$$

Lemma 6. *For every vectors* $a, b \in \mathbb{B}$ *we have*

$$gyr[a \oplus_E b, b] = gyr[a, b]. \tag{256}$$

Proof. We have by (241),

$$\begin{aligned} gyr[a \oplus_E b, b] &= I + \tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)}(a \oplus_E b) \oplus (a \oplus_E b)^\top \\ &+ \tilde{C}_{(a\oplus_E b)\,b}(a \oplus_E b)b^\top + \tilde{C}_{b\,(a\oplus_E b)}b(a \oplus_E b)^\top + \tilde{C}_{b\,b}bb^\top \end{aligned} \tag{257}$$

where

$$
\begin{aligned}
\tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} &= -\frac{\gamma^2_{a\oplus_E b}}{\gamma_{a\oplus_E b}+1}\frac{\gamma_b - 1}{\gamma_{a\oplus_E b}\gamma_b(1+(a\oplus_E b)^\top b)+1} \\
\tilde{C}_{(a\oplus_E b)\,b} &= \frac{\gamma_{a\oplus_E b}\gamma_b + 2(a\oplus_E b)^\top b\frac{\gamma^2_{a\oplus_E b}\gamma^2_b}{(\gamma_{a\oplus_E b}+1)(\gamma_b+1)}}{\gamma_{a\oplus_E b}\gamma_b(1+(a\oplus_E b)^\top b)+1} \\
\tilde{C}_{b\,(a\oplus_E b)} &= -\frac{\gamma_{a\oplus_E b}\gamma_b}{\gamma_{a\oplus_E b}\gamma_b(1+(a\oplus_E b)^\top b)+1} \\
\tilde{C}_{b\,b} &= -\frac{\gamma^2_b}{\gamma b+1}\frac{\gamma_{a\oplus_E b}-1}{\gamma_{a\oplus_E b}\gamma_b(1+(a\oplus_E b)^\top b)+1}\,.
\end{aligned}
\tag{258}
$$

By straightforward calculations we have the following identities,

$$
\begin{aligned}
(a\oplus_E b)(a\oplus_E b)^\top = \frac{(1+a^\top b\frac{\gamma_a}{\gamma_a+1})^2}{(1+a^\top b)^2}aa^\top + \frac{1+a^\top b\frac{\gamma_a}{\gamma_a+1}}{1+a^\top b}(ab^\top + ba^\top) \\
+\frac{1}{\gamma^2_a(1+a^\top b)^2}bb^\top
\end{aligned}
\tag{259}
$$

$$
(a\oplus_E b)b^\top = \frac{1+a^\top b\frac{\gamma_a}{\gamma_a+1}}{1+a^\top b}ab^\top + \frac{1}{\gamma_a(1+a^\top b)}bb^\top \tag{260}
$$

$$
\gamma_{a\oplus_E b} = \gamma_a\gamma_b(1+a^\top b) \tag{261}
$$

and

$$
\gamma_{a\oplus_E b}\gamma_b(1+(a\oplus_E b)^\top b)+1 = \frac{\gamma^2_b}{\gamma_a+1}(1+\gamma_a(1+a^\top b))^2\,. \tag{262}
$$

Hence, we have

$$
\tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} = -\frac{\gamma^2_a(1+a^\top b)^2}{\gamma_a\gamma_b(1+a^\top b)+1}\frac{(\gamma_b-1)(\gamma_a-1)}{(1+\gamma_a(a+a^\top b))^2} \tag{263}
$$

$$
\begin{aligned}
&\tilde{C}_{(a\oplus_E b)\,b} = \\
&\frac{\gamma_a(1+a^\top b)[(\gamma_a+1)(\gamma_b-1)(1+\gamma^2_a\gamma_b(1+a^\top b))+2\gamma^2_b(1+\gamma_a(1+a^\top b))^2]}{(1+\gamma_a\gamma^2_b(1+a^\top b))(\gamma_b+1)(1+\gamma_a(1+a^\top b))^2}
\end{aligned}
\tag{264}
$$

$$
\tilde{C}_{b\,(a\oplus_E b)} = -\frac{\gamma_a(\gamma_a+1)(1+a^\top b)}{(1+\gamma_a(1+a^\top b))^2} \tag{265}
$$

and

$$
\tilde{C}_{b\,b} = -\frac{(\gamma_a+1)(\gamma_a\gamma_b(1+a^\top b)-1)}{(\gamma_b+1)(1+\gamma_a(1+a^\top b))^2}\,. \tag{266}
$$

Therefore,

$$
gyr[a\oplus_E b, b] = I + D_{aa}aa^\top + D_{ab}ab^\top + D_{ba}ba^\top + D_{bb}bb^\top\,, \tag{267}
$$

where the coefficients are evaluated straightforwardly,

$$\begin{aligned}
D_{aa} &= \tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} \frac{(1+a^\top b\frac{\gamma_a}{\gamma_a+1})^2}{(1+a^\top b)^2} = C_{aa} \\
D_{ab} &= \tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} \frac{1+a^\top b\frac{\gamma_a}{\gamma_a+1}}{1+a^\top b} + \tilde{C}_{(a\oplus_E b)\,b} \frac{1+a^\top b\frac{\gamma_a}{\gamma_a+1}}{1+a^\top b} = C_{ab} \\
D_{ba} &= \tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} \frac{1+a^\top b\frac{\gamma_a}{\gamma_a+1}}{1+a^\top b} + \tilde{C}_{b\,(a\oplus_E b)} \frac{1+a^\top b\frac{\gamma_a}{\gamma_a+1}}{1+a^\top b} = C_b \\
D_{bb} &= \tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} \frac{1}{\gamma_a^2(1+a^\top b)^2} + (\tilde{C}_{(a\oplus_E b)\,(a\oplus_E b)} \\
&\quad + \tilde{C}_{b\,(a\oplus_E b)}) \frac{1}{\gamma_a(1+a^\top b)} + \tilde{C}_{b\,b} = C_{bb}\,.
\end{aligned} \tag{268}$$

Thus,

$$gyr[a \oplus_E b, b] = I + C_{aa}aa^\top + C_{ab}ab^\top + C_{ba}ba^\top + C_{bb}bb^\top = gyr[a,b]. \tag{269}$$

The proof of the Lemma is, thus, complete. □

8.6. Gyrocommutative Law

We are now in the position to prove the gyrocommutative law (214) of Einstein addition.

Lemma 7. *For every vectors $a, b \in \mathbb{B}$ we have*

$$a \oplus_E b = gyr[a,b](b \oplus_E a)\,. \tag{270}$$

Proof. Equation (225) with $c = b \oplus_E a$ yields the equation

$$(a \oplus_E b) \oplus_E d = a \oplus_E (b \oplus_E (b \oplus_E a)\,, \tag{271}$$

where $d = Mc$.

The value of d exists and is unique owing to the left cancellation law. Furthermore, we show that $d = a \oplus_E b$ satisfies (271). Indeed, for every vector $v \in \mathbb{B}$ we have

$$v \oplus_E v = \frac{1}{1+v^\top v}[(1+\frac{v^\top v}{\|v\|^2})v] = \frac{2}{1+\|v\|^2}v\,. \tag{272}$$

In particular,

$$(a \oplus_E b) \oplus_E (a \oplus_E b) = \frac{2}{1+\|a \oplus_E b\|^2}(a \oplus_E b)\,. \tag{273}$$

We now evaluate the right-hand side of (271), using the identities in (236) and in (223),

$$\begin{aligned}
b^\top(b \oplus_E a) &= \frac{\|b\|^2 + a^\top b}{1+a^\top b} \\
\|a \oplus_E b\|^2 &= 1 - \frac{(1-\|a\|^2)(1-\|b\|^2)}{(1-a^\top b)^2}\,.
\end{aligned} \tag{274}$$

Then,

$$\begin{aligned} b \oplus_E (b \oplus_E a) &= \frac{1}{1 + b^\top (b \oplus_E a)} \Big[(1 + \frac{b^\top (b \oplus_E a)}{\|b\|^2}) b \\ &\quad + \sqrt{1 - \|b\|^2} (b \oplus_E a - \frac{b^\top (b \oplus_E a)}{\|b\|^2} b) \Big] \\ &= \frac{1 + a^\top b}{1 + 2a^\top b | \|b\|^2} \Big[\frac{1}{1 + a^\top b} (1 + a^\top b + \frac{\|b\|^2 + a^\top b}{\|b\|^2}) b \\ &\quad + \frac{\sqrt{1 - \|b\|^2}}{1 + a^\top b} [(1 + \frac{a^\top b}{\|b\|^2}) b + \sqrt{1 - \|b\|^2} (a - \frac{a^\top b}{\|b\|^2} b) - \frac{\|b\|^2 + a^\top b}{\|b\|^2}] \Big] \\ &= \frac{1}{1 + 2a^\top b + \|b\|^2} \Big[(1 - \|b\|^2) a + 2(1 + a^\top b) b \Big] . \end{aligned} \tag{275}$$

Using the formula

$$a^\top (b \oplus_E (b \oplus_E a)) = \frac{(1 - \|b\|^2) \|a\|^2 + 2a^\top b (1 + a^\top b)}{1 + 2a^\top b + \|b\|^2} a^\top (b \oplus_E (b \oplus_E a)) , \tag{276}$$

we get

$$\begin{aligned} a \oplus_E (b \oplus_E (b \oplus_E a)) &= \frac{1}{1 + a^\top (b \oplus_E (b \oplus_E a))} \Big[(1 \\ &\quad + \frac{a^\top (b \oplus_E (b \oplus_E a))}{\|a\|^2}) a + \sqrt{1 - \|a\|^2} [b \oplus_E (b \oplus_E a) - \frac{a^\top (b \oplus_E (b \oplus_E a))}{\|a\|^2} a] \Big] \\ &= \frac{1}{1 + 2a^\top b + \|b\|^2 + (1 - \|b\|^2) \|a\|^2 + 2a^\top b (1 + a^\top b)} \Big[(1 + 2a^\top b + \|b\|^2 \\ &\quad + [(1 - \|b\|^2) \|a\|^2 + 2a^\top b (1 + a^\top b)] a + \sqrt{1 - \|a\|^2} [(1 - \|b\|^2) a + 2(1 + a^\top b) b \\ &\quad - ((1 - \|b\|^2) \|a\|^2 + 2a^\top b (1 + a^\top b)) \frac{a}{\|a\|^2}] \Big] \\ &= \frac{2(1 + a^\top b)}{2(1 + a^\top b)^2 - (1 - \|a\|^2)(1 - \|b\|^2)} \Big[(1 + \frac{a^\top b}{\|b\|^2}) a + \sqrt{1 - \|a\|^2} (b - \frac{a^\top b}{\|b\|^2}) \Big] \\ &= \frac{2}{1 - \|a \oplus_E b\|^2} a \oplus_E b \\ &= (a \oplus_E b) \oplus_E (a \oplus_E b) . \end{aligned} \tag{277}$$

Therefore, Equation (271) can be written as

$$(a \oplus b) \oplus d = (a \oplus b) \oplus (a \oplus b). \tag{278}$$

Now we add $-(a \oplus b)$ from the left to both sides of (278), obtaining

$$d = a \oplus b. \tag{279}$$

Recalling that $d = Mc = gyr[a,b]((b \oplus a)$, we get the assertion (270) of the lemma. The proof of the Lemma is, thus, complete. □

8.7. Gyrations Preserve Einstein Addition and Multiplication

For any $a, b \in \mathbb{B}$, the operator $c \to gyr[a,b]c$ is linear, being given by a multiplication by a unitary matrix. Interestingly, it is also *gyrolinear* in the sense that

$$gyr[a,b]\Big[(t_1 \otimes_E c_1) \oplus_E (t_2 \otimes_E c_2)\Big] = (t_1 \otimes_E (gyr[a,b]c_1)) \oplus_E (t_2 \otimes_E (gyr[a,b]c_2)) \tag{280}$$

for all $c_1, c_2 \in \mathbb{B}$ and $t_1, t_2 \in [-\infty, \infty)$.

A result that generalizes (280) is stated in the following Lemma.

Lemma 8. *For every unitary matrix U, for every vectors $c_1, c_2 \in \mathbb{B}$, and every numbers $t_1, t_2 \in \mathbb{R}$ we have*

$$U\Big[(t_1 \otimes_E c_1) \oplus_E (t_2 \otimes_E c_2)\Big] = (t_1 \otimes_E (Uc_1)) \oplus_E (t_2 \otimes_E (Uc_2))\,. \tag{281}$$

Proof. The proof follows from properties of Einstein addition and multiplication. For every $c_1, c_2 \in \mathbb{B}$

$$\begin{aligned}
&(Uc_1) \oplus_E (Uc_2) \\
&= \frac{1}{1+(Uc_1)^\top (Uc_2)}\{[1 + \frac{(Uc_1)^\top (Uc_2)}{\|Uc_1\|^2}(1-\sqrt{1-\|Uc_1\|^2})]Uc_1 \\
&+ \sqrt{1-\|Uc_1\|^2}Uc_2\} \\
&= U\Big[\frac{1}{1+c_1^\top c_2}\{[1+\frac{c_1^\top c_2}{\|c_1\|^2}(1-\sqrt{1-\|c_1\|^2})]c_1 + \sqrt{1-\|c_1\|^2}c_2\}\Big] \\
&= U(c_1 \oplus_E c_2)\,,
\end{aligned} \tag{282}$$

and for every vector $c \in \mathbb{B}$ and positive number t,

$$\begin{aligned}
t \otimes_E (Uc) &= \operatorname{atanh}(t\ \tanh(\|Uc\|))\frac{Uc}{\|Uc\|} \\
&= U\Big[\operatorname{atanh}(t\ \tanh(\|c\|))\frac{c}{\|c\|}\Big] \\
&= U[t \otimes_E c]\,.
\end{aligned} \tag{283}$$

The proof of the Lemma is, thus, complete. □

The following two corollaries are immediate consequences of Lemmas 8 and 4.

Corollary 2. *For every unitary matrix U, for every vectors $a, b, c_1, c_2 \in \mathbb{B}$, and every numbers $t_1, t_2 \in \mathbb{R}$ we have*

$$gyr[a,b]\Big[(t_1 \otimes_E c_1) \oplus_E (t_2 \otimes_E c_2)\Big] = (t_1 \otimes_E (gyr[a,b]c_1)) \oplus_E (t_2 \otimes_E (gyr[a,b]c_2))\,. \tag{284}$$

Corollary 3. *For every vectors $a, b, u, v \in \mathbb{B}$ we have*

$$(gyr[u,v]a)^\top (gyr[u,v]b) = a^\top b. \tag{285}$$

8.8. Gyrogroups and Gyrovector Spaces

Assume $\oplus_E$, $\otimes_E$ are the Einstein addition and scalar multiplication respectively.

Theorem 4. *The groupoid* $(\mathbb{B}, \oplus_E)$ *is a gyrocommutative gyrogroup.*

Proof. We need to check properties (G1)–(G6) in Definitions 1 and 2. Property (G1) follows from (45), property (G2) follows from (46), property (G3) follows from (240), property (G4) follows from (241), property (G5) follows from (256), and property (G6) follows from (270). □

Theorem 5. *The triple* $(\mathbb{B}, \oplus, \otimes)$ *is a gyrovector space.*

Proof. We need to check properties (V0)–(V8) in Definition 3. Property (V0) follows from (285), property (V1) follows from (105), property (V2) follows from (103), property (V3) follows from (104), property (V4) follows from (106), property (V5) follows from (280), property (V6) follows from (253), property (V7) follows from (123), and property (V8) follows from (127). □

8.9. Einstein Coaddition

In Section 8.1 we have found a solution to the equation

$$a \oplus_E x = b \tag{286}$$

in $\mathbb{B}$ for the unknown x.

We now consider the similar equation

$$x \oplus_E a = b \tag{287}$$

in $\mathbb{B}$ for the unknown x.

It is known ([1] Chap. 1, Section 6) that the unique solution of (287) is given by

$$x = b \ominus_E gyr[b, a]a\,. \tag{288}$$

Motivated by (288) we introduce the Einstein coaddition

$$a \boxplus_E b = a \oplus_E gyr[a, -b]b \tag{289}$$

and use the notation

$$a \boxminus_E b = a \boxplus_E (-b)\,, \tag{290}$$

so that the solution of equation (287) is given by

$$x = b \boxminus_E a\,. \tag{291}$$

Accordingly, in this section we calculate

$$a \boxminus_E b = a \ominus_E gyr[a, b]b\,. \tag{292}$$

By the representation of $gyr[a, b]$ in (241), we have

$$gyr[a, b] = I + C_{aa}aa^\top + C_{ab}ab^\top + C_{ba}ba^\top + C_{bb}bb^\top\,, \tag{293}$$

where the numbers C_{aa}, C_{ab}, C_{ba} and C_{bb} are given by (242).

Let

$$\Delta = 1 + a^\top b + \sqrt{(1 - \|a\|^2)(1 - \|b\|^2)}\,. \tag{294}$$

Then,

$$gyr[a,b]b = D_a a + D_b b\,, \tag{295}$$

where

$$\begin{aligned} D_a &= C_{aa} a^\top b + C_{ab} \|b\|^2 \\ &= \frac{1}{\Delta}\left[\|b\|^2 + \frac{a^\top b}{\|a\|^2}(1-\sqrt{1-\|a\|^2})(1-\sqrt{1-\|b\|^2})\right] \end{aligned} \tag{296}$$

and

$$\begin{aligned} D_b &= 1 + C_{ba} a^\top b + C_{bb} \|b\|^2 \\ &= \frac{\sqrt{1-\|a\|^2} + \sqrt{1-\|b\|^2}}{\Delta}. \end{aligned} \tag{297}$$

We have

$$a \ominus c = \frac{1}{1-a^\top c}\left[(1-\frac{a^\top c}{\|a\|^2})a + \sqrt{1-\|a\|^2}(\frac{a^\top c}{\|a\|^2}a - c)\right] \tag{298}$$

for any $a, c \in \mathbb{B}$.

Calculating (298) with $c = gyr[a,b]b$, noticing that

$$\begin{aligned} 1 - a^\top gyr[a,b]b &= \frac{1}{\Delta}\left[1 - \|a\|^2\|b\|^2 + (1-a^\top b)\sqrt{1-\|a\|^2}\sqrt{1-\|b\|^2}\right] \\ \frac{a^\top gyr[a,b]b}{\|a\|^2} - gyr[a,b]b &= \frac{\sqrt{1-\|a\|^2}+\sqrt{1-\|b\|^2}}{\Delta}\left[\frac{a^\top b}{\|a\|^2}a - b\right], \end{aligned} \tag{299}$$

we have

$$a \ominus_E gyr[a,b]b = \frac{(\sqrt{1-\|a\|^2}+\sqrt{1-\|b\|^2})(a\sqrt{1-\|b\|^2} - b\sqrt{1-\|a\|^2})}{1-\|a\|^2\|b\|^2 + (1-a^\top b)\sqrt{1-\|a\|^2}\sqrt{1-\|b\|^2}}\,. \tag{300}$$

Using the standard notation for the Lorentz factor in (254), we have the equation

$$a \boxplus b = \frac{\gamma_a a + \gamma_b b}{\gamma_a + \gamma_b - \frac{\gamma_{a\oplus b}+1}{\gamma_a+\gamma_b}}, \tag{301}$$

which is well known ([5] p. 92). An extension of (301) to more than two summands is presented in ([5] p. 425). It is clear from (301) that Einstein coaddition in $\mathbb{B}$ is commutative,

$$a \boxplus b = b \boxplus a\,. \tag{302}$$

In order to calculate the metric tensor for the binary operation $\boxplus$ in $\mathbb{B}$ we obtain the equation

$$(-x) \boxplus (x + \Delta x) = \left[I + \frac{xx^\top}{1-\|x\|^2}\right]\Delta x + o(\|\Delta x\|\,. \tag{303}$$

Hence, the metric tensor for the binary operation $\boxplus$ has the form

$$G_\boxplus(x) = \left[I + \frac{xx^\top}{1-\|x\|^2}\right]^2 = \left[I - \frac{xx^\top}{\|x\|^2}\right] + \frac{1}{(1-\|x\|^2)^2}\frac{xx^\top}{\|x\|^2}\,. \tag{304}$$

The metric tensor $G_{\boxplus}(x)$ possesses the form of (6) with

$$m_0(\|x\|^2) = 1$$
$$m_1(\|x\|^2) = \frac{1}{(1-\|x\|^2)^2}\,. \tag{305}$$

We should note that the function m_1 is the same as for Einstein addition and Möbius addition. Hence, the distances for all these three binary operations are the same (115):

$$\|b \oplus_E (-a)\|_E = \|b \oplus_M (-a)\|_M = \|b \boxplus (-a)\|_{\boxplus} = \text{atanh}(\|b-a\|)\,, \tag{306}$$

where $\|\cdot\|$ is the Euclidean norm. Moreover, the multiplication by a number is the same for all these three binary operations,

$$t \otimes_E a = t \otimes_M a = t \otimes_{\boxplus} a\,. \tag{307}$$

We can use also the standard Euclidean norm $\|\cdot\|$. Notice that for numbers (that is, for $n = 1$) all binary operations coincide,

$$x \oplus_E y = x \oplus_M y = x \boxplus y = \frac{x+y}{1+xy} = \tanh(\text{atanh}(x) + \text{atanh}(y))\,. \tag{308}$$

Therefore, if $\oplus$ is one of the operations $\oplus_E$, $\oplus_M$, $\boxplus$, then for all $a, b \in \mathbb{B}$,

$$\|a \oplus b\| \le \|a\| \oplus \|b\|\,, \tag{309}$$

and the equality occurs if and only if for some nonnegative number t we have $a = tb$ or $b = ta$.

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