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A Gyrogeometric Mean in the Einstein Gyrogroup

Takuro Honma [1] and Osamu Hatori [2,*]

[1] Niigata Pref. Takada Senior High School, Yasuduka Branch School, Yasuduka 942-0411, Japan; tak.hon1090@gmail.com
[2] Institute of Science and Technology, Niigata University, Niigata 950-2181, Japan
* Correspondence: hatori@math.sc.niigata-u.ac.jp

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Abstract: In this paper, we define a gyrogeometric mean on the Einstein gyrovector space. It satisfies several properties one would expect for means. For example, it is permutation-invariant and left-translation invariant. It is already known that the Einstein gyrogroup is a gyrocommutative gyrogroup. We give an alternative proof which depends only on an elementary calculation.

Keywords: gyrogroup; Einstein velocity addition; gyrogeometric mean

1. Introduction

Einstein addition is a binary operation that stems from his velocity composition law of relativistic admissible velocities. Einstein addition is neither commutative nor associative. Ungar initiated the study of gyrogroups and gyrovector spaces [1] associated with the Einstein addition in the theory of special relativity. A gyrocommutative gyrogroup is a gyrogroup which has weak associativity and commutativity. It is a generalization of a commutative group.

Let $\mathbb{V}$ be a real inner product space. For a positive real number s we denote $\mathbb{V}_s$ the s-ball of $\mathbb{V}$, i.e.,

$$\mathbb{V}_s = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < s\}.$$

The Einstein addition $\oplus_E$ on $\mathbb{V}_s$ is a binary operation on $\mathbb{V}_s$ given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\},$$

where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{s^2}}}$$

in $\mathbb{V}_s$, where $\cdot$ and $\|\cdot\|$ are the inner product and the norm of $\mathbb{V}$ respectively. By the definition of $\oplus_E$, $\mathbf{u} \oplus_E \mathbf{v} \in \mathbb{V}_s$ for every pair $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$ by Theorem 3.46 and the identity (3.189) in [1].

In [1] (p. 88) Ungar described that "one can show by computer algebra that Einstein addition in the ball is a gyrocommutative gyrogroup operation, giving rise to the Einstein ball gyrogroup $(\mathbb{V}_1, \oplus_E)$." On the other hand, Suksumran and Wiboonton [2] gave a proof applying the theory of Clifford algebras, without using computer algebras. We give an elementary and direct proof in Section 6, which is lengthy but just by a simple calculation without applying any substantial theory of mathematics.

In the following up to Section 5 we assume $s = 1$ just for simplicity. The Einstein scalar multiplication $\otimes_E$ on $(\mathbb{V}_1, \oplus_E)$ is given by the equation

$$\begin{aligned} r \otimes_E \mathbf{a} &= \frac{(1+\|\mathbf{a}\|)^r - (1-\|\mathbf{a}\|)^r}{(1+\|\mathbf{a}\|)^r + (1-\|\mathbf{a}\|)^r} \frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \tanh(r \tanh^{-1} \|\mathbf{a}\|) \frac{\mathbf{a}}{\|\mathbf{a}\|}, \end{aligned}$$

where $r \in \mathbb{R}, \mathbf{a} \in \mathbb{V}_1 \setminus \{\mathbf{0}\}$; and $r \otimes_E \mathbf{0} = \mathbf{0}$. By Theorem 6.84 in [1], $(\mathbb{V}_1, \oplus_E, \otimes_E)$ is a gyrovector space, which is called an Einstein gyrovector space.

Ungar [1] (pp. 172–173) defined the gyromidpoint $\mathbf{P}^m_{\mathbf{ab}}$ of two elements in a gyrovector space. For $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$ we have

$$\mathbf{P}^m_{\mathbf{ab}} = \frac{\gamma_{\mathbf{a}} \mathbf{a} + \gamma_{\mathbf{b}} \mathbf{b}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}}. \tag{1}$$

Ungar also defined the gyrocentroid $\mathbb{C}^m_{\mathbf{abc}}$ of three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$ as

$$\mathbb{C}^m_{\mathbf{abc}} = \frac{\gamma_{\mathbf{a}} \mathbf{a} + \gamma_{\mathbf{b}} \mathbf{b} + \gamma_{\mathbf{c}} \mathbf{c}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}} + \gamma_{\mathbf{c}}}.$$

The gyromidpoint corresponds to the average of two velocities in the special theory of relativity. On the other hand the gyrocentroid $\mathbb{C}^m_{\mathbf{abc}}$ does not satisfy a certain desirable property one would expected for *means*; by a simple calculation we have $\mathbb{C}^m_{\mathbf{abc}} = \frac{\gamma_{\mathbf{c}}}{\gamma_{\mathbf{c}}+2} \mathbf{c} \neq \frac{1}{3} \otimes_E \mathbf{c}$ for $\mathbf{a} = \mathbf{b} = \mathbf{0}$ and $\mathbf{c} \neq 0$. In this paper, we propose an alternative definition of the mean of three or more elements, the gyrogeometric mean, and show that it has several properties one would expect for means. The gyrogeometric mean corresponding to the average of the velocities in the special relativity. It is symmetric in the sense that permutation-invariant by the definition of the gyrogeometric mean. It is translation invariant (Proposition 5). The main idea of the definitions come from the geometric mean for positive definite matrices by Bhatia and Holbrook [3] and Ando, Li and Mathias [4].

2. The Metric Space $(\mathbb{V}_1, \oplus_E, \otimes_E)$

We define the set $\|\mathbb{V}_1\| = \{\pm\|\mathbf{v}\| : \mathbf{v} \in \mathbb{V}_1\} \subset \mathbb{R}$ which coincides with the open interval $(-1, 1)$. $\|\mathbb{V}_1\|$ admits the addition $\oplus'$ and the scalar multiplication $\otimes'$ given by the following:

$$\begin{aligned} a \oplus' b &= \frac{a+b}{1+ab} \\ r \otimes' a &= \tanh(r \tanh^{-1} a) \end{aligned}$$

where $a, b \in \|\mathbb{V}_1\|$ and $r \in \mathbb{R}$. Please note that the triplet $(\|\mathbb{V}_1\|, \oplus', \otimes')$ is a real one-dimensional space.

The gyrometric is defined by

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \ominus_E \mathbf{b}\| \in \|\mathbb{V}_1\|,$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$ and $\mathbf{a} \ominus_E \mathbf{b} = \mathbf{a} \oplus_E (-\mathbf{b})$. The gyrometric needs not be a metric. It satisfies the following [1] (p. 158).

Proposition 1. *(1) For every pair $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$, $d(\mathbf{a}, \mathbf{b}) \geq 0$ The equality $d(\mathbf{a}, \mathbf{b}) = 0$ holds if and only if $\mathbf{a} = \mathbf{b}$.*

(2) $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{b}, \mathbf{a})$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$.

(3) The gyrotriangle inequality:

$$d(\mathbf{a}, \mathbf{b}) \leq d(\mathbf{a}, \mathbf{c}) \oplus' d(\mathbf{c}, \mathbf{b})$$

holds for any $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{V}_1$.

We define the metric δ on $\mathbb{V}_1$ induced by the gyrometric d. Put the map $f : \|\mathbb{V}_1\| \to \mathbb{R}$ by $f(x) = \tanh^{-1}(x)$. For any $a, b \in \|\mathbb{V}_1\|$ and $r \in \mathbb{R}$, the map f satisfies the following.

(F1) $f(a \oplus' b) = f(a) + f(b)$
(F2) $f(r \otimes' a) = rf(a)$

Let the map δ on $\mathbb{V}_1$ be given by

$$\delta(\mathbf{a}, \mathbf{b}) = f(d(\mathbf{a}, \mathbf{b}))$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$.

Lemma 1. *The inequality*

$$\frac{1}{\gamma_{\mathbf{a}}(1 - \mathbf{a} \cdot \mathbf{b})} \|\mathbf{b} - \mathbf{a}\| \leq \|\mathbf{a} \ominus_E \mathbf{b}\| \leq \frac{1}{1 - \mathbf{a} \cdot \mathbf{b}} \|\mathbf{b} - \mathbf{a}\|$$

holds for every pair $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$.

Proof. Recall the equations (3.177) and (3.178) [1] (pp. 88–89):

$$\begin{aligned} \gamma_{\mathbf{a} \oplus_{\mathbf{E}} \mathbf{b}} &= \gamma_{\mathbf{a}} \gamma_{\mathbf{b}} (1 + \mathbf{a} \cdot \mathbf{b}), \\ \gamma_{\mathbf{a} \ominus_{\mathbf{E}} \mathbf{b}} &= \gamma_{\mathbf{a}} \gamma_{\mathbf{b}} (1 - \mathbf{a} \cdot \mathbf{b}). \end{aligned}$$

By $\gamma_{\mathbf{a} \oplus_{\mathbf{E}} \mathbf{b}} = \frac{1}{\sqrt{1 - \|\mathbf{a} \ominus_E \mathbf{b}\|}}$ we have

$$\begin{aligned} \|\mathbf{a} \ominus_E \mathbf{b}\| &= \sqrt{1 - \frac{1}{\gamma_{\mathbf{a} \oplus_{\mathbf{E}} \mathbf{b}}^2}} \\ &= \sqrt{1 - \frac{1}{\gamma_{\mathbf{a}} \gamma_{\mathbf{b}} (1 - \mathbf{a} \cdot \mathbf{b})}} \\ &= \sqrt{1 - \frac{(1 - \|\mathbf{a}\|^2)(1 - \|\mathbf{b}\|^2)}{(1 - \mathbf{a} \cdot \mathbf{b})^2}} \\ &= \frac{\sqrt{(1 - \mathbf{a} \cdot \mathbf{b})^2 - (1 - \|\mathbf{a}\|^2)(1 - \|\mathbf{b}\|^2)}}{1 - \mathbf{a} \cdot \mathbf{b}} \\ &= \frac{\sqrt{\|\mathbf{b} - \mathbf{a}\|^2 - \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2}}{1 - \mathbf{a} \cdot \mathbf{b}}. \end{aligned}$$

Since $\mathbf{a} \cdot \mathbf{b} \leq \|\mathbf{a}\| \|\mathbf{b}\|$ then we have

$$\|\mathbf{a} \ominus_E \mathbf{b}\| \leq \frac{1}{1 - \mathbf{a} \cdot \mathbf{b}} \|\mathbf{b} - \mathbf{a}\|.$$

Next we calculate $\|\mathbf{a} \ominus_E \mathbf{b}\|^2 - \left\{ \frac{1}{\gamma_{\mathbf{a}}(1 - \mathbf{a} \cdot \mathbf{b})} \|\mathbf{b} - \mathbf{a}\| \right\}^2$.

$$\begin{aligned} & \frac{\|\mathbf{b} - \mathbf{a}\|^2 - \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2}{(1 - \mathbf{a} \cdot \mathbf{b})^2} - \frac{1 - \|\mathbf{a}\|^2}{(1 - \mathbf{a} \cdot \mathbf{b})^2} \|\mathbf{b} - \mathbf{a}\|^2 \\ = & \frac{1}{(1 - \mathbf{a} \cdot \mathbf{b})^2} \left\{ \|\mathbf{b} - \mathbf{a}\|^2 - \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2 - (1 - \|\mathbf{a}\|^2) \|\mathbf{b} - \mathbf{a}\|^2 \right\} \\ = & \frac{1}{(1 - \mathbf{a} \cdot \mathbf{b})^2} \left\{ (\mathbf{a} \cdot \mathbf{b})^2 + \|\mathbf{a}\|^4 - 2\|\mathbf{a}\|^2 (\mathbf{a} \cdot \mathbf{b}) \right\} \\ = & \frac{1}{(1 - \mathbf{a} \cdot \mathbf{b})^2} (\|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b})^2 \\ \geq & \; 0. \end{aligned}$$

Thus, we have the desired inequalities and conclude the proof. □

Proposition 2. $(\mathbb{V}_1, \delta)$ *is a complete metric space.*

Proof. We first prove that $(\mathbb{V}_1, \delta)$ is a metric space. By (1) and (2) of Proposition 1, it is trivial that $\delta(\mathbf{a}, \mathbf{b}) = 0 \Leftrightarrow \mathbf{a} = \mathbf{b}$ and $\delta(\mathbf{a}, \mathbf{b}) = \delta(\mathbf{b}, \mathbf{a})$ for every $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$. By (3) of Proposition 1 and the monotonicity of f, the inequality $f(d(\mathbf{a}, \mathbf{b})) \leq f(d(\mathbf{a}, \mathbf{c}) \oplus d(\mathbf{c}, \mathbf{b}))$ for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$. By (F1) we have

$$\delta(\mathbf{a}, \mathbf{b}) \leq \delta(\mathbf{a}, \mathbf{c}) + \delta(\mathbf{c}, \mathbf{b}).$$

As $\mathbb{V}$ is complete, we have by Lemma 1 and the definition of $\delta(\cdot, \cdot)$ that $(\mathbb{V}_1, \delta)$ is complete. □

We recall the gyroline and the gyrosegment [1] (Definition 6.19). Let $\mathbf{a}, \mathbf{b}$ be elements of $\mathbb{V}_1$. The gyroline through $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$L(\mathbf{a}, \mathbf{b}) = \{\mathbf{a} \oplus_E t \otimes_E (\ominus_E \mathbf{a} \oplus_E \mathbf{b}) \ : \ t \in \mathbb{R}\}.$$

A gyrosegment with endpoints $\mathbf{a}$ and $\mathbf{b}$ is denoted by

$$S(\mathbf{a}, \mathbf{b}) = \{\mathbf{a} \oplus_E t \otimes_E (\ominus_E \mathbf{a} \oplus_E \mathbf{b}) \ : \ 0 \leq t \leq 1\}.$$

The point $\mathbf{a} \#_t \mathbf{b} = \mathbf{a} \oplus_E t \otimes_E (\ominus_E \mathbf{a} \oplus_E \mathbf{b})$ is called the gyro t-point on a gyroline or gyrosegment. We abbreviate $\mathbf{a} \#_{\frac{1}{2}} \mathbf{b}$ by $\mathbf{a} \# \mathbf{b}$. Please note that $\mathbf{a} \# \mathbf{b} = \mathbf{b} \# \mathbf{a}$ for every pair $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$.

Theorem 1. *For any* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$ *we have*

$$d(\mathbf{a} \# \mathbf{b}, \mathbf{a} \# \mathbf{c}) \leq \frac{1}{2} \otimes d(\mathbf{b}, \mathbf{c}). \tag{2}$$

Proof. To begin with the proof of the inequality (2), we show an equation related to the gyrometric and gamma factor. Recall the equations (3.197) and (6.266) [1] (pp. 93, 209):

$$\begin{aligned} \gamma_{\mathbf{a} \boxplus_{\mathbf{E}} \mathbf{b}} &= \frac{(\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}})^2}{\gamma_{\mathbf{a} \ominus \mathbf{b}} + 1} - 1, \\ \gamma_{\frac{1}{2} \otimes_{\mathbf{E}} \mathbf{a}} &= \sqrt{\frac{1 + \gamma_{\mathbf{a}}}{2}}. \end{aligned}$$

Hence

$$\gamma_{\mathbf{a} \# \mathbf{b}} = \frac{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}}{\sqrt{2(\gamma_{\mathbf{a} \ominus \mathbf{b}} + 1)}} \tag{3}$$

holds. By a simple calculation, we have

$$1 - (\mathbf{a} \# \mathbf{b}) \cdot (\mathbf{a} \# \mathbf{c}) = \frac{1 + \gamma_{\mathbf{a} \ominus \mathbf{b}} + \gamma_{\mathbf{b} \ominus \mathbf{c}} + \gamma_{\mathbf{c} \ominus \mathbf{a}}}{(\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}})(\gamma_{\mathbf{a}} + \gamma_{\mathbf{c}})}. \tag{4}$$

Hence we have

$$\begin{aligned} \gamma_{(\mathbf{a} \# \mathbf{b}) \ominus_{\mathbf{E}} (\mathbf{a} \# \mathbf{c})} &= \gamma_{\mathbf{a} \# \mathbf{b}} \gamma_{\mathbf{a} \# \mathbf{c}} (1 - (\mathbf{a} \# \mathbf{b}) \cdot (\mathbf{a} \# \mathbf{c})) \\ &= \frac{1 + \gamma_{\mathbf{a} \ominus \mathbf{b}} + \gamma_{\mathbf{b} \ominus \mathbf{c}} + \gamma_{\mathbf{c} \ominus \mathbf{a}}}{2\sqrt{(\gamma_{\mathbf{a} \ominus \mathbf{b}} + 1)(\gamma_{\mathbf{a} \ominus \mathbf{c}} + 1)}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} d^2(\mathbf{a}\#\mathbf{b}, \mathbf{a}\#\mathbf{c}) &= 1 - \frac{1}{\gamma_{(\mathbf{a}\#\mathbf{b})\ominus_{\mathbf{E}}(\mathbf{a}\#\mathbf{c})}{}^2} \\ &= 1 - \frac{4(\gamma_{\mathbf{a}\ominus\mathbf{b}}+1)(\gamma_{\mathbf{a}\ominus\mathbf{c}}+1)}{(1+\gamma_{\mathbf{a}\ominus\mathbf{b}}+\gamma_{\mathbf{b}\ominus\mathbf{c}}+\gamma_{\mathbf{c}\ominus\mathbf{a}})^2}. \end{aligned} \tag{6}$$

We also have

$$\begin{aligned} \frac{1}{2}\otimes d(\mathbf{b},\mathbf{c}) &= \tanh\Big(\frac{1}{2}\tanh^{-1} d(\mathbf{b},\mathbf{c})\Big) \\ &= \frac{\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}}{1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}} d(\mathbf{b},\mathbf{c}) \\ &= \sqrt{\frac{\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}-1}{1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}}}, \end{aligned}$$

where $\gamma_{\mathbf{d}(\mathbf{b},\mathbf{c})} = \gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}$. Hence we have

$$\Big(\frac{1}{2}\otimes d(\mathbf{b},\mathbf{c})\Big)^2 - d^2(\mathbf{a}\#\mathbf{b}, \mathbf{a}\#\mathbf{c}) \tag{7}$$

$$\begin{aligned} &= \frac{\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}-1}{1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}} - \Big(1 - \frac{4(\gamma_{\mathbf{a}\ominus\mathbf{b}}+1)(\gamma_{\mathbf{a}\ominus\mathbf{c}}+1)}{(1+\gamma_{\mathbf{a}\ominus\mathbf{b}}+\gamma_{\mathbf{b}\ominus\mathbf{c}}+\gamma_{\mathbf{c}\ominus\mathbf{a}})^2}\Big) \\ &= \frac{4(\gamma_{\mathbf{a}\ominus\mathbf{b}}+1)(\gamma_{\mathbf{a}\ominus\mathbf{c}}+1)}{(1+\gamma_{\mathbf{a}\ominus\mathbf{b}}+\gamma_{\mathbf{b}\ominus\mathbf{c}}+\gamma_{\mathbf{c}\ominus\mathbf{a}})^2} - \frac{2}{1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}} \\ &= \frac{2\Big\{2(\gamma_{\mathbf{a}\ominus\mathbf{b}}+1)(\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}+1)(\gamma_{\mathbf{c}\ominus\mathbf{a}}+1) - (1+\gamma_{\mathbf{a}\ominus\mathbf{b}}+\gamma_{\mathbf{b}\ominus\mathbf{c}}+\gamma_{\mathbf{c}\ominus\mathbf{a}})^2\Big\}}{(1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}})(1+\gamma_{\mathbf{a}\ominus\mathbf{b}}+\gamma_{\mathbf{b}\ominus\mathbf{c}}+\gamma_{\mathbf{c}\ominus\mathbf{a}})^2} \end{aligned}$$

$$= \frac{2\Big\{2\gamma_{\mathbf{a}\ominus\mathbf{b}}\gamma_{\mathbf{b}\ominus\mathbf{c}}\gamma_{\mathbf{c}\ominus\mathbf{a}} - (\gamma_{\mathbf{a}\ominus\mathbf{b}}{}^2+\gamma_{\mathbf{b}\ominus\mathbf{c}}{}^2+\gamma_{\mathbf{c}\ominus\mathbf{a}}{}^2)+1\Big\}}{(1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}})(1+\gamma_{\mathbf{a}\ominus\mathbf{b}}+\gamma_{\mathbf{b}\ominus\mathbf{c}}+\gamma_{\mathbf{c}\ominus\mathbf{a}})^2}. \tag{8}$$

Let $\mathbf{A} = \ominus_E\mathbf{c}\oplus_E\mathbf{a}, \mathbf{B} = \ominus_E\mathbf{c}\oplus_E\mathbf{b}$. It is well defined by $\|(\ominus_E\mathbf{c}\oplus_E\mathbf{a})\ominus_E(\ominus_E\mathbf{c}\oplus_E\mathbf{b})\| = \|\mathrm{gyr}[\ominus_{\mathbf{E}}\mathbf{c},\mathbf{a}](\mathbf{a}\ominus_E\mathbf{b})\| = \|\mathbf{a}\ominus_E\mathbf{b}\|$. We calculate the numerator of (8);

$$\begin{aligned} & 2\gamma_{\mathbf{a}\ominus\mathbf{b}}\gamma_{\mathbf{b}\ominus\mathbf{c}}\gamma_{\mathbf{c}\ominus\mathbf{a}} - (\gamma_{\mathbf{a}\ominus\mathbf{b}}{}^2+\gamma_{\mathbf{b}\ominus\mathbf{c}}{}^2+\gamma_{\mathbf{c}\ominus\mathbf{a}}{}^2)+1 \\ =\; & 2\gamma_{\mathbf{A}\ominus_{\mathbf{E}}\mathbf{B}}\gamma_{\mathbf{A}}\gamma_{\mathbf{B}} - (\gamma_{\mathbf{A}\ominus_{\mathbf{E}}\mathbf{B}}{}^2+\gamma_{\mathbf{A}}{}^2+\gamma_{\mathbf{B}}{}^2)+1 \\ =\; & 2\gamma_{\mathbf{A}}{}^2\gamma_{\mathbf{B}}{}^2(1-\mathbf{A}\cdot\mathbf{B}) - (\gamma_{\mathbf{A}}{}^2\gamma_{\mathbf{B}}{}^2(1-\mathbf{A}\cdot\mathbf{B})^2+\gamma_{\mathbf{A}}{}^2+\gamma_{\mathbf{B}}{}^2)+1 \\ =\; & \gamma_{\mathbf{A}}{}^2\gamma_{\mathbf{B}}{}^2(1-(\mathbf{A}\cdot\mathbf{B})^2) - (\gamma_{\mathbf{A}}{}^2+\gamma_{\mathbf{B}}{}^2)+1 \\ =\; & \frac{1-(\mathbf{A}\cdot\mathbf{B})^2}{(1-\|\mathbf{A}\|^2)(1-\|\mathbf{B}\|^2)} - \Big(\frac{1}{(1-\|\mathbf{A}\|^2)}+\frac{1}{(1-\|\mathbf{B}\|^2)}\Big)+1 \\ =\; & \frac{\|\mathbf{A}\|^2\|\mathbf{B}\|^2-(\mathbf{A}\cdot\mathbf{B})^2}{(1-\|\mathbf{A}\|^2)(1-\|\mathbf{B}\|^2)} \\ \geq\; & 0. \end{aligned}$$

We conclude a proof of Theorem 1. □

By Theorem 1 and the monotonicity of $f(x) = \tanh^{-1}(x)$, we have

$$\delta(\mathbf{a}\#\mathbf{b}, \mathbf{a}\#\mathbf{c}) \leq \frac{1}{2}\delta(\mathbf{b},\mathbf{c}). \tag{9}$$

By the triangle inequality, we have

$$\begin{aligned} \delta(\mathbf{a}\#\mathbf{b},\mathbf{c}\#\mathbf{d}) &\leq \delta(\mathbf{a}\#\mathbf{b},\mathbf{a}\#\mathbf{d})+\delta(\mathbf{a}\#\mathbf{d},\mathbf{c}\#\mathbf{d}) \\ &\leq \frac{1}{2}\delta(\mathbf{b},\mathbf{d})+\frac{1}{2}\delta(\mathbf{a},\mathbf{c}). \end{aligned} \tag{10}$$

Moreover, since the map $g(t)=\delta(\mathbf{a}\#_t\mathbf{b},\mathbf{c}\#_t\mathbf{d})$ is continuous, we infer that g is convex, i.e.,

$$\delta(\mathbf{a}\#_t\mathbf{b},\mathbf{c}\#_t\mathbf{d}) \leq (1-t)\delta(\mathbf{a},\mathbf{c})+t\delta(\mathbf{b},\mathbf{d}). \tag{11}$$

Letting $\mathbf{a}=\mathbf{c}$ we have

$$\delta(\mathbf{a}\#_t\mathbf{b},\mathbf{a}\#_t\mathbf{c}) \leq t\delta(\mathbf{b},\mathbf{c}). \tag{12}$$

3. The Gyroconvex Set and the Gyroconvex Hull in a Gyrovector Space

We define a gyroconvex set and a gyroconvex hull.

Definition 1. *Let A be a subset of $\mathbb{V}_1$. We say that A is gyroconvex if $S(\mathbf{a},\mathbf{b})\subset A$ for any $\mathbf{a},\mathbf{b}\in A$. Let X be a non-empty subset of $\mathbb{V}_1$.*

$$\mathrm{conv}(X)=\cap\{C\subset\mathbb{V}_1 : X\subset C \textit{ and } C \textit{ is gyroconvex set}\}.$$

We call $\mathrm{conv}(X)$ *the gyroconvex hull of X.*

Please note that the gyroconvex hull of a non-empty set $X\subset\mathbb{V}$ is gyroconvex.

Lemma 2. *Let $a,b\in\mathbb{V}_1$. Then the gyrosegment $S(\mathbf{a},\mathbf{b})$ is gyroconvex. The gyroconvex hull* $\mathrm{conv}(\{\mathbf{a},\mathbf{b}\})$ *coincides with $S(\mathbf{a},\mathbf{b})$.*

Proof. Let $\mathbf{P}_j$ be an arbitrary point in $S(\mathbf{a},\mathbf{b})$ for $j=1,2$. There exists $0\leq t_j\leq 1$ such that

$$\mathbf{P}_j=\mathbf{a}\oplus_E t_j\otimes_E(\ominus_E\mathbf{a}\oplus_E\mathbf{b})$$

for $j=1,2$. We may assume that $t_1\leq t_2$. where $0\leq t_1\leq t_2\leq 1$. Then we have $t\in[0,\ 1]$, $\mathbf{P}_1\#_t\mathbf{P}_2\in S(\mathbf{a},\mathbf{b})$. In fact, by the Equation (6.63) in [1] (p. 167) we have

$$\mathbf{P}_1\oplus_E t\otimes_E(\ominus_E\mathbf{P}_1\oplus_E\mathbf{P}_2)=\mathbf{a}\oplus_E(t_1+(-t_1+t_2)t)\otimes_E(\ominus_E\mathbf{a}\oplus_E\mathbf{b}).$$

Since $t_1\leq t_1+(-t_1+t_2)t\leq t_2$, we have $\mathbf{P}_1\#_t\mathbf{P}_2\in S(\mathbf{a},\mathbf{b})$. Thus, $S(\mathbf{P}_1,\mathbf{P}_2)\subset S(\mathbf{a},\mathbf{b})$ for every pair $\mathbf{P}_1$ and $\mathbf{P}_2$ in $S(\mathbf{a},\mathbf{b})$. Thus, $S(\mathbf{a},\mathbf{b})$ is gyroconvex. □

Let C_0 be a non-empty subset of $\mathbb{V}_1$. We define a sequence $\{C_n\}$ of a non-empty subset of $\mathbb{V}_1$ by induction. Suppose that C_{n-1} is defined. Put

$$C_n=\bigcup_{\mathbf{a},\mathbf{b}\in C_{n-1}} S(\mathbf{a},\mathbf{b}).$$

Proposition 3. *Let C_0 be a non-empty subset of $\mathbb{V}_1$. Then*

$$\mathrm{conv}(C_0)=\bigcup_{n=0}^{\infty} C_n.$$

Proof. We prove that $\cup_{k=0}^{\infty}C_n$ is gyroconvex. Let $\mathbf{a},\mathbf{b}\in\cup_{k=0}^{\infty}C_n$. Since $C_k\subset C_{k+1}$ for every $k\in\mathbb{N}\cup\{0\}$, there exists a positive integer n_0 with $\mathbf{a},\mathbf{b}\in C_{n_0}$. Then by the definition of C_{n_0+1} we have $S(\mathbf{a},\mathbf{b})\subset C_{n_0+1}\subset\cup_{k=0}^{\infty}C_n$. Thus, $\cup_{n=0}^{\infty}C_n$ is a gyroconvex set. As $C_0\subset\cup_{k=0}^{\infty}C_n$, we have $\mathrm{conv}(C_0)\subset\cup_{n=0}^{\infty}C_n$.

We prove $\cup_{n=0}^{\infty} C_n \subset \mathrm{conv}(C_0)$. For any $\mathbf{a}, \mathbf{b} \in C_0$, $S(\mathbf{a}, \mathbf{b}) \subset \mathrm{conv}(C_0)$. Hence $C_1 \subset \mathrm{conv}(X)$. Similarly, assuming that $C_n \subset \mathrm{conv}(C_0)$ for any $n \in \mathbb{N} \cup \{0\}$ we have $C_{n+1} \subset \mathrm{conv}(C_0)$. So for arbitrary nonnegative integer n, $C_n \subset \mathrm{conv}(C_0)$; $\cup_{n=0}^{\infty} C_n \subset \mathrm{conv}(C_0)$. □

4. The Gyrogeometric Mean

Let $\phi \neq X \subset \mathbb{V}_1$. We define $\mathrm{diam}(X) = \sup\{\delta(\mathbf{x}, \mathbf{y}) \; : \; \mathbf{x}, \mathbf{y} \in X\}$. First we prove the following.

Proposition 4. *Suppose that $\mathbf{x}_0, \mathbf{y}_0, \mathbf{x}_1, \mathbf{y}_1 \in \mathbb{V}_1$. If the inequality $\mathrm{diam}(\{\mathbf{x}_0, \mathbf{y}_0, \mathbf{x}_1, \mathbf{y}_1\}) \leq M$ holds for a positive real number M, then the inequality $\delta(\mathbf{x}, \mathbf{y}) \leq M$ holds for arbitrary points $\mathbf{x} \in S(\mathbf{x}_0, \mathbf{x}_1)$ and $\mathbf{y} \in S(\mathbf{y}_0, \mathbf{y}_1)$.*

Proof. Put $\mathbf{x} = \mathbf{x}_0 \#_t \mathbf{x}_1$ and $\mathbf{y} = \mathbf{y}_0 \#_s \mathbf{y}_1$, where $0 \leq s \leq t \leq 1$. We have

$$
\begin{aligned}
& d(\mathbf{y}_0 \#_t \mathbf{x}_1, \mathbf{y}_0 \#_s \mathbf{x}_1) \\
= \; & \|\{\mathbf{y}_0 \oplus_E t \otimes_E (\ominus_E \mathbf{y}_0 \oplus_E \mathbf{x}_1)\} \ominus_E \{\mathbf{y}_0 \oplus_E s \otimes_E (\ominus_E \mathbf{y}_0 \oplus_E \mathbf{x}_1)\}\| \\
= \; & \|\mathrm{gyr}[\ominus_E \mathbf{y_0}, t \otimes_E (\ominus_E \mathbf{y_0} \oplus_E \mathbf{x_1})]\{t \otimes_E (\ominus_E \mathbf{y}_0 \oplus_E \mathbf{x}_1) \ominus_E s \otimes_E (\ominus_E \mathbf{y}_0 \oplus_E \mathbf{x}_1)\}\| \\
= \; & (t - s) \otimes \| \ominus_E \mathbf{y}_0 \oplus_E \mathbf{x}_1\|.
\end{aligned}
$$

Applying this equality, we have by (11) and (12) that

$$
\begin{aligned}
\delta(\mathbf{x}, \mathbf{y}) & \leq \delta(\mathbf{x}, \mathbf{y}_0 \#_t \mathbf{x}_1) + \delta(\mathbf{y}_0 \#_t \mathbf{x}_1, \mathbf{y}_0 \#_s \mathbf{x}_1) + \delta(\mathbf{y}_0 \#_s \mathbf{x}_1, \mathbf{y}) \\
& = \delta(\mathbf{x}_0 \#_t \mathbf{x}_1, \mathbf{y}_0 \#_t \mathbf{x}_1) + (t - s)\delta(\mathbf{y}_0, \mathbf{x}_1) + \delta(\mathbf{y}_0 \#_s \mathbf{x}_1, \mathbf{y}_0 \#_s \mathbf{y}_1) \\
& \leq (1 - t)\delta(\mathbf{x}_0, \mathbf{y}_0) + (t - s)\delta(\mathbf{y}_0, \mathbf{x}_1) + s\delta(\mathbf{x}_1, \mathbf{y}_1) \\
& \leq (1 - t)M + (t - s)M + sM = M.
\end{aligned}
$$

□

Lemma 3. *Let $X \subset \mathbb{V}$ be a non-empty set. Then*

$$\mathrm{diam}(X) = \mathrm{diam}(\mathrm{conv}(X)) = \mathrm{diam}(\overline{\mathrm{conv}(X)}).$$

Proof. First we prove $\mathrm{diam}(X) = \mathrm{diam}(\mathrm{conv}(X))$. By Proposition 3, $\mathrm{conv}(X) = \bigcup_{n=0}^{\infty} C_n$ where $C_0 = X$. For any positive integer k let $\mathbf{x}, \mathbf{y}$ be arbitrary points in C_k. Then there exist $\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}_0, \mathbf{y}_1 \in C_{k-1}$ such that $\mathbf{x} \in S(\mathbf{x}_0, \mathbf{x}_1), \mathbf{y} \in S(\mathbf{y}_0, \mathbf{y}_1)$. Put $M = \mathrm{diam}(C_{k-1})$ then by Proposition 4.1 we have

$$\delta(\mathbf{x}, \mathbf{y}) \leq M = \mathrm{diam}(C_{k-1}),$$

whence

$$\mathrm{diam}(C_k) \leq \mathrm{diam}(C_{k-1}).$$

Thus, for arbitrary $n \in \mathbb{N}$, $\mathrm{diam}(C_n) \leq \mathrm{diam}(C_0) = \mathrm{diam}(X)$. It follows that for any $\mathbf{x}, \mathbf{y} \in \mathrm{conv}(X) = \bigcup_{n=0}^{\infty} C_n$, we have

$$\delta(\mathbf{x}, \mathbf{y}) \leq \mathrm{diam}(X).$$

Therefore

$$\mathrm{diam}(\mathrm{conv}(X)) \leq \mathrm{diam}(X).$$

The converse inequality is trivial, hence we have $\mathrm{diam}(\mathrm{conv}(X)) = \mathrm{diam}(X)$.

Next we prove $\mathrm{diam}(\mathrm{conv}(X)) = \mathrm{diam}(\overline{\mathrm{conv}(X)})$. For any pair $\mathbf{x}, \mathbf{y} \in \overline{\mathrm{conv}(X)}$, there exist sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ in $\mathrm{conv}(X)$ such that $\mathbf{x}_n, \mathbf{y}_n$ converge to $\mathbf{x}, \mathbf{y}$ respectively. Letting $n \to \infty$ for $\delta(\mathbf{x}_n, \mathbf{y}_n) \leq \mathrm{diam}(\mathrm{conv}(X))$, we have $\delta(\mathbf{x}, \mathbf{y}) \leq \mathrm{diam}(X)$. Thus, $\mathrm{diam}(\overline{\mathrm{conv}(X)}) \leq \mathrm{diam}(\mathrm{conv}(X))$. The converse inequality is trivial, hence $\mathrm{diam}(\mathrm{conv}(X)) = \mathrm{diam}(\overline{\mathrm{conv}(X)})$ holds. □

Lemma 4. *Suppose that K is a gyroconvex subset of $\mathbb{V}$. Then the closure $\overline{K}$ of K is gyroconvex.*

Proof. For any $\mathbf{x}, \mathbf{y} \in \overline{K}$, there exist $\{\mathbf{x}_n\}, \{\mathbf{y}_n\} \subset K$ such that $\mathbf{x}_n, \mathbf{y}_n$ converge to $\mathbf{x}, \mathbf{y}$ respectively. We show $\mathbf{x}_n \#_t \mathbf{y}_n$ converges to $\mathbf{x} \#_t \mathbf{y}$ for arbitrary $0 \leq t \leq 1$. By (11) we have

$$\delta(\mathbf{x}_n \#_t \mathbf{y}_n, \mathbf{x} \#_t \mathbf{y}) \leq (1-t)\delta(\mathbf{x}_n, \mathbf{x}) + t\delta(\mathbf{y}_n, \mathbf{y}).$$

By $n \to \infty$, then $\delta(\mathbf{x}_n \#_t \mathbf{y}_n, \mathbf{x}\#\mathbf{y}) \to 0$. Thus, $\mathbf{x} \#_t \mathbf{y} \in \overline{K}$. Hence $\overline{K}$ is gyroconvex. □

Let n be a positive integer. Let X_n be the set of all subsets of $\mathbb{V}_1$ whose number of elements is exactly n. We define, by induction, the sequence $\{G_n\}_{n=2}^{\infty}$ of the maps $G_n : X_n \to \mathbb{V}_1$ which satisfy the following two conditions (p_n) and (q_n);

(p_n) $G_n(\Delta) \in \overline{\mathrm{conv}(\Delta)}$ for every $\Delta \in X_n$,

(q_n) $\delta(G_n(\Delta), G_n(\Delta')) \leq \frac{1}{n}\sum_{i=1}^{n} \delta(\mathbf{a}_i, \mathbf{a}'_i)$ for every pair $\Delta = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\Delta' = \{\mathbf{a}'_1, \ldots, \mathbf{a}'_n\}$ in X_n.

First, put $G_2(\{\mathbf{a}, \mathbf{b}\}) = \mathbf{a}\#\mathbf{b}$ for $\{\mathbf{a}, \mathbf{b}\} \in X_2$. As $\mathrm{conv}(\{\mathbf{a}, \mathbf{b}\}) = S(\mathbf{a}, \mathbf{b})$ by Lemma 2 we obtain that $G_2(\{\mathbf{a}, \mathbf{b}\}) \in \mathrm{conv}(\{\mathbf{a}, \mathbf{b}\})$; (p_2) holds. Let $\Delta = \{\mathbf{a}, \mathbf{b}\}$, $\Delta' = \{\mathbf{c}, \mathbf{d}\} \in X_2$. Then by (10) we have

$$\delta(G_2(\Delta), G_2(\Delta')) = \delta(\mathbf{a}\#\mathbf{b}, \mathbf{c}\#\mathbf{d}) \leq \frac{1}{2}\{\delta(\mathbf{a}, \mathbf{c}) + \delta(\mathbf{b}, \mathbf{d})\},$$

which is (q_2).

Suppose now that the map $G_k : X_k \to \mathbb{V}_1$ which satisfies (p_k) and (q_k) is defined. We will define $G_{k+1} : X_{k+1} \to \mathbb{V}_1$ which satisfies (p_{k+1}) and (q_{k+1}). Let $\Delta^0 = \{\mathbf{a}_1^0, \ldots, \mathbf{a}_{k+1}^0\} \in X_{k+1}$. For a positive integer m we define $\Delta^m \in X_{k+1}$ which satisfies that $\Delta^m \subset \overline{\mathrm{conv}(\Delta^{m-1})}$ by induction on m. For every $1 \leq i \leq k+1$, put

$$\mathbf{a}_i^1 = G_k(\{\mathbf{a}_1^0, \ldots, \mathbf{a}_{i-1}^0, \mathbf{a}_{i+1}^0, \ldots, \mathbf{a}_{k+1}^0\}).$$

Please note that $\mathbf{a}_i^1$ is well defined since $\{\mathbf{a}_1^0, \ldots, \mathbf{a}_{i-1}^0, \mathbf{a}_{i+1}^0, \ldots, \mathbf{a}_{k+1}^0\} \in X_k$ and we have assumed that the map G_k is defined. By the condition (p_k) we have that

$$\mathbf{a}_i^1 \in \overline{\mathrm{conv}(\{\mathbf{a}_1^0, \ldots, \mathbf{a}_{i-1}^0, \mathbf{a}_{i+1}^0, \ldots, \mathbf{a}_{k+1}^0\})}$$

for every $1 \leq i \leq k+1$. Put $\Delta^1 = \{\mathbf{a}_1^1, \ldots, \mathbf{a}_{k+1}^1\}$. Then $\Delta^1 \in X_{k+1}$ and $\Delta^1 \subset \overline{\mathrm{conv}(\Delta^0)}$ since $\{\mathbf{a}_1^0, \ldots, \mathbf{a}_{i-1}^0, \mathbf{a}_{i+1}^0, \ldots, \mathbf{a}_{k+1}^0\} \subset \Delta^0$ for every $1 \leq i \leq k+1$. Suppose that $\Delta^l = \{\mathbf{a}_1^l, \ldots, \mathbf{a}_{k+1}^l\} \in X_{k+1}$ such that $\Delta^l \subset \overline{\mathrm{conv}(\Delta^{l-1})}$ is defined. For every $1 \leq i \leq k+1$, put

$$\mathbf{a}_i^{l+1} = G_k(\{\mathbf{a}_1^l, \ldots, \mathbf{a}_{i-1}^l, \mathbf{a}_{i+1}^l, \ldots, \mathbf{a}_{k+1}^l\}).$$

As in the same way as the above, $\mathbf{a}_i^{l+1}$ is well defined for $1 \leq i \leq k+1$, and $\Delta^{l+1} = \{\mathbf{a}_1^{l+1}, \ldots, \mathbf{a}_{k+1}^{l+1}\} \in X_{k+1}$ satisfies that $\Delta^{l+1} \subset \overline{\mathrm{conv}(\Delta^l)}$. Hence, by induction, we have defined a sequence $\{\Delta^m\}_{m=1}^{\infty} \subset X_{k+1}$ such that $\Delta^m \subset \overline{\mathrm{conv}(\Delta^{m-1})}$. Applying (q_k) for $\Delta = \{\mathbf{a}_1^m, \ldots, \mathbf{a}_{i-1}^m, \mathbf{a}_{i+1}^m, \ldots, \mathbf{a}_{k+1}^m\}$ and $\Delta' = \{\mathbf{a}_1^m, \ldots, \mathbf{a}_{j-1}^m, \mathbf{a}_{j+1}^m, \ldots, \mathbf{a}_{k+1}^m\}$, we infer that

$$\begin{aligned}
\delta(\mathbf{a}_i^{m+1}, \mathbf{a}_j^{m+1}) &= \delta(G_k(\Delta), G_k(\Delta')) \\
&= \delta(G_k(\{\mathbf{a}_1^m, \ldots, \mathbf{a}_{i-1}^m, \mathbf{a}_{i+1}^m, \ldots, \mathbf{a}_{k+1}^m\}), G_k(\{\mathbf{a}_1^m, \ldots, \mathbf{a}_{j-1}^m, \mathbf{a}_{j+1}^m, \ldots, \mathbf{a}_{k+1}^m\})) \\
&= \delta(G_k(\{\mathbf{a}_1^m, \ldots, \mathbf{a}_{i-1}^m, \mathbf{a}_{i+1}^m, \ldots, \mathbf{a}_{j-1}^m, \mathbf{a}_{j+1}^m, \ldots, \mathbf{a}_{k+1}^m, \mathbf{a}_j^m\}), \\
&\qquad G_k(\{\mathbf{a}_1^m, \ldots, \mathbf{a}_{i-1}^m, \mathbf{a}_{i+1}^m, \ldots, \mathbf{a}_{j-1}^m, \mathbf{a}_{j+1}^m, \ldots, \mathbf{a}_{k+1}^m, \mathbf{a}_i^m\})) \\
&\leq \frac{1}{k}\delta(\mathbf{a}_j^m, \mathbf{a}_i^m) = \frac{1}{k}\delta(\mathbf{a}_i^m, \mathbf{a}_j^m)
\end{aligned}$$

Then by Lemma 3 we obtain

$$\operatorname{diam}(\overline{\operatorname{conv}(\Delta^{m+1})}) \leq \frac{1}{k}\operatorname{diam}(\overline{\operatorname{conv}(\Delta^{m})})$$

for every positive integer m. By Cantor's intersection theorem there exists a unique $M_\infty \in \mathbb{V}_1$ with

$$\{M_\infty\} = \cap_m \overline{\operatorname{conv}(\Delta^m)}.$$

As $a_i^m \in \overline{\operatorname{conv}(\Delta^{m-1})}$ for every $1 \leq i \leq k+1$, we infer that $\lim_{m\to\infty} \mathbf{a}_i^m = M_\infty$ for every $1 \leq i \leq k+1$. Put $G_{k+1}(\Delta^0) = M_\infty$. Then the map $G_{k+1} : X_{k+1} \to \mathbb{V}_1$ is well defined, and $G_{k+1}(\Delta^0) = M_\infty \in \overline{\operatorname{conv}(\Delta^0)}$; G_{k+1} satisfies the condition (p_n).

Next we prove that the map G_{k+1} satisfies the condition (q_{k+1});

$$\delta(G_{k+1}(\Delta), G_{k+1}(\Delta')) \leq \frac{1}{k+1}\Big(\sum_{j=1}^{k+1}\delta(\mathbf{a}_j, \mathbf{a'}_j)\Big),$$

where $\Delta = \{\mathbf{a}_1^0, \dots, \mathbf{a}_{k+1}^0\}, \Delta' = \{\mathbf{a'}_1^0, \dots, \mathbf{a'}_{k+1}^0\} \in X_{k+1}$. Let m be a positive integer. We define $\mathbf{a}_i^m$ and $\mathbf{a'}_i^m$ for every $1 \leq i \leq k+1$ as in the same way as before. As (q_k) holds for G_k, we have

$$\begin{aligned}
&\delta(\mathbf{a}_i^m, \mathbf{a'}_i^m)\\
&\quad = \delta(G_k(\{\mathbf{a}_1^{m-1}, \mathbf{a}_2^{m-1}, \cdots, \mathbf{a}_{i-1}^{m-1}, \mathbf{a}_{i+1}^{m-1}, \cdots, \mathbf{a}_{k+1}^{m-1}\}),\\
&\quad G_k(\{\mathbf{a'}_1^{m-1}, \mathbf{a'}_2^{m-1}, \cdots, \mathbf{a'}_{i-1}^{m-1}, \mathbf{a'}_{i+1}^{m-1}, \cdots, \mathbf{a'}_{k+1}^{m-1}\}))\\
&\qquad \leq \frac{1}{k}\Big(\sum_{j=1, j\neq i}^{k+1}\delta(\mathbf{a}_j^{m-1}, \mathbf{a'}_j^{m-1})\Big).
\end{aligned}$$

By summing up the above inequalities with respect to $1 \leq i \leq k+1$ we have

$$\sum_{i=1}^{k+1}\delta(\mathbf{a}_i^m, \mathbf{a'}_i^m) \leq \sum_{i=1}^{k+1}\left(\frac{1}{k}\Big(\sum_{j=1, j\neq i}^{k+1}\delta(\mathbf{a}_j^{m-1}, \mathbf{a'}_j^{m-1})\Big)\right) = \sum_{j=1}^{k+1}\delta(\mathbf{a}_j^{m-1}, \mathbf{a'}_j^{m-1}).$$

for every positive integer m. Thus, we have

$$\sum_{i=1}^{k+1}\delta(\mathbf{a}_i^m, \mathbf{a'}_i^m) \leq \sum_{j=1}^{k+1}\delta(\mathbf{a}_j, \mathbf{a'}_j).$$

Letting $m \to \infty$, since $\lim_{m\to\infty} \mathbf{a}_i^m = G_{k+1}(\Delta), \lim_{m\to\infty} \mathbf{a'}_i^m = G_{k+1}(\Delta')$, we have

$$\sum_{i=1}^{k+1}\delta(G_{k+1}(\Delta), G_{k+1}(\Delta')) \leq \sum_{j=1}^{k+1}\delta(\mathbf{a}_j, \mathbf{a'}_j),$$

hence

$$\delta(G_{k+1}(\Delta)), G_{k+1}(\Delta')) \leq \frac{1}{k+1}(\sum_{j=1}^{k+1}\delta(\mathbf{a}_j, \mathbf{a'}_j)).$$

So, the condition (q_{k+1}) holds for the map G_{k+1}. We conclude that the map $G_n : X_n \to \mathbb{V}_1$ which satisfies the conditions (p_n) and (q_n) are defined by induction.

By applying the maps G_n we define the gyrogeometric mean of n elements in $\mathbb{V}_1$.

Definition 2. *Let $\Delta = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{V}_1$. We call that $G_n(\Delta)$ the gyrogeometric mean of Δ.*

Due to the definition, the gyrogeometric mean of $\{\mathbf{a}, \mathbf{b}\} \subset \mathbb{V}_1$ is $\mathbf{a}\#\mathbf{b}$. The gyrocentroid $\mathbb{C}^m_{\mathbf{abc}}$ is defined by applying the internal division points on the usual lines which makes the inconvenience such as $\mathbb{C}^m_{\mathbf{abc}} = \frac{\gamma_{\mathbf{c}}}{\gamma_{\mathbf{c}}+2}\mathbf{c} \neq \frac{1}{3} \otimes_E \mathbf{c}$ for $\mathbf{a} = \mathbf{b} = 0$. The gyrogeometric mean is defined by applying the gyrolines and it resolve the inconvenience.

5. Properties of the Gyrogeometric Mean

The gyrogeometric mean satisfies certain desirable properties one would expect for *means* in general. For example, the permutation invariance and the left-translation invariance would be expected properties. It is trivial that the gyrogeometric mean is permutation-invariant. We prove that the gyrogeometric mean is left-translation invariant.

Recall that X_n is the set of all n-points subset of $\mathbb{V}_1$ for a positive integer n.

Proposition 5. *Let* $\mathbf{x} \in \mathbb{V}_1$ *and* $D_n = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\} \in X_n$. *Put* $\mathbf{x} \oplus_E D_n = \{\mathbf{x} \oplus_E \mathbf{a}_1, \mathbf{x} \oplus_E \mathbf{a}_2, \cdots, \mathbf{x} \oplus_E \mathbf{a}_n\}$. *Then the following holds:*

$$G_n(\mathbf{x} \oplus_E D_n) = \mathbf{x} \oplus_E G_n(D_n). \tag{13}$$

Proof. We prove the equality (13) by induction on n. For $n = 2$, $G_2(\{\mathbf{a}_1, \mathbf{a}_2\}) = \mathbf{P}^m_{\mathbf{a}_1\mathbf{a}_2}$. By Theorem 6.37 in [1] (p. 175) we have

$$G_2(\{\mathbf{x} \oplus_E \mathbf{a}_1, \mathbf{x} \oplus_E \mathbf{a}_2\}) = \mathbf{x} \oplus_E G_2(\{\mathbf{a}_1, \mathbf{a}_2\}).$$

Assume that (13) holds for $n = k$. Let $D^m_{k+1} = \{\mathbf{a}^m_1, \mathbf{a}^m_2, \cdots, \mathbf{a}^m_{k+1}\}$ where $\mathbf{a}^m_i = G_k(\{\mathbf{a}^{m-1}_1, \mathbf{a}^{m-1}_2, \cdots, \mathbf{a}^{m-1}_{i-1}, \mathbf{a}^{m-1}_{i+1}, \cdots, \mathbf{a}_{k+1}\})$ for $1 \leq i \leq k+1$. By the assumption we have

$$\begin{aligned} G_k(\{\mathbf{x} \oplus_E \mathbf{a}^{m-1}_1, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_2, \cdots, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_{i-1}, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_{i+1}, \cdots, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_{k+1}\}) \\ = \mathbf{x} \oplus_E G_k(\{\mathbf{a}^{m-1}_1, \mathbf{a}^{m-1}_2, \cdots, \mathbf{a}^{m-1}_{i-1}, \mathbf{a}^{m-1}_{i+1}, \cdots, \mathbf{a}^{m-1}_{k+1}\}) \\ = \mathbf{x} \oplus_E \mathbf{a}^m_i \end{aligned}$$

for every $1 \leq i \leq k+1$. Then for $\mathbf{x} \oplus_E D_{k+1} = \{\mathbf{x} \oplus_E \mathbf{a}_1, \mathbf{x} \oplus_E \mathbf{a}_2, \cdots, \mathbf{x} \oplus_E \mathbf{a}_{k+1}\}$ we have

$$\begin{aligned} &\mathbf{x} \oplus_E D^m_{k+1} \\ &\quad = \{G_k(\{\mathbf{x} \oplus_E \mathbf{a}^{m-1}_1, \cdots, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_{i-1}, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_{i+1}, \cdots, \mathbf{x} \oplus_E \mathbf{a}^{m-1}_{k+1}\}) : 1 \leq i \leq k+1\} \\ &\quad = \{\mathbf{x} \oplus_E \mathbf{a}^m_i : i = 1, 2, \cdots, k+1\}. \end{aligned}$$

We prove that $\mathbf{x} \oplus_E D^m_{k+1} \to \{\mathbf{x} \oplus_E G_{k+1}(D_{k+1})\}$ as $m \to \infty$. By a simple calculation we have

$$\begin{aligned} &d(\mathbf{x} \oplus_E \mathbf{a}^m_i, \mathbf{x} \oplus_E G_{k+1}(D_{k+1})) \\ &\quad = \|\mathrm{gyr}[\mathbf{x}, \mathbf{a^m_i}](\mathbf{a}^m_i \ominus_E G(D_{k+1}))\| = \|\mathbf{a}^m_i \ominus_E G(D_{k+1})\| \to 0 \end{aligned}$$

as $m \to \infty$. Hence we have $\delta(\mathbf{x} \oplus_E \mathbf{a}^m_i, \mathbf{x} \oplus_E G_{k+1}(D_{k+1})) \to 0$ as $m \to \infty$. Thus, $\mathbf{x} \oplus_E \mathbf{a}^m_i \to \mathbf{x} \oplus_E G(D_{k+1})$ as $m \to \infty$ for $1 \leq i \leq k+1$. We conclude that $G_n(\mathbf{x} \oplus_E D_n) = \mathbf{x} \oplus_E G_n(D_n)$. $\square$

For $\mathbf{a} = \mathbf{b} = \mathbf{0}$ and $\mathbf{c}$ in $\mathbb{V}_1$, $G_3(\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}) = \frac{1}{3} \otimes_E \mathbf{c}$. More generally, the gyrogeometric mean satisfies the following.

Proposition 6. *Let* $\Delta = \{t_1 \otimes_E \mathbf{a}, t_2 \otimes_E \mathbf{a}, \cdots, t_n \otimes_E \mathbf{a}\} \subset \mathbb{V}_1$.

$$G_n(\Delta) = \frac{t_1 + t_2 + \cdots + t_n}{n} \otimes_E \mathbf{a}$$

In the case of $n = 2$, it is proved by the following calculation.

$$\begin{aligned}
& G_2(t_1 \otimes_E \mathbf{a}, t_2 \otimes_E \mathbf{a}) \\
= \; & t_1 \otimes_E \mathbf{a} \# t_2 \otimes_E \mathbf{a} \\
= \; & \frac{1}{2} \otimes_E \{(t_1 \otimes_E \mathbf{a}) \boxplus_E (t_2 \otimes_E \mathbf{a})\} \\
= \; & \frac{1}{2} \otimes_E \{(t_1 \otimes_E \mathbf{a}) \oplus_E \mathrm{gyr}[\mathbf{t_1} \otimes_\mathbf{E} \mathbf{a}, \ominus_\mathbf{E} \mathbf{t_2} \otimes_\mathbf{E} \mathbf{a}](t_2 \otimes_E \mathbf{a})\} \\
= \; & \frac{1}{2} \otimes_E \{(t_1 \otimes_E \mathbf{a}) \oplus_E (t_2 \otimes_E \mathbf{a})\} \\
= \; & \frac{1}{2} \otimes_E \{(t_1 + t_2) \otimes \mathbf{a}\} \\
= \; & \frac{t_1 + t_2}{2} \otimes_E \mathbf{a}.
\end{aligned}$$

Proposition 6 is proved by induction on n.

In Section 6 $\mathbb{V}_s = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < s\}$ with appropriate operation is a gyrocommutative gyrogroup, which is also called the Einstein gyrogroup. The gyrogeometric mean is defined for $\mathbb{V}_s$ similarly. If $s \to \infty$ or $\mathbf{v} \in \mathbb{V}_s$ such that $\|\mathbf{v}\|$ is small enough, $\gamma_\mathbf{v} \to 1$. So, in the case,

$$G_n(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}) \to \frac{\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n}{n}$$

is hold. It is simply proved by induction.

6. Proof that $(\mathbb{V}_s, \oplus_E)$ Is a Gyrocommutative Gyrogroup

A magma $(G, \oplus)$ is a non-empty set G with a binary operation $\oplus$. A magma $(G, \oplus)$ is a gyrogroup if its binary operation $\oplus$ satisfies the following axioms (G1) through (G5):

(G1) There exists a left identity $\mathbf{0}$ in G such that

$$\mathbf{0} \oplus \mathbf{a} = \mathbf{a}$$

for all $\mathbf{a} \in G$.

(G2) For each $\mathbf{a} \in G$ there exists a left inverse $\ominus\mathbf{a} \in G$ such that

$$\ominus\mathbf{a} \oplus \mathbf{a} = \mathbf{0}.$$

(G3) For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in G$ there exists a unique element $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{c} \in G$ such that the binary operation obeys the left gyroassociative law

$$\mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c}) = (\mathbf{a} \oplus \mathbf{b}) \oplus \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{c}.$$

(G4) The map $\mathrm{gyr}[\mathbf{a}, \mathbf{b}] : G \mapsto G$ given by $\mathbf{c} \mapsto \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{c}$ is an automorphism of the magma $(G, \oplus)$. It is called a gyroautomorphism. $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ generated by $\mathbf{a}, \mathbf{b} \in G$ is called a gyration.

(G5) The gyroautomorphism $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ generated by any $\mathbf{a}, \mathbf{b} \in G$ satisfies the left loop property:

$$\mathrm{gyr}[\mathbf{a}, \mathbf{b}] = \mathrm{gyr}[\mathbf{a} \oplus \mathbf{b}, \mathbf{b}].$$

The gyrogroup $(G, \oplus)$ is called gyrocommutative if the following (G6) holds for every pair $\mathbf{a}, \mathbf{b} \in G$

(G6) $\mathbf{a} \oplus \mathbf{b} = \mathrm{gyr}[\mathbf{a}, \mathbf{b}](\mathbf{b} \oplus \mathbf{a})$.

We prove that the Einstein gyrogroup $(\mathbb{V}_s, \oplus_E)$ is in fact a gyrocommutative gyrogroup only by simple calculations. Proof of (G1) and (G2) are simple and omitted.

We prove (G3). We prove that $\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$ holds for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_s$. First, we prove the left cancellation law which is given by the equation

$$\ominus_E \mathbf{a} \oplus_E (\mathbf{a} \oplus_E \mathbf{b}) = \mathbf{b},$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$. Put $D_{\mathbf{uv}} = 1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$ and put

$$\begin{aligned} \mathbf{x} &= \mathbf{a} \oplus_E \mathbf{b} \\ &= \frac{1}{D_{\mathbf{ab}}} \Big\{ \mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}} \mathbf{b} + \frac{1}{s^2} \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} (\mathbf{a} \cdot \mathbf{b}) \mathbf{a} \Big\}. \end{aligned}$$

Put $1 + \frac{1}{s^2}((-\mathbf{a}) \cdot \mathbf{x}) = D_{(-\mathbf{a})\mathbf{x}}$. We have

$$\begin{aligned} \ominus_E \mathbf{a} \oplus_E \mathbf{x} &= \frac{1}{D_{(-\mathbf{a})\mathbf{x}}} \Big\{ (-\mathbf{a}) + \frac{1}{\gamma_{\mathbf{a}}} \mathbf{x} + \frac{1}{s^2} \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} (\mathbf{a} \cdot \mathbf{x}) \mathbf{a} \Big\} \\ &= \frac{1}{D_{(-\mathbf{a})\mathbf{x}}} \Big\{ (-\mathbf{a}) + \frac{1}{\gamma_{\mathbf{a}} D_{\mathbf{ab}}} \Big(\frac{1 + \gamma_{\mathbf{a}} D_{\mathbf{ab}}}{1 + \gamma_{\mathbf{a}}} \mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}} \mathbf{b} \Big) + \frac{1}{s^2} \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} (\mathbf{a} \cdot \mathbf{x}) \mathbf{a} \Big\}. \end{aligned}$$

We compute,

$$\begin{aligned} D_{(-\mathbf{a})\mathbf{x}} &= 1 + \frac{1}{s^2} ((-\mathbf{a}) \cdot \mathbf{x}) \\ &= 1 - \frac{1}{D_{\mathbf{ab}} s^2} \Big\{ \|\mathbf{a}\|^2 + \frac{1}{\gamma_{\mathbf{a}}} (\mathbf{a} \cdot \mathbf{b}) + \frac{1}{s^2} \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} (\mathbf{a} \cdot \mathbf{b}) \|\mathbf{a}\|^2 \Big\} \\ &= 1 - \frac{1}{D_{\mathbf{ab}}} \Big\{ \frac{\gamma_{\mathbf{a}}^2 - 1}{\gamma_{\mathbf{a}}^2} + \frac{1}{\gamma_{\mathbf{a}}} \frac{(\mathbf{a} \cdot \mathbf{b})}{s^2} + \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} \frac{(\mathbf{a} \cdot \mathbf{b})}{s^2} \frac{\gamma_{\mathbf{a}}^2 - 1}{\gamma_{\mathbf{a}}^2} \Big\} \\ &= 1 - \frac{1}{D_{\mathbf{ab}}} \Big\{ 1 - \frac{1}{\gamma_{\mathbf{a}}^2} + \frac{1}{\gamma_{\mathbf{a}}} (D_{\mathbf{ab}} - 1) + \frac{\gamma_{\mathbf{a}} - 1}{\gamma_{\mathbf{a}}} (D_{\mathbf{ab}} - 1) \Big\} \\ &= \frac{1}{\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}}}, \end{aligned}$$

and

$$\frac{\mathbf{a} \cdot \mathbf{x}}{s^2} = 1 - D_{(-\mathbf{a})\mathbf{x}} = \frac{\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}} - 1}{\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}}}.$$

Hence we have

$$\begin{aligned} \ominus_E \mathbf{a} \oplus_E (\mathbf{a} \oplus_E \mathbf{b}) &= \ominus_E \mathbf{a} \oplus_E \mathbf{x} \\ &= \gamma_{\mathbf{a}}^2 D_{\mathbf{ab}} \Big\{ (-\mathbf{a}) + \frac{1}{\gamma_{\mathbf{a}} D_{\mathbf{ab}}} \Big(\frac{1 + \gamma_{\mathbf{a}} D_{\mathbf{ab}}}{1 + \gamma_{\mathbf{a}}} \mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}} \mathbf{b} \Big) \\ &\quad + \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} \frac{\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}} - 1}{\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}}} \mathbf{a} \Big\} \\ &= -\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}} \mathbf{a} + \gamma_{\mathbf{a}} \Big(\frac{1 + \gamma_{\mathbf{a}} D_{\mathbf{ab}}}{1 + \gamma_{\mathbf{a}}} \mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}} \mathbf{b} \Big) + \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} (\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}} - 1) \mathbf{a} \\ &= \mathbf{b} + \Big(-\gamma_{\mathbf{a}}^2 D_{\mathbf{ab}} + \frac{\gamma_{\mathbf{a}} + \gamma_{\mathbf{a}}^2 D_{\mathbf{ab}}}{1 + \gamma_{\mathbf{a}}} + \frac{\gamma_{\mathbf{a}}^3 D_{\mathbf{ab}} - \gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} \Big) \mathbf{a} \\ &= \mathbf{b}. \end{aligned}$$

Next, we prove the following equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus_E (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E (\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w})). \tag{14}$$

It is known in [5] ((2.84), (2.85)) that the Equation (14) can be rewritten as

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w} + \frac{\mathrm{A}\mathbf{u} + \mathrm{B}\mathbf{v}}{\mathrm{D}} \tag{15}$$

by applying computer algebra, where

$$\begin{aligned} A &= -\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w}) + \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w}) + \frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \\ B &= -\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\Big\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{w}) + (\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\Big\} \\ D &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}) + 1. \end{aligned}$$

We prove (15) without applying computer algebra. Put

$$\begin{aligned} \mathbf{z} &= \ominus_E(\mathbf{u}\oplus_E\mathbf{v}) \\ &= -\frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\Big\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\Big\} \\ &= -\frac{1+\gamma_{\mathbf{u}}D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_{\mathbf{u}})}\mathbf{u} - \frac{1}{\gamma_{\mathbf{u}}D_{\mathbf{uv}}}\mathbf{v}, \\ \mathbf{x} &= (\mathbf{v}\oplus_E\mathbf{w}) \\ &= \frac{1+\gamma_{\mathbf{v}}D_{\mathbf{vw}}}{D_{\mathbf{vw}}(1+\gamma_{\mathbf{v}})}\mathbf{v} + \frac{1}{\gamma_{\mathbf{v}}D_{\mathbf{vw}}}\mathbf{w}. \end{aligned}$$

Put also

$$\begin{aligned} \mathbf{y} &= \mathbf{u}\oplus_E(\mathbf{v}\oplus_E\mathbf{w}) \\ &= \mathbf{u}\oplus_E\mathbf{x} \\ &= \frac{1+\gamma_{\mathbf{u}}D_{\mathbf{ux}}}{D_{\mathbf{ux}}(1+\gamma_{\mathbf{u}})}\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}D_{\mathbf{ux}}}\mathbf{x} \\ &= \frac{1+\gamma_{\mathbf{u}}D_{\mathbf{ux}}}{D_{\mathbf{ux}}(1+\gamma_{\mathbf{u}})}\mathbf{u} + \frac{1+\gamma_{\mathbf{v}}D_{\mathbf{vw}}}{D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{v}})}\mathbf{v} + \frac{1}{D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\mathbf{w}. \end{aligned}$$

Then $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}$ is given by the following:

$$\begin{aligned} \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} &= \mathbf{z}\oplus_E\mathbf{y} \\ &= \frac{1+\gamma_{\mathbf{z}}D_{\mathbf{zy}}}{D_{\mathbf{zy}}(1+\gamma_{\mathbf{z}})}\mathbf{z} + \frac{1}{\gamma_{\mathbf{z}}D_{\mathbf{zy}}}\mathbf{y} \\ &= -\frac{(1+\gamma_{\mathbf{z}}D_{\mathbf{zy}})(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})}{D_{\mathbf{zy}}D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}})(1+\gamma_{\mathbf{u}})}\mathbf{u} - \frac{1+\gamma_{\mathbf{z}}D_{\mathbf{zy}}}{D_{\mathbf{zy}}D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}})\gamma_{\mathbf{u}}}\mathbf{v} \\ &\quad + \frac{1+\gamma_{\mathbf{u}}D_{\mathbf{ux}}}{D_{\mathbf{zy}}D_{\mathbf{ux}}\gamma_{\mathbf{z}}(1+\gamma_{\mathbf{u}})}\mathbf{u} + \frac{1+\gamma_{\mathbf{v}}D_{\mathbf{vw}}}{D_{\mathbf{zy}}D_{\mathbf{ux}}\gamma_{\mathbf{z}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{v}})}\mathbf{v} \\ &\quad + \frac{1}{D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{z}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\mathbf{w}. \end{aligned}$$

We will calculate each coefficient of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of the equation above.

We prove that the coefficient of $\mathbf{w}$ is 1, i.e., $D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\gamma_{\mathbf{z}}$ is 1. The equation $\frac{\mathbf{a}\cdot\mathbf{b}}{s^2} = D_{\mathbf{ab}} - 1$ holds for all $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$. Applying this equation, we have

$$\begin{aligned}
D_{\mathbf{ux}} &= 1 + \frac{\mathbf{u}\cdot\mathbf{x}}{s^2} \\
&= 1 + \frac{1}{s^2}\left\{\frac{1+\gamma_{\mathbf{v}}D_{\mathbf{vw}}}{D_{\mathbf{vw}}(1+\gamma_{\mathbf{v}})}(\mathbf{u}\cdot\mathbf{v}) + \frac{1}{\gamma_{\mathbf{v}}D_{\mathbf{vw}}}(\mathbf{u}\cdot\mathbf{w})\right\} \\
&= \frac{D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}}) + \gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})(D_{\mathbf{uv}}-1) + (1+\gamma_{\mathbf{v}})(D_{\mathbf{uw}}-1)}{D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}})} \\
&= \frac{{\gamma_{\mathbf{v}}}^2 D_{\mathbf{uv}}D_{\mathbf{vw}} + \gamma_{\mathbf{v}}D_{\mathbf{uv}} + \gamma_{\mathbf{v}}D_{\mathbf{vw}} + \gamma_{\mathbf{v}}D_{\mathbf{uw}} + D_{\mathbf{uw}} - 2\gamma_{\mathbf{v}} - 1}{D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}})}, \\
D_{\mathbf{zy}} &= 1 + \frac{\mathbf{z}\cdot\mathbf{y}}{s^2} \\
&= 1 - \frac{1}{s^2}\left\{\frac{(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1+\gamma_{\mathbf{u}}D_{\mathbf{ux}})}{D_{\mathbf{uv}}D_{\mathbf{ux}}(1+\gamma_{\mathbf{u}})^2}\|\mathbf{u}\|^2 + \frac{(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})}{D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})}(\mathbf{u}\cdot\mathbf{v})\right. \\
&\qquad + \frac{1+\gamma_{\mathbf{u}}D_{\mathbf{uv}}}{D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}})}(\mathbf{u}\cdot\mathbf{w}) + \frac{1+\gamma_{\mathbf{u}}D_{\mathbf{ux}}}{D_{\mathbf{uv}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{u}})}(\mathbf{u}\cdot\mathbf{v}) \\
&\qquad \left. + \frac{1+\gamma_{\mathbf{v}}D_{\mathbf{vw}}}{D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}{\gamma_{\mathbf{u}}}^2(1+\gamma_{\mathbf{v}})}\|\mathbf{v}\|^2 + \frac{\mathbf{v}\cdot\mathbf{w}}{D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}{\gamma_{\mathbf{u}}}^2\gamma_{\mathbf{v}}}\right\}.
\end{aligned}$$

Multiplying $D_{\mathbf{ux}}D_{\mathbf{vw}}$ from the right-hand side of the last equation, and applying the gamma factor, we have

$$\begin{aligned}
&D_{\mathbf{zy}}(D_{\mathbf{ux}}D_{\mathbf{vw}}) \\
&= D_{\mathbf{ux}}D_{\mathbf{vw}} - \frac{(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1+\gamma_{\mathbf{u}}D_{\mathbf{ux}})(\gamma_{\mathbf{u}}-1)D_{\mathbf{vw}}}{D_{\mathbf{uv}}(1+\gamma_{\mathbf{u}}){\gamma_{\mathbf{u}}}^2} \\
&\quad - \frac{(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})(D_{\mathbf{uv}}-1)}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})} - \frac{(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(D_{\mathbf{uw}}-1)}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}})} \\
&\quad - \frac{(1+\gamma_{\mathbf{u}}D_{\mathbf{ux}})(D_{\mathbf{uv}}-1)D_{\mathbf{vw}}}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{u}})} - \frac{(1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})(\gamma_{\mathbf{v}}-1)}{D_{\mathbf{uv}}{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2} - \frac{D_{\mathbf{vw}}-1}{D_{\mathbf{uv}}{\gamma_{\mathbf{u}}}^2\gamma_{\mathbf{v}}}.
\end{aligned}$$

Dividing the common denominator $D_{\mathbf{uv}}{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})$ and multiplying $\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\gamma_{\mathbf{z}}$ to $D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}$, where $\gamma_{\mathbf{z}} = \gamma_{\ominus_{\mathbf{E}}(\mathbf{u}\oplus_{\mathbf{E}}\mathbf{v})} = \gamma_{\mathbf{u}\oplus_{\mathbf{E}}\mathbf{v}} = D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}$, we have

$$\begin{aligned}
&D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\gamma_{\mathbf{z}} \\
&= \frac{1}{(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})}\Big\{D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}}) \\
&- (1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1+\gamma_{\mathbf{u}}D_{\mathbf{ux}})(\gamma_{\mathbf{u}}-1)D_{\mathbf{vw}}{\gamma_{\mathbf{v}}}^2(1+\gamma_{\mathbf{v}}) \\
&- (1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})(D_{\mathbf{uv}}-1)\gamma_{\mathbf{u}}{\gamma_{\mathbf{v}}}^2 - (1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(D_{\mathbf{uw}}-1)(1+\gamma_{\mathbf{v}})\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} \\
&- (1+\gamma_{\mathbf{u}}D_{\mathbf{ux}})(D_{\mathbf{uv}}-1)D_{\mathbf{vw}}\gamma_{\mathbf{u}}{\gamma_{\mathbf{v}}}^2 - (1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})(\gamma_{\mathbf{v}}-1)(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}}) \\
&- (D_{\mathbf{vw}}-1)\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})\Big\}.
\end{aligned} \tag{16}$$

We compute $\{\cdot\}$ of the Equation (16).

$$
\begin{aligned}
\{\cdot\} &= D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^3\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 \\
&+ D_{\mathbf{uv}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^3\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{vw}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{vw}}\gamma_{\mathbf{v}}{}^2 \\
&- D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 \\
&- D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 \\
&- D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^3\gamma_{\mathbf{v}}{}^2 + \underline{D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3} - D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^3\gamma_{\mathbf{v}}{}^3 \\
&+ \underline{D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2} + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 \\
&- D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{uv}}^2\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{uv}}^2D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 \\
&+ \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}} + D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 \\
&- D_{\mathbf{uw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} - D_{\mathbf{uw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{uw}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}} - D_{\mathbf{uw}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 \\
&+ D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 \\
&- D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 \\
&- D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 - \gamma_{\mathbf{v}}{}^2 + 1 - \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + \gamma_{\mathbf{u}} - D_{\mathbf{vw}}\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{vw}}\gamma_{\mathbf{v}} \\
&- D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}{}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{vw}}\gamma_{\mathbf{v}} - D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} \\
&- D_{\mathbf{vw}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2.
\end{aligned}
$$

Applying

$$
\begin{aligned}
D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}}) &= \gamma_{\mathbf{v}}{}^2D_{\mathbf{uv}}D_{\mathbf{vw}} + \gamma_{\mathbf{v}}D_{\mathbf{uv}} + \gamma_{\mathbf{v}}D_{\mathbf{vw}} \\
&+ \gamma_{\mathbf{v}}D_{\mathbf{uw}} + D_{\mathbf{uw}} - 2\gamma_{\mathbf{v}} - 1
\end{aligned} \tag{17}
$$

for underline items, we infer that

$$
\begin{aligned}
&\underline{D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3} + \underline{D_{\mathbf{uv}}D_{\mathbf{vw}}D_{\mathbf{ux}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2} \\
&= D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}}) \\
&= D_{\mathbf{uv}}^2D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^3 + D_{\mathbf{uv}}^2\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uw}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 \\
&+ D_{\mathbf{uw}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}} - 2D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{uv}}\gamma_{\mathbf{u}}{}^2.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\{\cdot\} &= -D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}})(\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}} - \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}) \\
&+ 2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 - D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 - D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^3 \\
&+ \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} - D_{\mathbf{uw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} - D_{\mathbf{uw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 + D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}})\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}} \\
&+ 1 + \gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} \\
&= (1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}}),
\end{aligned}
$$

hence we have $D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\gamma_{\mathbf{z}} = 1$.

Next, we prove that coefficient of $\mathbf{u}$ is $\frac{A}{D}$.

We have $1+\gamma_{\mathbf{z}} = 1 + D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} = D$. Then we compute the coefficient of $\mathbf{u}$ applying the equation $D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\gamma_{\mathbf{z}} = 1$.

$$
\begin{aligned}
&-\frac{(1+\gamma_{\mathbf{z}}D_{\mathbf{zy}})(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})}{D_{\mathbf{zy}}D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}})(1+\gamma_{\mathbf{u}})} + \frac{1+\gamma_{\mathbf{u}}D_{\mathbf{ux}}}{D_{\mathbf{zy}}D_{\mathbf{ux}}\gamma_{\mathbf{z}}(1+\gamma_{\mathbf{u}})} \\
&= \frac{-(1+\gamma_{\mathbf{z}}D_{\mathbf{zy}})(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})\gamma_{\mathbf{z}}D_{\mathbf{ux}} + (1+\gamma_{\mathbf{u}}D_{\mathbf{ux}})(1+\gamma_{\mathbf{z}})D_{\mathbf{uv}}}{D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{uv}}D\gamma_{\mathbf{z}}(1+\gamma_{\mathbf{u}})} \\
&= \frac{D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}{DD_{\mathbf{uv}}(1+\gamma_{\mathbf{u}})}\Big\{-(D_{\mathbf{ux}}\gamma_{\mathbf{z}} + D_{\mathbf{ux}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{z}} + D_{\mathbf{zy}}D_{\mathbf{ux}}{\gamma_{\mathbf{z}}}^2 + D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{uv}}{\gamma_{\mathbf{z}}}^2\gamma_{\mathbf{u}}) \\
&+ D_{\mathbf{uv}} + D_{\mathbf{uv}}\gamma_{\mathbf{z}} + D_{\mathbf{ux}}D_{\mathbf{uv}}\gamma_{\mathbf{u}} + D_{\mathbf{ux}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}\gamma_{\mathbf{z}}\Big\} \\
&= \frac{D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}{DD_{\mathbf{uv}}(1+\gamma_{\mathbf{u}})}\Big\{D_{\mathbf{uv}} + D_{\mathbf{uv}}^2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} + D_{\mathbf{ux}}D_{\mathbf{uv}}\gamma_{\mathbf{u}}(1-\gamma_{\mathbf{v}}) - \frac{D_{\mathbf{uv}}}{D_{\mathbf{vw}}} - \frac{D_{\mathbf{uv}}^2\gamma_{\mathbf{u}}}{D_{\mathbf{vw}}}\Big\} \\
&= \frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}{D(1+\gamma_{\mathbf{u}})}\Big\{D_{\mathbf{vw}} + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} - 1 - \gamma_{\mathbf{u}}D_{\mathbf{uv}} \\
&+ \gamma_{\mathbf{u}}(1-\gamma_{\mathbf{v}})\frac{{\gamma_{\mathbf{v}}}^2D_{\mathbf{uv}}D_{\mathbf{vw}} + \gamma_{\mathbf{v}}D_{\mathbf{uv}} + \gamma_{\mathbf{v}}D_{\mathbf{vw}} + \gamma_{\mathbf{v}}D_{\mathbf{uw}} + D_{\mathbf{uw}} - 2\gamma_{\mathbf{v}} - 1}{\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}})}\Big\} \\
&= \frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}{D(1+\gamma_{\mathbf{u}})}\Big\{-\frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{v}}}(\gamma_{\mathbf{v}}-1)D_{\mathbf{uw}} + (1 - \frac{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}-1)}{1+\gamma_{\mathbf{v}}})D_{\mathbf{vw}} \\
&- (1+\frac{\gamma_{\mathbf{v}}-1}{1+\gamma_{\mathbf{v}}})\gamma_{\mathbf{u}}D_{\mathbf{uv}} + (1-\frac{\gamma_{\mathbf{v}}-1}{1+\gamma_{\mathbf{v}}})\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}D_{\mathbf{vw}} - 1 + \frac{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}-1)(2\gamma_{\mathbf{v}}+1)}{\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{v}})}\Big\}
\end{aligned}
$$

By $D_{\mathbf{uv}} = 1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}$,

$$
\begin{aligned}
&= \frac{1}{D}\Big\{-\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{u}\cdot\mathbf{v}) - \frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{v}}-1) \\
&+ \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}-\gamma_{\mathbf{u}}\gamma_{\mathbf{v}})}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{v}\cdot\mathbf{w}) + \frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}-\gamma_{\mathbf{u}}\gamma_{\mathbf{v}})}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)} \\
&- \frac{1}{s^2}\frac{2{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v}) - \frac{1}{s^2}\frac{2{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)} \\
&+ \frac{2{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(1 + \frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + \frac{(\mathbf{v}\cdot\mathbf{w})}{s^2} + \frac{1}{s^4}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})) \\
&- \frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{u}}} + \frac{{\gamma_{\mathbf{u}}}^2(2{\gamma_{\mathbf{v}}}^2-\gamma_{\mathbf{v}}-1)}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}\Big\} \\
&= \frac{1}{D}\Big\{-\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{v}) + \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w}) + \frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\Big\} \\
&= \frac{A}{D}.
\end{aligned}
$$

Finally, we prove that the coefficient of $\mathbf{v}$ is $\frac{B}{D}$.

$$
\begin{aligned}
&-\frac{1+\gamma_{\mathbf{z}}D_{\mathbf{zy}}}{D_{\mathbf{zy}}D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}})\gamma_{\mathbf{u}}} + \frac{1+\gamma_{\mathbf{v}}D_{\mathbf{vw}}}{D_{\mathbf{zy}}D_{\mathbf{ux}}\gamma_{\mathbf{z}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{v}})} \\
&= \frac{-(1+\gamma_{\mathbf{z}}D_{\mathbf{zy}})D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{z}}(1+\gamma_{\mathbf{v}}) + (1+\gamma_{\mathbf{v}}D_{\mathbf{vw}})D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}})}{D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{uv}}D_{\mathbf{vw}}D\gamma_{\mathbf{z}}\gamma_{\mathbf{u}}(1+\gamma_{\mathbf{v}})} \\
&= \frac{\gamma_{\mathbf{v}}}{DD_{\mathbf{uv}}(1+\gamma_{\mathbf{v}})}\Big\{-D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{z}}(1+\gamma_{\mathbf{v}}) - D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}{\gamma_{\mathbf{z}}}^2(1+\gamma_{\mathbf{v}}) \\
&+ D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}}) + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{z}})\Big\}.
\end{aligned}
$$

Using (16) and $D_{\mathbf{zy}}D_{\mathbf{ux}}D_{\mathbf{vw}}\gamma_{\mathbf{z}} = \frac{1}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}$, then we have

$$
\begin{aligned}
&= \frac{\gamma_{\mathbf{v}}}{DD_{\mathbf{uv}}(1+\gamma_{\mathbf{v}})}\Big\{-(\gamma_{\mathbf{z}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}D_{\mathbf{vw}} + \gamma_{\mathbf{z}}D_{\mathbf{uv}} + \gamma_{\mathbf{z}}D_{\mathbf{vw}} + \gamma_{\mathbf{z}}D_{\mathbf{uw}} + \frac{\gamma_{\mathbf{z}}}{\gamma_{\mathbf{v}}}D_{\mathbf{uw}} \\
&- \frac{\gamma_{\mathbf{z}}}{\gamma_{\mathbf{v}}}(2\gamma_{\mathbf{v}}+1)) - \frac{\gamma_{\mathbf{z}}(1+\gamma_{\mathbf{v}})}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} + D_{\mathbf{uv}}(1+\gamma_{\mathbf{z}}) + D_{\mathbf{uv}}D_{\mathbf{vw}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{z}})\Big\} \\
&= \frac{1}{DD_{\mathbf{uv}}(1+\gamma_{\mathbf{v}})}\Big\{{\gamma_{\mathbf{v}}}^2(1+\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + \frac{(\mathbf{v}\cdot\mathbf{w})}{s^2} + \frac{1}{s^4}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})) \\
&- \gamma_{\mathbf{u}}{\gamma_{\mathbf{v}}}^2(1+\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + \frac{(\mathbf{v}\cdot\mathbf{w})}{s^2} + \frac{1}{s^4}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})) \\
&- \gamma_{\mathbf{u}}{\gamma_{\mathbf{v}}}^2(1+\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + \frac{(\mathbf{u}\cdot\mathbf{w})}{s^2} + \frac{1}{s^4}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})) \\
&- \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + \frac{(\mathbf{u}\cdot\mathbf{w})}{s^2} + \frac{1}{s^4}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})) \\
&+ \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(2\gamma_{\mathbf{v}}+1)(1+\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2}) - {\gamma_{\mathbf{v}}}^2(1+\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2})\Big\} \\
&= -\frac{\gamma_{\mathbf{v}}}{s^2D(1+\gamma_{\mathbf{v}})}\Big\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{w}) + (\gamma_{\mathbf{v}}-1)\gamma_{\mathbf{u}}(\mathbf{v}\cdot\mathbf{w})\Big\} \\
&= \frac{B}{D}.
\end{aligned}
$$

Hence $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w} + \frac{\mathrm{Au+Bv}}{\mathrm{D}}$ holds. By applying the left cancellation law for the Equation (14), we obtain (G3).

We prove (G4). We prove that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is automorphism for every pair $\mathbf{u},\mathbf{v} \in \mathbb{V}$. To prove (G4), we first show the gyration preserves, the inner product of $\mathbb{V}$ and the norm. So, we compute

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$$

for all $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in \mathbb{V}_s$. By applying the Equation (15), we have

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a} = \mathbf{a} + \frac{A_{\mathbf{a}}\mathbf{u} + B_{\mathbf{a}}\mathbf{v}}{D}$$

and

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{b} + \frac{A_{\mathbf{b}}\mathbf{u} + B_{\mathbf{b}}\mathbf{v}}{D}$$

respectively, where

$$
\begin{aligned}
A_{\mathbf{a}} &= -\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{a}) + \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{a}) + \frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a}), \\
B_{\mathbf{a}} &= -\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\Big\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{a}) + (\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{a})\Big\}.
\end{aligned}
$$

The terms $A_{\mathbf{b}}$ and $B_{\mathbf{b}}$ are defined in the similar way an $A_{\mathbf{a}}$ and $B_{\mathbf{b}}$respectively. Then we have

$$
\begin{aligned}
\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} &= \mathbf{a}\cdot\mathbf{b} + \frac{A_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{b}) + B_{\mathbf{a}}(\mathbf{v}\cdot\mathbf{b})}{D} + \frac{A_{\mathbf{b}}(\mathbf{u}\cdot\mathbf{a}) + B_{\mathbf{b}}(\mathbf{v}\cdot\mathbf{a})}{D} \\
&+ \frac{1}{D^2}\{A_{\mathbf{a}}A_{\mathbf{b}}\|\mathbf{u}\|^2 + A_{\mathbf{a}}B_{\mathbf{b}}(\mathbf{u}\cdot\mathbf{v}) + A_{\mathbf{b}}B_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{v}) + B_{\mathbf{a}}B_{\mathbf{b}}\|\mathbf{v}\|^2\},
\end{aligned}
\tag{18}
$$

We show that terms other than $\mathbf{a}\cdot\mathbf{b}$ of the right-hand side of the Equation (18) equal to zero.

$$\begin{aligned}
&A_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{b})+B_{\mathbf{a}}(\mathbf{v}\cdot\mathbf{b})\\
=&-\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\\
&+\frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\\
&-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})-\frac{1}{s^2}\frac{{\gamma_{\mathbf{v}}}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}).\\
&A_b(\mathbf{u}\cdot\mathbf{a})+B_b(\mathbf{v}\cdot\mathbf{a})\\
=&-\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&+\frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})-\frac{1}{s^2}\frac{{\gamma_{\mathbf{v}}}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}).\\
&A_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{b})+B_{\mathbf{a}}(\mathbf{v}\cdot\mathbf{b})+A_{\mathbf{b}}(\mathbf{u}\cdot\mathbf{a})+B_{\mathbf{b}}(\mathbf{v}\cdot\mathbf{a})\\
=&-\frac{2}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})+\frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&-\frac{2}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})+\frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}).
\end{aligned}$$

Then we compute each terms of the sum
$A_{\mathbf{a}}A_{\mathbf{b}}\|\mathbf{u}\|^2+A_{\mathbf{a}}B_{\mathbf{b}}(\mathbf{u}\cdot\mathbf{v})+A_{\mathbf{b}}B_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{v})+B_{\mathbf{a}}B_{\mathbf{b}}\|\mathbf{v}\|^2$.

$$\begin{aligned}
&A_{\mathbf{a}}A_{\mathbf{b}}\|\mathbf{u}\|^2\\
&=\|\mathbf{u}\|^2\Big\{\frac{1}{s^4}\frac{{\gamma_{\mathbf{u}}}^4}{(\gamma_{\mathbf{u}}+1)^2}(\gamma_{\mathbf{v}}-1)^2(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})-\frac{1}{s^4}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&-\frac{2}{s^6}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\\
&-\frac{1}{s^4}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})+\frac{1}{s^4}{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&+\frac{2}{s^6}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&-\frac{2}{s^6}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\\
&+\frac{2}{s^6}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\\
&+\frac{4}{s^8}\frac{{\gamma_{\mathbf{u}}}^4{\gamma_{\mathbf{v}}}^4}{(\gamma_{\mathbf{u}}+1)^2(\gamma_{\mathbf{v}}+1)^2}(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\Big\}.
\end{aligned}\tag{19}$$

By $\frac{\|\mathbf{u}\|^2}{s^2} = \frac{{\gamma_\mathbf{u}}^2-1}{{\gamma_\mathbf{u}}^2}$ this equation is rewritten in the following.

$$\begin{aligned}
&A_\mathbf{a} A_\mathbf{b} \|\mathbf{u}\|^2 \\
=&\frac{1}{s^2}\frac{{\gamma_\mathbf{u}}^2}{(\gamma_\mathbf{u}+1)}(\gamma_\mathbf{v}-1)^2(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) - \frac{1}{s^2}\gamma_\mathbf{u}\gamma_\mathbf{v}(\gamma_\mathbf{u}-1)(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\
&-\frac{2}{s^4}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)}(\gamma_\mathbf{u}-1)(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) \\
&-\frac{1}{s^2}\gamma_\mathbf{u}\gamma_\mathbf{v}(\gamma_\mathbf{u}-1)(\gamma_\mathbf{v}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) + \frac{1}{s^2}{\gamma_\mathbf{v}}^2({\gamma_\mathbf{u}}^2-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\
&+\frac{4}{s^4}\frac{\gamma_\mathbf{u}{\gamma_\mathbf{v}}^3}{(\gamma_\mathbf{v}+1)}(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\
&-\frac{2}{s^4}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)}(\gamma_\mathbf{u}-1)(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})) \\
&+\frac{4}{s^6}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^4}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)^2}(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}).
\end{aligned} \tag{20}$$

Then we obtain

$$\begin{aligned}
&A_\mathbf{a} B_\mathbf{b}(\mathbf{u}\cdot\mathbf{v}) \\
=&\frac{1}{s^4}\frac{{\gamma_\mathbf{u}}^3\gamma_\mathbf{v}}{(\gamma_\mathbf{u}+1)^2}(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) \\
&+\frac{1}{s^4}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}-1)}(\gamma_\mathbf{u}-1)(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\
&-\frac{1}{s^4}{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) - \frac{1}{s^4}\frac{\gamma_\mathbf{u}{\gamma_\mathbf{v}}^3}{(\gamma_\mathbf{v}+1)}(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\
&-\frac{2}{s^6}\frac{{\gamma_\mathbf{u}}^3{\gamma_\mathbf{v}}^3}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)}(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) \\
&-\frac{2}{s^6}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^4}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)^2}(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}). \\
&A_b B_a(\mathbf{u}\cdot\mathbf{v}) \\
=&\frac{1}{s^4}\frac{{\gamma_\mathbf{u}}^3\gamma_\mathbf{v}}{(\gamma_\mathbf{u}+1)^2}(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) \\
&+\frac{1}{s^4}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}-1)}(\gamma_\mathbf{u}-1)(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{b})(\mathbf{v}\cdot\mathbf{a}) \\
&-\frac{1}{s^4}{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{a}) - \frac{1}{s^4}\frac{\gamma_\mathbf{u}{\gamma_\mathbf{v}}^3}{(\gamma_\mathbf{v}+1)}(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\
&-\frac{2}{s^6}\frac{{\gamma_\mathbf{u}}^3{\gamma_\mathbf{v}}^3}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)}(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{a}) \\
&-\frac{2}{s^6}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^4}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)^2}(\gamma_\mathbf{u}-1)(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}).
\end{aligned} \tag{21}$$

Calculating $B_{\mathbf{a}}B_{\mathbf{b}}\|\mathbf{v}\|^2$ in a way similar to the calculation of $A_{\mathbf{a}}A_{\mathbf{b}}\|\mathbf{u}\|^2$, we have

$$\begin{aligned} &B_{\mathbf{a}}B_{\mathbf{b}}\|\mathbf{v}\|^2 \\ =&\frac{1}{s^2}\gamma_{\mathbf{u}}{}^2(\gamma_{\mathbf{v}}{}^2-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}) \\ &+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b}) \\ &+\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)^2(\gamma_{\mathbf{v}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}). \end{aligned} \tag{22}$$

Hence, comparing the Equations (19) with (22) we have

$$\begin{aligned} &A_{\mathbf{a}}A_{\mathbf{b}}\|\mathbf{u}\|^2+A_{\mathbf{a}}B_{\mathbf{b}}(\mathbf{u}\cdot\mathbf{v})+A_{\mathbf{b}}B_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{v})+B_{\mathbf{a}}B_{\mathbf{b}}\|\mathbf{v}\|^2 \\ =&\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\Big\{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1) \\ &\quad+(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)+2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\Big\} \\ &\quad-\frac{1}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\Big\{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1) \\ &\quad+(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)+2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\Big\} \\ &\quad+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\Big\{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1) \\ &\quad+(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)+2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\Big\} \\ &\quad-\frac{1}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b}\Big\{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1) \\ &\quad+(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)+2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\Big\} \\ =&-D(A_{\mathbf{a}}(\mathbf{u}\cdot\mathbf{b})+B_{\mathbf{a}}(\mathbf{v}\cdot\mathbf{b})+A_{\mathbf{b}}(\mathbf{u}\cdot\mathbf{a})+B_{\mathbf{b}}(\mathbf{v}\cdot\mathbf{a})). \end{aligned}$$

We conclude that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b}=\mathbf{a}\cdot\mathbf{b}$.

To prove that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is a homomorphism for all $\mathbf{u},\mathbf{v}\in\mathbb{V}_s$, we show

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{x}\oplus_E\mathbf{y})=\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}\oplus_E\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y} \tag{23}$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{V}_s$. Applying (15) we have

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{x}\oplus_E\mathbf{y})=\mathbf{x}\oplus_E\mathbf{y}+\frac{1}{D}(A_{\mathbf{x}\oplus_E\mathbf{y}}\mathbf{u}+B_{\mathbf{x}\oplus_E\mathbf{y}}\mathbf{v}).$$

Put

$$\mathbf{x}\oplus_E\mathbf{y}=\frac{1+\gamma_{\mathbf{x}}D_{\mathbf{xy}}}{D_{\mathbf{xy}}(1+\gamma_{\mathbf{x}})}\mathbf{x}+\frac{1}{\gamma_{\mathbf{x}}D_{\mathbf{xy}}}\mathbf{y}=E_{\mathbf{xy}}\mathbf{x}+F_{\mathbf{xy}}\mathbf{y}.$$

By a simple calculation we infer that

$$\begin{aligned} A_{\mathbf{x}\oplus_E\mathbf{y}}&=E_{\mathbf{xy}}A_{\mathbf{x}}+F_{\mathbf{xy}}A_{\mathbf{y}} \\ B_{\mathbf{x}\oplus_E\mathbf{y}}&=E_{\mathbf{xy}}B_{\mathbf{x}}+F_{\mathbf{xy}}B_{\mathbf{y}}. \end{aligned}$$

We have

$$\begin{aligned}
&\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{x} \oplus_E \mathbf{y}) \\
&= E_{\mathbf{xy}}\mathbf{x} + F_{\mathbf{xy}}\mathbf{y} + \frac{1}{D}\Big\{(E_{\mathbf{xy}}A_{\mathbf{x}} + F_{\mathbf{xy}}A_{\mathbf{y}})\mathbf{u} + (E_{\mathbf{xy}}B_{\mathbf{x}} + F_{\mathbf{xy}}B_{\mathbf{y}})\mathbf{v})\Big\} \\
&= E_{\mathbf{xy}}\Big\{\mathbf{x} + \frac{1}{D}(A_{\mathbf{x}}\mathbf{u} + B_{\mathbf{x}}\mathbf{v})\Big\} + F_{\mathbf{xy}}\Big\{\mathbf{y} + \frac{1}{D}(A_{\mathbf{y}}\mathbf{u} + B_{\mathbf{y}}\mathbf{v})\Big\} \\
&= E_{\mathbf{xy}}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} + F_{\mathbf{xy}}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}.
\end{aligned} \tag{24}$$

Then the right-hand side of the Equation (24) is rewritten as the following equation.

$$\begin{aligned}
&\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} \oplus_E \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y} \\
&= E_{\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} + F_{\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}.
\end{aligned}$$

Since $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ preserves the inner product and the norm of $\mathbb{V}$, we have

$$\begin{aligned}
\gamma_{\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}} &= \gamma_{\mathbf{x}} \\
D_{\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}} &= D_{\mathbf{xy}},
\end{aligned}$$

so that

$$E_{\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}} = E_{\mathbf{xy}}$$

and

$$F_{\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}} = F_{\mathbf{xy}}.$$

Hence $\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{x} \oplus_E \mathbf{y}) = \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} \oplus_E \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{y}$. We conclude that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is a homomorphism.

We observe that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is bijective for every pair of $\mathbf{u},\mathbf{v} \in \mathbb{V}_s$. To prove this, we compute $\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) = \mathbf{w}$ for every $\mathbf{w} \in \mathbb{V}_s$. We denote

$$\begin{aligned}
A_{\mathbf{ab}}(\mathbf{c}) &= -\frac{1}{s^2}\frac{{\gamma_{\mathbf{a}}}^2}{\gamma_{\mathbf{a}}+1}(\gamma_{\mathbf{b}}-1)(\mathbf{a}\cdot\mathbf{c}) + \frac{1}{s^2}\gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(\mathbf{b}\cdot\mathbf{c}) + \frac{2}{s^4}\frac{{\gamma_{\mathbf{a}}}^2{\gamma_{\mathbf{b}}}^2}{(\gamma_{\mathbf{a}}+1)(\gamma_{\mathbf{b}}+1)}(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c}) \\
B_{\mathbf{ab}}(\mathbf{c}) &= -\frac{1}{s^2}\frac{\gamma_{\mathbf{b}}}{\gamma_{\mathbf{b}}+1}\Big\{\gamma_{\mathbf{a}}(\gamma_{\mathbf{b}}+1)(\mathbf{a}\cdot\mathbf{c}) + (\gamma_{\mathbf{a}}-1)\gamma_{\mathbf{b}}(\mathbf{b}\cdot\mathbf{c})\Big\},
\end{aligned}$$

where $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{V}_s$. Then applying (15), we have

$$\begin{aligned}
&\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) \\
&=\mathrm{gyr}[\mathbf{u},\mathbf{v}]\Big(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D}\Big) \\
&=\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D} \\
&\quad + \frac{1}{D}\Big\{A_{\mathbf{uv}}(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D})\mathbf{u} + B_{\mathbf{uv}}(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D})\mathbf{v}\Big\}.
\end{aligned}$$

We compute

$$\begin{aligned}
&A_{\mathbf{uv}}(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D}) \\
&= -\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D})) \\
&\quad + \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D})) \\
&\quad + \frac{2}{s^4}\frac{{\gamma_{\mathbf{u}}}^2{\gamma_{\mathbf{v}}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot(\mathbf{w} + \frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v} + B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D})) \\
&=A_{\mathbf{uv}}(\mathbf{w}) + \frac{1}{D}A_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{v}) + \frac{1}{D}B_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{u})
\end{aligned}$$

and

$$
\begin{aligned}
&B_{\mathbf{uv}}(\mathbf{w}+\frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v}+B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D})\\
=&-\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\Big\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot(\mathbf{w}+\frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v}+B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D}))\\
&+(\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot(\mathbf{w}+\frac{A_{\mathbf{vu}}(\mathbf{w})\mathbf{v}+B_{\mathbf{vu}}(\mathbf{w})\mathbf{u}}{D}))\Big\}\\
&=B_{\mathbf{uv}}(\mathbf{w})+\frac{1}{D}A_{\mathbf{vu}}(\mathbf{w})B_{\mathbf{uv}}(\mathbf{v})+\frac{1}{D}B_{\mathbf{vu}}(\mathbf{w})B_{\mathbf{uv}}(\mathbf{u}).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w})\\
=&\mathbf{w}+\Big\{\frac{B_{\mathbf{vu}}(\mathbf{w})}{D}+\frac{A_{\mathbf{uv}}(\mathbf{w})}{D}+\frac{1}{D^2}(A_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{v})+B_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{u}))\Big\}\mathbf{u}\\
&+\Big\{\frac{A_{\mathbf{vu}}(\mathbf{w})}{D}+\frac{B_{\mathbf{uv}}(\mathbf{w})}{D}+\frac{1}{D^2}(A_{\mathbf{vu}}(\mathbf{w})B_{\mathbf{uv}}(\mathbf{v})+B_{\mathbf{vu}}(\mathbf{w})B_{\mathbf{uv}}(\mathbf{u}))\Big\}\mathbf{v}.
\end{aligned}
$$

We show that the coefficients of $\mathbf{u}$ and $\mathbf{v}$ vanish. We compute the coefficient of $\mathbf{u}$.

$$
B_{\mathbf{vu}}(\mathbf{w})+A_{\mathbf{uv}}(\mathbf{w})=-\frac{2}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}).
$$

We also have

$$
\begin{aligned}
&A_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{v})+B_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{u})\\
=&\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{v})-\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\|\mathbf{v}\|^2\\
&-\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w})\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})\|\mathbf{v}\|^2\\
&-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{w})\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{v})+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\|\mathbf{v}\|^2\\
&+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{w})\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})\|\mathbf{v}\|^2\\
&-\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{v})\\
&+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\|\mathbf{v}\|^2\\
&+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})\|\mathbf{v}\|^2\\
&+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)\|\mathbf{u}\|^2-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v})\\
&-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})^2\\
&+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)\|\mathbf{u}\|^2\\
&-\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v})\\
&-\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})^2.
\end{aligned}
$$

Applying the gamma identity, we have the following.

$$
\begin{aligned}
&A_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{v}) + B_{\mathbf{vu}}(\mathbf{w})A_{\mathbf{uv}}(\mathbf{u}) \\
=&\frac{1}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\Big(-\gamma_{\mathbf{u}}\gamma_{\mathbf{v}} + \gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} - 1 - 2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2}\Big) \\
&-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + \frac{1}{s^2}\Big(\gamma_{\mathbf{u}}{}^2(\gamma_{\mathbf{v}}{}^2-1) + \frac{\gamma_{\mathbf{u}}{}^2(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1)^2}{\gamma_{\mathbf{u}}+1}\Big) \\
&+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w}) \\
=&\frac{1}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\Big(-(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1) - (\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1) - 2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2}\Big) \\
&+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})\Big((\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}+1) + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2}\Big) \\
&= -D(B_{\mathbf{vu}}(\mathbf{w}) + A_{\mathbf{uv}}(\mathbf{w})).
\end{aligned}
$$

So, the coefficient of $\mathbf{u}$ vanishes.

We compute the coefficient of $\mathbf{v}$.

$$
A_{\mathbf{vu}}(\mathbf{w}) + B_{\mathbf{uv}}(\mathbf{w}) = -\frac{2}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w}) + \frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w}).
$$

We also have

$$
\begin{aligned}
&A_{\mathbf{vu}}(\mathbf{w})B_{\mathbf{uv}}(\mathbf{v}) + B_{\mathbf{vu}}(\mathbf{w})B_{\mathbf{uv}}(\mathbf{u}) \\
=&\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v}) \\
&+\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)\|\mathbf{v}\|^2 \\
&-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v}) - \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)\|\mathbf{v}\|^2 \\
&-\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v}) \\
&-\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)\|\mathbf{v}\|^2 \\
&+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\|\mathbf{u}\|^2 + \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{u}\cdot\mathbf{v}) \\
&+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\|\mathbf{u}\|^2 \\
&+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}{}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{u}\cdot\mathbf{v}) \\
&= \frac{2}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{\gamma_{\mathbf{v}}+1}(\gamma_{\mathbf{u}}-1)(\mathbf{v}\cdot\mathbf{w})\Big\{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2} + (\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1) + (\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)\Big\} \\
&-\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}{}^2\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})\Big\{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1) \\
&+(\gamma_{\mathbf{u}}-1)(\gamma_{\mathbf{v}}-1) + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\frac{(\mathbf{u}\cdot\mathbf{v})}{s^2}\Big\} \\
&= -D(A_{\mathbf{vu}}(\mathbf{w}) + B_{\mathbf{uv}}(\mathbf{w})).
\end{aligned}
$$

So, the coefficient of $\mathbf{v}$ vanishes. Thus, $\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) = \mathbf{w}$ holds for every $\mathbf{w} \in \mathbb{V}_s$. Changing $\mathbf{u}$ and $\mathbf{v}$, $\mathrm{gyr}[\mathbf{v},\mathbf{u}](\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}) = \mathbf{w}$ also holds for every $\mathbf{w} \in \mathbb{V}_s$. We conclude that $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is bijective. Thus, $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is an automorphism; a proof of (G4) is complete.

To prove (G5) we first observe $\mathrm{gyr}[\mathbf{u} \oplus_\mathbf{E} \mathbf{v}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}]$ for every pair $\mathbf{u},\mathbf{v} \in \mathbb{V}_s$. Let $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}_s$. Applying (15) we have that

$$\begin{aligned}\mathrm{gyr}[\mathbf{u} \oplus_\mathbf{E} \mathbf{v}, \mathbf{v}]\mathbf{w} =& \mathbf{w} + \frac{A'\mathbf{u} \oplus_E \mathbf{v} + B'\mathbf{v}}{D'} \\ =& \mathbf{w} + \frac{A'}{D'}\frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}\mathbf{u} + \frac{1}{D'}\Big(\frac{A'}{D_{\mathbf{uv}}\gamma_\mathbf{u}} + B'\Big)\mathbf{v},\end{aligned}$$

where

$$\begin{aligned}A' =& -\frac{1}{s^2}\frac{{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}}^2}{(\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}+1)}(\gamma_\mathbf{v}-1)(\mathbf{u} \oplus_E \mathbf{v}\cdot\mathbf{w}) + \frac{1}{s^2}\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}\gamma_\mathbf{v}(\mathbf{v}\cdot\mathbf{w}) \\ &+ \frac{2}{s^4}\frac{{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}}^2{\gamma_\mathbf{v}}^2}{(\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}+1)(\gamma_\mathbf{v}+1)}(\mathbf{u} \oplus_E \mathbf{v}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \\ B' =& -\frac{1}{s^2}\frac{\gamma_\mathbf{v}}{\gamma_\mathbf{v}+1}\Big\{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}(\gamma_\mathbf{v}+1)(\mathbf{u} \oplus_E \mathbf{v}\cdot\mathbf{w}) + (\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}-1)\gamma_\mathbf{v}(\mathbf{v}\cdot\mathbf{w})\Big\} \\ D' =& \gamma_{(\mathbf{u}\oplus_\mathbf{E}\mathbf{v})\oplus_\mathbf{E}\mathbf{v}} + 1 \\ =& \gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}\gamma_\mathbf{v}(1 + \frac{(\mathbf{u} \oplus_E \mathbf{v})\cdot\mathbf{v}}{s^2}) + 1.\end{aligned}$$

We prove that $\mathrm{gyr}[\mathbf{u} \oplus_\mathbf{E} \mathbf{v}, \mathbf{v}]\mathbf{w} = \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}$ for every $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}_s$. We have

$$\begin{aligned}&(\mathbf{u} \oplus_E \mathbf{v})\cdot\mathbf{w} = \frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}(\mathbf{u}\cdot\mathbf{w}) + \frac{1}{D_{\mathbf{uv}}\gamma_\mathbf{u}}(\mathbf{v}\cdot\mathbf{w}), \\ &(\mathbf{u} \oplus_E \mathbf{v})\cdot\mathbf{v} = \frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}(\mathbf{u}\cdot\mathbf{v}) + \frac{1}{D_{\mathbf{uv}}\gamma_\mathbf{u}}\|\mathbf{v}\|^2, \\ &\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}} + 1 = D.\end{aligned}$$

Then A' is computed as in the following.

$$\begin{aligned}A' =& \frac{1}{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}+1}\Big\{-{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}}^2(\gamma_\mathbf{v}-1)\frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}\frac{(\mathbf{u}\cdot\mathbf{w})}{s^2} - {\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}}^2(\gamma_\mathbf{v}-1)\frac{1}{D_{\mathbf{uv}}\gamma_\mathbf{u}}\frac{(\mathbf{v}\cdot\mathbf{w})}{s^2} \\ &+ \frac{1}{s^2}\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}\gamma_\mathbf{v}(\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}+1)(\mathbf{v}\cdot\mathbf{w}) + \frac{2}{s^4}\frac{{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{v}+1)}\frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \\ &+ \frac{2}{s^4}\frac{{\gamma_{\mathbf{u}\oplus_\mathbf{E}\mathbf{v}}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{v}+1)}\frac{1}{D_{\mathbf{uv}}\gamma_\mathbf{u}}(\mathbf{v}\cdot\mathbf{w})\|\mathbf{v}\|^2\Big\} \\ =& \frac{1}{D}\Big\{-{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2D_{\mathbf{uv}}^2(\gamma_\mathbf{v}-1)\frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}\frac{(\mathbf{u}\cdot\mathbf{w})}{s^2} - {\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2D_{\mathbf{uv}}^2(\gamma_\mathbf{v}-1)\frac{1}{D_{\mathbf{uv}}\gamma_\mathbf{u}}\frac{(\mathbf{v}\cdot\mathbf{w})}{s^2} \\ &+ \frac{1}{s^2}\gamma_\mathbf{u}{\gamma_\mathbf{v}}^2D_{\mathbf{uv}}(\gamma_\mathbf{u}\gamma_\mathbf{v}D_{\mathbf{uv}}+1)(\mathbf{v}\cdot\mathbf{w}) + \frac{2}{s^4}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^4D_{\mathbf{uv}}^2}{(\gamma_\mathbf{v}+1)}\frac{1+\gamma_\mathbf{u} D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_\mathbf{u})}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \\ &+ \frac{2}{s^2}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2D_{\mathbf{uv}}^2}{(\gamma_\mathbf{v}+1)}\frac{1}{D_{\mathbf{uv}}\gamma_\mathbf{u}}\frac{{\gamma_\mathbf{v}}^2-1}{{\gamma_\mathbf{v}}^2}(\mathbf{v}\cdot\mathbf{w})\Big\} \\ =& \frac{1}{D}\Big\{-D_{\mathbf{uv}}{\gamma_\mathbf{v}}^2(1+\gamma_\mathbf{u}D_{\mathbf{uv}})\frac{1}{s^2}\frac{{\gamma_\mathbf{u}}^2}{\gamma_\mathbf{u}+1}(\gamma_\mathbf{v}-1)(\mathbf{u}\cdot\mathbf{w}) \\ &+ D_{\mathbf{uv}}{\gamma_\mathbf{v}}^2(1+\gamma_\mathbf{u}D_{\mathbf{uv}})\frac{1}{s^2}\gamma_\mathbf{u}\gamma_\mathbf{v}(\mathbf{v}\cdot\mathbf{w}) \\ &+ D_{\mathbf{uv}}{\gamma_\mathbf{v}}^2(1+\gamma_\mathbf{u}D_{\mathbf{uv}})\frac{2}{s^4}\frac{{\gamma_\mathbf{u}}^2{\gamma_\mathbf{v}}^2}{(\gamma_\mathbf{u}+1)(\gamma_\mathbf{v}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\Big\} \\ =& D_{\mathbf{uv}}{\gamma_\mathbf{v}}^2(1+\gamma_\mathbf{u}D_{\mathbf{uv}})\frac{A}{D}.\end{aligned} \quad (25)$$

D' can be computed as in the following.

$$\begin{aligned} D' =&\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2 D_{\mathbf{uv}}\Big\{1+\frac{1+\gamma_{\mathbf{u}}D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_{\mathbf{u}})}(D_{\mathbf{uv}}-1)+\frac{1}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}}\frac{\gamma_{\mathbf{v}}{}^2-1}{\gamma_{\mathbf{v}}{}^2}\Big\}+1 \\ =&\frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}^2}(2D_{\mathbf{uv}}-1+\gamma_{\mathbf{u}}D_{\mathbf{uv}}^2)}{1+\gamma_{\mathbf{u}}}+\gamma_{\mathbf{v}}{}^2 \\ =&\frac{\gamma_{\mathbf{v}}{}^2(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})^2}{1+\gamma_{\mathbf{u}}} \end{aligned}$$

Hence $\frac{A'}{D'}\frac{1+\gamma_{\mathbf{u}}D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_{\mathbf{u}})}=\frac{A}{D}$.

Next, we compute the coefficient of $\mathbf{v}$. B' can be computed as in the following.

$$\begin{aligned} B' =& -\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\Big\{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}(\gamma_{\mathbf{v}}+1)\frac{1+\gamma_{\mathbf{u}}D_{\mathbf{uv}}}{D_{\mathbf{uv}}(1+\gamma_{\mathbf{u}})}(\mathbf{u}\cdot\mathbf{w}) \\ &+\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}(\gamma_{\mathbf{v}}+1)\frac{1}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}}(\mathbf{v}\cdot\mathbf{w})+(\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\Big\} \\ =& -\frac{1}{s^2}\gamma_{\mathbf{v}}{}^2(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})\Big\{\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{w})+\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot\mathbf{w})\Big\}. \end{aligned}$$

The Equation (25) is rewritten by $A'\frac{1}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}}=\frac{A}{D}\frac{\gamma_{\mathbf{v}}{}^2(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})}{\gamma_{\mathbf{u}}}$. Then

$$\begin{aligned} &A'\frac{1}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}}+B' \\ =&\frac{\gamma_{\mathbf{v}}{}^2(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})}{D}\Big\{-\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})+\frac{1}{s^2}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w}) \\ &+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})-D\Big(\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{w})+\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot\mathbf{w})\Big)\Big\} \\ =&\frac{\gamma_{\mathbf{v}}{}^2(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})}{D(1+\gamma_{\mathbf{u}})}\Big\{-\frac{1}{s^2}(\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}-1)+(\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}+1)\gamma_{\mathbf{u}})(\mathbf{u}\cdot\mathbf{w}) \\ &+\frac{1}{s^2}\Big(\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}})-(\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}D_{\mathbf{uv}}+1)\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}(1+\gamma_{\mathbf{u}})\Big)(\mathbf{v}\cdot\mathbf{w}) \\ &+\frac{2}{s^2}\frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}{}^2}{(\gamma_{\mathbf{v}}+1)}(D_{\mathbf{uv}}-1)(\mathbf{v}\cdot\mathbf{w})\Big\} \\ =&\frac{\gamma_{\mathbf{v}}{}^2(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})}{D(1+\gamma_{\mathbf{u}})}\Big\{-\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(\mathbf{u}\cdot\mathbf{w}) \\ &+\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}{}^2}{1+\gamma_{\mathbf{v}}}(1+\gamma_{\mathbf{u}}D_{\mathbf{uv}})(1-\gamma_{\mathbf{u}})(\mathbf{v}\cdot\mathbf{w})\Big\} \\ =&\frac{D'}{D}B. \end{aligned}$$

Hence $\frac{1}{D'}(A'\frac{1}{D_{\mathbf{uv}}\gamma_{\mathbf{u}}}+B')=\frac{B}{D}$. We conclude that $\mathrm{gyr}[\mathbf{u}\oplus_{\mathbf{E}}\mathbf{v},\mathbf{v}]\mathbf{w}=\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}_s$, so (G5) holds.

We conclude that $(\mathbb{V}_s,\oplus_E)$ is a gyrogroup. Finally, we prove that it is gyrocommutative.

We prove (G6). We prove that $\mathbf{a}\oplus\mathbf{b}=\mathrm{gyr}[\mathbf{a},\mathbf{b}](\mathbf{b}\oplus\mathbf{a})$ for all $\mathbf{a},\mathbf{b}\in\mathbb{V}_s$. Gyroautomorphic inverse property defined by Ungar in [1] (Definition 3.1, p. 51) is given by the equation

$$\ominus_E(\mathbf{a}\oplus_E\mathbf{b})=\ominus_E\mathbf{a}\ominus_E\mathbf{b},$$

where $\mathbf{a}, \mathbf{b}$ are arbitrary elements in $\mathbb{V}_s$. According to Theorem 3.2 in [1] (p. 51), $(\mathbb{V}_s, \oplus_E)$ is gyrocommutative if and only if it has the gyroautomorphic inverse property. So, we observe the gyroautomorphic inverse property:

$$\begin{aligned} \ominus_E(\mathbf{a} \oplus_E \mathbf{b}) &= -(\mathbf{a} \oplus_E \mathbf{b}) \\ &= -\frac{1}{1+\frac{\mathbf{a}\cdot\mathbf{b}}{s^2}}\Big\{\mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}}\mathbf{b} + \frac{1}{s^2}\frac{\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})\mathbf{a}\Big\}, \\ \ominus_E\mathbf{a} \ominus_E \mathbf{b} &= (-\mathbf{a}) \oplus_E (-\mathbf{b}) \\ &= \frac{1}{1+\frac{\mathbf{a}\cdot\mathbf{b}}{s^2}}\Big\{-\mathbf{a} - \frac{1}{\gamma_{\mathbf{a}}}\mathbf{b} + \frac{1}{s^2}\frac{\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})(-\mathbf{a})\Big\} \\ &= \ominus_E(\mathbf{a} \oplus_E \mathbf{b}). \end{aligned}$$

Hence $(\mathbb{V}_s, \oplus_E)$ is gyrocommutative.

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