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On a Semigroup Problem II

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Abstract: We consider the following semigroup problem: is the closure of a semigroup S in a topological vector space X a group when S does not lie on "one side" of any closed hyperplane of X? Whereas for finite dimensional spaces, the answer is positive, we give a new example of infinite dimensional spaces where the answer is negative.

Keywords: semigroup; group; linear topological space; separation by functionals

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1. Introduction

We consider the following question.

Semigroup problem: Let X be a topological vector space (TVS) over the real field $\mathbb{R}$, and let $S \subset X$ be a semigroup (By a semigroup in a vector space X, we mean a subset $S \subset X$ that is closed under addition. It does not have to contain the zero vector). Let X_0 be the topological closure of the linear span of S (that is, the linear span of S is dense in X_0, and the complement of X_0 is open). Assume that S is not separated by any continuous linear functional in the (real) dual of X_0: for any $\phi \in (X_0)^* \setminus \{0\}$, there are $x_1, x_2 \in S$ such that $\phi(x_1) > 0$ and $\phi(x_2) < 0$. Does it follow that the closure of S is a group?

The topological vector spaces considered here are over the field $\mathbb{R}$ of real numbers, except in Section 3.3, where the field is C. As stated in the problem, we are considering only $\mathbb{R}$-linear continuous functionals, into $\mathbb{R}$.

If $X = \mathbb{R}^n$ is endowed with the Euclidean topology, then the semigroup problem has an affirmative answer [1,2]. A similar problem, in which separation by linear functionals is replaced by separation by certain maximal semigroups, was investigated for several classes of finite dimensional non-compact Lie groups such as Euclidean groups [1], nilpotent groups [3], and solvable groups [4].

Whereas the semigroup problem is of independent interest with respect to testing what symmetries a system possesses, it is also relevant for the construction of topologically transitive extensions of hyperbolic dynamical systems [5]. See [6] for an up-to-date review of extensions by finite dimensional Lie groups.

Much less is known if the fiber is an infinite dimensional topological group. We show in [7] (but see also [8]) that if X is $\mathbb{R}^\omega$, the countably infinite direct product of lines, then the semigroup problem has a positive answer, and we also show that if X is $\mathbb{R}^\infty$, the countably infinite direct sum of lines, then the semigroup problem has a negative answer. We also show that if X is an infinite dimensional Banach space or an infinite dimensional Fréchet space different from $\mathbb{R}^\omega$, then there are semigroups $S \subset X$ for which the semigroup problem has a negative answer. These results have consequences for the topological transitivity of extensions of hyperbolic sets with fiber $\mathbb{R}^\omega$ and $\mathbb{R}^\infty$; see the above cited papers.

In this paper, we discuss the case of not necessarily Hausdorff finite dimensional topological vector spaces (Section 2), infinite dimensional normed spaces (Section 3.2), and a particular class of quasi-normed spaces (Section 3.3), which includes $L^p(0,1)$ with $0 < p < 1$. See Section 4 for how the answer to the semigroup problem is relevant to the transitivity of skew extensions. Open problems are mentioned at the end.

The definition and standard facts about a TVS can be found in [9]. However, we do not require the topology to be Hausdorff (this only matters in Section 2).

2. Finite Dimensional Topological Vector Spaces

Recall the following:

Theorem 1 ([1,2])**.** *Let X be $\mathbb{R}^n$ endowed with the Euclidean topology. Assume that the semigroup $S \subset X$ is not separated by any non-zero linear functional. Then, the closure of S is a group.*

This shows that the semigroup problem has an affirmative answer in any finite dimensional Hausdorff TVS, as any two Hausdorff TVS of the same finite dimension are isomorphic.

Consider now the non-Hausdorff case. Recall that a topological vector space X is called topologically trivial if the only nonempty open set in X is X itself. The following result can be found in [10]; it follows by considering the connected components of the TVS.

Theorem 2 ([10] Section 7, Problem A)**.** *Let X be a TVS of finite dimension d, not necessarily Hausdorff. Then, X is isomorphic to the Cartesian product $\mathbb{R}^m \times V$, where $m \leq d$, and V is a topologically trivial TVS of dimension $d - m$.*

We can now prove the following:

Theorem 3. *Let X be a finite dimensional TVS, not necessarily Hausdorff. Let $S \subset X$ be a semigroup that is not separated by any non-zero continuous linear functional. Then, the closure of S is a group.*

Proof. Following Theorem 2, write $X = \mathbb{R}^m \times V$ as the direct sum $\mathbb{R}^m \oplus V$ (this is a linear homeomorphism), and consider the linear projection $\pi_{\mathbb{R}} : X \to \mathbb{R}^m$, which is continuous. Each linear continuous map $\psi : \mathbb{R}^m \to \mathbb{R}$ extends to a continuous linear map $\psi_X : X \to \mathbb{R}$ by $\psi_X(x,v) := \psi(x) = \psi \circ \pi_{\mathbb{R}}$.

Because V has the trivial topology, any closed set F of X is of the form $F = F + V$, and the projection $\pi_{\mathbb{R}}$ is a closed map.

Consider the projection $S_{\mathbb{R}}$ of S onto the $\mathbb{R}^m$-component of X. Then, the closure of S in X is equal to $\operatorname{clos}_{\mathbb{R}^m}(S_{\mathbb{R}}) \oplus V$.

Now, $S_{\mathbb{R}} \subset \mathbb{R}^m$ is not separated by any functional on $\mathbb{R}^n$ that is non-zero on its closure. Hence, by Theorem 1 (applied to the closed linear span of $S_{\mathbb{R}}$), $\operatorname{clos}_{\mathbb{R}^m}(S_{\mathbb{R}})$ is a group, and denote it as $G_{\mathbb{R}}$.

Thus, the closure of S in X is $G_{\mathbb{R}} + V$, so it is a group as well. □

3. Infinite Dimensional Topological Vector Spaces

We start with some general facts about (not necessarily complete) normed spaces. These definitions are also valid for quasi-normed spaces, introduced in Section 3.3.

Definition 1. *A sequence $(x_n)_{n=1}^\infty$ in a (quasi-)normed space X is called a Schauder basis of X if for every $x \in X$, there exists a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty a_n x_n$. A sequence $(x_n)_{n=1}^\infty \subset X$, which is a Schauder basis of its closed linear span, is called a basic sequence. Given a space with a Schauder basis $(x_n)_n$, the linear operators:*

$$P_n : X \to X, \quad P_n\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{n} a_i x_i \tag{1}$$

are called canonical projections, and the maps:

$$\sum_{i=1}^{\infty} a_i x_i \mapsto a_n$$

are called coordinate functions.

Remark 1. *Without loss of generality, we will assume that the vectors in a basic sequence have each unit norm.*

Clearly, the coordinate function being continuous is equivalent to the canonical projections being continuous. If X is a Banach space with a Schauder basis, then canonical projections are continuous ([11] Chapter V, page 32), and therefore, the uniform boundedness principle implies that the canonical projections are uniformly bounded,

$$\sup_{n\geq 1} \|P_n\| < \infty. \tag{2}$$

For notational simplicity, we might identify a (finite) sum $\sum_k a_k x_k$ with the sequence $(a_n)_{n\geq 1}$.

3.1. The Semigroup S

This construction was introduced in [7]. Given a family $\{x_n\}_{n\geq 1} \subset X$, let $S \subset X$ be the semigroup generated by elements $s_k(p) \in X$ of the form:

$$s_k(p) := p \sum_{\ell=1}^{k-1} x_\ell + x_k = (p, p, \ldots, p, 1, 0, 0, \ldots)$$

for any integer p. That is,

$$s_k(p)|_n = \begin{cases} p, & \text{if } 1 \leq n \leq k-1, \\ 1, & \text{if } n = k, \\ 0, & \text{if } n > k. \end{cases} \tag{3}$$

with $p \in \mathbb{Z}$. Therefore, $s_k(p)$ is zero beyond the k-th entry; the k-th entry is one; and the first $k-1$ entries are equal to p, an arbitrary integer.

3.2. Normed Spaces

By a result attributed to Mazur (and Banach [12]), any Banach space has a basic sequence for a closed non-zero subspace. For normed spaces, the following is due to Day [13].

Theorem 4. *[13] Every normed infinite dimensional space X contains an infinite dimensional closed subspace with a Schauder basis, for which the canonical projections $(P_n)_n$ are uniformly bounded operators.*

More precisely: there are $x_i \in X$, $x_i^* \in X^*$, $i \geq 1$, orthogonal (that is, $x_i^*(x_j) = \delta_{i,j}$, the Kronecker symbol (the Kronecker symbol $\delta_{i,j}$ equals one if $i = j$ and zero otherwise), with $\|x_i\| = \|x_i^*\| = 1$ for each i, and such that if L denotes the closed span of the set $\{x_i\}_i$, then the canonical projection $P_n : L \to L$, $P_n(x) = \sum_{k=1}^{n} x_k^*(x) x_k$, satisfies $\|P_n\| \leq 1 + \frac{1}{n}$ for $n \geq 1$.

Theorem 5 (Normed spaces)**.** *Assume X is an infinite-dimensional normed vector space. Then, there is a semigroup $S \subset X$ for which the semigroup problem fails. That is, S is not one side of any hyperplane, but its closure is not a group.*

Proof of Theorem 5. We use for S the semigroup described in Section 3.1. The conclusion follows from Propositions 1 and 2. $\square$

Proposition 1. *The semigroup S is not separated by any non-zero continuous linear functional on L, the closed linear span of S.*

Proof. For $\phi \in L^*$, an arbitrary non-zero functional, denote $\phi_n = \phi(x_n)$. Since $\phi \neq 0$, not all the ϕ_n's are zero. Depending on the non-zero entries in ϕ, we have:

Case 1: Only one non-zero ϕ_n. Assume $\phi_k \neq 0$ and $\phi_n = 0$ if $n \neq k$. Consider an element of type s_k and the elements of type s_{k+1}. Then, $\phi(s_k(p)) = \phi_k$, $\phi(s_{k+1}(p)) = \phi_k p$ with $p \in \mathbb{Z}$ arbitrary, so ϕ does not separate S.

Case 2: More than one non-zero ϕ_n. Assume the first two non-zero ϕ_n's are ϕ_{k_1} and ϕ_{k_2}, $k_1 < k_2$. Consider the elements $s_{k_2}(p) \in X$. Since $\phi(s_{k_2}(p)) = p\phi_{k_1} + \phi_{k_2}$ with $p \in \mathbb{Z}$ arbitrary, ϕ does not separate S. □

We now show that the closure of S is not a group.

Proposition 2. *If the canonical projections are uniformly bounded, then the element $0 \in L$ does not belong to the closure of S.*

Proof. The elements of S are finite linear combinations, with positive integer coefficients, of the elements $s_k(p)$.

First, the vector zero does not belong to S: indeed, if:

$$\sum_{n=1}^{N} a_n s_{k_n}(p_n) = 0$$

with all $a_n > 0$, then considering only those n for which k_n equals $K := \max\{k_\ell : 1 \leq \ell \leq N\}$, we have (because the "leading coefficient" of $s_k(p)$ is one):

$$\sum_{\{n:k_n=K\}} a_n = 0,$$

a contradiction.

Next, note that any vector in S has as its first non-zero coefficient an integer. We claim that all such vectors have norm bounded away from zero.

Indeed, let $x = px_k + \sum_{\ell=k+1}^{\infty} a_\ell x_\ell \in S$, with $p \in \mathbb{Z} \setminus \{0\}$. Then, $|p| = \|P_k(x)\| \leq \|P_k\|\|x\| \leq M\|x\|$ because $M := \sup_k \|P_k\| < \infty$, by the hypothesis. Therefore, $\|x\| \geq p/M \geq 1/M$. □

3.3. Quasi-Normed Spaces

Recall that a quasi-norm on the linear spaces X is a map $\| \, . \, \| : X \to \mathbb{R}$ that satisfies the requirements of a norm, except that the triangle inequality is replaced by:

$$\|x + y\| \leq C(\|x\| + \|y\|) \text{ for } x, y \in X. \tag{4}$$

A quasi-norm is called a p-norm if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, $0 < p < 1$. Each locally bounded (a TVS X is locally bounded if the origin has a bounded neighborhood: there exists an open set $0 \in U \subset X$ such that for each open set $0 \in V \subset X$, $U \subset tV$ for some scalar t) F-space (an F-space is a TVS with the topology given by a translation invariant metric in which it is complete; if the topology is also locally convex, then the space is called Fréchet) can be given a p-norm that induces the same topology, and the converse is also true [14,15].

For example, the spaces $L^p([0,1])$ with $0 < p < 1$ are quasi-normed (and complete), with the usual L^p-norm being a p-norm. Moreover, when considered as complex-valued functions, then it is also plurisubharmonic [16]; see Definition 2. However, this space has a trivial continuous dual. In order to overcome this, we use the following result of Kalton [17].

Theorem 6. *([17] Theorem 4.4) Let X be an F-space. Then, the following are equivalent:*

1. *X contains no basic sequence (see Definition 1)*
2. *Every closed subspace of X with a separating dual is finite-dimensional.*

Consider from now on topological vector spaces over the field of complex numbers.

Definition 2. *For a complex vector space X, a quasi-norm is called plurisubharmonic if for each $x, y \in X$:*

$$\|x\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| d\theta.$$

A quasi-normed complex vector space is called A-convex (from "analytically convex") if it admits an equivalent quasi-norm that is plurisubharmonic.

Actually, every A-convex quasi-normed space can be given an equivalent plurisubharmonic p-norm, with $0 < p < 1$; see [18].

Tam [19] proved that plurisubharmonic quasi-normed spaces contain a basic sequence:

Theorem 7 ([19])**.** *Every A-convex quasi-normed space (equivalently, any plurisubharmonic quasi-normed space) contains basic sequences.*

More precisely: there is a basic sequence $\{x_n\}_{n \geq 1}$ consisting of unit vectors such that the canonical projections are uniformly bounded (the argument in [19] page 71, Theorem [11] page 38, Lemma 2), which implies the desired conclusion, including (2).

Theorem 8 (Plurisubharmonic spaces)**.** *Let X be a complete plurisubharmonic quasi-normed infinite-dimensional space. Then, there is a semigroup $S \subset X$ such that the closed linear span L of S has a separating dual, S is separated by each continuous non-zero functional on L, and the closure of S does not contain zero, so it is not a group.*

Proof. By Tam's Theorem 7, X contains a basic sequence. Therefore, by Kalton's Theorem 6, X contains an infinite dimensional closed subspace X_0 that has a separating dual.

Since X_0 still satisfies the hypothesis of Tam's theorem, Theorem 7 implies that there is a basic sequence $\{x_n\}_{n \geq 1}$ in X_0. Consider the semigroup defined in Section 3.1, and let L be its closed linear span. Then, L still has a separating dual.

Now, Propositions 1 and 2 can be applied, yielding the conclusion. □

4. Conclusions

In this paper, we discuss a semigroup problem that is relevant for the construction of stably topological transitive extensions of hyperbolic dynamical systems [7,20]. Here, we greatly extend our results in [7] by showing that the semigroup problem has a negative answer for a larger class of infinite dimensional topological vector spaces. We observe that a positive answer to the semigroup problem helps in the construction of stably transitive extensions of hyperbolic dynamical systems, but it is not necessary. Therefore our negative results here and in [7] leave open the question of the existence of stably transitive extensions for the respective fibers.

We do not know the answer to the semigroup problem in the following quasi-normed spaces: Kalton constructed in [12] a quasi-Banach space with no basic sequences; the spaces L^p/H^p, $0 < p < 1$, which are not plurisubharmonic [21] (see [18] Section 1).

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