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Differential Geometry and Binary Operations

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Abstract: We derive a large set of binary operations that are algebraically isomorphic to the binary operation of the Beltrami–Klein ball model of hyperbolic geometry, known as the Einstein addition. We prove that each of these operations gives rise to a gyrocommutative gyrogroup isomorphic to Einstein gyrogroup, and satisfies a number of nice properties of the Einstein addition. We also prove that a set of cogyrolines for the Einstein addition is the same as a set of gyrolines of another binary operation. This operation is found directly and it turns out to be commutative. The same results are obtained for the binary operation of the Beltrami–Poincare disk model, known as Möbius addition. We find a canonical representation of metric tensors of binary operations isomorphic to the Einstein addition, and a canonical representation of metric tensors defined by cogyrolines of these operations. Finally, we derive a formula for the Gaussian curvature of spaces with canonical metric tensors. We obtain necessary and sufficient conditions for the Gaussian curvature to be equal to zero.

Keywords: differential geometry; canonical metric tensor; binary operation; Einstein addition; gyrogroup; gyrovector space; curvature

1. Introduction

The theory of gyrogroups and gyrovector spaces has been intensively developed over recent years. The structure of gyrovector subspaces and orthogonal gyrodecomposition are studied in [1]. Topological gyrogroups are the subject of investigations in [2]. Article [3] is devoted to metric properties of gyrovector spaces. Several geometric inequalities in gyrovector spaces are established in [4]. Algebraic properties of gyrogroups in Hilbert spaces are investigated in [5]. An introduction to a theory of harmonic analysis on gyrogroups is presented in [6]. A study of isometries in generalized gyrovector spaces is presented in [7]. Gyrogroup actions are studied in [8]. An application of Einstein bi-gyrogroups to quantum multi-particle entanglement is presented in [9]. Several recent studies of gyrogroups and gyrovector spaces are presented in [10–12]. A number of fundamental results concerning gyrovector spaces and bi-gyrovector spaces are presented in [13–20]. The main concrete examples of gyrogroups and gyrovector spaces are those induced by the Einstein addition and by Möbius addition. Interestingly, (i) Einstein gyrovector spaces are based on the Einstein addition, and they provide the algebraic setting for the Klein ball model of hyperbolic geometry. Similarly, (ii) Möbius gyrovector spaces are based on Möbius addition, and they provide the algebraic setting for the Poincaré ball model of hyperbolic geometry, just as (iii) vector spaces form the algebraic setting for the common model of Euclidean geometry.

Recently, we developed in [21] a differential geometry approach to the theory of gyrogroups and gyrovector spaces based on local properties of underlying binary operations and, particularly, on properties of canonical metric tensors (see Definition 1) of corresponding Riemannian manifolds. It turned out to be possible to restore Einstein addition and Möbius addition from corresponding canonical metric tensors using standard tools of differential geometry. These are the parallel transport and the geodesics. Among important properties of the resulting Einstein and Möbius gyrogroups and gyrovector spaces are the left cancellation law, the existence of gyrations, the gyrocommutative law, and the left reduction law. These were proved using the differential geometry approach. Moreover, we found in [21] a gyrogroup and a gyrovector space in the ball $\mathbb{B}$, which turn out to be a group and a vector space isomorphic to the Euclidean group and space. Here we may note that any group and vector space is a gyrogroup and gyrovector space with trivial gyrations.

A gyration is a groupoid automorphism that emerges as a mathematical extension by abstraction of the special relativistic effect known as Thomas precession. It gives rise to the prefix "gyro" that we extensively use in the resulting gyroformalism. We, accordingly, prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and in nonassociative algebra. Our gyroterminology thus conveys a world of meaning in an elegant and memorable fashion. Thus, for instance, the Einstein addition and Möbius addition in the ball are neither commutative nor associative. However, they are both gyrocommutative and gyroassociative, giving rise to gyrogroups and gyrovector spaces [20].

The new results presented in this paper split up into three classes:

Class 1: Einstein addition and Möbius addition are isomorphic to each other, giving rise to an isomorphism between corresponding gyrogroups and gyrovector spaces. There exists a one-parameter set of binary operations that are isomorphic to the Einstein addition, and which generate gyrogroups and gyrovector spaces isomorphic to Einstein ones. Möbius addition is one of these operations. We consider the following problem. Are there operations that generate gyrogroups and gyrovector spaces isomorphic to Einstein ones, which are other than those belonging to the one parameter set? In this paper we show that there is a large class of such operations parametrized by a function $\tilde{\varphi}$ satisfying some mild conditions. All such operations are described in terms of corresponding canonical metric tensors.

Class 2: Each binary operation in $\mathbb{B}$ that we study in this paper defines sets of lines called gyrolines and cogyrolines. Gyrolines and cogyrolines are well studied for the cases of the Einstein addition and Möbius addition. We encounter here the following problem. Does the set of cogyrolines of an operation parametrized by a function $\tilde{\varphi}$ coincide with the set of gyrolines of some other operations? If the answer is yes, then how can we get such operations? In this paper we prove that such operations exist, and find the canonical metric tensors of these operations.

Class 3: It is known that the Gaussian curvature of the gyrovector space generated by Einstein addition is -1, and by Möbius addition is -4. What can we say about the Gaussian curvature of the gyrovector spaces generated by the operations found in Class 2? We provide an answer to this question. We prove that the Gaussian curvature of corresponding gyrovector spaces is equal to zero.

In this paper we extend the study of the differential geometry of binary operations in the ball that we initiated in [21]. The organization of the paper is the following. In Section 2 we present a short description of important results in [21], following which we introduce a set of operations isomorphic to Einstein addition. We, then, find the canonical metric tensors of these operations, enabling us to formulate an operation of scalar multiplication determined uniquely by these operations. We, thus, get the corresponding gyrovector spaces. In Section 3 we establish important properties of these operations that correspond to similar properties of Einstein gyrogroups. Section 4 is devoted to gyrolines and cogyrolines. We find the differential equations of the sets of gyrolines and the sets of cogyrolines for the cases of Einstein and Möbius additions. Remarkably, the operations, which we find using the sets of cogyrolines of Einstein and Möbius additions, are coincident. Moreover, they turn out to be exactly the operation that we have encountered in [21]. We also find the corresponding operations for an arbitrary function $\tilde{\varphi}$. In Section 5 we employ Brioschi formula [22] to calculate the Gaussian curvature of line elements in manifolds generated by the operations corresponding to cogyrolines. We prove that this curvature is always equal to zero. Finally, in Section 6 we present an interesting open problem.

2. Main Definitions, Procedures and Assumptions

Let $\mathbb{B}$ be the open unit ball in the n-dimensional Euclidean space $\mathbb{R}^n$,

$$\mathbb{B} = \{x \in \mathbb{R}^n \ : \ \|x\| < 1\}. \tag{1}$$

We seek binary operations $\oplus$ in $\mathbb{B}$ that are invariant under unitary transformations, that is, for every vectors $a, b \in \mathbb{B}$ and a unitary $n \times n$-matrix U

$$(Ua) \oplus (Ub) = U(a \oplus b). \tag{2}$$

Assuming that the function $f(a, b) = a \oplus b$ is differentiable, we introduce the matrix-function $G(x)$ given by

$$g(x) = \frac{\partial f(-x, y)}{\partial y}|_{y=x} \tag{3}$$

and

$$G(x) = g(x)^\top g(x), \tag{4}$$

where $\top$ denotes transposition.

The matrix-function G is viewed as a metric tensor in $\mathbb{B}$. We assume that this function has the canonical form (5) in the following formal definition.

Definition 1. (Canonical Metric Tensor). *The $n \times n$ matrix function $G(x)$,*

$$G(x) = m_0(\|x\|^2)\,[I - \frac{xx^\top}{\|x\|^2}] + m_1(\|x\|^2)\,\frac{xx^\top}{\|x\|^2}, \tag{5}$$

$x \in \mathbb{B}$, where m_0 and m_1 are scalar functions satisfying Assumptions 1 and 2 below, is said to be the canonical metric tensor in $\mathbb{B}$ parametrized by m_0 and m_1.

Assumption 1. *The functions m_0 and m_1 are differentiable, positive, and $m_0(0) = m_1(0) = 1$.*

Assumption 2. *The function m_1 satisfies the condition*

$$\int_0^\infty \sqrt{m_1(s^2)}ds = \infty. \tag{6}$$

Then G is also differentiable, and invariant under unitary transformations, that is, for all $x \in \mathbb{B}$ and $n \times n$-matrices U such that $U^\top U = I$ we have

$$G(Ux) = UG(x)U^\top. \tag{7}$$

Having such a matrix G we can restore the binary operation $\oplus$ using the following procedure that we introduced in [21]. Let $a, b \in \mathbb{B}$. If $b = 0$, then $a \oplus b = 0$. If $a = 0$, then $a \oplus b = b$. Otherwise we perform the following four steps that lead to $a \oplus b$.

Step 1. We calculate the vector

$$X_0 = \frac{b}{\|b\|}\int_0^{\|b\|} \sqrt{m_1(s^2)}ds. \tag{8}$$

Step 2. We calculate

$$X(1) = \frac{1}{\sqrt{m_0(\|a\|^2)}}(I - \frac{aa^\top}{\|a\|^2})X_0 + \frac{1}{\sqrt{m_1(\|a\|^2)}}\frac{aa^\top}{\|a\|^2}X_0. \tag{9}$$

Step 3. We find a solution $x(\cdot)$ of the differential equation

$$\ddot{x} + \frac{m_0'}{m_0} 2(x^\top \dot{x})\dot{x} + \left[(\frac{m_1'}{m_1} - \frac{2m_0'}{m_0}) \frac{(x^\top \dot{x})^2}{\|x\|^2} + \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1} (\|\dot{x}\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2}) \right] x = 0\,, \tag{10}$$

with the initial values $x(0) = a$, $\dot{x}(0) = X_1$. Here $m_j = m_j(\|x\|^2)$ and $m_j' = \frac{dm_j(u)}{du}|_{u=\|x\|^2}$, $j = 0, 1$.

Step 4. Then $a \oplus b = x(1)$.

For such binary operations $\oplus$ we defined in [21] an operation of scalar multiplication $\otimes$ satisfying the following properties: for all $a \in \mathbb{B}$ and numbers t_1, t_2 we have

$$\begin{aligned} (t_1 \otimes a) \oplus (t_2 \otimes a) &= (t_1 + t_2) \otimes a \\ (t_1 \otimes (t_2 \otimes a)) &= (t_1 t_2) \otimes a \\ 1 \otimes a &= a. \end{aligned} \tag{11}$$

The operation $\otimes$ is unique and is defined in [21] as follows. We introduce the following strictly increasing function h: $\mathbb{R} \to \mathbb{R}$,

$$h(p) = \int_0^p \sqrt{m_1(s^2)} ds, \tag{12}$$

and denote by h^{-1} the function inverse to h. Then for all $t \in \mathbb{R}$, $a \in \mathbb{B}$,

$$t \otimes a = \frac{a}{\|a\|} h^{-1}(t\, h(\|a\|)). \tag{13}$$

We pay special attention to the binary operation $\oplus_E$ in $\mathbb{B}$ of the Beltrami–Klein ball model of hyperbolic geometry, known as the Einstein addition. For all $a, b \in \mathbb{B}$

$$a \oplus_E b = \frac{1}{1 + a^\top b} [(1 + \frac{a^\top b}{\|a\|^2}) a + \sqrt{1 - \|a\|^2} (b - \frac{a^\top b}{\|a\|^2} a))]. \tag{14}$$

It is shown in [21] that this operation enjoys the following nice properties:

1. Left Cancellation Law:

$$a \oplus_E ((-a) \oplus_E b) = b, \qquad \forall a, b \in \mathbb{B}\,. \tag{15}$$

2. Existence of Gyrations: for every $a, b \in \mathbb{B}$ there exists a unitary matrix denoted by $gyr[a, b]$ such that for all $c \in \mathbb{B}$ we have the following gyroassociative law:

$$a \oplus_E (b \oplus_E c) = (a \oplus_E b) \oplus_E (gyr[a, b]c), \qquad \forall a, b, c \in \mathbb{B}. \tag{16}$$

3. Gyrocommutative Law:

$$a \oplus_E b = gyr[a, b](b \oplus_E a), \qquad \forall a, b \in \mathbb{B}\,, \tag{17}$$

such that

$$\|a \oplus_E b\| = \|b \oplus_E a\|\,. \tag{18}$$

4. Left Reduction Property:

$$gyr[a \oplus_E b, b] = gyr[a, b], \qquad \forall a, b \in \mathbb{B}\,. \tag{19}$$

The operation $\oplus_E$, along with the corresponding scalar multiplication $\otimes_E$ and gyrations $gyr[a, b]$, forms a gyrocommutative gyrogroup and a gyrovector space, as shown in [21].

In this paper we show that there exists a large class of binary operations $\oplus$ in $\mathbb{B}$ that satisfy properties (15)–(19). These operations are isomorphic to the Einstein addition $\oplus_E$, and are parametrized by special functions φ.

We now introduce the set of gyrolines

$$Q(a,b) = \{a \oplus (t \otimes b) \ : \ t \in \mathbb{R}\}, \tag{20}$$

and the set of cogyrolines

$$S(a,b) = \{(t \otimes a) \oplus b \ : \ t \in \mathbb{R}\}. \tag{21}$$

We find a binary operation $\oplus_{co}$ for which the set of gyrolines coincides with the set of cogyrolines of the Einstein addition. The same results are obtained for Möbius addition, which is isomorphic to Einstein addition.

Finally, for a curvature of the manifold $\mathbb{B}$ with canonical metric tensors G generated by binary operations $\oplus$ we calculate the Gaussian curvature in terms of coefficients of G. We show that the Gaussian curvature of Einstein and Möbius additions are constant, and the Gaussian curvature of Einstein and Möbius coadditions are zero.

3. New Binary Operations that Give Rise to Gyrogroups

3.1. A Family of Binary Operations

Every binary operation for which Properties (15)–(19) hold, determines its gyrocommutative gyrogroup structure [13], presented in [21]. We extend the study of Einstein and Möbius addition as follows.

Einstein addition and Möbius addition are isomorphic to each other (in the sense of gyrogroups) since they are related by the identities in ([15] Equations (6.325)),

$$\begin{aligned} \frac{1}{2} \otimes_E [(2 \otimes_E a) \oplus_E (2 \otimes_E b)] &= a \oplus_M b \\ 2 \otimes_M [(\frac{1}{2} \otimes_M a) \oplus_M (\frac{1}{2} \otimes_M b)] &= a \oplus_E b\,. \end{aligned} \tag{22}$$

Owing to the isomorphism between Einstein and Möbius addition, it is obvious that, like the Einstein addition, Möbius addition also satisfies Properties (15)–(19), and therefore forms a gyrogroup.

Instead of the number 2 in (22) it is possible to place any positive number t, thus obtaining from the Einstein addition $\oplus_E$ a new binary operation $\oplus_t$ in $\mathbb{B}$, given by the equation

$$\frac{1}{t} \otimes_E [(t \otimes_E a) \oplus_E (t \otimes_E b)] = a \oplus_t b. \tag{23}$$

When $t = 1$, the binary operation $\oplus_t$ descends to the Einstein addition and when $t = 2$, the binary operation $\oplus_t$ descends to Möbius addition.

It seems natural that for every $t \in \mathbb{R}_+ = (0, \infty)$ the ball $\mathbb{B}$ with the binary operation $\oplus_t$ forms a gyrocommutative gyrogroup (that is, it satisfies Properties (15)–(19)). We prove below that this is, indeed, the case.

More generally, we construct in this section a large family of binary operations (parametrized by a function $\tilde{\varphi}$) that satisfy Properties (15)–(19).

3.2. Operations Parametrized by Functions φ

Let us consider an arbitrary bijection $\tilde{\varphi}$: $[0,1) \to [0,1)$, which is differentiable, strictly increasing, and satisfies $\tilde{\varphi}'(0) > 0$. Since $\tilde{\varphi}$ is an increasing bijection, we have $\tilde{\varphi}(0) = 0$.

Then there exists an inverse smooth bijection $\tilde{\varphi}^{-1}$: $[0,1) \to [0,1)$, and $(\tilde{\varphi}^{-1})'(0) = \frac{1}{\tilde{\varphi}'(0)} > 0$.

We now define a function $\varphi\colon \mathbb{B} \to \mathbb{B}$ as follows. We set

$$\varphi(0) = 0, \tag{24}$$

and for every $x \in \mathbb{B}\backslash\{0\}$ we set

$$\varphi(x) = \tilde{\varphi}(\|x\|)\frac{x}{\|x\|}\,. \tag{25}$$

The function φ is differentiable everywhere in $\mathbb{B}$ including zero, since $\tilde{\varphi}$ is differentiable, and Equation (24) holds. Moreover, the function φ is a smooth bijection $\mathbb{B} \to \mathbb{B}$, there exists an inverse bijection $\varphi^{-1}\colon \mathbb{B} \to \mathbb{B}$ and, as it may be checked directly,

$$\varphi^{-1}(x) = \tilde{\varphi}^{-1}(\|x\|)\frac{x}{\|x\|}\,. \tag{26}$$

Now we introduce a new operation determined by the function $\tilde{\varphi}$. For every $a, b \in \mathbb{B}$ we define

$$a \oplus_\varphi b = \varphi^{-1}(\varphi(a) \oplus_E \varphi(b)), \tag{27}$$

where $\oplus_E$ is the Einstein addition. Obviously, this operation is isomorphic to the Einstein addition. Still, it is necessary to prove that the gyration operator is actually an operator of multiplication by a unitary matrix. We also prove below that the operation $\oplus_t$ is a special case of the operation $\oplus_\varphi$.

3.3. The Canonical Metric Tensor

Let us find the canonical metric tensor determined by the operation $\oplus_\varphi$. We have

$$\tilde{\varphi}(\|x+\Delta x\|) - \tilde{\varphi}(\|x\|) = \tilde{\varphi}'(\|x\|)\frac{x^\top \Delta x}{\|x\|} + o(\|\Delta x\|), \tag{28}$$

$$\varphi(x+\Delta x) - \varphi(x) = \left[\frac{\tilde{\varphi}(\|x\|)}{\|x\|}(I - \frac{xx^\top}{\|x\|^2}) + \tilde{\varphi}'(\|x\|)\frac{xx^\top}{\|x\|^2}\right]\Delta x + o(\Delta x), \tag{29}$$

$$\begin{aligned}(-\varphi(x)) \oplus_E \varphi(x+\Delta x) = \frac{1}{1-\tilde{\varphi}(\|x\|)^2}\left[\frac{\tilde{\varphi}(\|x\|)}{\|x\|}\sqrt{1-\tilde{\varphi}(\|x\|)^2}(I - \frac{xx^\top}{\|x\|^2})\right.\\ \left. + \tilde{\varphi}'(\|x\|)\frac{xx^\top}{\|x\|^2}\right]\Delta x + o(\Delta x)\,.\end{aligned} \tag{30}$$

Denoting the coefficient of Δx in (30) by $g(x)$, we obtain

$$(-x) \oplus_\varphi (x+\Delta x) = \frac{1}{\tilde{\varphi}'(0)} g(x)\Delta x + o(\Delta x), \tag{31}$$

and the canonical metric tensor is

$$G_\varphi(x) = \frac{g(x)g(x)^\top}{\tilde{\varphi}'(0)^2} \frac{1}{\tilde{\varphi}'(0)^2}\left[\frac{(\tilde{\varphi}(\|x\|))^2}{\|x\|^2(1-\tilde{\varphi}(\|x\|)^2)}(I - \frac{xx^\top}{\|x\|^2}) + \frac{(\tilde{\varphi}'(\|x\|)^2}{(1-\tilde{\varphi}(\|x\|)^2)^2}\frac{xx^\top}{\|x\|^2}\right]. \tag{32}$$

Hence, we have in (32) the canonical metric tensor (5), parametrized by m_0 and m_1, given by

$$\begin{aligned} m_{0,\varphi}(r^2) &= \left[\frac{\tilde{\varphi}(r)}{\tilde{\varphi}'(0) r\sqrt{1-\tilde{\varphi}(r)^2}}\right]^2 \\ m_{1,\varphi}(r^2) &= \left[\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}'(0)(1-\tilde{\varphi}(r)^2)}\right]^2. \end{aligned} \tag{33}$$

Noticing that $m_{0,\varphi}(0) = 1$ and $m_{1,\varphi}(0) = 1$, the second equation in (33) may be solved for $\tilde{\varphi}$. We further notice that $\tilde{\varphi}(0) = 0$. Hence,

$$\tilde{\varphi}(r) = \tanh\left[\tilde{\varphi}'(0)\int_0^r \sqrt{m_{1,\varphi}(s^2)}ds\right]. \tag{34}$$

Now we consider a set of functions $\tilde{\varphi}$ such that functions $m_{1,\varphi}$ are equal to the same function, which we denote by m_1. Due to (34) all such functions $\tilde{\varphi}$ may be parametrized by a number $t = \tilde{\varphi}'(0) \in (0, \infty)$. We denote such functions by $\tilde{\varphi}_t$:

$$\tilde{\varphi}_t(r) = \tanh\left[t\int_0^r \sqrt{m_1(s^2)}ds\right]. \tag{35}$$

As we show below in Section 3.7, for each $t > 0$ and the same m_1 we can find a function $\tilde{\varphi}_t$ and a corresponding function

$$m_{0,\varphi_t}(r^2) = \left[\frac{\tilde{\varphi}_t(r)}{tr\sqrt{1-\tilde{\varphi}_t(r)^2}}\right]^2 \tag{36}$$

such that a binary operation generated by the canonical metric tensor

$$G_{\varphi_t}(x) = m_{0,\varphi_t}(r^2)\left[I - \frac{xx^\top}{\|x\|^2}\right] + m_1(r^2)\frac{xx^\top}{\|x\|^2} \tag{37}$$

is a gyrogroup operation, satisfying Properties (15)–(19).

Example 1. *Let*

$$m_1(r^2) = \frac{1}{(1-r^2)^2}. \tag{38}$$

Then, by (35),

$$\tilde{\varphi}_t(r) = \tanh(t\,\mathrm{atanh}(r)) \tag{39}$$

and by (36),

$$m_{0,\varphi_t}(r^2) = \frac{\tilde{\varphi}_t(r)^2}{t^2r^2(1-\tilde{\varphi}_t(r)^2)}. \tag{40}$$

For $t = 1$ we get

$$\begin{aligned}\tilde{\varphi}_1(r) &= r \\ m_{0,\varphi_1}(r^2) &= \frac{1}{1-r^2} \\ G_{\varphi_1}(x) &= \frac{1}{1-r^2}\left[I - \frac{xx^\top}{\|x\|^2}\right] + \frac{1}{(1-r^2)^2}\frac{xx^\top}{\|x\|^2},\end{aligned} \tag{41}$$

which is the canonical metric tensor for Einstein addition as shown in [21].

For $t = 2$ we get

$$\begin{aligned}\tilde{\varphi}_2(r) &= \frac{2r}{1+r^2} \\ m_{0,\varphi_2}(r^2) &= \frac{1}{(1-r^2)^2} \\ G_{\varphi_2}(x) &= \frac{1}{(1-r^2)^2}\left[I - \frac{xx^\top}{\|x\|^2}\right] + \frac{1}{(1-r^2)^2}\frac{xx^\top}{\|x\|^2},\end{aligned} \tag{42}$$

which is the canonical metric tensor for Möbius addition as shown in [21].

3.4. Multiplication of Vectors by Numbers

The function h in (12) has the form

$$\begin{aligned} h_\varphi(p) &= \int_0^p \sqrt{m_1(s^2)}ds \\ &= \int_0^p \frac{\tilde{\varphi}'(s)}{\tilde{\varphi}'(0)(1-\tilde{\varphi}(s)^2)}ds \\ &= \frac{1}{\tilde{\varphi}'(0)}\Big[\operatorname{atanh}(\tilde{\varphi}(p)) - \operatorname{atanh}(\tilde{\varphi}(0))\Big] \\ &= \frac{\operatorname{atanh}(\tilde{\varphi}(p))}{\tilde{\varphi}'(0)}. \end{aligned} \tag{43}$$

Therefore

$$\begin{aligned} t \otimes_\varphi (a) &= \frac{a}{\|a\|} h_\varphi^{-1}(t h_\varphi(\|a\|)) \\ &= \frac{a}{\|a\|}\tilde{\varphi}^{-1}(\tanh(t\operatorname{atanh}(\tilde{\varphi}(\|a\|)))) \\ &= \varphi^{-1}(t \otimes_E \varphi(a)). \end{aligned} \tag{44}$$

3.5. Relations between the Functions m_0 and m_1

In this subsection we explore the relations between the functions $m_{0,t}$ and m_1 for which the corresponding tensor G determines a gyrocommutative gyrogroup operation.

Let us fix a smooth positive function m_1 such that $m_1(0) = 1$ and m_1 is a bijection $[0,1) \to [1,\infty)$. We choose an arbitrary positive number t and define

$$\tilde{\varphi}_t(r) = \tanh\left[t\int_0^r \sqrt{m_1(s^2)}ds\right] \tag{45}$$

and

$$m_{0,\varphi_t}(r^2) = \left[\frac{\tilde{\varphi}_t(r)}{rt\sqrt{1-\tilde{\varphi}_t(r)^2}}\right]^2. \tag{46}$$

The pair of functions $(m_{0,t}, m_1)$ determines a canonical metric tensor (37) and a binary operation of a gyrogroup satisfying properties (15)–(19). Then

$$\tilde{\varphi}_t(r) = \frac{rt\sqrt{m_{0,\varphi_t}(r^2)}}{\sqrt{1+r^2t^2m_{0,\varphi_t}(r^2)}} \tag{47}$$

and

$$\begin{aligned} \sqrt{m_1(r^2)} &= \frac{d}{dr}\frac{\operatorname{atanh}(\tilde{\varphi}_t(r))}{t} \\ &= \frac{1}{t}\frac{d}{dr}\ln\left[\sqrt{1+r^2t^2m_{0,\varphi_t}(r^2)} + rt\sqrt{m_{0,\varphi_t}(r^2)}\right]. \end{aligned} \tag{48}$$

Let a pair of smooth functions $m_{0,t}$ and m_1 satisfy (48), and such that the function m_1 is increasing and $m_1(0) = 1$. Then this pair determines a gyrocommutative gyrogroup operation in $\mathbb{B}$, as we will show in Section 3.7.

3.6. Unitary Gyration Operator

For every binary operation $\oplus_n$ isomorphic to Einstein addition $\oplus$ it is possible to introduce the gyration operator $gyr[a,b]\colon \mathbb{B} \times \mathbb{B} \to \mathbb{B}$. In general this operator need not be linear. Remarkably, however, the gyration operator for the operation $\oplus_\varphi$ turns out to be linear, as we will see in Lemma 1.

Lemma 1. *For every function $\tilde{\varphi}$ introduced in (45), the gyration operator $gyr_\varphi[d,e]$,*

$$gyr_\varphi[d,e]f = -(d \oplus_\varphi e) \oplus_\varphi (d \oplus_\varphi (e \oplus_\varphi f)), \tag{49}$$

is a linear operator $\mathbb{B} \to \mathbb{B}$. The matrix of this operator is unitary. Moreover,

$$gyr_\varphi[a,b]c = gyr[\varphi(a), \varphi(b)]c\,, \tag{50}$$

for all $a, b, c \in \mathbb{B}$.

Proof. We use (26), and the fact that the matrix of the gyration operator $gyr[a,b]$ for Einstein addition $\oplus$ is unitary. For every function $\tilde{\varphi}$ described in Section 3.2, and vectors $a, b, c \in \mathbb{B}$ we have

$$\begin{aligned}
gyr_\varphi[a,b]c &= -(a \oplus_\varphi b) \oplus_\varphi (a \oplus_\varphi (b \oplus_\varphi c)) \\
&= \varphi^{-1}\Big[-(\varphi(a) \oplus_E \varphi(b)) \oplus_E (\varphi(a) \oplus_E (\varphi(b) \oplus_E \varphi(c)))\Big] \\
&= \varphi^{-1}\Big[gyr[\varphi(a), \varphi(b)]\varphi(c)\Big] \\
&= \tilde{\varphi}^{-1}(\|\varphi(c)\|)\frac{gyr[\varphi(a), \varphi(b)]\varphi(c)}{\|\varphi(c)\|} \\
&= \tilde{\varphi}^{-1}(\tilde{\varphi}(\|c\|))\frac{gyr[\varphi(a), \varphi(b)]\tilde{\varphi}(\|c\|)c}{\tilde{\varphi}(\|c\|)\|c\|} \\
&= gyr[\varphi(a), \varphi(b)]c\,.
\end{aligned} \tag{51}$$

Hence, the operator $gyr_\varphi[a,b]$ is linear, and its matrix representation is the same as the matrix representation of the operator $gyr[\varphi(a), \varphi(b)]$ for the Einstein addition. This matrix is unitary. Therefore, the matrix of the linear operator $gyr_\varphi[a,b]$ is also unitary. The proof of the Lemma is, thus, complete. □

The gyrolinearity of the operation $\oplus_\varphi$ follows from the fact that the matrix $gyr[\varphi(a), \varphi(b)]$ is unitary.

3.7. Special Properties of Operations Parametrized by Functions φ

Theorem 1. *The operation $\oplus_\varphi$ has the same properties as those of Einstein addition:*

1. Left cancellation law:

$$a \oplus_\varphi ((-a) \oplus_\varphi b) = b \qquad \forall a, b \in \mathbb{B}\,. \tag{52}$$

2. Existence of gyrations: for every $a, b \in \mathbb{B}$ there exists a unitary matrix denoted by $gyr_\varphi[a, b]$ such that for all $c \in \mathbb{B}$ we have the following gyroassociative law:

$$a \oplus_\varphi (b \oplus_\varphi c) = (a \oplus_\varphi b) \oplus_\varphi (gyr_\varphi[a, b]c), \quad \forall a, b, c \in \mathbb{B}. \tag{53}$$

3. For all $a, b \in \mathbb{B}$ we have the following gyrocommutative law:

$$a \oplus_\varphi b = gyr_\varphi[a, b](b \oplus_\varphi a), \quad \forall a, b \in \mathbb{B}, \tag{54}$$

implying

$$\|a \oplus_\varphi b\| = \|b \oplus_\varphi a\|\,. \tag{55}$$

4. Reduction property:

$$gyr_\varphi[a \oplus_\varphi b, b] = gyr_\varphi[a, b], \quad \forall a, b \in \mathbb{B}. \tag{56}$$

5. Linearity of gyrations with respect to addition and multiplication:

$$gyr_\varphi[a, b]\Big[(t_1 \otimes_\varphi c_1) \oplus_\varphi (t_2 \otimes_\varphi c_2)\Big] = (t_1 \otimes_\varphi (gyr_\varphi[a, b]c_1)) \oplus_\varphi (t_2 \otimes_\varphi (gyr_\varphi[a, b]c_2)) \tag{57}$$

for all $c_1, c_2 \in \mathbb{B}$ and $t_1, t_2 \in [0, \infty)$.

Proof. The proof follows straightforwardly from the definition of the operation $\oplus_\varphi$ given in (27). We have,

$$\begin{aligned} a \oplus_\varphi ((-a) \oplus_\varphi b) &= \varphi^{-1}[\varphi(a) \oplus_E \varphi(\varphi^{-1}(\varphi(-a) \oplus_E \varphi(b))] \\ &= \varphi^{-1}[\varphi(a) \oplus_E ((-\varphi(a)) \oplus_E \varphi(b))] \\ &= \varphi^{-1}[\varphi(b)] = b\,. \end{aligned} \tag{58}$$

Hence, Property 1 is satisfied.
We have,

$$\begin{aligned} a \oplus_\varphi (b \oplus_\varphi c) &= \varphi^{-1}(\varphi(a) \oplus_E \varphi(\varphi^{-1}(\varphi(b) \oplus_E \varphi(c))) \\ &= \varphi^{-1}[\varphi(a) \oplus_E (\varphi(b) \oplus_E \varphi(c))] \\ &= \varphi^{-1}[(\varphi(a) \oplus_E \varphi(b)) \oplus_E (gyr[\varphi(a), \varphi(b)]\varphi(c))] \\ &= \varphi^{-1}[\varphi(a) \oplus_E \varphi(b)] \oplus_\varphi \varphi^{-1}(gyr[\varphi(a), \varphi(b)]\varphi(c)) \\ &= (a \oplus_\varphi b) \oplus_\varphi (gyr_\varphi[a, b]c)\,. \end{aligned} \tag{59}$$

Hence, Property 2 holds.
We have,

$$\begin{aligned} a \oplus_\varphi b &= \varphi^{-1}(\varphi(a) \oplus_E \varphi(b)) \\ &= \varphi^{-1}(gyr[\varphi(a), \varphi(b)](\varphi(b) \oplus_E \varphi(a))) \\ &= gyr_\varphi[a, b]\varphi^{-1}(\varphi(b) \oplus_E \varphi(a)) \\ &= gyr_\varphi[a, b](b \oplus_\varphi a). \end{aligned} \tag{60}$$

Thus, Property 3 is valid.
We have,

$$\begin{aligned} gyr_\varphi[a \oplus_\varphi b, b] &= gyr[\varphi(a \oplus_\varphi b), \varphi(b)] \\ &= gyr[(\varphi \oplus_E \varphi(b)), \varphi(b)] \\ &= gyr[\varphi(a), \varphi(b)] = gyr_\varphi[a, b]\,. \end{aligned} \tag{61}$$

Thus, Property 4 is valid.

Finally, we have,

$$
\begin{aligned}
& gyr_\varphi[a,b]\Big[(t_1 \otimes_\varphi c_1) \oplus_\varphi (t_2 \otimes_\varphi c_2)\Big] \\
&= gyr[\varphi(a),\varphi(b)]\varphi^{-1}\Big[\varphi(\varphi^{-1}(t_1 \otimes_E \varphi(c_1))) \oplus_E \varphi(\varphi^{-1}(t_2 \otimes_E \varphi(c_2)))\Big] \\
&= \varphi^{-1}\Big[gyr[\varphi(a),\varphi(b)][(t_1 \otimes_E \varphi(c_1)) \oplus_E (t_2 \otimes_E \varphi(c_2))]\Big] \\
&= \varphi^{-1}\Big[(t_1 \otimes_E (gyr[\varphi(a),\varphi(b)]\varphi(c_1))) \oplus_E (t_2 \otimes_E (gyr[\varphi(a),\varphi(b)]\varphi(c_2)))\Big] \\
&= \varphi^{-1}\Big[\varphi(t_1 \otimes_\varphi (gyr_\varphi[a,b]c_1)) \oplus_E \varphi(t_2 \otimes_\varphi (gyr_\varphi[a,b]c_2))\Big] \\
&= (t_1 \otimes_\varphi (gyr_\varphi[a,b]c_1)) \oplus_\varphi (t_2 \otimes_\varphi (gyr_\varphi[a,b]c_2))\,.
\end{aligned} \tag{62}
$$

Thus, Property 5 is valid, and the proof of the Theorem is complete. □

We now check properties of gyrocommutative gyrogroups for the groupoid $(\mathbb{B}, \oplus_\varphi)$.

1. From the following three results, (i) identity (52) of Theorem 1, (ii) $\varphi(0) = 0$ (see (24)), and (iii) $0 \oplus_E a = a$ for all $a \in \mathbb{B}$, we obtain the existence of a left identity, that is, for all $a \in \mathbb{B}$

$$0 \oplus_\varphi a = a. \tag{63}$$

2. From identity (52) of Theorem 1 with $b = 0$ and (63) we obtain the existence of a left inverse, that is, for all $a \in \mathbb{B}$

$$(-a) \oplus_\varphi a = 0. \tag{64}$$

3. Identity (53) of Theorem 1 implies that the binary operation $\oplus_\varphi$ obeys the left gyroassociative law, that is, for all $a, b, c \in \mathbb{B}$

$$a \oplus_\varphi (b \oplus_\varphi c) = (a \oplus_\varphi b) \oplus_\varphi gyr_\varphi[a,b]c. \tag{65}$$

4. From statement 2 of Theorem 1 we see that $gyr_\varphi[a,b]$ is a unitary matrix for all $a, b \in \mathbb{B}$. Therefore, the mapping $c \to gyr_\varphi[a,b]c$ is invertible. Identity (57) with $t_1 = t_2 = 1$ shows that this mapping is an automorphism of the groupoid $(\mathbb{B}, \oplus_\varphi)$.

5. Identity (56) of Theorem 1 implies that the operator gyr_φ possesses the left reduction property.

Hence, as shown in [21], the groupoid $(\mathbb{B}, \oplus_\varphi)$ is a gyrogroup.

Finally, identity (54) of Theorem 1 implies that the groupoid $(\mathbb{B}, \oplus_\varphi)$ is gyrocommutative so that, by [21], it is a gyrocommutative gyrogroup.

3.8. The Canonical Metric Tensor For Coaddition

Let $\boxplus_\varphi$ be the binary operation such that for every $a, b \in \mathbb{B}$ the solution x of the equation

$$x \oplus_\varphi a = b \tag{66}$$

is given by

$$x = b \boxplus_\varphi (-a)\,. \tag{67}$$

The binary operation $\boxplus_\varphi$ turns out to be

$$\begin{aligned} a \boxplus_\varphi b &= a \oplus_\varphi gyr_\varphi[a, -b]b \\ &= \varphi^{-1}\Big[\varphi(a) \oplus \varphi(gyr[\varphi(a), -\varphi(b)]b)\Big] \\ &= \varphi^{-1}\Big[\varphi(a) \oplus gyr[\varphi(a), -\varphi(b)]\varphi(b)\Big] \\ &= \varphi^{-1}\Big[\varphi(a) \boxplus \varphi(b)\Big]. \end{aligned} \tag{68}$$

Then,

$$(-x) \boxplus_\varphi (x + \Delta x) = \varphi^{-1}\Big[(-\varphi(x) \boxplus \varphi(x + \Delta x)\Big]. \tag{69}$$

Noticing that

$$\begin{aligned} \varphi(x + \Delta x) - \varphi(x) &= \tilde{\varphi}(\|x + \Delta x\|)\frac{x + \Delta x}{\|x + \Delta x\|} - \tilde{\varphi}(x)\frac{x}{\|x\|} \\ = \frac{1}{\tilde{\varphi}'(0)}&\left[\frac{\tilde{\varphi}(\|x\|)}{\|x\|}(I - \frac{xx^\top}{\|x\|^2}) + \frac{\tilde{\varphi}'(\|x\|)}{1 - \tilde{\varphi}(\|x\|)^2}\frac{xx^\top}{\|x\|^2}\right]\Delta x + o(\|\Delta x\|), \end{aligned} \tag{70}$$

we see that the canonical metric tensor in the space with the binary operation $\boxplus_\varphi$ is given by

$$G_{\boxplus_\varphi}(x) = \left[\frac{\tilde{\varphi}(\|x\|)}{\tilde{\varphi}'(0)\|x\|}\right]^2 (I - \frac{xx^\top}{\|x\|^2}) + \left[\frac{\tilde{\varphi}'(\|x\|)}{\tilde{\varphi}'(0)(1 - \tilde{\varphi}(\|x\|)^2)}\right]^2 \frac{xx^\top}{\|x\|^2}. \tag{71}$$

Thus, we have the canonical metric tensor G with

$$\begin{aligned} m_{0,\boxplus_\varphi}(\|x\|^2) &= \left[\frac{\tilde{\varphi}(\|x\|)}{\tilde{\varphi}'(0)\|x\|}\right]^2 \\ m_{1,\boxplus_\varphi}(\|x\|^2) &= \left[\frac{\tilde{\varphi}'(\|x\|)}{\tilde{\varphi}'(0)(1 - \tilde{\varphi}(\|x\|)^2)}\right]^2. \end{aligned} \tag{72}$$

In particular, for the trivial case with linear $\tilde{\varphi}$ (i.e., when $\tilde{\varphi}(\|x\|) = \|x\|$) we have

$$\begin{aligned} \boxplus_\varphi &= \boxplus \\ m_{0,\boxplus_\varphi}(\|x\|^2) &= 1 \\ m_{1,\boxplus_\varphi}(\|x\|^2) &= \frac{1}{(1 - \|x\|^2)^2}. \end{aligned} \tag{73}$$

Noticing that if $\tilde{\varphi}(r) = \tanh(t \operatorname{atanh}(r))$ for some positive number t, as for the cases of Einstein and Möbius additions, then we see that

$$m_{1,\boxplus_\varphi}(\|x\|^2) = \frac{1}{(1 - \|x\|^2)^2}. \tag{74}$$

4. Gyrolines and Cogyrolines

Let us consider a Riemannian manifold with a canonical metric tensor G in $\mathbb{B}$,

$$G(x) = \{g_{ij}(x)\}_{i,j=1}^{n} = m_0(\|x\|^2)(I - \frac{xx^\top}{\|x\|^2}) + m_1(\|x\|^2)\frac{xx^\top}{\|x\|^2}. \tag{75}$$

The geodesics in this manifold are solutions of the second order differential Equation (10), that is,

$$\ddot{x} + \frac{m_0'}{m_0}2(x^\top\dot{x})\dot{x} + \left[(\frac{m_1'}{m_1} - \frac{2m_0'}{m_0})\frac{(x^\top\dot{x})^2}{\|x\|^2} + \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1}(\|\dot{x}\|^2 - \frac{(x^\top\dot{x})^2}{\|x\|^2})\right]x = 0. \tag{76}$$

We denote by $\oplus$ the binary operation introduced in (14) and assume that Assumption 2 for the function m_1 holds, that is,

$$\int_0^\infty \sqrt{m_1(s^2)}ds = \infty. \tag{77}$$

Then, the product $t\otimes$ in (13) is well defined and belongs to $\mathbb{B}$ for every $a \in \mathbb{B}$, $t \in \mathbb{R}$, and $a \oplus b$ is also well defined and belongs to $\mathbb{B}$ for all $a, b \in \mathbb{B}$.

Definition 2. *For every $a, b \in \mathbb{B}$ such that $b \neq 0$ the curve*

$$Q_g(a,b) = \{a \oplus (t \otimes b) \quad : \quad t \in \mathbb{R}\} \tag{78}$$

is called a gyroline.

For the Einstein addition $\oplus_E$ the gyrolines are Euclidean intervals in $\mathbb{B}$. For Möbius addition the gyrolines are circular arcs that intersect the boundary of $\mathbb{B}$ orthogonally. Every gyroline is a geodesic in a Riemannian manifold $\mathbb{B}$ with a canonical metric tensor G. Notice that in order to get gyrolines from a binary operation $a \oplus b$ we multiply the second vector b by numbers t, as in (78).

Definition 3. *For every $a, b \in \mathbb{B}$ such that $a \neq 0$ the curve*

$$Q(a,b) = \{(t \otimes a) \oplus b \quad : \quad t \in \mathbb{R}\} \tag{79}$$

is called a cogyroline.

In this section, we face the following problem. Is it possible to find a canonical metric tensor G_{co} such that cogyrolines are geodesics in the Riemannian manifold with a canonical metric tensor G_{co}?

If b is parallel to a, then cogyrolines coincide with gyrolines and are segments of Euclidean lines: $Q(a,0) = \{at \;:\; t \in \mathbb{R},\; \|at\| < 1\}$. We, therefore, assume that b is not parallel to a.

4.1. Einstein Cogyrolines

In this section, we consider the Einstein addition $\oplus_E$ and Einstein multiplication $\otimes_E$.

4.1.1. Elliptic Curves

For every $b \in \mathbb{B}$ not parallel to $a \in \mathbb{B}$ we define

$$b_\perp = b - \frac{a^\top b}{\|a\|^2}a. \tag{80}$$

Obviously

$$(b_\perp)^\top a = 0. \tag{81}$$

We notice that every cogyroline lies in the two-dimensional plane $P(a,b) \subset \mathbb{B}$ that contains a and b.

Theorem 2. *For every $a \in \mathbb{B}\backslash\{0\}$ and b not parallel to a the cogyroline lies in an ellipse:*

$$Q(a,b) = P(a,b) \cap \{y \in \mathbb{B} \ : \ \left[\frac{y^\top a}{\|a\|}\right]^2 + c\left[\frac{y^\top b_\perp}{\|b_\perp\|}\right]^2 = 1\}, \tag{82}$$

where

$$c = \frac{1 - \frac{(a^\top b)^2}{\|a\|^2}}{\|b_\perp\|^2} = \frac{\|a\|^2 - (a^\top b)^2}{\|a\|^2\|b\|^2 - (a^\top b)^2} > 1\,. \tag{83}$$

Proof. By (13) with $h(t) = \tanh(t)$ we have

$$t \otimes_E a = g(t)a\,, \tag{84}$$

where

$$g(t) = \frac{\operatorname{atanh}(t\ \tanh(\|a\|))}{\|a\|}\,. \tag{85}$$

For $x(t) = (t \otimes_E a) \oplus_E b$ we have, by (84), (85) and (14),

$$x(t) = \frac{1}{1 + g(t)a^\top b}\left[(g(t) + \frac{a^\top b}{\|a\|^2})a + \sqrt{1 - g(t)^2\|a\|^2}(b - \frac{a^\top b}{\|a\|^2}a)\right]. \tag{86}$$

Therefore

$$\begin{aligned} \frac{x(t)^\top a}{\|a\|} &= \frac{g(t)\|a\| + \frac{a^\top b}{\|a\|}}{1 + g(t)a^\top b} \\ \frac{x(t)^\top b_\perp}{\|b_\perp\|^2} &= \frac{\sqrt{1 - g(t)^2\|a\|^2}}{1 + g(t)a^\top b}\,, \end{aligned} \tag{87}$$

and for all $d \in \mathbb{R}$,

$$\left[\frac{x(t)^\top a}{\|a\|}\right]^2 + d\left[\frac{x(t)^\top b_\perp}{\|b_\perp\|^2}\right]^2 = \frac{g(t)^2\|a\|^2 + 2g(t)a^\top b + \frac{(a^\top b)^2}{\|a\|^2} + d - d\,g(t)^2\|a\|^2}{(1 + g(t)a^\top b)^2}\,. \tag{88}$$

Setting $d = 1 - \frac{(a^\top b)^2}{\|a\|^2}$, we have

$$\left[\frac{x(t)^\top a}{\|a\|}\right]^2 + d\left[\frac{x(t)^\top b_\perp}{\|b_\perp\|^2}\right]^2 = \frac{g(t)^2\|a\|^2(1-d) + 2g(t)a^\top b + d + \frac{(a^\top b)^2}{\|a\|^2}}{(1 + g(t)a^\top b)^2} = 1\,. \tag{89}$$

We now set $c = \frac{d}{\|b_\perp\|^2}$. Then

$$\left[\frac{x(t)^\top a}{\|a\|}\right]^2 + c\left[\frac{x(t)^\top b_\perp}{\|b_\perp\|}\right]^2 = 1 \tag{90}$$

and

$$c = \frac{1 - \frac{(a^\top b)^2}{\|a\|^2}}{\|b_\perp\|^2} = \frac{\|a\|^2 - (a^\top b)^2}{\|a\|^2\|b\|^2 - (a^\top b)^2} > 1\,. \tag{91}$$

The proof of the Theorem is, thus, complete. □

We notice that the derivative $\frac{d}{dt}(t \otimes_E a) \oplus_E b$ is not equal to zero for all t. Therefore for every $a, b \in \mathbb{B}$ the function $x(t) = (t \otimes_E a) \oplus_E b$ tends to $\pm\frac{a}{\|a\|}$ as $t \to \pm\infty$, and the corresponding cogyroline is a half of ellipse (semi-ellipse), represented in (82).

4.1.2. The Canonical Metric Tensor for Cogyrolines

Let us find the canonical metric tensor for which every geodesic lies on some ellipse (82).

We consider the following second order differential equation

$$\ddot{x} + \frac{2(x^\top \dot{x})}{1 - \|x\|^2}\dot{x} + \left[\|\dot{x}\|^2 + \frac{(x^\top \dot{x})^2}{1 - \|x\|^2}\right]x = 0\,. \tag{92}$$

Theorem 3. *Every solution of Equation (92) lies on a cogyroline. Every cogyroline is a set of all points of a solution $x(\cdot)$ of (92) defined on $\mathbb{R}$.*

Proof. Let x be a solution of (92). If the vectors $x(t)$ and $\dot{x}(t)$ are parallel for any point $t \in \mathbb{R}$, then $x(t)$ belongs to the ray $\{as \,:\, 0 < s < \infty\}$ for all $t \in \mathbb{R}$, and the set of points $\{x(t) \,:\, t \in \mathbb{R}\}$ coincides with the interval with endpoints $\pm a/\|a\|$, which in turn coincides with a cogyroline.

We assume that the vectors $x(t)$ and $\dot{x}(t)$ are not parallel for all $t \in \mathbb{R}$, and denote by P the two-dimensional plane that contains the vectors $x(\bar{t})$ and $\dot{x}(\bar{t})$ for some $\bar{t} \in \mathbb{R}$. Then $x(t) \in P$ for all $t \in \mathbb{R}$. Introducing an orthogonal basis of P, let $x_1(\cdot)$, $x_2(\cdot)$ be coordinates of $x(\cdot)$ in this basis. We denote by $y(\cdot)$ the 2-vector function $col(x_1(\cdot), x_2(\cdot))$. Then $y(\cdot)$ satisfies Equation (92).

Let us consider the functions

$$\begin{aligned} p_1 &= \dot{x}_2(1 - \|x\|^2) + x_2\, x^\top \dot{x} \\ p_2 &= \dot{x}_1(1 - \|x\|^2) + x_1\, x^\top \dot{x}\,. \end{aligned} \tag{93}$$

Then,

$$\begin{aligned} &\dot{p}_1 p_2 - \dot{p}_2 p_1 = \left[\ddot{x}_2(1 - \|x\|^2) - 2\dot{x}_2(x^\top \dot{x}) + \dot{x}_2 x^\top \dot{x} + x_2\|\dot{x}\|^2 + x_2 x^\top \ddot{x}\right] p_2 \\ &- \left[\ddot{x}_1(1 - \|x\|^2) - 2\dot{x}_1(x^\top \dot{x}) + \dot{x}_1 x^\top \dot{x} + x_1\|\dot{x}\|^2 + x_1 x^\top \ddot{x}\right] p_1 \\ &= \ddot{x}_2(1 - \|x\|^2)\dot{x}_1 - \ddot{x}_1(1 - \|x\|^2)\dot{x}_2 + (x_2\dot{x}_1 - x_1\dot{x}_2)[(1 - \|x\|^2)\|\dot{x}\|^2 + (x^\top \dot{x})^2] \\ &= \dot{x}_1(1 - \|x\|^2)[\ddot{x}_2 + x_2(\|\dot{x}\|^2 + \frac{(x^\top \dot{x})^2}{1 - \|x\|^2})] - \dot{x}_2(1 - \|x\|^2)[\ddot{x}_1 + x_1(\|\dot{x}\|^2 + \frac{(x^\top \dot{x})^2}{1 - \|x\|^2})] = 0\,. \end{aligned} \tag{94}$$

Since the vectors x and $\dot{x}$ are not parallel, at least one of p_1 and p_2 is not equal to zero. Hence, there exists a number φ such that

$$\begin{aligned} \sin\varphi &= \frac{p_1}{\sqrt{p_1^2 + p_2^2}} \\ \cos\varphi &= \frac{p_2}{\sqrt{p_1^2 + p_2^2}}\,. \end{aligned} \tag{95}$$

Then

$$q_1 = x_1 \sin\varphi - x_2 \cos\varphi = \frac{(1-\|x\|^2)(x_1\dot{x}_2 - x_2\dot{x}_1)}{\sqrt{p_1^2+p_2^2}}$$
$$\dot{q}_1 = \dot{x}_1 \sin\varphi - \dot{x}_2 \cos\varphi = -\frac{(x^\top \dot{x})(x_1\dot{x}_2 - x_2\dot{x}_1)}{\sqrt{p_1^2+p_2^2}}\,. \tag{96}$$

The value $q_1(t)$ is not equal to zero for all t since the vectors $x(t)$ and $\dot{x}(t)$ are not parallel. We define a function α:

$$\alpha = \frac{1-\|x\|^2}{q_1^2}\,, \tag{97}$$

so that

$$\dot{\alpha} = \frac{-2(x^\top \dot{x})q_1 - (1-\|x\|^2)2\dot{q}_1}{q_1^3} = 0\,. \tag{98}$$

Therefore, the function α is constant, $\alpha(t) \equiv \alpha_0$.

We now define a unitary matrix U,

$$U = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}. \tag{99}$$

Then, $Ux = (0, q_1)^\top$, and the equation $\alpha_0 q_1^2 = 1 - \|x\|^2$ is equivalent to

$$(Ux)^\top \begin{pmatrix} 1 & 0 \\ 0 & 1+\alpha_0 \end{pmatrix} (Ux) = 1\,. \tag{100}$$

We denote by a a vector in $\mathbb{B}$ parallel to $(\cos\varphi, \sin\varphi)^\top$, and denote

$$a = \frac{1}{2}\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \quad \text{and} \quad b = \frac{1}{\sqrt{1+a}}\begin{pmatrix} \sin\varphi \\ -\cos\varphi \end{pmatrix}. \tag{101}$$

Then $a^\top b = 0$ and

$$\left[\frac{x(t)^\top a}{\|a\|}\right]^2 + (1+\alpha_0)\left[\frac{x(t)^\top b}{\|b\|}\right]^2 = 1\,. \tag{102}$$

Therefore the whole solution $x(\cdot)$ lies on the same cogyroline determined by the vectors $a, b \in \mathbb{B}$ and the number α_0.

Since $\|x(t)\| \to 1$ as $t \to \pm\infty$, the set of points $\{x(t) \ : \ t \in \mathbb{R}\}$ is a cogyroline. The proof of the Theorem is, thus, complete. □

In order to find a canonical metric tensor for which the solutions of Equation (92) are geodesics, we compare Equation (92) with

$$\ddot{x} + \frac{m_0'}{m_0}2(x^\top \dot{x})\dot{x} + \left[(\frac{m_1'}{m_1} - \frac{2m_0'}{m_0})\frac{(x^\top \dot{x})^2}{\|x\|^2} + \frac{m_1 - m_0 - \|x\|^2 m_0'}{\|x\|^2 m_1}(\|\dot{x}\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2})\right]x = 0\,. \tag{103}$$

We need to find functions $m_0 = m_{0,co,E}$ and $m_1 = m_{1,co,E}$ such that for all numbers $z \geq 0$

$$\begin{aligned} \frac{m'_{0,co,E}(z)}{m_{0,co,E}(z)} &= \frac{1}{1-z} \\ \frac{m_{1,co,E}(z) - m_{0,co,E}(z) - z\, m'_{0,co,E}(z)}{z\, m_{1,co,E}(z)} &= 1 \\ \frac{m'_{1,co,E}(z)}{m_{1,co,E}(z)} - 2\frac{m'_{0,co,E}(z)}{m_{0,co,E}(z)} - \frac{m_{1,co,E}(z) - m_{0,co,E}(z) - z\, m'_{0,co,E}(z)}{z\, m_{1,co,E}(z)} &= \frac{z}{1-z}\,. \end{aligned} \tag{104}$$

The elegant solution to the equations in (104) for the unknowns $m_{0,co,E}$ and $m_{1,co,E}$ is

$$\begin{aligned} m_{0,co,E}(z) &= \frac{1}{1-z} \\ m_{1,co,E}(z) &= \frac{1}{(1-z)^3}\,. \end{aligned} \tag{105}$$

Let us consider the canonical metric tensor parametrized by the functions $m_{0,co,E}$ and $m_{1,co,E}$ in (105),

$$G_{co_E}(x) = \frac{1}{1-\|x\|^2}(I - \frac{xx^\top}{\|x\|^2}) + \frac{1}{(1-\|x\|^2)^3}\frac{xx^\top}{\|x\|^2}. \tag{106}$$

The geodesics of the Riemannian manifold with the canonical metric tensor (106) satisfy Equation (92). Hence, every geodesic in this manifold is a cogyroline, and every cogyroline is a geodesic in the Riemannian manifold with the canonical metric tensor $G_{co,E}$.

The canonical metric tensor G_E for the Einstein addition $\oplus_E$ is parametrized by the functions $m_{0,E}$ and $m_{1,E}$ given by [21]

$$\begin{aligned} m_{0,E}(z) &= \frac{1}{1-z} \\ m_{1,E}(z) &= \frac{1}{(1-z)^2}\,. \end{aligned} \tag{107}$$

The functions m_0 are the same for the tensors G_E and $G_{co,E}$. The distinction lies in the function m_1.

4.1.3. A Binary Operation for Einstein Cogyrolines

The functions $m_{0,co,E}$ and $m_{1,co,E}$ satisfy Assumptions 1 and 2. Hence, we follow the four steps that lead to a binary operation $\oplus_{co,E}$ in $\mathbb{B}$ for which the canonical metric tensor is $G_{co,E}$. We assume $a, b \in \mathbb{B}$. If a and b are parallel, then $a \oplus_{co,E} b$ can be defined using multiplication of vectors by numbers in (13). We now assume that the vectors a and b are not parallel. In particular, they are not zero vectors. We now follow the four steps that lead to the binary operation $a \oplus_{co,E} b$ in $\mathbb{B}$ when a and b are not parallel.

Step 1. We evaluate the integral

$$\begin{aligned} \dot{r}(0) &= \frac{1}{\|b\|}\int_0^{\|b\|}\sqrt{m_{1,co,E}(s^2)}ds \\ &= \frac{1}{\|b\|}\int_0^{\|b\|}\sqrt{\frac{1}{(1-s^2)^3}}ds = \frac{1}{\|b\|}\frac{\|b\|}{\sqrt{1-\|b\|^2}} = \frac{1}{\sqrt{1-\|b\|^2}}\,. \end{aligned} \tag{108}$$

Step 2. We perform a parallel transport of the vector $X_0 = \dot{r}(0)b$ along the interval $\{at : 0 \le t \le 1\}$. The vector parallel to X_0 at the point a is

$$X_1 = \sqrt{1 - \|a\|^2}\left[I - \frac{aa^\top}{\|a\|^2}\right]X_0 + (1 - \|a\|^2)^{3/2}\frac{aa^\top}{\|a\|^2}X_0. \tag{109}$$

Step 3. We find a solution $x(\cdot)$ of Equation (92),

$$\ddot{x} + \frac{2(x^\top \dot{x})}{1 - \|x\|^2}\dot{x} + \left[\|\dot{x}\|^2 + \frac{(x^\top \dot{x})^2}{1 - \|x\|^2}\right]x = 0 \tag{110}$$

with initial data $x(0) = a$, $\dot{x}(0) = X_1$.

We seek a solution having the form

$$x(t) = \tilde{a}\sin\varphi(t) + \tilde{b}\ d\ \cos\varphi(t), \tag{111}$$

where $\{\tilde{a}, \tilde{b}\}$ is an orthonormal basis in the plane containing the vectors $x(0)$ and $\dot{x}(0)$, d being a number, $d \in [0, 1)$, and φ is a scalar function to be determined. Then

$$\begin{aligned} \dot{x} &= \dot{\varphi}(\tilde{a}\cos\varphi - \tilde{b}d\sin\varphi) \\ \ddot{x} &= \ddot{\varphi}(\tilde{a}\cos\varphi - \tilde{b}d\sin\varphi) - \dot{\varphi}^2(\tilde{a}\sin\varphi + \tilde{b}\ d\ \cos\varphi) \end{aligned} \tag{112}$$

and

$$\begin{aligned} \|x\|^2 &= \sin^2\varphi + d^2\cos^2\varphi \\ \|\dot{x}\|^2 &= \dot{\varphi}^2(\cos^2\varphi + d^2\sin^2\varphi) \\ x^\top\dot{x} &= \dot{\varphi}(1 - d^2)\sin\varphi\cos\varphi\,. \end{aligned} \tag{113}$$

We notice that

$$\begin{aligned} 1 - \|x\|^2 &= (1 - d^2)\cos^2\varphi \\ \frac{2(x^\top\dot{x})}{1 - \|x\|^2} &= 2\dot{\varphi}\tan\varphi \\ \|\dot{x}\|^2 + \frac{(x^\top\dot{x})^2}{1 - \|x\|^2} &= \dot{\varphi}^2\,. \end{aligned} \tag{114}$$

Therefore Equation (92) takes the form

$$\tilde{a}[\ddot{\varphi} + 2\dot{\varphi}^2\tan\varphi]\cos\varphi - \tilde{b}d\sin\varphi[\ddot{\varphi} + 2\dot{\varphi}^2\tan\varphi] = 0\,. \tag{115}$$

This equation is obviously equivalent to

$$\ddot{\varphi} + 2\dot{\varphi}^2\tan\varphi = 0\,, \tag{116}$$

the general solution of which is

$$\tan\varphi(t) = C_0 t + C_1\,, \tag{117}$$

where C_1, C_0 are arbitrary constants. Equation (111) shows that a general solution of Equation (92) is

$$x(t) = \frac{C_0 t + C_1}{\sqrt{1 + (C_0 t + C_1)^2}}\tilde{a} + \frac{d}{\sqrt{1 + (C_0 t + C_1)^2}}\tilde{b}\,. \tag{118}$$

The initial conditions are

$$\begin{aligned} x(0) &= \frac{C_1\tilde{a} + d\tilde{b}}{\sqrt{1+C_1^2}}\tilde{a} \\ \dot{x}(0) &= \frac{C_0}{(1+C_1^2)^{3/2}}\left[\tilde{a} - dC_1\tilde{b}\right], \end{aligned} \tag{119}$$

where

$$\begin{aligned} C_0 &= \frac{\sqrt{1-\|a\|^2}}{\sqrt{1-\|b\|^2}}\frac{\|b\|^2}{\sqrt{\|b\|^2-(a^\top b)^2}}, \\ C_1 &= \frac{a^\top b}{\sqrt{\|b\|^2-(a^\top b)^2}}, \\ d &= \frac{\sqrt{\|a\|^2\|b\|^2-(a^\top b)^2}}{\sqrt{\|b\|^2-(a^\top b)^2}}, \\ \tilde{a} &= \frac{b}{\|b\|}, \\ \tilde{b} &= \frac{a\|b\| - \frac{a^\top b}{\|b\|}b}{\sqrt{\|a\|^2\|b\|^2-(a^\top b)^2}}. \end{aligned} \tag{120}$$

According to Theorem 3, the function x determines a cogyroline for Einstein addition.

Step 4. Define

$$\begin{aligned} a \oplus_{co,E} b &= x(1) \\ &= \frac{(C_0+C_1)\tilde{a} + d\tilde{b}}{\sqrt{1+(C_0+C_1)^2}} \\ &= \frac{a\sqrt{1-\|b\|^2} + b\sqrt{1-\|a\|^2}}{\sqrt{1-\|a\|^2\|b\|^2 + 2a^\top b\sqrt{(1-\|a\|^2)(1-\|b\|^2)}}}. \end{aligned} \tag{121}$$

If we use the standard notation $\gamma_x = (1-\|x\|^2)^{-1/2}$ for all vectors $x \in \mathbb{B}$, then (121) may be written in the symmetric form

$$a \oplus_{co,E} b = \frac{\gamma_a a + \gamma_b b}{\sqrt{1+\|\gamma_a a + \gamma_b b\|^2}}. \tag{122}$$

This operation is obviously commutative. It has been studied in [21].

The binary operation $\oplus_{co,E}$ determines the canonical metric tensor $G_{co,E}$.

To define an operation of multiplication of a vector by a number, as it is shown in [21], we have to calculate the function $h(p)$,

$$h(p) = \int_0^p \sqrt{m_{1,co,E}(s^2)}ds = \int_0^p \frac{ds}{(1-s^2)^{3/2}} = \frac{p}{\sqrt{1-p^2}}, \tag{123}$$

where $m_{1,co,E}$ is given by (105).

Then

$$t \otimes_{co,E} a = \frac{a}{\|a\|}h^{-1}(t\, h(\|a\|)) = a\frac{t}{\sqrt{1+(t^2-1)\|a\|^2}}. \tag{124}$$

In particular, if $b = \lambda a$, then $a \oplus_{co,E} b = (1+\lambda) \otimes_{co,E} a$.

For every $a, b \in \mathbb{B}$, $b \neq 0$, the cogyroline $S(a,b)$ of the Einstein addition is given by

$$S(a,b) = \{(t \otimes_E a) \oplus_E b \;:\; t \in \mathbb{R}\}. \tag{125}$$

This curve is also a gyroline for the addition $\oplus_{co}$,

$$S(a,b) = \{b \oplus_{co,E} (\tau \otimes_{co,E} a) \;:\; \tau \in \mathbb{R}\}. \tag{126}$$

4.1.4. Distance and Norm For Cogyrolines

We can define a cogyronorm as a norm $\|\cdot\|_{\oplus_{co,E}}$ as it is described in [21]. In particular, if $a, b \in \mathbb{B}$, then for the function

$$h(p) = \frac{p}{\sqrt{1-p^2}} \tag{127}$$

we have

$$\begin{aligned} h(\|a\|) + h(\|b\|) &\geq h(\|a \oplus_{co,E} b\|) \\ \|a\| \oplus_{co,E} \|b\| &\geq \|a \oplus_{co,E} b\|, \end{aligned} \tag{128}$$

and the equalities in each line is attained if and only if there exists a non negative number λ such that $a = \lambda b$ or $b = \lambda a$. Here $\|\cdot\|$ is the Euclidean norm.

The distance between points a and b is given by

$$dist_{co,E}(a,b) = h(\|\ominus a \oplus_{co,E} b\|) = \frac{\|\ominus a \oplus_{co,E} b\|}{\sqrt{1 - \|\ominus a \oplus_{co,E} b\|^2}} = \gamma_{\ominus a \oplus_{co,E} b} \|\ominus a \oplus_{co,E} b\|. \tag{129}$$

For arbitrary three points $a, b, c \in \mathbb{B}$ we have the triangle inequality

$$dist_{co,E}(a,b) + dist_{co,E}(b,c) \geq dist_{co,E}(a,c) \tag{130}$$

where equality is attained if and only if these points lie on the same cogyroline, and b is between a and c. Hence, we can define the cogyronorm as follows.

$$\|a\|_{co,E} = dist_{co,E}(0,a) = h(\|a\|) = \frac{\|a\|}{\sqrt{1-\|a\|^2}}. \tag{131}$$

For this norm we have

$$\|t \otimes_{co,E} a\|_{co,E} = |t|\|a\|_{co,E} \tag{132}$$

for all $a \in \mathbb{B}$, $t \in \mathbb{R}$, and

$$\|a\|_{co,E} + \|b\|_{co,E} \geq \|a \oplus_{co,E} b\|_{co} \tag{133}$$

for all $a, b \in \mathbb{B}$.

4.2. M öbius Cogyrolines

In this section, we consider cogyrolines for the Möbius addition $\oplus_M$:

$$a \oplus_M b = \frac{(1 + 2a^\top b + \|b\|^2)a + (1 - \|a\|^2)b}{1 + 2a^\top b + \|a\|^2\|b\|^2}. \tag{134}$$

4.2.1. Circular Arcs

For every $a, b \in \mathbb{B}$ if $P(a,b)$ is a two-dimensional plane that contains both a and b, then the cogyroline

$$S(a,b) = \{(t \otimes_M a) \oplus_M b \;:\; t \in \mathbb{R}\} \tag{135}$$

lies in P. If $a = 0$, then the cogyroline is a point b. If b is parallel to a, that is, there exists a number λ such that $b = \lambda a$, then the cogyroline is a segment $\{as \ : \ \|as\| < 1\}$.

Theorem 4. *For every $a \in \mathbb{B}\backslash\{0\}$ and $b \in \mathbb{B}$ not parallel to a the cogyroline $S(a,b)$ is an arc of a circle that intersects the unit circle at centrally symmetric points, that is, for the vectors*

$$b_\perp = b - \frac{a^\top b}{\|a\|^2} a \tag{136}$$

and

$$y = \frac{1 - \|b\|^2}{\|b_\perp\|^2} b_\perp \tag{137}$$

we have

$$S(a,b) = P(a,b) \cap \{w \in \mathbb{B} \ : \ \|w\|^2 + y^\top w = 1, \ \ y^\top a = 0, \ \ \|y\| > 1\}\,. \tag{138}$$

Proof. Since the right-hand side of (138) is a curve in $\mathbb{B}$ that does not intersect itself, and that connects the points a and $-a$, it is sufficient to prove that for every point $x \in S(a,b)$ we have

$$\|x\|^2 + y^\top x = 1\,. \tag{139}$$

We notice that (139) is equivalent to the equation

$$\|x + \frac{1}{2} y\|^2 = 1 + \frac{1}{4}\|y\|^2\,, \tag{140}$$

which determines a circle with radius $\sqrt{1 + \frac{1}{4}\|y\|^2}$.

To verify (139) we assume that $x \in S(a,b)$. Then, for some $t \in \mathbb{R}$ we have

$$\begin{aligned} x &= (t \otimes_M a) \oplus_M b \\ &= \frac{(1 + 2(t \otimes_M a)^\top b + \|b\|^2)(t \otimes_M a) + (1 - \|t \otimes_M a\|^2) b}{1 + 2(t \otimes_M a)^\top b + \|t \otimes_M a\|^2 \|b\|^2}\,. \end{aligned} \tag{141}$$

Let

$$g(t) = \frac{\tanh(t \operatorname{atanh}(\|a\|))}{\|a\|}\,. \tag{142}$$

Then, $t \otimes a = g(t) a$, and

$$x = \frac{(1 + 2g(t) a^\top b + \|b\|^2) g(t) a + (1 - \|g(t) a\|^2) b}{1 + 2g(t) a^\top b + \|g(t) a\|^2 \|b\|^2}\,. \tag{143}$$

Let us now calculate $\|x\|^2$,

$$\|x\|^2 = \Big[\|a\|^2 g(t)^2 (1 + 2g(t)a^\top b + \|b\|^2)^2$$
$$+ 2a^\top b g(t)(1 + 2g(t)a^\top b + \|b\|^2)(1 - g(t)^2\|a\|^2)$$
$$+ \|b\|^2 (1 - g(t)^2\|a\|^2)^2\Big] \frac{1}{(1 + 2g(t)a^\top b + \|g(t)a\|^2\|b\|^2)^2} \tag{144}$$
$$= \frac{(1 + 2g(t)a^\top b + g(t)^2\|a\|^2\|b\|^2)(2g(t)a^\top b + g(t)^2\|a\|^2 + \|b\|^2)}{(1 + 2g(t)a^\top b + \|g(t)a\|^2\|b\|^2)^2}$$
$$= \frac{2g(t)a^\top b + g(t)^2\|a\|^2 + \|b\|^2}{1 + 2g(t)a^\top b + \|g(t)a\|^2\|b\|^2}.$$

Noticing that $y^\top b = 1 - \|b\|^2$, we have

$$y^\top x = \frac{(1 - g(t)^2\|a\|^2)(1 - \|b\|^2)}{1 + 2g(t)a^\top b + \|g(t)a\|^2\|b\|^2}. \tag{145}$$

Adding Equations (144) and (145) yields (139), that is,

$$\|x\|^2 + y^\top x = 1, \tag{146}$$

and the proof of the Theorem is complete. □

4.2.2. The Canonical Metric Tensor for Cogyrolines

As in Section 4.1.2, we consider the second order differential equation

$$\ddot{x} + \frac{4(x^\top \dot{x})}{1 - \|x\|^2}\dot{x} + \frac{2\|\dot{x}\|^2}{1 + \|x\|^2} x = 0. \tag{147}$$

Theorem 5. *For every solution $x(\cdot)$ of Equation (147) the set of its points $\{x(t) \; : \; t \in \mathbb{R}\}$ is a cogyroline. Every cogyroline is a set of points of a solution $x(\cdot)$ of (147) defined on $\mathbb{R}$.*

Proof. Let $x_0 \in \mathbb{B}$ and $x_0' \in \mathbb{R}^n$ be arbitrary non parallel vectors. Below we prove that a solution $x(\cdot)$ of (147) with initial conditions $x(0) = x_0$ and $\dot{x}(0) = x_0'$ lies on a circular arc $S(a,b)$ defined in (138) for any $a, b \in \mathbb{B}$. Since there are no stationary points of Equation (147) in $\mathbb{B}$, and every solution $x(\cdot)$ with initial conditions $x(0) \in \mathbb{B}$ can't reach the points $\pm a$, these would imply that the set of all points of the solution $x(\cdot)$ coincides with $S(a,b)$, and the statement of the theorem holds.

For a curve $y(t) = (t \otimes_M a) \oplus_M b$ we use representation (143) to get the initial conditions

$$y(0) = b$$
$$\dot{y}(0) = \frac{dy}{dg}\Big|_{g=0} \frac{dg}{dt}\Big|_{t=0} = [(1 + \|b\|^2)a - 2a^\top b\, b]\mathrm{atanh}(\|a\|). \tag{148}$$

We consider a solution $x(\cdot)$ of (147) with initial conditions $x(0) = x_0$ and $\dot{x}(0) = x_0'$, and choose vectors a, b such that

$$x_0 = x(0) = b$$
$$x_0' = \dot{x}(0) = \dot{y}(0) = [(1 + \|b\|^2)I - 2bb^\top]a\ \mathrm{atanh}(\|a\|). \tag{149}$$

Since

$$\det[(1+\|b\|^2)I - 2bb^\top] = (1+\|b\|^2)^{n-1}(1-\|b\|^2) \neq 0\,, \tag{150}$$

where $n = dim(x)$, we can define a vector w,

$$w = [(1+\|b\|^2)I - 2bb^\top]^{-1}x_0'\,. \tag{151}$$

Let a number δ be such that

$$\|w\| = \delta\ \text{atanh}(|\delta|)\,. \tag{152}$$

Set $\tilde{a} = \frac{\delta}{\|w\|}w$, $a = \frac{\tilde{a}}{2\|\tilde{a}\|}$. Then for the solution $x(\cdot)$ we have $x(0) = b \in S(a,b)$, and $\dot{x}(0)$ is parallel to the vector tangent to the curve $S(a,b)$.

Now we can define vectors $b_\perp = b - \frac{a^\top b}{\|a\|^2}a$ and $y = \frac{1-\|b\|^2}{\|b_\perp\|^2}b_\perp$. Noticing that $b_\perp^\top b = \|b_\perp\|^2$ and $a^\top b_\perp = 0$, we have

$$\|x(0)\|^2 + y^\top x(0) = \|b\|^2 + \frac{1-\|b\|^2}{\|b_\perp\|^2}b_\perp^\top b = 1 \tag{153}$$

and

$$2x(0)^\top \dot{x}(0) + y^\top \dot{x}(0) = (2b^\top + \frac{1-\|b\|^2}{\|b_\perp\|^2}b_\perp^\top)[(1+\|b\|^2)a - 2a^\top b\ b]\text{atanh}(\|a\|) = 0\,. \tag{154}$$

According to Theorem 4, a solution x lies on the circular arc $S(a,b)$ if and only if

$$\|x(t)\|^2 + y^\top x(t) = 1 \tag{155}$$

for all $t \in \mathbb{R}$. To prove the theorem it is sufficient to show that (155) holds, which we accomplish in (164).

The vector y belongs to the two dimensional plane that contains $x(0)$ and $\dot{x}(0)$. Hence, there exist numbers $\tilde{\alpha}$ and $\tilde{\beta}$ such that $y = \alpha x(0) + \beta \dot{x}(0)$. Solving Equations (153) and (154) for these numbers yields

$$\begin{aligned} \tilde{\alpha} &= \frac{\|\dot{x}(0)\|^2(1-\|x(0)\|^2) + 2(x(0)^\top \dot{x}(0))^2}{\|x(0)\|^2\|\dot{x}(0)\|^2 - (x(0)^\top \dot{x}(0))^2} \\ \tilde{\beta} &= \frac{-(x(0)^\top \dot{x}(0))(1+\|x(0)\|^2)}{\|x(0)\|^2\|\dot{x}(0)\|^2 - (x(0)^\top \dot{x}(0))^2}\,. \end{aligned} \tag{156}$$

We now introduce the functions

$$\begin{aligned} \alpha(t) &= \frac{\|\dot{x}(t)\|^2(1-\|x(t)\|^2) + 2(x(t)^\top \dot{x}(t))^2}{\|x(t)\|^2\|\dot{x}(t)\|^2 - (x(t)^\top \dot{x}(t))^2} \\ \beta(t) &= \frac{-(x(t)^\top \dot{x}(t))(1+\|x(t)\|^2)}{\|x(t)\|^2\|\dot{x}(t)\|^2 - (x(t)^\top \dot{x}(t))^2}\,. \end{aligned} \tag{157}$$

If $\alpha(t)x(t) + \beta(t)\dot{x}(t) = const = y$ for all $t \in \mathbb{R}$, then $\|x(t)\|^2 + y^\top x(t) = 1$ for all $t \in \mathbb{R}$. Furthermore, we drop for clarity the argument (t). We denote by p and q the numerator and denominator of the sum $\alpha x + \beta \dot{x}$,

$$\frac{p}{q} = \frac{[\|\dot{x}\|^2(1-\|x\|^2) + 2(x^\top \dot{x})^2]x - (x^\top \dot{x})(1+\|x\|^2)\dot{x}}{\|x(t)\|^2\|\dot{x}(t)\|^2 - (x(t)^\top \dot{x}(t))^2}\,. \tag{158}$$

We need to show that $\dot{p}q - \dot{q}p = 0$. We have

$$\begin{aligned}
\dot{p} &= [2\dot{x}^\top \ddot{x}(1-\|x\|^2) - \|\dot{x}\|^2 2(x^\top \dot{x}) + 4(x^\top \dot{x})(\|\dot{x}\|^2 + \dot{x}^\top \ddot{x})]x \\
&\quad + [\|\dot{x}\|^2 + 2(x^\top \dot{x})^2]\dot{x} - (\|\dot{x}\|^2 + x^\top \ddot{x})(1+\|x\|^2)\dot{x} \\
&\quad - 2(x^\top \dot{x})^2 \dot{x} - (x^\top \dot{x})(1+\|x\|^2)\ddot{x} \\
&= [2(1-\|x\|^2)(\dot{x}^\top \ddot{x}) + 2(x^\top \dot{x})\|\dot{x}\|^2 + 4(x^\top \dot{x})(x^\top \ddot{x})]x \\
&\quad - [(1+\|x\|^2)(x^\top \ddot{x}) + 2\|x\|^2\|\dot{x}\|^2]\dot{x} - [(x^\top \dot{x})(1+\|x\|^2)]\ddot{x}
\end{aligned} \tag{159}$$

$$\begin{aligned}
\dot{q} &= 2(x^\top \dot{x})\|\dot{x}\|^2 + 2(\dot{x}^\top \ddot{x})\|x\|^2 - 2(x^\top \dot{x})[\|\dot{x}\|^2 + (x^\top \ddot{x})] \\
&= 2(\dot{x}^\top \ddot{x})\|x\|^2 - 2(x^\top \dot{x})(x^\top \ddot{x})\,.
\end{aligned}$$

Therefore,

$$\dot{p}q - \dot{q}p = Ax + B\dot{x} + C\ddot{x}\,, \tag{160}$$

where

$$\begin{aligned}
A &= 2(\dot{x}^\top \ddot{x})[-(1+\|x\|^2)(x^\top \dot{x})^2] + 2(x^\top \ddot{x})[(x^\top \dot{x})\|\dot{|}\|^2(1+\|x\|^2)] \\
&\quad + 2(x^\top \dot{x})\|\dot{x}\|^2(\|x\|^2\|\dot{x}\|^2 - (x^\top \dot{x})^2) \\
B &= (\dot{x}^\top \ddot{x})[2(x^\top \dot{x})\|x\|^2(1+\|x\|^2] - (x^\top \ddot{x})[(1+\|x\|^2)(\|x\|^2\|\dot{x}\|^2 + (x^\top \dot{x})^2)] \\
&\quad - 2\|x\|^2\|\dot{x}\|^2(\|x\|^2\|\dot{x}\|^2 - (x^\top \dot{x})^2) \\
C &= -(x^\top \dot{x})(1+\|x\|^2)(\|x\|^2\|\dot{x}\|^2 - (x^\top \dot{x})^2)\,.
\end{aligned} \tag{161}$$

Since x is a solution of Equation (147), we have

$$\ddot{x} = -\frac{-4(x^\top \dot{x})}{1-\|x\|^2}\dot{x} - \frac{2\|\dot{x}\|^2}{1+\|x\|^2}x\,. \tag{162}$$

Straightforwardly, with the value of $\ddot{x}$ in (162), we have

$$Ax + \mathbb{B}\dot{x} + C\ddot{x} = 0\,. \tag{163}$$

Hence,

$$\|x(t)\|^2 + y^\top x(t) = 1 \tag{164}$$

for all $t \in \mathbb{R}$, and the proof of Theorem 5 is complete. □

Equation (147) has the same form as (5) with $m_0 = m_{0,co,E}$, $m_1 = m_{1,co,E}$ if for all $x \in \mathbb{B}$ the following equations hold

$$\begin{aligned}
\frac{m'_{0,co,E}}{m_{0,co,E}} &= \frac{2}{1-\|x\|^2}, \\
\frac{m_{1,co,E} - m_{0,co,E} - \|x\|^2 m'_{0,co,E}}{m_{1,co,E}} &= \frac{2\|x\|^2}{1+\|x\|^2}, \\
\frac{m'_{1,co,E}}{m_{1,co,E}} - 2\frac{m'_{0,co,E}}{m_{0,co,E}} - \frac{2}{1+\|x\|^2} &= 0\,.
\end{aligned} \tag{165}$$

The equations in (165) possess the solution

$$m_{0,co,E}(\|x\|^2) = \frac{1}{(1-\|x\|^2)^2}$$
$$m_{1,co,E}(\|x\|^2) = \frac{(1+\|x\|^2)^2}{(1-\|x\|^2)^4}. \tag{166}$$

Let us introduce the canonical metric tensor parametrized by m_0 and m_1 in (166),

$$G_{co,M}(x) = \frac{1}{(1-\|x\|^2)^2}(I - \frac{xx^\top}{\|x\|^2}) + \frac{(1+\|x\|^2)^2}{(1-\|x\|^2)^4}\frac{xx^\top}{\|x\|^2}. \tag{167}$$

The geodesics of the Riemannian manifold with metric tensor (167) satisfy Equation (147). Hence, every geodesic in this manifold is a cogyroline, and every cogyroline is a geodesic in the Riemannian manifold with the metric tensor $G_{co,M}$.

We notice that the canonical metric tensor G_M for Möbius addition is parametrized by the functions m_0 and m_1 given by

$$m_{0,M}(\|x\|^2) = \frac{1}{(1-\|x\|^2)^2}$$
$$m_{1,M}(\|x\|^2) = \frac{1}{(1-\|x\|^2)^2}, \tag{168}$$

where, remarkably, $m_0 = m_1$, as shown in [21].

The functions m_0 are the same for the tensors G_M and $G_{co,M}$. Again, as for the case of the Einstein addition, the difference lies in the function m_1.

4.2.3. A Binary Operation for M öbius cogyrolines

In this section, we introduce a new binary operation $\oplus_{co,M}$ such that every Möbius cogyroline is a gyroline for this operation, and vice versa, every gyroline for this operation is a Möbius cogyroline. According to Theorem 5 it is sufficient to find a smooth binary operation satisfying Condition (2) with a canonical metric tensor (4) equal to $G_{co,M}$ given in (167).

Introduce an operation $\oplus_{co,M}$: $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$,

$$a \oplus_{co,M} b = \frac{2(\gamma_a^2 a + \gamma_b^2 b)}{1 + \sqrt{1 + 4\|\gamma_a^2 a + \gamma_b^2 b\|^2}}, \tag{169}$$

where $\gamma_x = (1-\|x\|^2)^{-1/2}$ for all $x \in \mathbb{B}$. This operation is well defined, smooth, and satisfies the invariance condition (2).

Theorem 6. *The canonical metric tensor (4) of the operation $\oplus_{co,M}$ coincides with $G_{co,M}$.*

Proof. In order to use the formula (4) we consider the first two terms of the Taylor series at a point x of the following function:

$$(-x) \oplus_{co,M} (x + \Delta x) = \frac{2xx^\top + (1-\|x\|^2)I}{(1-\|x\|^2)^2}\Delta x + o(\|\Delta x\|). \tag{170}$$

Therefore, the matrix $g(x)$ in (3) is equal to

$$g(x) = \frac{2xx^\top + (1-\|x\|^2)I}{(1-\|x\|^2)^2}, \tag{171}$$

and we get the canonical metric tensor

$$G(x) = g(x)^\top g(x) = \frac{1}{(1-\|x\|^2)^2}\left[I - \frac{xx^\top}{\|x\|^2}\right] + \frac{(1+\|x\|^2)^2}{(1-\|x\|^2)^4}\frac{xx^\top}{\|x\|^2}, \tag{172}$$

which coincides with the metric tensor $G_{co,M}$ in (167). The theorem is thus proved. □

Notice that the operation $\oplus_{co,M}$ is commutative.

To define an operation of multiplication of a vector by a number we have to calculate the function

$$h(p) = \int_0^p \sqrt{m_{1,co,M}(s^2)}ds = \int_0^p \frac{1+s^2}{(1-s^2)^2}ds = \frac{p}{1-p^2}\,. \tag{173}$$

Then

$$t \otimes_{co,M} a = \frac{a}{\|a\|}h^{-1}(t\,h(\|a\|)) = \frac{2ta}{1-\|a\|^2+\sqrt{(1-\|a\|^2)^2+4t^2\|a\|^2}}\,. \tag{174}$$

In particular, if $b = \lambda a$ with a real number λ, then $a \oplus_{co,M} b = (1+\lambda) \otimes_{co,M} a$.

For every $a, b \in \mathbb{B}$, $b \neq 0$, the cogyroline $S(a,b)$ of Möbius addition is given by

$$S(a,b) = \{(t \otimes_M a) \oplus_M b \;:\; t \in \mathbb{R}\}\,. \tag{175}$$

This curve is also a gyroline for the addition $\oplus_{co,M}$,

$$S(a,b) = \{b \oplus_{co,M} (\tau \otimes_{co,M} a) \;:\; \tau \in \mathbb{R}\}\,. \tag{176}$$

4.2.4. Distance and Norm For Cogyrolines

We can define a cogyronorm as a norm $\|\cdot\|_{\oplus_{co,M}}$ as it is described in [21]. For the function h in (173), we have,

$$\begin{aligned} h(\|a\|) + h(\|b\|) &\geq h(\|a \oplus_{co,M} b\|) \\ \|a\| \oplus_{co,M} \|b\| &\geq \|a \oplus_{co,M} b\|\,. \end{aligned} \tag{177}$$

Equalities in (177) are attained if and only if there exists a non negative number λ such that $a = \lambda b$ or $b = \lambda a$. Here $\|\cdot\|$ is the Euclidean norm.

4.3. Cogyrolines in Spaces Parametrized by Functions φ

Consider again an arbitrary bijection $\tilde{\varphi}$: $[0,1) \to [0,1)$, which is differentiable, strictly increasing, and $\tilde{\varphi}'(0) > 0$. Following (25) for every $x \in B$ we define

$$\varphi(x) = \tilde{\varphi}(\|x\|)\frac{x}{\|x\|}. \tag{178}$$

In Section 3, we introduced a canonical metric tensor G_φ, corresponding functions m_0, m_1, and a binary operation $\oplus_\varphi$, which has the same properties as Einstein operation $\oplus_E$. In this section we consider cogyrolines in the space with canonical metric tensor G_φ. The cogyrolines with parallel $a \neq 0$ and b are intervals of the form $\{t\frac{a}{\|a\|} \;:\; |t| < 1\}$. Hence further in this section we assume that a and b are not parallel. Recall that we have the Einstein addition if $\tilde{\varphi}(s) = s$, and the Möbius addition if $\tilde{\varphi}(s) = \frac{2s}{1+s^2}$.

4.3.1. A Relation with Gyrolines in the Space with Einstein Addition

The cogyroline in the space with canonical metric tensor G_φ and corresponding binary operation $\oplus_\varphi$ and scalar multiplication $\otimes_\varphi$ (see Section 3) is defined as the set

$$S(a,b) = \{(t \otimes_\varphi a) \oplus_\varphi b \ : \ t \in \mathbb{R}\}\,, \tag{179}$$

where a and b are arbitrary points in the open ball $\mathbb{B}$.

According to the definition of the operations with subindex φ, the set (179) coincides with the set

$$S(a,b) = \{\varphi^{-1}[t \otimes_E \varphi(a)) \oplus_E \varphi(b)] \ : \ t \in \mathbb{R}\}\,. \tag{180}$$

As we have seen above for non parallel points a and b, the set (181),

$$S_E(a,b) = \{(t \otimes_E a) \oplus_E b \ : \ t \in \mathbb{R}\}\,, \tag{181}$$

is the following semiellipse in the plane $P(a,b)$ containing a and b:

$$S_E(a,b) = P(a,b) \cap \{y \in \mathbb{B} \ : \ \left[\frac{y^\top a}{\|a\|}\right]^2 + c\left[\frac{y^\top b_\perp}{\|b_\perp\|}\right]^2 = 1,\ \ y^\top b_\perp > 0\}, \tag{182}$$

where $b_\perp = b - \frac{a^\top b}{\|a\|^2} a$, and

$$c = \frac{\|a\|^2 - (a^\top b)^2}{\|a\|^2\|b\|^2 - (a^\top b)^2} > 1\,. \tag{183}$$

The set of such semiellipses parametrized by points $a, b \in B$ and numbers $c > 1$ coincides with the set of all cogyrolines in the space with the binary operation $\oplus_E$.

In Section 4.1.3 we proved (see (126)) that there exists a binary operation $\oplus_{co,E}$ with scalar multiplication $\otimes_{co,E}$ such that every cogyroline is a gyroline in the space with the binary operation $\oplus_{co,E}$:

$$S_E(a,b) = \{b \oplus_{co,E} (\tau \otimes_{co,E} a) \ : \ \tau \in \mathbb{R}\}\,. \tag{184}$$

Every such a line is the set of all the points of some geodesic in the space with the canonical metric tensor (106) parametrized by the functions

$$m_{0,co,E}(r^2) = \frac{1}{1-r^2} \qquad \text{and} \qquad m_{1,co,E}(r^2) = \frac{1}{(1-r^2)^3}\,. \tag{185}$$

Such geodesics satisfy the second order differential Equation (92),

$$\ddot{x} + \frac{2(x^\top \dot{x})}{1-\|x\|^2}\dot{x} + \left[\|\dot{x}\|^2 + \frac{(x^\top \dot{x})^2}{1-\|x\|^2}\right]x = 0\,. \tag{186}$$

In this section, we are going to find a canonical metric tensor $G_{co,\varphi}$ parametrized by functions $m_{0,co,\varphi}$ and $m_{1,co,\varphi}$, and an equation for geodesics such that every cogyroline in the space with a canonical metric tensor G_φ is a gyroline in the space with the canonical metric tensor $G_{co,\varphi}$.

4.3.2. Description of the Set of Cogyrolines

From (180) and (181), it follows that every cogyroline is an image of a semiellipse $S_E(a,b)$ under the mapping φ^{-1}.

Let us assume $y \in S_E(a,b)$ and $z = \varphi^{-1}y \in S(a,b)$. Then $y = \varphi(z)$, and

$$1 = \left[\frac{y^\top a}{\|a\|}\right]^2 + c\left[\frac{y^\top b_\perp}{\|b_\perp\|}\right]^2 = \left[\frac{\varphi(z)^\top a}{\|a\|}\right]^2 + c\left[\frac{\varphi(z)^\top b_\perp}{\|b_\perp\|}\right]^2 \\ = \frac{\tilde{\varphi}(\|z\|)}{\|z\|}\{\left[\frac{z^\top a}{\|a\|}\right]^2 + c\left[\frac{z^\top b_\perp}{\|b_\perp\|}\right]^2\}. \tag{187}$$

The vectors $\frac{a}{\|a\|}$, $\frac{b_\perp}{\|b_\perp\|}$ form an orthonormal basis in the plane $P(a,b)$. Therefore

$$\left[\frac{z^\top a}{\|a\|}\right]^2 + \left[\frac{z^\top b_\perp}{\|b_\perp\|}\right]^2 = \|z\|^2, \tag{188}$$

and (187) is equivalent to

$$(c-1)\left[\frac{z^\top b_\perp}{\|b_\perp\|}\right]^2 = \|z\|^2\left[\frac{1}{\tilde{\varphi}(\|z\|)^2} - 1\right]. \tag{189}$$

Introduce a vector $d = \sqrt{c-1}\frac{b_\perp}{\|b_\perp\|}$. Assume $z(\cdot)$ is a cogyroline which belongs to the set $S(a,b)$. Denote

$$f(s) = s\sqrt{\frac{1}{\tilde{\varphi}(s)^2} - 1}. \tag{190}$$

Then for all $t \in \mathbb{R}$

$$z(t)^\top d = f(\|z(t)\|). \tag{191}$$

Moreover, Equation (191) parametrizes all the cogyrolines by a non zero n-vector d as follows. For every cogyroline there exists d such that (191) holds for all $t \in \mathbb{R}$, and for every $d \in \mathbb{R}^n \backslash \{0\}$ for every cogyroline for which (191) holds for at least one number t, it holds for all $t \in \mathbb{R}$.

4.3.3. Differential Equations for Geodesics

For the sake of clarity we drop the argument t. Differentiating the equation

$$z^\top d = f(\|z\|) \tag{192}$$

we have

$$\dot{z}^\top d = f'(\|z\|)\frac{z^\top \dot{z}}{\|z\|} \tag{193}$$

and

$$\ddot{z}^\top d = f''(\|z\|)\frac{(z^\top \dot{z})^2}{\|z\|^2} + f'(\|z\|)\left[\frac{\|\dot{z}\|^2 + z^\top \ddot{z}}{\|z\|} - \frac{(z^\top \dot{z})^2}{\|z\|^3}\right]. \tag{194}$$

The curve z lies in the plane $S(a,b)$, and the vectors z and $\dot{z}$ are non parallel. Therefore there exist functions g_0, g_1 such that

$$\ddot{z} = g_0 z + g_1 \dot{z}. \tag{195}$$

Multiplying this equation by d from the right yields

$$f''(\|z\|)\frac{(z^\top \dot{z})^2}{\|z\|^2} + f'(\|z\|)\left[\frac{\|\dot{z}\|^2 + z^\top \ddot{z}}{\|z\|} - \frac{(z^\top \dot{z})^2}{\|z\|^3}\right] = \ddot{z}^\top d = g_0 z^\top d + g_1 \dot{z}^\top d. \tag{196}$$

Therefore

$$\ddot{z}^\top(d - f'(\|z\|)\frac{z}{\|z\|}) = f''(\|z\|)\frac{(z^\top \dot{z})^2}{\|z\|^2} + f'(\|z\|)\left[\frac{\|\dot{z}\|^2}{\|z\|} - \frac{(z^\top \dot{z})^2}{\|z\|^3}\right] \tag{197}$$

and

$$\begin{aligned} & g_0 z^\top (d - f'(\|z\|)\frac{z}{\|z\|}) + g_1 \dot{z}^\top (d - f'(\|z\|)\frac{z}{\|z\|}) \\ & = f''(\|z\|)\frac{(z^\top \dot{z})^2}{\|z\|^2} + f'(\|z\|)\left[\frac{\|\dot{z}\|^2}{\|z\|} - \frac{(z^\top \dot{z})^2}{\|z\|^3}\right]. \end{aligned} \tag{198}$$

Noticing that by (193) the coefficient of g_1 in (198) is equal to zero, and solving (198) for g_0 yields

$$g_0 = \frac{f''(\|z\|)\frac{(z^\top \dot{z})^2}{\|z\|^2} + f'(\|z\|)\left[\frac{\|\dot{z}\|^2}{\|z\|} - \frac{(z^\top \dot{z})^2}{\|z\|^3}\right]}{f(\|z\|) - \|z\| f'(\|z\|)}. \tag{199}$$

Equation (10) has the form

$$\ddot{z} + \frac{m_0'}{m_0} 2(z^\top \dot{z})\dot{z} + [(\frac{m_1'}{m_1} - \frac{2m_0'}{m_0})\frac{(z^\top \dot{z})^2}{\|z\|^2} + \frac{m_1 - m_0 - \|z\|^2 m_0'}{\|z\|^2 m_1}(\|\dot{z}\|^2 - \frac{(z^\top \dot{z})^2}{\|z\|^2})]z = 0. \tag{200}$$

Equations (195) and (200) coincide with $m_0 = m_{0,co,\varphi}$, $m_1 = m_{1,co,\varphi}$ if

$$g_1 = -\frac{m_{0,co,\varphi}'}{m_{0,co,\varphi}} 2(z^\top \dot{z}) \tag{201}$$

and

$$\frac{m_{1,co,\varphi}'}{m_{1,co,\varphi}} - \frac{2m_{0,co,\varphi}'}{m_{0,co,\varphi}} - \frac{m_{1,co,\varphi} - m_{0,co,\varphi} - \|z\|^2 m_{0,co,\varphi}'}{\|z\|^2 m_{1,co,\varphi}} = \frac{-f''(\|z\|) + \frac{f'(\|z\|)}{\|z\|}}{f(\|z\|) - \|z\| f'(\|z\|)} \tag{202}$$

and

$$\frac{m_{1,co,\varphi} - m_{0,co,\varphi} - \|z\|^2 m_{0,co,\varphi}'}{\|z\|^2 m_{1,co,\varphi}} = \frac{-\frac{f'(\|z\|)}{\|z\|}}{f(\|z\|) - \|z\| f'(\|z\|)}. \tag{203}$$

We add Equations (202)–(203), and solve Equation (203) for $m_{1,co,\varphi}$. Then we get the following system of equations for $m_{0,co,\varphi}$ and $m_{1,co,\varphi}$ as functions of f:

$$\begin{aligned} \frac{m_{1,co,\varphi}'}{m_{1,co,\varphi}} - \frac{2m_{0,co,\varphi}'}{m_{0,co,\varphi}} &= \frac{-f''(\|z\|)}{f(\|z\|) - \|z\| f'(\|z\|)} \\ m_{1,co,\varphi} &= \frac{(m_{0,co,\varphi} + \|z\|^2 m_{0,co,\varphi}')(f(\|z\|) - \|z\| f'(\|z\|))}{f(\|z\|)}. \end{aligned} \tag{204}$$

The solution of the system (204) is given by

$$\begin{aligned} m_{0,co,\varphi}(s)|_{s=\|z\|^2} &= \frac{C}{(f(\|z\|))^2} \\ m_{1,co,\varphi}(s)|_{s=\|z\|^2} &= \frac{C(f(\|z\|) - \|z\| f'(\|z\|))^2}{(f(\|z\|))^4}, \end{aligned} \tag{205}$$

where C is a positive constant.

Let $G_{co,\varphi}$ be the canonical metric tensor parametrized by the functions

$$\begin{aligned} m_{0,co,\varphi}(\|z\|^2) &= \frac{C}{(f(\|z\|))^2} \\ m_{1,co,\varphi}(\|z\|^2) &= \frac{C(f(\|z\|) - \|z\| f'(\|z\|))^2}{(f(\|z\|))^4} \end{aligned} \tag{206}$$

and let $\oplus_{co,\varphi}$ be the binary operation in the space with this canonical metric tensor. Then the set of cogyrolines $S(a,b)$ in (179) coincides with the set of gyrolines in the space with the canonical metric tensor $G_{co,\varphi}$.

Furthermore, let us normalize the functions $m_{0,co,\varphi}$ and $m_{1,co,\varphi}$ such that their values at zero are equal to one. To this end we need to choose $C = \frac{1}{\tilde{\varphi}'(0)^2}$. Then

$$\begin{aligned} m_{0,co,\varphi}(r^2) &= \left[\frac{\tilde{\varphi}(r)}{\tilde{\varphi}'(0) r \sqrt{1-\tilde{\varphi}(r)^2}}\right]^2 \\ m_{1,co,\varphi}(r^2) &= \frac{\tilde{\varphi}'(r)^2}{\tilde{\varphi}'(0)^2 (1-\tilde{\varphi}(r)^2)^3} . \end{aligned} \tag{207}$$

We now recall the values of these functions in (33) for the space with the binary operation $\oplus_\varphi$:

$$\begin{aligned} m_{0,\varphi}(r^2) &= \left[\frac{\tilde{\varphi}(r)}{\tilde{\varphi}'(0) r \sqrt{1-\tilde{\varphi}(r)^2}}\right]^2 \\ m_{1,\varphi}(r^2) &= \left[\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}'(0)(1-\tilde{\varphi}(r)^2)}\right]^2 . \end{aligned} \tag{208}$$

Obviously, the functions $m_{0,\varphi}$ and $m_{0,co,\varphi}$ coincide, while the functions $m_{1,\varphi}$ and $m_{1,co,\varphi}$ are different.

In particular, if $\tilde{\varphi}(s) = s$, we get the functions for Einstein cogyrolines:

$$\begin{aligned} m_{0,co,E}(r^2) &= \frac{1}{1-r^2} \\ m_{1,co,E}(r^2) &= \frac{1}{(1-r^2)^3} . \end{aligned} \tag{209}$$

If $\tilde{\varphi}(s) = \frac{2s}{1+s^2}$, we get the functions for Möbius cogyrolines:

$$\begin{aligned} m_{0,co,M}(r^2) &= \frac{1}{(1-r^2)^2} \\ m_{1,co,M}(r^2) &= \frac{(1+r^2)^2}{(1-r^2)^4} . \end{aligned} \tag{210}$$

5. Curvature

Every geodesic in a manifold with canonical metric tensor

$$G(x) = m_0(\|x\|^2)\left[I - \frac{xx^\top}{\|x\|^2}\right] + m_1(\|x\|^2)\frac{xx^\top}{\|x\|^2}, \tag{211}$$

parametrized by the functions m_0 and m_1, lies in a two dimensional plane containing $x(0)$ and $\dot{x}(0)$. Hence, let us calculate the Gaussian curvature of geodesics in a two dimensional space $\mathbb{R}^2$. In this section we assume that the second derivatives of the functions m_0 and m_1 exist and they are continuous.

5.1. Brioschi Formula

Let us denote two dimensional vectors by $(u,v)^\top$, where u and v are scalars. We use the standard notation for Riemannian line elements,

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2 . \tag{212}$$

Since $ds^2 = (du, dv)G(x)(du, dv)^\top$, we have

$$\begin{aligned} E &= m_0 + (m_1 - m_0)\frac{u^2}{u^2+v^2} = \frac{v^2 m_0 + u^2 m_1}{u^2+v^2} \\ F &= (m_1 - m_0)\frac{uv}{u^2+v^2} \\ G &= m_0 + (m_1 - m_0)\frac{v^2}{u^2+v^2} = \frac{u^2 m_0 + v^2 m_1}{u^2+v^2}\,. \end{aligned} \tag{213}$$

It should be noted that it is always clear from the context whether $G = G(u, v)$ represents the scalar function in (212) or the canonical metric tensor $G = G(x)$ in (211).

According to Brioschi formula the Gaussian curvature for a curve with the Riemannian line element (212) is given by [22],

$$K = \frac{A - B}{(EG - F^2)^2}\,, \tag{214}$$

where

$$A = \det \begin{pmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix} \tag{215}$$

and

$$B = \det \begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}. \tag{216}$$

Here, a lower index of E, F, G means a derivative with respect to a corresponding variable as, for instance, $E_u = \frac{\partial E}{\partial u}$.

5.2. Calculating the Curvature

In this section, we present a list of results of direct calculations of elements of determinants in Formula (214) as well as the value of K in terms of the functions $m_0(u^2+v^2)$ and $m_1(u^2+v^2)$. For the sake of clarity we omit the argument $u^2 + v^2$ of these functions.

Theorem 7. *The Gaussian curvature K of a manifold with the canonical metric tensor (211) is given by*

$$K = \frac{({m_0'}^2 m_1 + m_0 m_0' m_1' - 2 m_0 m_1 m_0'')\|x\|^2 - 3 m_0 m_1 m_0' + m_0^2 m_1'}{(m_0 m_1)^2}\,. \tag{217}$$

Proof. The definitions in (213) imply the following formulas.

$$EG - F^2 = m_0 m_1, \tag{218}$$

$$\begin{aligned} \frac{1}{2}E_u &= u m_1' - (m_1' - m_0')\frac{uv^2}{u^2+v^2} + (m_1 - m_0)\frac{uv^2}{(u^2+v^2)^2}, \\ \frac{1}{2}E_v &= v m_0' + (m_1' - m_0')\frac{u^2 v}{u^2+v^2} - (m_1 - m_0)\frac{u^2 v}{(u^2+v^2)^2}, \\ \frac{1}{2}G_u &= u m_0' + (m_1' - m_0')\frac{uv^2}{u^2+v^2} - (m_1 - m_0)\frac{uv^2}{(u^2+v^2)^2}, \\ \frac{1}{2}G_v &= v m_1' - (m_1' - m_0')\frac{u^2 v}{u^2+v^2} + (m_1 - m_0)\frac{u^2 v}{(u^2+v^2)^2}, \end{aligned} \tag{219}$$

$$\begin{aligned}\frac{1}{2}E_{vv} &= 2v^2m_0'' + (m_1'' - m_0'')\frac{2u^2v^2}{u^2+v^2}\\ &\quad + (m_1' - m_0')\frac{u^2(u^2-3v^2)}{(u^2+v^2)^2} - (m_1 - m_0)\frac{u^2(u^2-3v^2)}{(u^2+v^2)^3} + m_0'\\ \frac{1}{2}G_{uu} &= 2u^2m_0'' + (m_1'' - m_0'')\frac{2u^2v^2}{u^2+v^2}\\ &\quad + (m_1' - m_0')\frac{v^2(v^2-3u^2)}{(u^2+v^2)^2} - (m_1 - m_0)\frac{v^2(v^2-3u^2)}{(u^2+v^2)^3} + m_0',\end{aligned} \tag{220}$$

$$\begin{aligned}F_{uv} &= (m_1'' - m_0'')\frac{4u^2v^2}{u^2+v^2}\\ &= (m_1' - m_0')\frac{2v^4 - 4u^2v^2 + 2v^4}{(u^2+v^2)^2} + (m_1 - m_0)\frac{-u^4 + 6u^2v^2 - v^4}{(u^2+v^2)^3},\end{aligned} \tag{221}$$

$$-\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} = -2(u^2+v^2)m_0'' - 3m_0' + m_1', \tag{222}$$

$$\begin{aligned}F_u - \frac{1}{2}E_v &= -vm_0' + (m_1' - m_0')\frac{u^2v}{u^2+v^2} + (m_1 - m_0)\frac{v^3}{(u^2+v^2)^2}\\ F_v - \frac{1}{2}G_u &= -um_0' + (m_1' - m_0')\frac{uv^2}{u^2+v^2} + (m_1 - m_0)\frac{u^3}{(u^2+v^2)^2},\end{aligned} \tag{223}$$

$$\begin{aligned}&\det\begin{pmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v, \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix}\\ &= -2m_0m_1m_0''(u^2+v^2) - 3m_0m_1m_0' + m_1'm_0^2 + m_0m_0'm_1'(u^2+v^2)\\ &\quad - \frac{(2m_0m_0'm_1' - 4{m_0'}^2m_1 + m_0{m_0'}^2 + m_0m_1^2)u^2v^2}{u^2+v^2}\\ &\quad + \frac{m_0(m_1-m_0)(m_1'+m_0')2u^2v^2}{(u^2+v^2)^2} - \frac{m_0(m_1-m_0)^2u^2v^2}{(u^2+v^2)^3},\end{aligned} \tag{224}$$

$$\begin{aligned}&\det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}\\ &= -{m_0'}^2m_1(u^2+v^2) - \frac{(2m_0m_0'm_1' - 4{m_0'}^2m_1 + m_0{m_0'}^2 + m_0m_1^2)u^2v^2}{u^2+v^2}\\ &\quad + \frac{m_0(m_1-m_0)(m_1'+m_0')2u^2v^2}{(u^2+v^2)^2} - \frac{m_0(m_1-m_0)^2u^2v^2}{(u^2+v^2)^3}.\end{aligned} \tag{225}$$

Finally,

$$K = \frac{({m_0'}^2m_1 + m_0m_0'm_1' - 2m_0m_1m_0'')(u^2+v^2) - 3m_0m_1m_0' + m_0^2m_1'}{(m_0m_1)^2}. \tag{226}$$

Noticing that $u^2 + v^2 = \|x\|^2$, we have

$$K = \frac{(m_0'^2 m_1 + m_0 m_0' m_1' - 2m_0 m_1 m_0'')\|x\|^2 - 3m_0 m_1 m_0' + m_0^2 m_1'}{(m_0 m_1)^2}\,, \tag{227}$$

as desired. □

5.3. Gaussian Curvatures of Several Particular Spaces

For the space $\mathbb{B}$ with the Einstein addition $\oplus_E$ we have, by [21],

$$\begin{aligned} m_{0,E}(\|x\|^2) &= \frac{1}{1 - \|x\|^2} \\ m_{1,E}(\|x\|^2) &= \frac{1}{(1 - \|x\|^2)^2}\,. \end{aligned} \tag{228}$$

For the functions m_0 and m_1 in (228), the curvature formula (227) gives

$$K = -1\,. \tag{229}$$

For the space with Möbius addition we have, by [21],

$$\begin{aligned} m_{0,M}(\|x\|^2) &= \frac{1}{(1 - \|x\|^2)^2} \\ m_{1,M}(\|x\|^2) &= \frac{1}{(1 - \|x\|^2)^2}\,. \end{aligned} \tag{230}$$

For the functions m_0 and m_1 in (230), the curvature formula (227) gives

$$K = -4\,. \tag{231}$$

For the space with the set of geodesics that are cogyrolines for Einstein addition, that is, for the space with the canonical metric tensor (106), we have, by (105),

$$\begin{aligned} m_{0,co,E}(\|x\|^2) &= \frac{1}{1 - \|x\|^2} \\ m_{1,co,E}(\|x\|^2) &= \frac{1}{(1 - \|x\|^2)^3}\,. \end{aligned} \tag{232}$$

For the functions m_0 and m_1 in (232), the curvature formula (227) gives

$$K = 0\,. \tag{233}$$

For the space with the set of geodesics that are cogyrolines for Möbius addition (i.e., for the space with the canonical metric tensor (167)), we have, by (166),

$$\begin{aligned} m_{0,co,M}(\|x\|^2) &= \frac{1}{(1 - \|x\|^2)^2} \\ m_{1,co,M}(\|x\|^2) &= \frac{(1 + \|x\|^2)^2}{(1 - \|x\|^2)^4}\,. \end{aligned} \tag{234}$$

For the functions m_0 and m_1 in (234), the curvature formula (227) gives

$$K = 0\,. \tag{235}$$

5.4. Spaces with Zero Gaussian Curvature

In this subsection we find conditions on the functions m_0 and m_1 under which the Gaussian curvature K is equal to zero, and check these conditions for the canonical metric tensors generated by cogyrolines (208). Recall that according to Assumption 1 we have $m_0(0) = m_1(0) = 1$. Let us denote by z the argument $\|x\|^2$ in the curvature formula (227).

Theorem 8. *The Gaussian curvature of a canonical metric tensor G parametrized by functions m_0 and m_1 is equal to zero if and only if*

$$m_1(z)m_0(z) = [(zm_0(z))']^2\,. \tag{236}$$

Proof. According to (227) the Gaussian curvature K is equal to zero if and only if

$$\frac{m_1'}{m_1} = \frac{3m_0m_0' + 2zm_0m_0'' - z{m_0'}^2}{m_0^2 + zm_0m_0'}\,. \tag{237}$$

The right-hand side of (237) is equal to

$$2\frac{(m_0^2 + zm_0m_0')'}{m_0^2 + zm_0m_0'} - 3\frac{m_0'}{m_0} = \left[\ln\frac{(m_0^2 + zm_0m_0')^2}{m_0^3}\right]'\,. \tag{238}$$

Therefore Equation (237) may be integrated, obtaining

$$m_1 = C\frac{(m_0^2 + zm_0m_0')^2}{m_0^3} = C\frac{[(zm_0)']^2}{m_0}\,. \tag{239}$$

Using the normalization condition $m_0(0) = m_1(0) = 1$, we get $C = 1$. Hence, the space with the functions m_0 and m_1 parametrizing the canonical metric tensor G has the Gaussian curvature equal to zero if and only if

$$m_1(z)m_0(z) = [(zm_0(z))']^2\,. \tag{240}$$

□

We have seen that the curvature of spaces defined by cogyrolines of Einstein addition or Möbius additions are equal to zero. Let us check if the same property holds for all the spaces considered in Section 3.

Theorem 9. *Let G be a canonical metric tensor parametrized by the functions $m_{0,co,\varphi}$ and $m_{1,co,\varphi}$ given by* (207), *that is,*

$$\begin{aligned} m_{0,co,\varphi}(r^2) &= \frac{\tilde{\varphi}(r)^2}{\tilde{\varphi}'(0)^2 r^2 (1 - \tilde{\varphi}(r)^2)}\,, \\ m_{1,co,\varphi}(r^2) &= \frac{\tilde{\varphi}'(r)^2}{\tilde{\varphi}'(0)^2 (1 - \tilde{\varphi}(r)^2)^3}\,. \end{aligned} \tag{241}$$

Then the corresponding Gaussian curvature is equal to zero.

Proof. We need to check Condition (236). Since

$$zm_{0,co,\varphi}(z) = \frac{\tilde{\varphi}(\sqrt{z})^2}{\tilde{\varphi}'(0)^2(1-\tilde{\varphi}(\sqrt{z})^2)}, \tag{242}$$

we have

$$\left[(zm_{0,co,\varphi}(z))'\right]^2 = \left[\frac{\tilde{\varphi}(\sqrt{z})\tilde{\varphi}(\sqrt{z})'}{\sqrt{z}\tilde{\varphi}'(0)^2(1-\tilde{\varphi}(\sqrt{z})^2)}\right]^2 = m_{0,co,\varphi}(z)m_{1,co,\varphi}(z). \tag{243}$$

Hence, Equation (236) is satisfied, and the theorem is proved. □

Thus, for every function $\tilde{\varphi}$ the curvature of the space, defined by cogyrolines of spaces with a canonical metric tensor G_φ in (32), is equal to zero.

6. An Open Problem

It remains an open problem, as to whether there exist binary operations in $\mathbb{B}$ other than those described in Section 3, for which Properties (15)–(19) are satisfied, and the gyration operators are linear with unitary matrices.

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