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Existence and Uniqueness of the Solution for an Integral Equation with Supremum, via w -Distances

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Received: 2 September 2020; Accepted: 18 September 2020; Published: 20 September 2020

Abstract: Following the idea of T. Wongyat and W. Sintunavarat, we obtain some existence and uniqueness results for the solution of an integral equation with supremum. The paper ends with the study of Gronwall-type theorems, comparison theorems and a result regarding a Ulam–Hyers stability result for the corresponding fixed point problem.

Keywords: w -distance; integral equation with supremum; abstract Gronwall theorem; weakly Picard operator; Ulam–Hyers stability

1. Introduction

The object of investigation of this paper is the qualitative theory of integral equations with supremum. These equations arise naturally when solving real-world problems, for example in the study of systems with automatic regulation and automatic control, problems in control theory. These types of equations are characterized by the fact that the maximum values of some regulated state parameters depend on certain time intervals, see for example [1] and the references therein. Recently, the interest in differential equations with supremum has an intensive development (see [2–4]). The aim of this paper focuses on two aspects: one is to prove existence and uniqueness results using w -weak generalized contractions theorem; the other is to prove a Gronwall-type theorem and comparison theorems. Using this theory symmetry is important in determining the qualitative properties of the solution of the integral equation.

We consider the following class of integral equation with supremum

$$x(t) = \varphi(t) + \int_{\alpha}^t f(t, s, x(s), \sup_{\theta \in [\alpha, s]} x(\theta)) ds, \quad t \in [\alpha, \beta] \quad (1)$$

with α, β real and $\alpha < \beta$, the functions $\varphi \in C([\alpha, \beta], \mathbb{R})$, $f \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^2, \mathbb{R})$ are given. To prove our results, we shall use the w -weak generalized contractions theorem due to T. Wongyat and W. Sintunavarat [5] and we obtain an existence and uniqueness result for the solutions of this equation.

2. Preliminaries

We consider (\mathcal{T}, d) a metric space. In the sequel, we will use the following definitions and theorems, for details, see [5,6].

Definition 1. ([6]) Let (\mathcal{T}, d) a metric space and a function $q: \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$. We say that q is a w -distance on \mathcal{T} , if the below conditions hold, for all $x, y, z \in \mathcal{T}$:

- (1) $q(x, y) \leq q(x, z) + q(z, y)$;
- (2) $q(x, \cdot): \mathcal{T} \rightarrow [0, \infty)$ is lower semicontinuous;
- (3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

We recall that each metric on the nonempty set \mathcal{T} is a w -distance on \mathcal{T} .

Definition 2 ([5]). We say that the function $\psi: \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ is a w^0 -distance on \mathcal{T} , if it is a w -distance on \mathcal{T} with $q(x, x) = 0$, for all $x \in \mathcal{T}$.

Definition 3 ([5]). We say that the function $\psi: [0, \infty) \rightarrow [0, \infty)$ is an altering distance function, if the below assertions hold:

- (1) The function ψ is continuous and nondecreasing;
- (2) $\psi(t)$ is zero if and only if $t = 0$.

Definition 4 ([5]). Let (\mathcal{T}, d) be a metric space. We say that a w -distance q is a ceiling distance of d if and only if $q(x, y) \geq d(x, y)$, for all $x, y \in \mathcal{T}$.

Definition 5 ([5]). We consider q a w -distance on the metric space (\mathcal{T}, d) , the altering distance function $\psi: [0, \infty) \rightarrow [0, \infty)$, and the continuous function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(t)$ is zero if and only if $t = 0$. If the below inequality holds we say that the operator $A: \mathcal{T} \rightarrow \mathcal{T}$ is a w -generalized weak contraction mapping

$$\psi(q(A(x), A(y))) \leq \psi(m(x, y) - \phi(q(x, y))), \text{ for all } x, y \in \mathcal{T}, \quad (2)$$

where

$$m(x, y) := \max \left\{ q(x, y), \frac{q(x, A(y)) + q(A(x), y)}{2} \right\}. \quad (3)$$

If $q = d$, then we say that A is a generalized weak contraction mapping.

Now we consider (\mathcal{T}, d) a complete metric space. The following fixed point result of the equation $A(x) = x$, $x \in \mathcal{T}$ via w -distances represents the motivation of our work.

Theorem 1 ([5]). We consider $q: \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ a continuous w -distance on \mathcal{T} and a ceiling distance of d , the altering distance function $\psi: [0, \infty) \rightarrow [0, \infty)$, and the continuous function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(t)$ is zero if and only if $t = 0$. Let $A: \mathcal{T} \rightarrow \mathcal{T}$ a continuous operator such that

$$\psi(q(A(x), A(y))) \leq \psi(m(x, y) - \phi(q(x, y))), \text{ for all } x, y \in \mathcal{T}. \quad (4)$$

Then, A has a unique fixed point in \mathcal{T} and the sequence of successive approximations $\{x_n\}_{n \in \mathbb{N}}$, defined by $x_n = A^n(x_0)$, for each $x_0 \in \mathcal{T}$, for all $n \in \mathbb{N}$, converges to the unique fixed point of A .

For other fixed points results obtained employing the theory of w -distance, the reader is referred to [5,7–11].

In this paper, we emphasize some connection between w -generalized weak contraction mapping and the Picard operator theory.

In the sequel, we recall the following results (see [12–14]).

Let (\mathcal{T}, d) be a metric space. We say that the operator $A: \mathcal{T} \rightarrow \mathcal{T}$ is weakly a Picard operator (WPO) if the successive approximations sequence $\{A^n(x)\}_{n \in \mathbb{N}}$, converges for all $x \in \mathcal{T}$ and its limit (which generally depend on x) is a fixed point of A . If an operator A is WPO with $F_A = \{x^*\}$, then, we say that the operator A is a Picard operator (PO).

If $A: \mathcal{T} \rightarrow \mathcal{T}$ is a WPO, we can define the operator $A^\infty: \mathcal{T} \rightarrow F_A$, by $A^\infty(x) := \lim_{n \rightarrow +\infty} A^n(x)$.

Definition 6. Let A be a weakly Picard operator and $c > 0$. We say that the operator A is a c -weakly Picard operator if

$$d(x, A^\infty(x)) \leq cd(x, A(x)), \text{ for all } x \in \mathcal{T}.$$

If \mathcal{T} is a nonempty set, then (\mathcal{T}, d, \leq) is an ordered metric space, where \leq is a partial order relation on \mathcal{T} .

Now we present some properties regarding WPOs and POs.

Theorem 2 ([12]). (Characterization theorem) Let (\mathcal{T}, d) be a metric space. The operator $A: \mathcal{T} \rightarrow \mathcal{T}$ is WPO if there exists a partition of \mathcal{T} , $\mathcal{T} = \bigcup_{\lambda \in \Lambda} X_\lambda$, such that

- (a) $\mathcal{T}_\lambda \in I(A)$, for all $\lambda \in \Lambda$;
- (b) $A|_{\mathcal{T}_\lambda}: \mathcal{T}_\lambda \rightarrow \mathcal{T}_\lambda$ is PO, for all $\lambda \in \Lambda$.

Theorem 3 ([13]). (Abstract Gronwall Theorem) Let (\mathcal{T}, d, \leq) be an ordered metric space and we consider the operator $A: \mathcal{T} \rightarrow \mathcal{T}$. We suppose

- (i) The operator A is increasing with respect to \leq ;
- (ii) A is a Picard operator with $F_A = \{x^*\}$.

Then the below conclusions hold:

- (i) for $x \in \mathcal{T}$, $x \leq A(x) \Rightarrow x \leq x^*$,
- (ii) for $x \in \mathcal{T}$, $x \geq A(x) \Rightarrow x \geq x^*$.

Theorem 4 ([13]). (Abstract Comparison Lemma) Let (\mathcal{T}, d, \leq) be an ordered metric space and we consider the operators $A, B, C: \mathcal{T} \rightarrow \mathcal{T}$ with the properties:

- (i) $A \leq B \leq C$;
- (ii) A, B, C are WPOs;
- (iii) B is an increasing operator.

Then, for $x, y, z \in \mathcal{T}$, $x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

We present now the concept of Hyers–Ulam stability in the setting of metric spaces given by I.A. Rus in [15].

Definition 7. Let (\mathcal{T}, d) be a metric space and we consider the operator $A: \mathcal{T} \rightarrow \mathcal{T}$. Then, we say that the fixed point equation

$$x = A(x) \tag{5}$$

is Ulam–Hyers stable if there exists $c_A \in \mathbb{R}_+^*$ such that: for any $\varepsilon > 0$ and for each solution $y^* \in \mathcal{T}$ of (5), i.e., $d(y^*, A(y^*)) \leq \varepsilon$, there exists a solution x^* of (5) such that

$$d(y^*, x^*) \leq c_A \varepsilon.$$

We recall the following abstract result of the Ulam–Hyers stability of the fixed point Equation (5).

Theorem 5. (Ulam–Hyers stability, [15]) Let (\mathcal{T}, d) be a metric space. Suppose that $A: \mathcal{T} \rightarrow \mathcal{T}$ is a c -Picard operator. Then, Equation (5) is Ulam–Hyers stable.

For more results regarding WPOs and POs, see [3,4,14–16].

3. Main Result

Let the operator $A: C([\alpha, \beta], \mathbb{R}) \rightarrow C([\alpha, \beta], \mathbb{R})$ expressed by

$$A(x)(t) := \varphi(t) + \int_{\alpha}^t f(t, s, x(s), \sup_{\theta \in [\alpha, s]} x(\theta)) ds, \quad (6)$$

where $f \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^2, \mathbb{R})$, $x \in C([\alpha, \beta], \mathbb{R})$ and $t \in [\alpha, \beta]$.

Our first result is the following theorem.

Theorem 6. We consider the integral Equation (1) with α, β real and $\alpha < \beta$, the functions $\varphi \in C([\alpha, \beta], \mathbb{R})$, $f \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^2, \mathbb{R})$ are given. We assume the following:

- (i) The operator $A: C([\alpha, \beta], \mathbb{R}) \rightarrow C([\alpha, \beta], \mathbb{R})$ defined by (6) is continuous;
- (ii) The altering distance function $\psi: [0, \infty) \rightarrow [0, \infty)$ satisfies $\psi(t) < t$, for all $t > 0$, and the continuous function $\phi: [0, \infty) \rightarrow [0, \infty)$ satisfies $\phi(t)$ is zero if and only if $t = 0$;
- (iii) The below inequality holds

$$\begin{aligned} & \left| f(t, s, x(s), \sup_{\theta \in [\alpha, s]} x(\theta)) \right| + \left| f(t, s, y(s), \sup_{\theta \in [\alpha, s]} y(\theta)) \right| \\ & \leq \frac{\psi \left(|x(s)| + |y(s)| + \left| \sup_{\theta \in [\alpha, s]} x(\theta) \right| + \left| \sup_{\theta \in [\alpha, s]} y(\theta) \right| \right)}{\beta - \alpha} \\ & - \frac{\phi \left(\sup_{l \in [\alpha, \beta]} |x(l)| + \sup_{l \in [\alpha, \beta]} |y(l)| + \sup_{l \in [\alpha, \beta]} \left| \sup_{\theta \in [\alpha, l]} x(\theta) \right| + \sup_{l \in [\alpha, \beta]} \left| \sup_{\theta \in [\alpha, l]} y(\theta) \right| \right)}{\beta - \alpha} \\ & - \frac{2|\varphi(t)|}{\beta - \alpha}, \text{ for all } x, y \in C([\alpha, \beta], \mathbb{R}), t, s \in [\alpha, \beta]. \end{aligned}$$

Then the integral equation with supremum (1) has a unique solution and the sequence of successive approximations $\{x_n\}_{n \in \mathbb{N}}$, defined by $x_n = A^n(x_0)$, for each $x_0 \in C([\alpha, \beta], \mathbb{R})$, for all $n \in \mathbb{N}$, converges to the unique solution of Equation (1).

Proof. Let $\mathcal{T} = C([\alpha, \beta], \mathbb{R})$ and we consider the metric $d: \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ defined as below

$$d(x, y) := \sup_{t \in [\alpha, \beta]} |x(t) - y(t)|, \text{ for all } x, y \in C([\alpha, \beta], \mathbb{R}). \quad (7)$$

It is clear that (\mathcal{T}, d) is a complete metric space. We consider the function $q: C([\alpha, \beta], \mathbb{R}) \times C([\alpha, \beta], \mathbb{R}) \rightarrow [0, \infty)$ defined by:

$$q(x, y) := \sup_{t \in [\alpha, \beta]} |x(t)| + \sup_{t \in [\alpha, \beta]} |y(t)|, \text{ for all } x, y \in C([\alpha, \beta], \mathbb{R}). \quad (8)$$

We get that q is a w -distance on \mathcal{T} and also a ceiling distance of d .

We will show that A satisfies the contraction condition (4).

$$\begin{aligned}
& |A(x)(t)| + |A(y)(t)| \\
&= \left| \varphi(t) + \int_{\alpha}^t f(t, s, x(s), \sup_{\theta \in [\alpha, s]} x(\theta)) ds \right| + \left| \varphi(t) + \int_{\alpha}^t f(t, s, y(s), \sup_{\theta \in [\alpha, s]} y(\theta)) ds \right| \\
&\leq |\varphi(t)| + \left| \int_{\alpha}^t f(t, s, x(s), \sup_{\theta \in [\alpha, s]} x(\theta)) ds \right| + |\varphi(t)| \\
&+ \left| \int_{\alpha}^t f(t, s, y(s), \sup_{\theta \in [\alpha, s]} y(\theta)) ds \right| \\
&\leq 2|\varphi(t)| + \int_{\alpha}^t \left(\left| f(t, s, x(s), \sup_{\theta \in [\alpha, s]} x(\theta)) \right| + \left| f(t, s, y(s), \sup_{\theta \in [\alpha, s]} y(\theta)) \right| \right) ds \\
&\leq 2|\varphi(t)| + \frac{1}{\beta - \alpha} \int_{\alpha}^t \left[\psi \left(|x(s)| + |y(s)| + \left| \sup_{\theta \in [\alpha, s]} x(\theta) \right| + \left| \sup_{\theta \in [\alpha, s]} y(\theta) \right| \right) \right. \\
&\quad \left. - \phi \left(\sup_{l \in [\alpha, \beta]} |x(l)| + \sup_{l \in [\alpha, \beta]} |y(l)| + \sup_{l \in [\alpha, \beta]} \left| \sup_{\theta \in [\alpha, l]} x(\theta) \right| + \sup_{l \in [\alpha, \beta]} \left| \sup_{\theta \in [\alpha, l]} y(\theta) \right| \right) - 2|\varphi(t)| \right] ds \\
&\leq \psi(q(x, y) - \phi(q(x, y))).
\end{aligned}$$

We obtain that

$$\sup_{t \in [\alpha, \beta]} |Ax(t)| + \sup_{t \in [\alpha, \beta]} |Ay(t)| \leq \psi(q(x, y) - \phi(q(x, y)))$$

and using (8) we get

$$q(Ax, Ay) \leq \psi(q(x, y) - \phi(q(x, y))), \text{ for all } x, y \in \mathcal{T}.$$

Hence we have

$$\psi(q(Ax, Ay)) \leq q(Ax, Ay) \leq \psi(q(x, y) - \phi(q(x, y))), \text{ for all } x, y \in \mathcal{T}.$$

Therefore the condition (4) holds and thus we may conclude that A has a unique fixed point. So there exists a unique solution for the integral equation with supremum (1). \square

From the above theorem, the operator A defined in (6) is a PO. Now we establish a Gronwall-type theorem for Equation (1).

Theorem 7. We consider the integral Equation (1) with α, β real, $\alpha < \beta$, and the functions $\varphi \in C([\alpha, \beta], \mathbb{R})$, $f \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}, \mathbb{R})$ are given. We assume that the conditions (i)–(iii) from Theorem 6 hold. Furthermore, we suppose that

(iv) $f(t, s, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function with respect to the last argument, for all $t, s \in [\alpha, \beta]$.

Let $x^* \in C([\alpha, \beta], \mathbb{R})$ be the unique solution of the integral Equation (1). Then, the following conditions are satisfied:

(1) for all $x \in C([\alpha, \beta], \mathbb{R})$ with

$$x(t) \leq \varphi(t) + \int_{\alpha}^t f(t, s, \sup_{\theta \in [\alpha, s]} x(\theta)) ds, \quad t \in [\alpha, \beta],$$

we have $x \leq x^*$;

(2) for all $x \in C([\alpha, \beta], \mathbb{R})$ with

$$x(t) \geq \varphi(t) + \int_{\alpha}^t f(t, s, \sup_{\theta \in [\alpha, s]} x(\theta)) ds, \quad t \in [\alpha, \beta],$$

we have $x \geq x^*$.

Proof. From (iv), we have that the operator A defined in (6) is increasing with respect to the partial order.

By the proof of Theorem 6, it follows that A is a Picard operator. The conclusion of the theorem follows from Theorem 3. \square

We establish now a comparison theorem for Equation (1), using Theorem 4.

Theorem 8. We consider the integral Equation (1) with α, β real, $\alpha < \beta$, and we suppose that $\varphi_i \in C([\alpha, \beta], \mathbb{R})$ and $f_i \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}, \mathbb{R})$, $i = 1, 2, 3$ are given. We assume that the conditions (i)–(iii) from Theorem 6 hold. Furthermore, we suppose that

- (i) $\varphi_1 \leq \varphi_2 \leq \varphi_3$, $f_1 \leq f_2 \leq f_3$;
- (ii) φ_2, f_2 are increasing.

Let $x_i \in C([\alpha, \beta], \mathbb{R})$ be a solution of the equation

$$x_i(t) = \varphi_i(t) + \int_{\alpha}^t f_i(t, s, \sup_{\theta \in [\alpha, s]} x(\theta)) ds, \quad t \in [\alpha, \beta], \quad i = 1, 2, 3.$$

If $x_1(\alpha) \leq x_2(\alpha) \leq x_3(\alpha)$, then $x_1 \leq x_2 \leq x_3$.

Proof. The proof follows from the Theorem 4. \square

Now we prove a Ulam–Hyers stability result for the integral Equation (1).

Theorem 9. We consider the integral equation with supremum (1) and we suppose that all the conditions of Theorem 6 are satisfied. Then, the integral Equation (1) is Ulam–Hyers stable.

Proof. Applying Theorem 6 and Theorem 5 we get the conclusion of the theorem. \square

4. Conclusions

The purpose of this paper is to establish some fixed point results for generalized contraction operators with respect to w -distances. The operators considered here contain a supremum over a certain time interval. Section 3 begins with an existence and uniqueness theorem proved using the method of w -distances. Adding to the hypotheses that sustain the existence and uniqueness of the solution, the fact that f is an increasing function, we obtain Gronwall-type and comparison theorems. In the last part of the paper we study the Ulam–Hyers stability using Picard operators techniques. We define a fixed point equation from the integral equation with supremum. If the defined operator is c -weakly Picard we have Ulam–Hyers stability of the corresponding fixed point problem.

Author Contributions: Conceptualization, V.I. and D.O.; methodology, V.I. and D.O.; validation, V.I. and D.O.; formal analysis, V.I. and D.O.; investigation, V.I. and D.O.; resources, V.I. and D.O.; writing—original draft preparation, V.I. and D.O.; writing—review and editing, V.I. and D.O.; visualization, V.I. and D.O.; supervision, V.I. and D.O.; project administration, V.I. and D.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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