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Neutral Delay Differential Equations: Oscillation Conditions for the Solutions

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Abstract: The purpose of this article is to explore the asymptotic properties for a class of fourth-order neutral differential equations. Based on a comparison with the differential inequality of the first-order, we have provided new oscillation conditions for the solutions of fourth-order neutral differential equations. The obtained results can be used to develop and provide theoretical support for and to further develop the study of oscillation for a class of fourth-order neutral differential equations. Finally, we provide an illustrated example to demonstrate the effectiveness of our new criteria.

Keywords: oscillation; neutral differential equations; fourth-order

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1. Introduction

In recent years, there has been growing interest in exploring neutral differential equations, with applications in many areas, including fluid dynamics, physics, chemistry, and biology. Problems in these areas have often guided researchers and physicists to expend great efforts to investigate interesting phenomena, such as the effect of vibrating systems fixed to an elastic bar. Examples of such problems can be found in the Euler equations and the Taylor–Goldstein equation in fluid dynamics, and the perturbed vertical velocity in stratified flow with the effect of viscosity.

The oscillatory properties of neutral differential equations also play crucial roles in mechanical engineering, civil engineering, and application-oriented research—which can support research with the potential of developing the ship building, airplane, and rocket industries—along with the fabrication of microelectromechanical systems (MEMS) and gyroscopes; see [1–3]. In this paper, we aim to effectively study the oscillation criteria of the following equation.

$$L_x' + q(x)w^{(p_2-1)}(z(x)) = 0,\ x \geq x_0, \tag{1}$$

where

$$L_x := \xi(x)\left(y'''(x)\right)^{(p_1-1)}, \quad \text{so that} \quad y(x) := w(x) + r(x)w(\delta(x)). \tag{2}$$

The operator L_x is the canonical form if $\int_{x_0}^{\infty} \frac{1}{\xi^{1/(p_1-1)}(s)} \mathrm{d}s = \infty$; otherwise, it represents the noncanonical form.

Here, our significant novel outcomes are obtained by considering the following conditions:

S1: $r \in C[x_0, \infty),\ 0 \leq r(x) < r_0 < \infty$,
S2: $\delta, z, q \in C[x_0, \infty),\ q(x) > 0, \delta(x) \leq x, \lim_{x\to\infty} \delta(x) = \lim_{x\to\infty} z(x) = \infty$,

S3: $\xi \in C[x_0, \infty)$, $\xi(x) > 0$, $\xi'(x) \geq 0$, and

$$\int_{x_0}^{\infty} \frac{1}{\xi^{1/(p_1-1)}(s)} ds = \infty, \tag{3}$$

S4: $p_i > 1, \ i = 1, 2$ are constants and

$$p_1 := \begin{cases} 2 & \text{if } p_2 \leq 2; \\ 1 + 2^{\beta-1} & \text{if } p_2 > 2. \end{cases}$$

By a solution of Equation (1) we mean a function $w \in C^3[x, \infty), \ x \geq x_0$, which has the property $\xi(x)(y'''(x))^{\alpha} \in C^1[x_0, \infty)$, and satisfies Equation (1) on $[x_0, \infty)$.

Definition 1. *A solution of Equation (1) represents oscillatory behavior if it has arbitrarily large zeros on $[x_0, \infty)$. Otherwise, a solution can be known as nonoscillatory.*

Definition 2. *Equation (1) is known as oscillatory if each solution of it is oscillatory.*

Definition 3. *If the highest-order derivative of the unknown function occurs with and without delay, then the differential equation is defined as neutral.*

Over the past few years, the asymptotic behavior of solutions of differential equations has become a key research area in different disciplines. In particular, in mathematics, many researchers investigated the oscillatory properties of solutions to neutral differential equations; see [4–12]. For differential equations of neutral type, we present the following results that are closely related to our work. For example, Bazighifan [13], obtained oscillation conditions for solutions of Equation (1). In [14], the authors considered the equation

$$\left(\xi(x) F\left(y^{(n-1)}(x)\right)\right)' + p(x) F\left(y^{(n-1)}(x)\right) + q(x) F(y(z(x))) = 0,$$

where $F = |s|^{p-2}s$ and obtained some new oscillation conditions.

In [6,15] the authors studied oscillation for equations of Emden–Fowler neutral type where a criterion of Kamenev-type oscillation was found.

The authors in [16,17] considered the equation

$$y^{(r)}(x) + q(x) w(z(x)) = 0,$$

and proved that it is oscillatory if

$$\liminf_{x\to\infty} \int_{\sigma(x)}^{x} K(s) ds > 2^{(r-1)(r-2)} \frac{(r-1)}{e}, \tag{4}$$

and

$$\liminf_{x\to\infty} \int_{\sigma(x)}^{x} K(s) ds > \frac{(r-1)!}{e}, \tag{5}$$

where $K(t) := z^{r-1}(x)(1 - b(z(x))) q(x)$ and r is an even number.

In [18,19], the authors considered the equation

$$\left(\xi(x)\left(y^{(n-1)}(x)\right)^{\alpha}\right)' + q(x) w^{\alpha}(z(x)) = 0,$$

and it was found to be oscillatory if

$$\liminf_{x\to\infty} \int_{\delta^{-1}(z(x))}^{x} \frac{q(s)}{\xi(s)} \left(s^{n-1}\right)^{\alpha} ds > \left(\frac{1}{z_0} + \frac{p_0^{\alpha}}{z_0 \delta_0}\right) \frac{((n-1)!)^{\alpha}}{e}, \tag{6}$$

and
$$\liminf_{x\to\infty}\int_{\delta^{-1}(\eta(x))}^{x}\left(\frac{\left(\tau^{-1}(\eta(s))\right)^{n-1}}{\xi^{1/\alpha}(\delta^{-1}(\eta(s)))}\right)^{\alpha}q(s)P_n^{\alpha}(z(s))\mathrm{d}s>\frac{((n-1)!)^{\alpha}}{\mathrm{e}},\tag{7}$$
where $\eta\in C^1([x_0,\infty),\mathbb{R})$ and $\widehat{q}(x):=\min\{q(z^{-1}(x)),q(z^{-1}(\delta(x)))\}$.

Inspired by the work of [16–18], we concentrate on establishing the oscillatory properties of solutions of (1). We aim also to provide some examples that can ensure the validity of the proposed criteria.

2. Main Results

This section aims to present the main contribution in this article which lies in its potential to empower systematic analysis and understanding the oscillatory properties of solutions of (1).

Lemma 1 ([15])**.** *If u satisfies $u^{(i)}(x)>0$, $i=0,1,\ldots,j$, and $u^{(j+1)}(x)<0$ eventually, then for every $\varepsilon_1\in(0,1)$, $u(x)/u'(x)\geq\varepsilon_1 x/j$ eventually.*

Lemma 2 ([20] (Lemma 1 and 2))**.** *Let $m_1,m_2\geq 0$; then*
$$(m_1+m_2)^{\beta}\leq\begin{cases}2^{\beta-1}\left(m_1^{\beta}+m_2^{\beta}\right) & \text{for } \beta\geq 1;\\ m_1^{\beta}+m_2^{\beta} & \text{for } \beta\leq 1.\end{cases}$$

Lemma 3 (([21] Lemma 2.2.3))**.** *Let $u\in C^j([x_0,\infty),(0,\infty))$ and $u^{(j-1)}(x)u^{(j)}(x)\leq 0$. If $\lim_{x\to\infty}u(x)\neq 0$, then*
$$u(x)\geq\frac{\mu}{(j-1)!}x^{j-1}\left|u^{(j-1)}(x)\right| \text{ for } x\geq x_{\mu},$$
for every $\mu\in(0,1)$.

These are some of the important hypotheses of Equation (1):

Hypothesis 1. *w is an eventually positive solution of Equation (1).*

Hypothesis 2. *The inequality*
$$\eta'(x)+\frac{1}{(p_1-1)}\left(\frac{\mu x^3}{6\xi^{1/(p_1-1)}(x)}\right)^{(p_2-1)}\left(\frac{z_0\delta_0}{\delta_0+r_0^{(p_2-1)}}\right)^{(p_2-1)/(p_1-1)}\widehat{q}(x)\eta^{(p_2-1)/(p_1-1)}\left(\delta^{-1}(z(x))\right)\leq 0,\tag{8}$$
is oscillatory where
$$\widehat{q}(x):=\min\left\{q\left(z^{-1}(x)\right),q\left(z^{-1}(\delta(x))\right)\right\}.$$

Hypothesis 3. *The inequality*
$$\vartheta'(x)+\frac{1}{(p_1-1)}\left(\frac{\mu x^3}{6\xi^{1/(p_1-1)}(x)}\right)^{(p_2-1)}\left(\frac{z_0\delta_0}{\delta_0+r_0^{(p_2-1)}}\right)\widehat{q}(x)\vartheta^{(p_2-1)/(p_1-1)}(z(x))\leq 0\tag{9}$$
is oscillatory.

It can be noted that the first order of equations (H2) and (H3) has been discussed previously by different authors—for example, [10,22,23], and they proved that these equations are oscillatory. In our current study, we use them as a point of comparison so that (H2) and (H3) can be assumed to be oscillatory.

Theorem 1. *Let us assume that*

$$\left(z^{-1}(x)\right)' \geq z_0 > 0 \ \ and \ \ \delta'(x) \geq \delta_0 > 0. \tag{10}$$

If $(H2)$ *holds, then Equation (1) is oscillatory.*

Proof. Let x be a non-oscillatory solution of Equation (1) on $[x_0, \infty)$; we get $x > 0$. Then, there exists a $x_1 \geq x_0$ such that $w(x) > 0$, $w(\delta(x)) > 0$ and $w(z(x)) > 0$ for $x \geq x_1$. Since $\xi'(x) > 0$, we have

$$y(x) > 0,\ y'(x) > 0,\ y'''(x) > 0,\ y^{(4)}(x) < 0 \text{ and } \left(\xi(x)(y'''(x))^{(p_1-1)}\right)' \leq 0, \tag{11}$$

for $x \geq x_1$. From Equation (1), we get

$$\frac{1}{(z^{-1}(x))'}\left(\xi\left(z^{-1}(x)\right)\left(y'''\left(z^{-1}(x)\right)\right)^{(p_1-1)}\right)' + q\left(z^{-1}(x)\right)w^{(p_2-1)}(x) = 0. \tag{12}$$

By using Lemma 2 and the definition of y in (2), we get

$$\begin{aligned} y^{(p_2-1)}(x) &= (w(x) + r(x)w(\delta(x)))^{(p_2-1)} \\ &\leq (p_1-1)\left(w^{(p_2-1)}(x) + r_0^{(p_2-1)}w^{(p_2-1)}(\delta(x))\right). \end{aligned} \tag{13}$$

From Equations (12) and (13), we obtain

$$\begin{aligned} 0 &= \frac{1}{(z^{-1}(x))'}\left(\xi\left(z^{-1}(x)\right)\left(y'''\left(z_j^{-1}(x)\right)\right)^{(p_1-1)}\right)' + q\left(z^{-1}(x)\right)w^{(p_2-1)}(x) \\ &\quad + r_0^{(p_2-1)}\left(\frac{1}{(z^{-1}(\delta(x)))'}\left(\xi\left(z^{-1}(\delta(x))\right)\left(y'''\left(z^{-1}(\delta(x))\right)\right)^{(p_1-1)}\right)' + q\left(z^{-1}(\delta(x))\right)w^{(p_2-1)}(\delta(x))\right) \\ &= \frac{\left(\xi(z^{-1}(x))\,(y'''(z^{-1}(x)))^{(p_1-1)}\right)'}{(z^{-1}(x))'} + r_0^{(p_2-1)}\frac{\left(\xi(z^{-1}(\delta(x)))\,(y'''(z^{-1}(\delta(x))))^{(p_1-1)}\right)'}{(z^{-1}(\delta(x)))'} \\ &\quad + q\left(z^{-1}(x)\right)w^{(p_2-1)}(x) + r_0^{(p_2-1)}q\left(z^{-1}(\delta(x))\right)w^{(p_2-1)}(\delta(x)) \\ &\geq \frac{\left(\xi(z^{-1}(x))\,(y'''(z^{-1}(x)))^{(p_1-1)}\right)'}{(z^{-1}(x))'} + r_0^{(p_2-1)}\frac{\left(\xi(z^{-1}(\delta(x)))\,(y'''(z^{-1}(\delta(x))))^{(p_1-1)}\right)'}{(z^{-1}(\delta(x)))'} \\ &\quad + \frac{1}{(p_1-1)}\widehat{q}(x)y^{(p_2-1)}(x), \end{aligned}$$

which with (10) gives

$$\begin{aligned} &\frac{1}{z_0}\left(\xi\left(z^{-1}(x)\right)\left(y'''\left(z_j^{-1}(x)\right)\right)^{(p_1-1)}\right)' \\ &+ \frac{r_0^{(p_2-1)}}{z_0\delta_0}\left(\xi\left(z^{-1}(\delta(x))\right)\left(y'''\left(z^{-1}(\delta(x))\right)\right)^{(p_1-1)}\right)' + \frac{1}{(p_1-1)}\widehat{q}(x)y^{(p_2-1)}(x) \leq 0. \end{aligned} \tag{14}$$

Since $y'(x) > 0$, we find $\lim_{x\to\infty} y(x) \neq 0$, and by Lemma 3 we obtain

$$y(x) \geq \frac{\mu}{6}x^3 y'''(x). \tag{15}$$

Combining (14) and (15), we obtain

$$\frac{1}{z_0}\left(\xi\left(z^{-1}(x)\right)\left(y'''\left(z_j^{-1}(x)\right)\right)^{(p_1-1)}\right)' + \frac{r_0^{(p_2-1)}}{z_0\delta_0}\left(\xi\left(z^{-1}(\delta(x))\right)\left(y'''\left(z^{-1}(\delta(x))\right)\right)^{(p_1-1)}\right)'$$
$$+\frac{1}{(p_1-1)}\widehat{q}(x)\left(\frac{\mu}{6}x^3\right)^{(p_2-1)}(y'''(x))^{(p_2-1)} \leq 0. \quad (16)$$

If we set

$$\eta(x) := \frac{1}{z_0}\xi\left(z^{-1}(x)\right)\left(y'''\left(z_j^{-1}(x)\right)\right)^{(p_1-1)} + \frac{r_0^{(p_2-1)}}{z_0\delta_0}\xi\left(z^{-1}(\delta(x))\right)\left(y'''\left(z^{-1}(\delta(x))\right)\right)^{(p_1-1)},$$

then it is easy to see that

$$\eta\left(\delta^{-1}(z(x))\right) \leq \left(\frac{\delta_0 + r_0^{(p_2-1)}}{z_0\delta_0}\right)\xi(x)(y'''(x))^{(p_1-1)}.$$

From (16), we find

$$\eta'(x) + \frac{1}{(p_1-1)}\left(\frac{\mu x^3}{6\xi^{1/(p_1-1)}(x)}\right)^{(p_2-1)}\left(\frac{z_0\delta_0}{\delta_0 + r_0^{(p_2-1)}}\right)^{(p_2-1)/(p_1-1)}\widehat{q}(x)\eta^{(p_2-1)/(p_1-1)}\left(\delta^{-1}(z(x))\right) \leq 0,$$

which is a contradiction. Therefore, this contradiction completes the proof. □

Theorem 2. *Let (10) and (H3) hold; then (1) is oscillatory.*

Proof. It is known that (16) holds in the proof of Theorem 1. If we set $\vartheta(x) := \xi(z^{-1}(x))$ $(y'''(z^{-1}(x)))^{(p_1-1)}$, then ϑ is a positive solution of (9), which is a contradiction. This completes the proof. □

Corollary 1. *Let $p_1 = p_2$ and (10) hold. If $\xi(x) \leq x$ and*

$$\liminf_{x\to\infty}\int_{\xi(x)}^{x}\frac{s^{3(p_1-1)}}{\xi(s)}\widehat{q}(s)\mathrm{d}s > \left(\frac{\delta_0 + r_0^{(p_1-1)}}{z_0\delta_0}\right)\frac{(p_1-1)6^{(p_1-1)}}{\mathrm{e}}, \quad (17)$$

where $\xi(x) = \delta^{-1}(z(x))$ or $z(x)$, then Equation (1) is oscillatory.

Theorem 3. *Let $r_0 < 1$ and $z(x) \leq x$. For some $\mu \in (0,1)$ if*

$$\psi'(x) + (1-r_0)^{(p_2-1)}\left(\frac{\mu z^3(x)}{6\xi^{1/(p_1-1)}(z(x))}\right)^{(p_2-1)}q(x)\psi^{(p_2-1)/(p_1-1)}(z(x)) = 0 \quad (18)$$

is oscillatory, then Equation (1) is oscillatory.

Proof. It is known that (11) holds in the proof of Theorem 1. By the definition of y in (2), we find

$$\begin{aligned} w(x) &\geq y(x) - r_0 w(\delta(x)) \geq y(x) - r_0 y(\delta(x)) \\ &\geq (1-r_0)y(x), \end{aligned}$$

which with Equation (1) gives

$$\left(\xi(x)(y'''(x))^{(p_1-1)}\right)' + q(x)(1-r_0)^{(p_2-1)}y^{(p_2-1)}(z(x)) \leq 0. \quad (19)$$

From Lemma 3, we obtain

$$y(x) \geq \frac{\mu}{6} x^3 y'''(x). \tag{20}$$

Combining (19) and (20), we get

$$\left(\xi(x)\left(y'''(x)\right)^{(p_1-1)}\right)' + q(x)(1-r_0)^{(p_2-1)}\left(\frac{\mu}{6} z^3(x)\right)^{(p_2-1)}\left(y'''(z(x))\right)^{(p_2-1)} \leq 0.$$

If we set $\psi := \xi(y''')^{(p_1-1)}$, then the inequality

$$\psi'(x) + (1-r_0)^{(p_2-1)}\left(\frac{\mu z^3(x)}{6\xi^{1/(p_1-1)}(z(x))}\right)^{(p_2-1)} q(x)\psi^{(p_2-1)/(p_1-1)}(z(x)) \leq 0.$$

In view of ([10] Corollary 1), Equation (18) has a positive solution, which is a contradiction. This completes the proof. □

Corollary 2. *Let $p_1 = p_2$, $r_0 < 1$ and $z(x) \leq x$. If*

$$\liminf_{x\to\infty} \int_{z_j(x)}^{x} \frac{z^{3(p_1-1)}(s)}{\xi(z(s))} q(s) \mathrm{d}s > \frac{6^{(p_1-1)}}{(1-r_0)^{(p_1-1)}\mathrm{e}}, \tag{21}$$

then Equation (1) is oscillatory.

Lemma 4. *If $(H1)$ holds, then*

$$\begin{aligned} \phi_1'(x) \leq\ & \frac{\omega_1'(x)}{\omega_1(x)}\phi_1(x) - \omega_1(x)q(x)(1-r_0)^{(p_2-1)} y^{p_2-p_2}(x)\varepsilon_1\left(\frac{z_j(x)}{x}\right)^{3(p_2-1)} \\ & -(p_1-1)\mu_1 \frac{x^2}{2\xi^{1/(p_1-1)}(x)\omega_1^{1/(p_1-1)}(x)} \phi_1^{\frac{p_1}{(p_1-1)}}(x), \end{aligned} \tag{22}$$

for some $\mu_1, \varepsilon_1 \in (0,1)$ and every $M_1 > 0$, where

$$\Psi(x) := M_1^{p_2-p_1}\omega_1(x)q(x)(1-r_0)^{(p_2-1)}\left(\frac{z(x)}{x}\right)^{3(p_2-1)}.$$

Proof. Let $(H1)$ hold. In the case where $y''(x) > 0$, let

$$\phi_1(x) := \omega_1(x)\frac{\xi(x)(y'''(x))^{(p_1-1)}}{y^{(p_1-1)}(x)} > 0.$$

From (19), we find

$$\begin{aligned} \phi_1'(x) \leq\ & \frac{\omega_1'(x)}{\omega_1(x)}\phi_1(x) - \omega_1(x)q(x)(1-r_0)^{(p_2-1)}\frac{y^{(p_2-1)}(z(x))}{y^{(p_1-1)}(x)} \\ & -(p_1-1)\omega_1(x)\frac{\xi(x)(y'''(x))^{(p_1-1)}}{y^{p_1}(x)} y'(x). \end{aligned} \tag{23}$$

Using Lemma 1, we obtain $y(x) \geq \frac{x}{3} y'(x)$, and hence,

$$\frac{y(z_j(x))}{y(x)} \geq \varepsilon_1 \frac{z^3(x)}{x^3}. \tag{24}$$

Using Lemma 3, we get
$$y'(x) \geq \frac{\mu_1}{2} x^2 y'''(x), \tag{25}$$
for all $\mu_1 \in (0,1)$. Thus, by (23)–(25), we obtain

$$\begin{aligned} \phi_1'(x) \quad &\leq \quad \frac{\varpi_1'(x)}{\varpi_1(x)}\phi_1(x) - \varpi_1(x)q(x)(1-r_0)^{(p_2-1)}y^{p_2-p_2}(x)\varepsilon_1\left(\frac{z_j(x)}{x}\right)^{3(p_2-1)} \\ & -(p_1-1)\mu_1 \frac{x^2}{2\xi^{1/(p_1-1)}(x)\varpi_1^{1/(p_1-1)}(x)}\phi_1^{\frac{p_1}{(p_1-1)}}(x). \end{aligned}$$

This completes the proof. □

Lemma 5. *If* $(H1)$ *holds, then*

$$\phi_2'(x) \leq -\Psi_1(x) + \frac{\varpi'(x)}{\varpi(x)}\vartheta(x) - \frac{1}{\varpi(x)}\phi_2^2(x), \tag{26}$$

for some $\varepsilon_1 \in (0,1)$ *and every* $M_2 > 0$*, where*

$$\Psi_1(x) := ((1-r_0)\varepsilon_1)^{(p_2-1)/(p_1-1)}\varpi(x)M_2^{(p_2-p_1)/(p_1-1)}\int_x^\infty \left(\frac{1}{\xi(t)}\int_t^\infty q(s)\frac{z^{(p_2-1)}(s)}{s^{(p_2-1)}}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}t.$$

Proof. Let $(H1)$ hold.In the case where $y''(x) < 0$, by integrating (19) from x to t, we find

$$\xi(t)\left(y'''(t)\right)^{(p_1-1)} - \xi(x)\left(y'''(x)\right)^{(p_1-1)} \leq -\int_x^t q(s)(1-r_0)^{(p_2-1)}y^{(p_2-1)}(z(s))\mathrm{d}s. \tag{27}$$

By Lemma 1, we get $y(x) \geq xy'(x)$, and hence,

$$y(z(x)) \geq \varepsilon_1 \frac{z(x)}{x} y(x). \tag{28}$$

For (27), letting $t \to \infty$ and using (28), we get

$$\xi(x)\left(y'''(x)\right)^{(p_1-1)} \geq ((1-r_0)\varepsilon_1)^{(p_2-1)}y^{(p_2-1)}(x)\int_x^\infty q(s)\frac{z_j^{(p_2-1)}(s)}{s^{(p_2-1)}}\mathrm{d}s. \tag{29}$$

By integrating (29) from x to ∞, we get

$$y''(x) \leq -((1-r_0)\varepsilon_1)^{(p_2-1)/(p_1-1)}y^{(p_2-1)/(p_1-1)}(x)\int_x^\infty \left(\frac{1}{\xi(t)}\int_t^\infty q(s)\frac{z^{(p_2-1)}(s)}{s^{(p_2-1)}}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}t, \tag{30}$$

for all $\varepsilon_1 \in (0,1)$. Now, we define

$$\phi_2(x) = \varpi(x)\frac{y'(x)}{y(x)}.$$

Then $\phi_2(x) > 0$ for $x \geq x_1$. By using (33) and (30), we obtain

$$\begin{aligned}\phi_2'(x) &= \frac{\varpi'(x)}{\varpi(x)}\phi_2(x)+\varpi(x)\frac{y''(x)}{y(x)}-\varpi(x)\left(\frac{y'(x)}{y(x)}\right)^2\\ &\le \frac{\varpi'(x)}{\varpi(x)}\phi_2(x)-\frac{1}{\varpi(x)}\phi_2^2(x)\\ &\quad -((1-r_0)\varepsilon_1)^{(p_2-1)/(p_1-1)}\varpi(x)y^{(p_2-1)/(p_1-2)}(x)\int_x^\infty\left(\frac{1}{\xi(t)}\int_t^\infty q(s)\frac{z^{(p_2-1)}(s)}{s^{(p_2-1)}}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}t.\end{aligned}$$

Thus, we find

$$\phi_2'(x)\le-\Psi_1(x)+\frac{\varpi'(x)}{\varpi(x)}\vartheta(x)-\frac{1}{\varpi(x)}\phi_2^2(x).$$

This completes the proof. □

Theorem 4. *Assume that $r_0 < 1$ and $z(x)\le x$. If there exists positive functions ϖ_1, $\varpi\in C^1([x_0,\infty))$ such that*

$$\int_{x_0}^\infty\left(\Psi(s)-\frac{2^{(p_1-1)}}{p_1^{p_1}}\frac{\xi(s)\left(\varpi_1'(s)\right)^{p_1}}{\mu_1^{(p_1-1)}s^{2(p_1-1)}\varpi_1^{(p_1-1)}(s)}\right)\mathrm{d}s=\infty, \tag{31}$$

and

$$\int_{x_0}^\infty\left(\Psi_1(s)-\frac{(\varpi'(s))^2}{4\varpi(s)}\right)\mathrm{d}s=\infty, \tag{32}$$

then (1) is oscillatory.

Proof. It is known that (11) and (19) hold in the proof of Theorem 3. From (11), we have that y'' is of one sign. From Lemma 4, we find that (22) holds.
Since $y'(x)>0$, there exists a $x_2\ge x_1$ and a constant $M>0$ such that

$$y(x)>M, \tag{33}$$

for all $x\ge x_2$. From the inequality

$$Ew-Fw^{(\alpha+1)/\alpha}\le\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}E^{\alpha+1}F^{-\alpha},\ \ F>0,$$

with $E=\varpi_1'(x)/\varpi_1(x)$, $F=(p_1-1)\mu x^2/2\xi^{1/(p_1-1)}(x)\varpi_1^{1/(p_1-1)}(x)$ and $w=\phi_1$, we find

$$\phi_1'(x)\le-\Psi(x)+\frac{2^{(p_1-1)}}{p_1^{p_1}}\frac{\xi(x)\left(\varpi_1'(x)\right)^{p_1}}{\mu_1^{(p_1-1)}x^{2(p_1-1)}\varpi_1^{(p_1-1)}(x)}.$$

This implies that

$$\int_{x_1}^x\left(\Psi(s)-\frac{2^{(p_1-1)}}{p_1^{p_1}}\frac{\xi(s)\left(\varpi_1'(s)\right)^{p_1}}{\mu_1^{(p_1-1)}s^{2(p_1-1)}\varpi_1^{(p_1-1)}(s)}\right)\mathrm{d}s\le\phi_1(x_1),$$

which contradicts (31). From Lemma 5, we find that (26) holds. This implies that

$$\phi_2'(x)\le-\Psi_1(x)+\frac{(\varpi'(x))^2}{4\varpi(x)}.$$

Then, we obtain

$$\int_{x_1}^{x}\left(\Psi_1(s)-\frac{(\varpi'(x))^2}{4\varpi(x)}\right)\mathrm{d}s \le \phi_2(x_1),$$

which contradicts (32). This completes the proof. □

3. Applications

This section presents some interesting examples and applications which are addressed based on above hypotheses to show some interesting results in this paper. Example 1 is presented to show how to investigate the problem of practical interest to the specific conditions $(0 \le r(x) < r_0 < 1)$, whereas Example 2 extends the study to be under this condition $(0 \le r(x) < r_0 < \infty)$.

Example 1. *Consider the following equation:*

$$(w(x)+(7/8)w(x/e))^{(4)}+q_0u^{-4}w\left(x/e^2\right)=0,\ u \ge 1, \tag{34}$$

where $q_0 > 0$ *is a constant, and*

$$p_1=2,\ \xi(x)=1,\ r(x)=7/8,\ \delta(x)=u/e,\ q(t)=q_0u^{-4},\ z(x)=x/e^2.$$

By applying the conditions (4)–(7) to the above Equation (34), we obtain the desired results in the following Table 1

Table 1. The values of q_0 for different conditions.

The condition	(4)	(5)	(6)	(7)
The criterion	$q_0 > 113,981.3$	$q_0 > 3561.9$	$q_0 > 3008.5$	$q_0 > 587.93$

Observe that, as shown in Table 1 the value of q_0 *for the condition (7) is smaller than other values of* q_0 *for other conditions. Hence, the condition (7) provides a better result than the results obtained by conditions (4)–(6) in [16–18]. However, these conditions for oscillation cannot be applied to the following example where we investigate the problem of practical interest under this condition* $(0 \le r(x) < r_0 < \infty)$.

Example 2. *Consider the differential equation*

$$\left(\left((w+r_0w(\varpi x))'''\right)^{(p_1-1)}\right)'+\frac{q_0}{x^{3p_1-2}}w(\lambda x)=0,\ x \ge 1, \tag{35}$$

where $\varpi,\lambda \in (0,1]$ *and* $r_0,q_0 > 0$. *Let* $\xi(x)=1$, $r(x)=r_0$, $\delta(x)=\varpi x$, $z(x)=\lambda x$ *and* $q(x)=q_0/x^{3p_1-2}$. *Hence, it is easy to see that*

$$\widehat{q}(x)=q_0\lambda^{3p_1-2}\frac{1}{x^{3p_1-2}}.$$

Using Corollary 1, Equation (35) is oscillatory if

$$q_0\ln\frac{1}{\lambda} > (p_1-1)\left(\frac{\varpi+r_0^{(p_1-1)}}{\varpi}\right)\frac{6^{(p_1-1)}}{\lambda^{3(p_1-1)}\mathrm{e}}. \tag{36}$$

From Corollary 2, if

$$q_0\ln\frac{1}{\lambda} > \frac{1}{(1-r_0)^{(p_1-1)}}\frac{6^{(p_1-1)}}{\lambda^{3(p_1-1)}\mathrm{e}}, \tag{37}$$

then (35) is oscillatory.

Finally, if we set $\varpi_1(s) := x^{3(p_1-1)}$ *and* $\varpi(x) := x^2$*, then we have*

$$\Psi(x) = q_0(1-r_0)^{(p_1-1)}\lambda^{3(p_1-1)}\frac{1}{s},$$

and

$$\Psi_1(x) := \frac{1}{2}\left(\frac{q_0}{3(p_1-1)}\right)^{1/(p_1-1)}(1-r_0)\lambda.$$

Using Theorem 4, Equation (35) is oscillatory if

$$q_0(1-r_0)^{(p_1-1)}\lambda^{3(p_1-1)} > 2^{(p_1-1)}3^{p_1}\left(\frac{(p_1-1)}{p_1}\right)^{p_1}, \tag{38}$$

and

$$q_0 > \left(\frac{2}{(1-r_0)\lambda}\right)^{(p_1-1)} 3(p_1-1). \tag{39}$$

4. Conclusions

This paper explored the oscillatory behavior of the fourth-order neutral differential equation. In particular, we addressed and investigated the oscillation criteria of Equation (1). We obtained different forms of conditions to expand the application area by using different methods. These new conditions complement several results in the literature. Additionally, some interesting examples were presented to investigate the proposed criteria. Further research based on the results of this article could extend the analysis herein to investigate a philos type oscillation criteria to ensure that every solution of the desired Equation (1) is oscillatory . Moreover, some oscillation criteria for (1) if $\int_{x_0}^{\infty} \frac{1}{\varsigma^{1/(p_1-1)}(s)} ds < \infty$, can be investigated.

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