symmetry

Article

Picture Fuzzy Geometric Aggregation Operators Based on a Trapezoidal Fuzzy Number and Its Application

Minxia Luo * and Huifeng Long

Department of Information and Computing Science, China Jiliang University, Hangzhou 310018, China; S1808070109@cjlu.edu.cn
* Correspondence: mxluo@cjlu.edu.cn; Tel.: +86-571-8687-5634

Abstract: The picture fuzzy set is a generation of an intuitionistic fuzzy set. The aggregation operators are important tools in the process of information aggregation. Some aggregation operators for picture fuzzy sets have been proposed in previous papers, but some of them are defective for picture fuzzy multi-attribute decision making. In this paper, we introduce a transformation method for a picture fuzzy number and trapezoidal fuzzy number. Based on this method, we proposed a picture fuzzy multiplication operation and a picture fuzzy power operation. Moreover, we develop the picture fuzzy weighted geometric ($PFWG$) aggregation operator, the picture fuzzy ordered weighted geometric ($PFOWG$) aggregation operator and the picture fuzzy hybrid geometric ($PFHG$) aggregation operator. The related properties are also studied. Finally, we apply the proposed aggregation operators to multi-attribute decision making and pattern recognition.

Keywords: picture fuzzy sets; aggregation operators; multi-attribute decision making; pattern recognition

check for updates

Citation: Luo, M.; Long, H. Picture Fuzzy Geometric Aggregation Operators Based on Trapezoidal Fuzzy Number and Its Application. *Symmetry* **2021**, *13*, 119. https://doi.org/10.3390/sym13010119

Received: 23 December 2020
Accepted: 9 January 2021
Published: 12 January 2021

1. Introduction

Multi-attribute decision making and pattern recognition problems are widely used in economy, politics, culture and other fields. Because of the ambiguity and complexity of information, it is very difficult for people to evaluate attributes with real numbers. Fuzzy set theory can be used to deal with fuzzy information effectively, which has attracted the attention of many scholars in fuzzy set theory, and it is applied in many fields. In the process of solving multi-attribute decision and pattern recognition problems, information aggregation is a common activity. Among which, weighted aggregation operators, ordered weighted aggregation operators and hybrid aggregation operators are three common aggregation operators.

1.1. Introduction for Picture Fuzzy Sets

Zadeh [1] proposed fuzzy set, which can deal with fuzziness information by membership degree. Fuzzy sets are limited in dealing with fuzziness and uncertainty of information. Atanassov [2,3] proposed intuitionistic fuzzy set, as directly extension of fuzzy set, which presents information by membership degree, non-membership degree, and the sum of two membership degrees less than or equal to one. Intuitionistic fuzzy sets can better describe fuzziness and uncertainty of information. Many scholars have done a lot of research for deal with uncertain of information under the intuitionistic fuzzy environment, such as: distance measures [4,5], similarity measures [6–8], aggregation operators [9–11], accuracy function and score function [12]. However, intuitionistic fuzzy sets are limited in dealing with inconsistency, uncertainty and fuzziness of information. For example, voting questions. The results of the vote were divided into yes, abstention, against and refusal. For solving these types of problem, Cuong and Kreinovich [13] proposed picture fuzzy set, as directly extension of intuitionistic fuzzy set, which presents information by positive membership degree, neutral membership degree, negative membership degree, and the sum of three membership degrees less than or equal to one. The picture fuzzy sets can dealing with inconsistency, uncertainty and fuzziness of information. Many scholars have also done a lot of research on the theory and application under the picture fuzzy environment. Son [14] proposed generalized picture fuzzy distance measure and applied to clustering problem of picture fuzzy sets. Dinh [15] proposed dissimilarity measure and distance measure of picture fuzzy sets. Guleria and Bajaj [16] proposed picture fuzzy probabilistic distance measure and applied to classification problems. Wei [17] proposed some similarity measures of picture fuzzy sets. Wei [18,19] proposed cross-entropy and 2-tuple linguistic aggregation operators for picture fuzzy and applied them to multi-attribute decision making. Joshi [20] propose information measure of picture fuzzy sets and applied them to decision making.

1.2. The Development State for Picture Fuzzy Aggregation Operators

In the process of information aggregation, the aggregation operators are important tools. Based on the family of t-norm and t-conorm, many aggregation operators of picture fuzzy sets have been proposed. Jana [21] proposed picture fuzzy Dombi aggregation operators. Wei [22] proposed picture fuzzy Hamacher aggregation operators and applied them to multi-attribute decision making. Jana [23] applied picture fuzzy Hamacher aggregation operators to assess the best enterprise. Garg [24] proposed some picture fuzzy aggregation operators based on a decreasing function generates the t-norm and t-conorm. Ashraf [25] proposed picture fuzzy aggregation operators by using algebraic and Einstein t-norm and t-conorm, applied them to multi-attribute decision making. Wang et al. [26] proposed some picture fuzzy geometric aggregation operators based on a view point of probability and applied them to multi-attribute decision making. Ju [27] proposed a picture fuzzy weighted interaction geometric operator and applied it to the selection of addresses. Tian et al. [28] proposed weighted geometric aggregation operators based on the Shapley fuzzy measure, fuzzy measure and power aggregation operator. Wang et al. [29] proposed Muirhead mean operators of picture fuzzy sets and applied them to assessment of financial investment risk. Xu et al. [30] propose a family of picture fuzzy Muirhead mean operators and applied them to multi-attribute decision making. Although picture fuzzy aggregation operators have been widely use in many ares, there are some drawbacks. For example, the positive membership degree of aggregate result is zero, though $n-1$ positive membership degree are not equal to zero [22,25,31] (see Example 1). The neutral membership degree of aggregate result is zero, though $n-1$ neutral membership degree are not equal to zero [26] (see Example 1). As a result, the best alternative cannot be chosen in the multi-attribute decision making problem (see Example 2).

1.3. Main Contributions for This Paper

In order to overcome these defects, first, we construct the transformation method of picture fuzzy number and trapezoidal fuzzy number. Second, we define picture fuzzy multiplication operation and power operation. Third, we propose picture fuzzy weighted geometric aggregation operator, picture fuzzy ordered weighted geometric aggregation operator and picture fuzzy hybrid geometric aggregation operator and apply them to multi-attribute decision making problem. The research steps of this paper are shown in Figure 1.

The rest are organized as follows. In Section 2, we review basic concepts and properties for picture fuzzy sets. In Section 3, we present a transformation method for picture fuzzy number and trapezoid fuzzy number, based on this method, we proposed a multiplication operation and a power operation for picture fuzzy sets. In Section 4, we present three novel picture fuzzy aggregation operators based on the new multiplication operation and power operation. In Section 5, we discuss the efficiency and reasonable of the proposed aggregation operators by numerical cases, and applied the aggregation operators to multi-attribute decision making. In Section 6, we applied the aggregation operators to pattern recognition. The final section is conclusion.

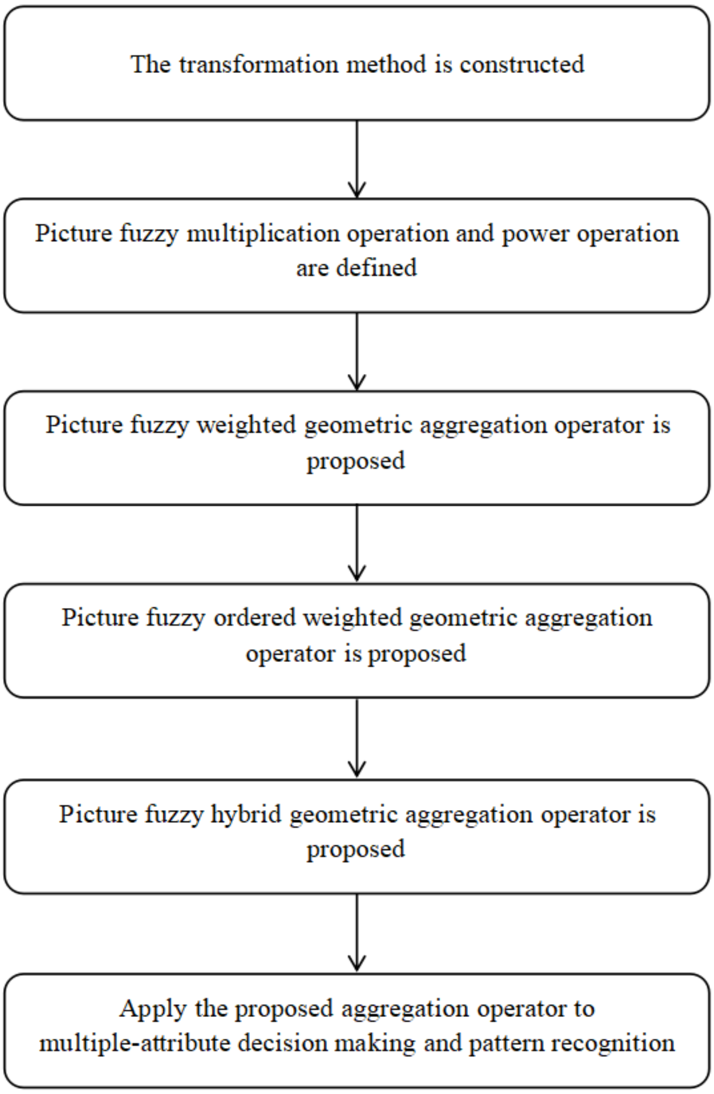

Figure 1. Research steps of this paper.

2. Basic Concepts and Properties for Picture Fuzzy Sets

In this section, we review some basic concepts and properties for picture fuzzy sets, which will be used in this paper.

Definition 1 ([2])**.** *An intuitionistic fuzzy set (IFS) A on universe X is an object of the form:*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called "degree of membership" and $\nu_A(x) \in [0,1]$ is called "degree of non-membership", and where $\mu_A(x)$, $\nu_A(x)$ satisfies $\mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$.

Definition 2 ([13])**.** *A picture fuzzy set (PFS) A on universe X is an object of the form:*

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called "degree of positive membership", $\eta_A(x) \in [0,1]$ is called "degree of neutral membership" and $\nu_A(x) \in [0,1]$ is called "degree of negative membership", and where $\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \forall x \in X$. For all $x \in X$, $\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is called "degree of refusal membership".

We denote the family of all the picture fuzzy subsets on universe X by $PFS(X)$.

The set of all picture fuzzy numbers (PFNs) is denoted by $D^* = \{\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle | \alpha \in [0,1]^3, \mu_\alpha + \eta_\alpha + \nu_\alpha \leq 1\}$. For $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle, \beta = \langle \mu_\beta, \eta_\beta, \nu_\beta \rangle \in D^*$, the order relation $\leq_{D^*}$ is defined as $\alpha \leq_{D^*} \beta$ if and only if $\mu_\alpha \leq \mu_\beta$, and $\mu_\alpha + \eta_\alpha \leq \mu_\beta + \eta_\beta$, and $\nu_\alpha \geq \nu_\beta$. $\alpha \vee \beta$ and $\alpha \wedge \beta$ are defined as follows [32]:

$$\begin{aligned}\alpha \vee \beta =& (\mu_\alpha \vee \mu_\beta, (\mu_\alpha + \eta_\alpha) \vee (\mu_\beta + \eta_\beta) - (\mu_\alpha \vee \mu_\beta), \nu_\alpha \wedge \nu_\beta),\\ \alpha \wedge \beta =& (\mu_\alpha \wedge \mu_\beta, (\mu_\alpha + \eta_\alpha) \wedge (\mu_\beta + \eta_\beta) - (\mu_\alpha \wedge \mu_\beta), \nu_\alpha \vee \nu_\beta).\end{aligned}$$

Then $(D^*, \leq_{D^*})$ is a complete lattice. The top element and bottom element are $1_{D^*} = (1,0,0)$ and $0_{D^*} = (0,0,1)$ on D^*, respectively.

Definition 3 ([33]). *Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number, show in Figure 2. Its membership function $\mu_{\tilde{a}}(x)$ is defined as follows:*

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & x < a_1, \\ \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2, \\ 1 & a_2 \leq x \leq a_3, \\ \frac{x-a_4}{a_3-a_4} & a_3 \leq x \leq a_4, \\ 0 & x > a_4. \end{cases}$$

If $a_1 = a_2$, then $\tilde{a}$ is a right-angled trapezoid fuzzy number (show in Figure 3). The set of all the tight angled trapezoid fuzzy number is denoted by C^*, i.e., $C^* = \{\tilde{a} = (a_2, a_2, a_3, a_4)\}$.

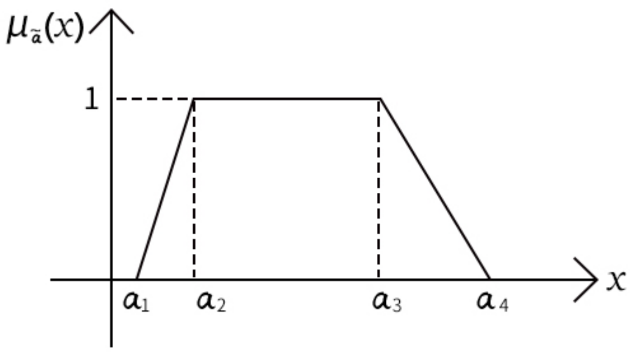

Figure 2. A trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$.

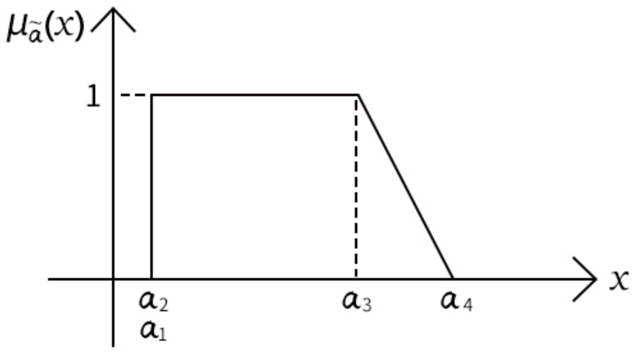

Figure 3. A right-angled trapezoidal fuzzy number $\tilde{a} = (a_2, a_2, a_3, a_4)$.

Definition 4 ([34]). *Let $\tilde{a} = (a_1, a_2, a_3, a_4), \tilde{b} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers, the multiplication $\odot$ is defined as $\tilde{a} \odot \tilde{b} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$.*

Proposition 1. *C^* with respect to $\odot$ is a Abel semigroup.*

Definition 5. *A mapping $\mathcal{A} : \cup_{n \in N^*} (D^*)^n \to D^*$ is called an aggregation operator on D^*, if it satisfies the some properties:*
(P1) $\mathcal{A}(\alpha) = \alpha(\forall \alpha \in D^*)$;
(P2) Let $\alpha_j, \beta_j \in D^*(j = 1, 2 \ldots, n)$, if $\alpha_j \leq \beta_j (j = 1, 2, \ldots, n)$, then $\mathcal{A}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathcal{A}(\beta_1, \beta_2, \ldots, \beta_n)$;
(P3) $\mathcal{A}(\underbrace{0_{D^*}, 0_{D^*}, \ldots, 0_{D^*}}_{n\ tims}) = 0_{D^*}$;
(P4) $\mathcal{A}(\underbrace{1_{D^*}, 1_{D^*}, \ldots, 1_{D^*}}_{n\ tims}) = 1_{D^*}$.

Remark 1. *If the PFS reduces to the IFS, the Definition 5 is Definition 4.2 in [35].*

Remark 2. *If the PFS reduces to the FS, the Definition 5 is Definition 2 in [36].*

Definition 6 ([31,37]). *Let $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle \in D^*$, the score function S and accuracy function H of the picture fuzzy number are defined as follows:*
$S(\alpha) = \mu_\alpha - \nu_\alpha, S(\alpha) \in [-1, 1]$.
$H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha, H(\alpha) \in [0, 1]$.

Definition 7 ([31]). *Let $\alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$ and $\beta = \langle \mu_\beta, \eta_\beta, \nu_\beta \rangle \in D^*$, if $S(\alpha) > S(\beta)$, then $\alpha > \beta$; if $S(\alpha) = S(\beta)$, then*
(1) If $H(\alpha) > H(\beta)$, then $\alpha > \beta$;
(2) If $H(\alpha) = H(\beta)$, then $\alpha = \beta$.

Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle (j = 1, 2, \ldots, n) \in D^*$. Some existing picture fuzzy weighted geometric ($PFWG$) aggregation operators, picture fuzzy ordered weighted geometric ($PFOWG$) aggregation operators and picture fuzzy hybrid geometric ($PFHG$) aggregation operators are shown in Table 1.

Table 1. Some existing aggregation operators.

References	Aggregation Operators
Jana et al. [21]	$PFWG_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \frac{1}{1+\{\sum_{j=1}^n \omega_j (\frac{1-\mu_{\alpha_j}}{\mu_{\alpha_j}})^\xi\}^{\frac{1}{\xi}}}, 1 - \frac{1}{1+\{\sum_{j=1}^n \omega_j (\frac{\eta_{\alpha_j}}{1-\eta_{\alpha_j}})^\xi\}^{\frac{1}{\xi}}}, 1 - \frac{1}{1+\{\sum_{j=1}^n \omega_j (\frac{\nu_{\alpha_j}}{1-\nu_{\alpha_j}})^\xi\}^{\frac{1}{\xi}}} \rangle.$
	$PFOWG_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \frac{1}{1+\{\sum_{j=1}^n w_j (\frac{1-\mu_{\alpha_{\sigma(j)}}}{\mu_{\alpha_{\sigma(j)}}})^\xi\}^{\frac{1}{\xi}}}, 1 - \frac{1}{1+\{\sum_{j=1}^n w_j (\frac{\eta_{\alpha_{\sigma(j)}}}{1-\eta_{\alpha_{\sigma(j)}}})^\xi\}^{\frac{1}{\xi}}}, 1 - \frac{1}{1+\{\sum_{j=1}^n w_j (\frac{\nu_{\alpha_{\sigma(j)}}}{1-\nu_{\alpha_{\sigma(j)}}})^\xi\}^{\frac{1}{\xi}}} \rangle.$
	$PFHG_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \frac{1}{1+\{\sum_{j=1}^n W_j (\frac{1-\mu_{\tilde{\alpha}_{\sigma(j)}}}{\mu_{\tilde{\alpha}_{\sigma(j)}}})^\xi\}^{\frac{1}{\xi}}}, 1 - \frac{1}{1+\{\sum_{j=1}^n W_j (\frac{\eta_{\tilde{\alpha}_{\sigma(j)}}}{1-\eta_{\tilde{\alpha}_{\sigma(j)}}})^\xi\}^{\frac{1}{\xi}}}, 1 - \frac{1}{1+\{\sum_{j=1}^n W_j (\frac{\nu_{\tilde{\alpha}_{\sigma(j)}}}{1-\nu_{\tilde{\alpha}_{\sigma(j)}}})^\xi\}^{\frac{1}{\xi}}} \rangle.$
Wei [22]	$PFWG_2(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \frac{\gamma \prod_{j=1}^n (\mu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n [1+(\gamma-1)(1-\mu_{\alpha_j})]^{\omega_j} + (\gamma-1) \prod_{j=1}^n (\mu_{\alpha_j})^{\omega_j}}, \frac{\prod_{j=1}^n [1+(\gamma-1)\eta_{\alpha_j}]^{\omega_j} - \prod_{j=1}^n (1-\eta_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n [1+(\gamma-1)\eta_{\alpha_j}]^{\omega_j} + (\gamma-1) \prod_{j=1}^n (1-\eta_{\alpha_j})^{\omega_j}}, \frac{\prod_{j=1}^n [1+(\gamma-1)\nu_{\alpha_j}]^{\omega_j} - \prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n [1+(\gamma-1)\nu_{\alpha_j}]^{\omega_j} + (\gamma-1) \prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}} \rangle.$
	$PFOWG_2(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \frac{\gamma \prod_{j=1}^n (\mu_{\alpha_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n [1+(\gamma-1)(1-\mu_{\alpha_{\sigma(j)}})]^{w_j} + (\gamma-1) \prod_{j=1}^n (\mu_{\alpha_{\sigma(j)}})^{w_j}}, \frac{\prod_{j=1}^n [1+(\gamma-1)\eta_{\alpha_{\sigma(j)}}]^{w_j} - \prod_{j=1}^n (1-\eta_{\alpha_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n [1+(\gamma-1)\eta_{\alpha_{\sigma(j)}}]^{w_j} + (\gamma-1) \prod_{j=1}^n (1-\eta_{\alpha_{\sigma(j)}})^{w_j}}, \frac{\prod_{j=1}^n [1+(\gamma-1)\nu_{\alpha_{\sigma(j)}}]^{w_j} - \prod_{j=1}^n (1-\nu_{\alpha_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n [1+(\gamma-1)\nu_{\alpha_{\sigma(j)}}]^{w_j} + (\gamma-1) \prod_{j=1}^n (1-\nu_{\alpha_{\sigma(j)}})^{w_j}} \rangle.$
	$PFHG_2(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle \frac{\gamma \prod_{j=1}^n (\mu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}}{\prod_{j=1}^n [1+(\gamma-1)(1-\mu_{\tilde{\alpha}_{\sigma(j)}})]^{W_j} + (\gamma-1) \prod_{j=1}^n (\mu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}}, \frac{\prod_{j=1}^n [1+(\gamma-1)\eta_{\tilde{\alpha}_{\sigma(j)}}]^{W_j} - \prod_{j=1}^n (1-\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j}}{\prod_{j=1}^n [1+(\gamma-1)\eta_{\tilde{\alpha}_{\sigma(j)}}]^{W_j} + (\gamma-1) \prod_{j=1}^n (1-\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j}}, \frac{\prod_{j=1}^n [1+(\gamma-1)\nu_{\tilde{\alpha}_{\sigma(j)}}]^{W_j} - \prod_{j=1}^n (1-\nu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}}{\prod_{j=1}^n [1+(\gamma-1)\nu_{\tilde{\alpha}_{\sigma(j)}}]^{W_j} + (\gamma-1) \prod_{j=1}^n (1-\nu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}} \rangle.$

Table 1. *Cont.*

References	Aggregation Operators
Ashraf et al. [25]	$PFWG_3(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\alpha_j})^{\omega_j},\prod_{j=1}^{n}(\eta_{\alpha_j})^{\omega_j},1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}\rangle$. $PFOWG_3(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\alpha_{\sigma(j)}})^{w_j},\prod_{j=1}^{n}(\eta_{\alpha_{\sigma(j)}})^{w_j},1-\prod_{j=1}^{n}(1-\nu_{\alpha_{\sigma(j)}})^{w_j}\rangle$. $PFHG_3(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\tilde{\alpha}_{\sigma(j)}})^{W_j},\prod_{j=1}^{n}(\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j},1-\prod_{j=1}^{n}(1-\nu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}\rangle$.
Wang et al. [26]	$PFWG_4(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\alpha_j}+\eta_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(\eta_{\alpha_j})^{\omega_j},\prod_{j=1}^{n}(\eta_{\alpha_j})^{\omega_j},1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}\rangle$. $PFOWG_4(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\alpha_{\sigma(j)}}+\eta_{\alpha_{\sigma(j)}})^{w_j}-\prod_{j=1}^{n}(\eta_{\alpha_{\sigma(j)}})^{w_j},\prod_{j=1}^{n}(\eta_{\alpha_{\sigma(j)}})^{w_j},1-\prod_{j=1}^{n}(1-\nu_{\alpha_{\sigma(j)}})^{w_j}\rangle$. $PFHG_4(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\tilde{\alpha}_{\sigma(j)}}+\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j}-\prod_{j=1}^{n}(\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j},\prod_{j=1}^{n}(\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j},1-\prod_{j=1}^{n}(1-\nu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}\rangle$.
Wei [31]	$PFWG_5(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\alpha_j})^{\omega_j},1-\prod_{j=1}^{n}(1-\eta_{\alpha_j})^{\omega_j},1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}\rangle$. $PFOWG_5(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\alpha_{\sigma(j)}})^{w_j},1-\prod_{j=1}^{n}(1-\eta_{\alpha_{\sigma(j)}})^{w_j},1-\prod_{j=1}^{n}(1-\nu_{\alpha_{\sigma(j)}})^{w_j}\rangle$. $PFHG_5(\alpha_1,\alpha_2,\ldots,\alpha_n)=\langle\prod_{j=1}^{n}(\mu_{\tilde{\alpha}_{\sigma(j)}})^{W_j},1-\prod_{j=1}^{n}(1-\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j},1-\prod_{j=1}^{n}(1-\nu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}\rangle$.

3. New Transformation Approach for Picture Fuzzy Sets

In this section, we introduce a transformation method between picture fuzzy numbers and trapezoidal fuzzy numbers.

Proposition 2. *Let* $\alpha=\langle\mu_\alpha,\eta_\alpha,\nu_\alpha\rangle,\beta=\langle\mu_\beta,\eta_\beta,\nu_\beta\rangle\in D^*,\otimes:(D^*)^2\to D^*$ *is defined as* $\alpha\otimes\beta=\langle 1-(1-\mu_\alpha)(1-\mu_\beta),(1-\mu_\alpha)(1-\mu_\beta)-(1-\mu_\alpha-\eta_\alpha)(1-\mu_\beta-\eta_\beta),(1-\mu_\alpha-\eta_\alpha)(1-\mu_\beta-\eta_\beta)-(1-\mu_\alpha-\eta_\alpha-\nu_\alpha)(1-\mu_\beta-\eta_\beta-\nu_\beta)\rangle$*, then* $(D^*,\otimes)$ *is a Abel semigroup.*

Definition 8. *Let* $\alpha=\langle\mu_\alpha,\eta_\alpha,\nu_\alpha\rangle\in D^*$, $\lambda\in[0,1]$*. The power operation is* $\alpha^\lambda=\big\langle 1-(1-\mu_\alpha)^\lambda,(1-\mu_\alpha)^\lambda-(1-\mu_\alpha-\eta_\alpha)^\lambda,(1-\mu_\alpha-\eta_\alpha)^\lambda-(1-\mu_\alpha-\eta_\alpha-\nu_\alpha)^\lambda\big\rangle$.

Theorem 1. *Semigroup* D^* *is isomorphic to semigroup* C^**.*

4. New Geometric Aggregation Operators for Picture Fuzzy Sets

In this section, we introduce the picture fuzzy weighted geometric ($PFWG$) aggregation operator, the picture fuzzy ordered weighted geometric ($PFOWG$) aggregation operator and the picture fuzzy hybrid geometric ($PFHG$) aggregation operator based on the picture fuzzy multiplication operation and the picture fuzzy power operation.

Theorem 2. *Let* $\alpha_j=\langle\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j}\rangle(j=1,2,\ldots,n)\in D^*$*, a mapping PFWG:* $\cup_{n\in N^*}(D^*)^n\to D^*$ *is defined as follow:*

$$\begin{aligned}PFWG(\alpha_1,\alpha_2,\ldots,\alpha_n)=&\alpha_1^{\omega_1}\otimes\alpha_2^{\omega_2}\otimes\ldots\otimes\alpha_n^{\omega_n}\\=&\langle 1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j},\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\mu_{\alpha_j}-\eta_{\alpha_j})^{\omega_j},\\&\prod_{j=1}^{n}(1-\mu_{\alpha_j}-\eta_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\mu_{\alpha_j}-\eta_{\alpha_j}-\nu_{\alpha_j})^{\omega_j}\rangle.\end{aligned}\tag{1}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ *is the weighting vector of* $\alpha_j(j=1,2,\ldots,n)$*, with* $\omega_j\in[0,1]$ *and* $\sum_{j=1}^{n}\omega_j=1$*. Then the mapping PFWG is an aggregation operator, and called picture fuzzy weighted geometric (PFWG) aggregation operator.*

Remark 3. *If $\omega = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then PFWG is called the picture fuzzy geometric average aggregation operator:*

$$\begin{aligned} PFWG(\alpha_1, \alpha_2, \ldots, \alpha_n) =& \langle 1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{1}{n}}, \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\frac{1}{n}} - \prod_{j=1}^{n}(1-\mu_{\alpha_j}-\eta_{\alpha_j})^{\frac{1}{n}}, \\ & \prod_{j=1}^{n}(1-\mu_{\alpha_j}-\eta_{\alpha_j})^{\frac{1}{n}} - \prod_{j=1}^{n}(1-\mu_{\alpha_j}-\eta_{\alpha_j}-\nu_{\alpha_j})^{\frac{1}{n}} \rangle. \end{aligned}$$

Theorem 3. *Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle (j = 1, 2, \ldots, n) \in D^*$, a mapping PFOWG: $\cup_{n \in N^*}$ $(D^*)^n \to D^*$ is defined as follow:*

$$\begin{aligned} PFOWG(\alpha_1, \alpha_2, \ldots, \alpha_n) =& \alpha_{\sigma(1)}^{w_1} \otimes \alpha_{\sigma(2)}^{w_2} \otimes \ldots \otimes \alpha_{\sigma(n)}^{w_n} \\ =& \langle 1 - \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{w_j}, \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}}-\eta_{\alpha_{\sigma(j)}})^{w_j}, \\ & \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}}-\eta_{\alpha_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}}-\eta_{\alpha_{\sigma(j)}}-\nu_{\alpha_{\sigma(j)}})^{w_j} \rangle. \end{aligned} \tag{2}$$

Where $\sigma(j)$ is a permutation of $(1, 2, \ldots, n)$, i.e., $\alpha_{\sigma(j-1)} \geq \alpha_{\sigma_{(j)}}$ $j = (1, 2, \ldots, n)$, α_{σ_j} is the jth largest of α_j in descending order. $w = (w_1, w_2, \ldots, w_n)$ is the weighting vector of PFOWG, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then the mapping PFOWG is an aggregation operator, and called picture fuzzy ordered weighted geometric (PFOWG) aggregation operator.

Theorem 4. *Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle (j = 1, 2, \ldots, n) \in D^*$, a mapping PFHG: $\cup_{n \in N^*}$ $(D^*)^n \to$ D^* is defined as follow:*

$$\begin{aligned} PFHG(\alpha_1, \alpha_2, \ldots, \alpha_n) =& \tilde{\alpha}_{\sigma(1)}^{W_1} \otimes \tilde{\alpha}_{\sigma(2)}^{W_2} \otimes \ldots \otimes \tilde{\alpha}_{\sigma(n)}^{W_n} \\ =& \langle 1 - \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}})^{W_j}, \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}})^{W_j} - \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}}-\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j}, \\ & \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}}-\eta_{\tilde{\alpha}_{\sigma(j)}})^{W_j} - \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}}-\eta_{\tilde{\alpha}_{\sigma(j)}}-\nu_{\tilde{\alpha}_{\sigma(j)}})^{W_j} \rangle. \end{aligned} \tag{3}$$

where $\sigma(j)$ is a permutation of $(1, 2, \ldots, n)$, i.e., $\alpha_{\sigma(j-1)} \geq \alpha_{\sigma_{(j)}}$ $j = (1, 2, \ldots, n)$. $\tilde{\alpha}_j = \alpha_j^{n\omega_j} (j = 1, 2, \ldots, n)$ is weighted α_j, n is the number of α_j, $\omega_j = (\omega_1, \omega_2, \ldots, \omega_n)(j = 1, 2, \ldots n)$ is the weighting vector of α_j, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. $\tilde{\alpha}_{\sigma(j)}$ is jth largest of $\tilde{\alpha}_j$ in descending order. $W_j = (W_1, W_2, \ldots, W_n)(j = 1, 2, \ldots n)$ is the weighting vector of PFHG, with $W_j \in [0, 1]$ and $\sum_{j=1}^{n} W_j = 1$. Then the mapping PFHG is an aggregation operator, and called picture fuzzy hybrid geometric (PFHG) aggregation operator.

Remark 4. *If the PFS reduces to the IFS, the PFWG aggregation operator reduces to IFWG aggregation operator; the PFOWG aggregation operator reduces to IFOWG aggregation operator; PFHG aggregation operator reduces to IFHG aggregation operator. These aggregation operators of intuitionistic fuzzy can be found in [38].*

5. Application to Multi-Criteria Decision Making

In this section, to demonstrate the effectiveness of the proposed aggregation operators, we use numerical example to compare with some existing aggregation operators. To demonstrate the practicability of the proposed aggregation operators, we give the algorithm for the multi-attribute decision making, and apply the proposed three aggregation operators to solve the multi-attribute decision making.

5.1. Numerical Example

Example 1. *Let* $\alpha_1 = \langle 0.1, 0.0, 0.2\rangle$, $\alpha_2 = \langle 0.2, 0.1, 0.0\rangle$, $\alpha_3 = \langle 0.0, 0.1, 0.2\rangle$ *and* $\alpha_4 = \langle 0.3, 0.3, 0.1\rangle$ *are four PFNs, the weighting vector of* $\alpha_j (j = 1, 2, 3, 4)$ *is* $\omega = (0.25, 0.25, 0.25, 0.25)$. *Assume that the weighting vector of PFOWG aggregation operator is* $w = (0.25, 0.25, 0.25, 0.25)$, *the weighting vector of PFHG aggregation operator is* $W = (0.25, 0.25, 0.25, 0.25)$. *Based on the PFWG, PFOWG and PFHG aggregation operators of some existing and proposed, the results of aggregation are shown in Table 2.*

Table 2. Results of aggregation.

	PFWG	*PFOWG*	*PFHG*
Jana et al. [21] ($\xi = 1$)	**Cannot be calculated**	**Cannot be calculated**	**Cannot be calculated**
Wei [22] ($\gamma = 2$)	$\langle \mathbf{0.0000}, 0.1269, 0.1258\rangle$	$\langle \mathbf{0.0000}, 0.1269, 0.1258\rangle$	$\langle \mathbf{0.0000}, 0.1269, 0.1258\rangle$
Ashraf et al. [25]	$\langle \mathbf{0.0000}, \mathbf{0.0000}, 0.1288\rangle$	$\langle \mathbf{0.0000}, \mathbf{0.0000}, 0.1288\rangle$	$\langle \mathbf{0.0000}, \mathbf{0.0000}, 0.1288\rangle$
Wang et al. [26]	$\langle 0.2060, \mathbf{0.0000}, 0.1288\rangle$	$\langle 0.2060, \mathbf{0.0000}, 0.1288\rangle$	$\langle 0.2060, \mathbf{0.0000}, 0.1288\rangle$
Wei [31]	$\langle \mathbf{0.0000}, 0.1322, 0.1288\rangle$	$\langle \mathbf{0.0000}, 0.1322, 0.1288\rangle$	$\langle \mathbf{0.0000}, 0.1322, 0.1288\rangle$
The proposed	$\langle 0.1574, 0.1525, 0.1237\rangle$	$\langle 0.1574, 0.1525, 0.1237\rangle$	$\langle 0.1574, 0.1525, 0.1237\rangle$

From Table 2, we can see that some existing aggregation operators have drawbacks. Wei [22], Ashraf et al. [25] and Wei [31]' positive membership degree of aggregate result is zero, though $n - 1$ *positive membership degree are not equal to zero. Wang et al. [26]' neutral membership degree of aggregate result is zero, though* $n - 1$ *neutral membership degree are not equal to zero. Moreover, Jana et al. [21] aggregation operators cannot be calculated, because the zero in the denominator. However, the proposed aggregation operators overcome this defect, positive membership degree and neutral membership degree of aggregate results are not equal to zero. Therefore, the proposed aggregation operators are more effective.*

5.2. Application to Multi-Criteria Decision Making

5.2.1. Algorithm for Multi-Criteria Decision Making

Suppose, there's m alternatives, which denoted by $A = \{A_1, A_2, \dots, A_m\}$. The alternatives have been evaluated by the PFNs under the set of attribute $C = \{C_1, C_2, \dots, C_n\}$. The weighting vector of attribute is $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, and $\omega_j \in [0, 1]$, $\sum_{i=1}^{n} \omega_j = 1$. Construct decision matrix $D = (\alpha_{ij})_{m\times n}$. Which is the best alternative?

A new multi-attribute decision making algorithm is shown in Figure 4.

Step 1: Normalized decision matrix:

$$r_{ij} = \begin{cases} \alpha_{ij}^c, & \text{for cost attribute } C_j \\ \alpha_{ij}, & \text{for benefit attribute } C_j \end{cases}$$

where $\alpha_{ij}^c = \langle \nu_{ij}, \eta_{ij}, \mu_{ij}\rangle$ is the complement of $\alpha_{ij} = \langle \mu_{ij}, \eta_{ij}, \nu_{ij}\rangle$.

Step 2: Calculating the aggregation values of each alternative $A_i (i = 1, 2, \dots, m)$ by using the aggregation operator *PFWG*(or *PFOWG*, or *PFHG*).

Step 3: Calculating the score function values $S(A_i)$ of aggregation alternative by Definition 6.

Step 4: Ranking the alternatives by Definition 7, the greatest score value is the best alternative.

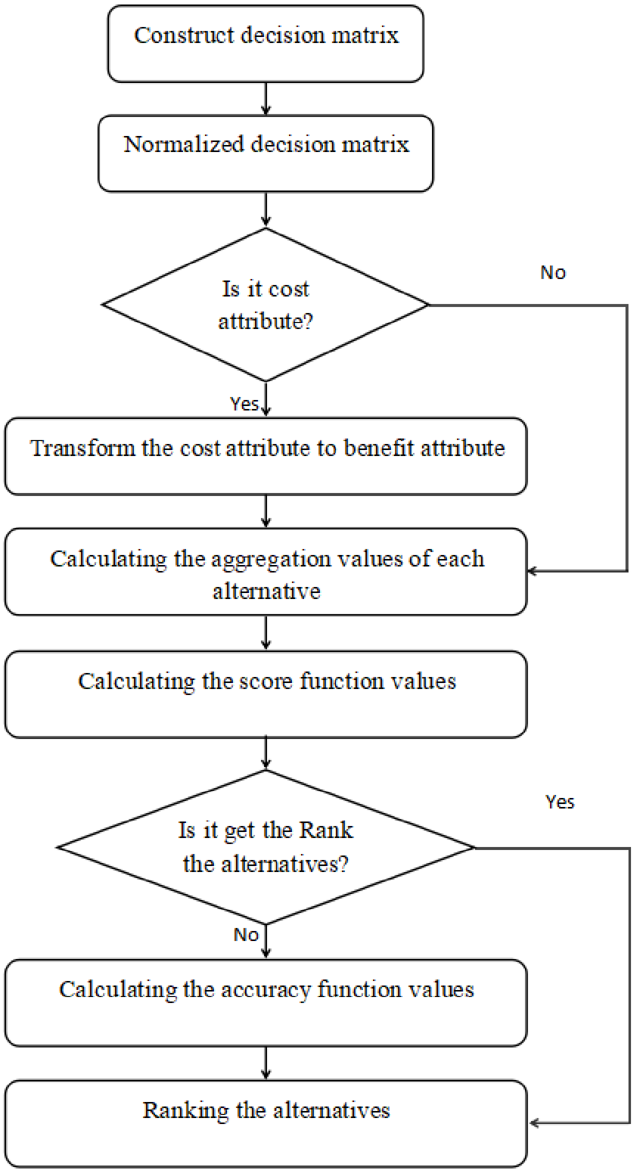

Figure 4. Multi-attribute decision making algorithm.

5.2.2. Application to Multi-Criteria Decision Making

Example 2. *Suppose an investment firm wants to choose a corporation to invest in. According to market research, there are three possible alternatives, which are denoted by* $A = \{A_1, A_2, A_3\}$*: mobile phone corporation* A_1*, television corporation* A_2*, computer corporation* A_3*. There are five attributes* $C = \{C_1, C_2, C_3, C_4, C_5\}$*: resources analysis* (C_1)*, economy analysis* (C_2)*, market analysis* (C_3)*, environmental analysis* (C_4)*, infrastructure analysis* (C_5)*, are considered. The weighting vector of the five attributes is* $\omega = (0.2, 0.3, 0.1, 0.1, 0.3)$*. The three possible alternatives be evaluated by the PFNs under the five attributes. The decision matrix constructed is shown in Table 3. Which company should the investment company choose to invest?*

Table 3. Picture fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle 0.1, 0.2, 0.1 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$	$\langle 0.0, 0.0, 1.0 \rangle$	$\langle 0.2, 0.2, 0.1 \rangle$	$\langle 0.2, 0.1, 0.4 \rangle$
A_2	$\langle 0.2, 0.0, 0.7 \rangle$	$\langle 0.4, 0.1, 0.1 \rangle$	$\langle 0.1, 0.5, 0.0 \rangle$	$\langle 0.3, 0.2, 0.4 \rangle$	$\langle 0.0, 0.0, 1.0 \rangle$
A_3	$\langle 0.3, 0.1, 0.6 \rangle$	$\langle 0.0, 0.0, 1.0 \rangle$	$\langle 0.2, 0.4, 0.0 \rangle$	$\langle 0.1, 0.4, 0.1 \rangle$	$\langle 0.4, 0.1, 0.4 \rangle$

We use the PFWG, PFOWG and PFHG aggregation operators to calculate, respectively. As follows:

(i) Using proposed PFWG aggregation operator.

***Step 1:** Since the five attributes are all benefit attributes, the picture fuzzy decision matrix is normalized decision matrix.*

***Step 2:** Calculating the aggregation values of each alternative by using the proposed PFWG aggregation operator, we have*

$$A_1 : \langle 0.2317, 0.2143, 0.5540\rangle, A_2 : \langle 0.2166, 0.1221, 0.6613\rangle, A_3 : \langle 0.2270, 0.1487, 0.6243\rangle.$$

***Step 3:** Calculating the score function values of each aggregation alternative by Definition 6, we have*

$$S(A_1) = -0.3223, S(A_2) = -0.4447, S(A_3) = -0.3973.$$

***Step 4:** Ranking the alternatives by Definition 7, the greatest score value is the best alternative. We have $S(A_1) > S(A_3) > S(A_2)$. Therefore*

$$A_1 > A_3 > A_2.$$

The best alternative is A_1, i.e., to invest the mobile phone company.

(ii) Using proposed PFOWG aggregation operator.

***Step 1:** Since the five attributes are all benefit attributes, the picture fuzzy decision matrix is the normalized decision matrix.*

First, calculating the score values of each criteria according to the Definition 6 and rank the criteria in order of the highest score values to the lowest, the picture fuzzy ordered decision matrix in Table 4.

Table 4. Picture fuzzy ordered decision matrix.

	C_1	C_2	C_3	C_4	C_5
A_{σ_1}	$\langle 0.4, 0.3, 0.2\rangle$	$\langle 0.2, 0.2, 0.1\rangle$	$\langle 0.1, 0.2, 0.1\rangle$	$\langle 0.2, 0.1, 0.5\rangle$	$\langle 0.0, 0.0, 1.0\rangle$
A_{σ_2}	$\langle 0.4, 0.1, 0.1\rangle$	$\langle 0.1, 0.5, 0.0\rangle$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.2, 0.0, 0.7\rangle$	$\langle 0.0, 0.0, 1.0\rangle$
A_{σ_3}	$\langle 0.2, 0.4, 0.0\rangle$	$\langle 0.4, 0.1, 0.4\rangle$	$\langle 0.1, 0.4, 0.1\rangle$	$\langle 0.3, 0.1, 0.6\rangle$	$\langle 0.0, 0.0, 1.0\rangle$

***Step 2:** Calculating the aggregation values by using the proposed PFOWG aggregation operator, where the weighting vector of PFOWG aggregation operator is $w = (0.2, 0.3, 0.1, 0.1, 0.3)$, we have*

$$A_1 : \langle 0.1829, 0.1892, 0.6279\rangle, A_2 : \langle 0.1745, 0.2221, 0.6034\rangle, A_3 : \langle 0.2166, 0.1839, 0.5995\rangle.$$

***Step 3:** Calculating the score function values of each aggregation alternative by Definition 6, we have*

$$S_1 = -0.4450, S_2 = -0.4289, S_3 = -0.3829.$$

***Step 4:** Ranking the alternatives by Definition 7, the greatest score value is the best alternative. We have $S(A_3) > S(A_2) > S(A_1)$. Therefore*

$$A_3 > A_2 > A_1.$$

The best alternative is A_3, i.e., to invest the computer company.

(iii) Using proposed PFHG aggregation operator.

***Step 1:** Since the five attributes are all benefit attributes, the picture fuzzy decision matrix is the normalized decision matrix.*

First, aggregation the evaluating PFNs of alternatives by using the proposed multiplication operation and power operation, $\widetilde{A}_{ij} = A_{ij}^{n\omega}(i = 1, 2, 3; j = 1, 2, 3, 4, 5)$, where $\omega =$

$(0.2, 0.3, 0.1, 0.1, 0.3)$ *is weighting vector of* $A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}$ *and* $n = 5$. *Then, calculating the score values of aggregation evaluating PFNs by Definition 6 and rank the criteria according to the Definition 7, the picture fuzzy ordered weighted decision matrix in Table 5.*

Table 5. Picture fuzzy ordered weighted decision matrix.

	$\widetilde{A}_{\sigma_1}$	$\widetilde{A}_{\sigma_2}$	$\widetilde{A}_{\sigma_3}$
C_1	$\langle 0.5352, 0.3005, 0.1327 \rangle$	$\langle 0.5352, 0.1112, 0.1006 \rangle$	$\langle 0.5352, 0.1112, 0.3219 \rangle$
C_2	$\langle 0.1056, 0.1198, 0.0675 \rangle$	$\langle 0.0513, 0.3162, 0.0000 \rangle$	$\langle 0.1056, 0.2620, 0.0000 \rangle$
C_3	$\langle 0.1000, 0.2000, 0.1000 \rangle$	$\langle 0.1633, 0.1296, 0.3909 \rangle$	$\langle 0.0513, 0.2416, 0.0747 \rangle$
C_4	$\langle 0.2845, 0.1298, 0.4214 \rangle$	$\langle 0.2000, 0.0000, 0.7000 \rangle$	$\langle 0.3000, 0.1000, 0.6000 \rangle$
C_5	$\langle 0.0000, 0.0000, 1.0000 \rangle$	$\langle 0.0000, 0.0000, 1.0000 \rangle$	$\langle 0.0000, 0.0000, 1.0000 \rangle$

Step 2: *Calculating the aggregation values of each alternative by using the proposed PFHG aggregation operator. Where the weighting vector of PFHG aggregation operator is* $W = (0.112, 0.236, 0.304, 0.236, 0.112)$ *based on the normal distribution in [39], we have*

$$A_1 : \langle 0.2000, 0.1918, 0.6082 \rangle, A_2 : \langle 0.1854, 0.1325, 0.6821 \rangle, A_3 : \langle 0.1913, 0.1714, 0.6373 \rangle.$$

Step 3: *Calculating the score function values of each aggregation alternative according to the Definition 6, we can get*

$$S(A_1) = -0.4082, S(A_2) = -0.4967, S(A_3) = -0.4460.$$

Step 4: *Ranking the alternatives according to the Definition 7, the greatest score value is the best alternative. We have* $S(A_1) > S(A_3) > S(A_2)$. *Therefore*

$$A_1 > A_3 > A_2.$$

The best alternative is A_1, *i.e., to invest the mobile phone company.*

(iv) Compare analysis with some existing aggregation operators, the results are summarized in Table 6.

Table 6. Comparison analysis results.

References	***PFWG***	***PFOWG***	***PFHG***
Jana et al. [21]	Cannot be calculated	Cannot be calculated	Cannot be calculated
Wei [22]	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$
Ashraf et al. [25]	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$
Wang et al. [26]	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$
Wei [31]	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$	$A_1 = A_2 = A_3$
The proposed operator	$A_1 > A_3 > A_2$	$A_3 > A_2 > A_1$	$A_1 > A_3 > A_2$

The best alternative are A_1 *by using the proposed PFWG and PFHG aggregation operators. i.e., to invest the mobile phone company. By using the PFOWG aggregation operator, we can get* $A_3 > A_2 > A_1$, *the best alternative is* A_3, *i.e., to invest the computer company. These three operators PFWG, PFOWG and PFHG provide different choices for decision makers. Decision makers can choose PFWG aggregation operator when only consider the self importance of each criteria. Decision makers can choose PFOWG aggregation operator when only consider the ordered position importance of each criteria. Decision makers can choose PFHG aggregation operator when both consider the self importance and the ordered position importance of each criteria.*

From Table 6, we can see that by using the aggregation operators in [22,25,26,31], the decision results are $A_1 = A_2 = A_3$, *which cannot rank for alternative* A_1, A_2 *and* A_3. *The aggregation operators of [21] cannot calculate the result of aggregation because the denominator appears 0 in the calculation process. However, the proposed aggregation operators can overcome this defect and* *obtain rank of the alternatives. Therefore, we proposed aggregation operators are not only effective, but also overcome the shortcomings of some aggregation operators.*

Example 3 ([31])**.** *One team plans to implement an enterprise resource planning (ERP) system. The first is to build a project team. Project term choose five potential ERP systems $A_i(i = 1, 2, \ldots, 5)$ are to be evaluated by the PFNs under the four attributes C_1: function and technology, C_2 strategic fitness, C_3: vendor′s ability, C_4: vendor′s reputation, whose weighting vector is $\omega = (0.2, 0.1, 0.3, 0.4)$. Suppose that the weighting vector of PFOWG aggregation operator is $w = (0.2, 0.1, 0.3, 0.4)$, the weighting vector of PFHG aggregation operator is $W = (0.140, 0.264, 0.332, 0.264)$, and construct the following matrix is shown in Table 7. Which ERP system is the most desirable?*

Table 7. Picture fuzzy decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\langle 0.53, 0.33, 0.09 \rangle$	$\langle 0.89, 0.08, 0.03 \rangle$	$\langle 0.42, 0.35, 0.18 \rangle$	$\langle 0.08, 0.89, 0.02 \rangle$
A_2	$\langle 0.73, 0.12, 0.08 \rangle$	$\langle 0.13, 0.64, 0.21 \rangle$	$\langle 0.03, 0.82, 0.13 \rangle$	$\langle 0.73, 0.15, 0.08 \rangle$
A_3	$\langle 0.91, 0.03, 0.02 \rangle$	$\langle 0.07, 0.09, 0.05 \rangle$	$\langle 0.04, 0.85, 0.10 \rangle$	$\langle 0.68, 0.26, 0.06 \rangle$
A_4	$\langle 0.85, 0.09, 0.05 \rangle$	$\langle 0.74, 0.16, 0.10 \rangle$	$\langle 0.02, 0.89, 0.05 \rangle$	$\langle 0.08, 0.84, 0.06 \rangle$
A_5	$\langle 0.90, 0.05, 0.02 \rangle$	$\langle 0.68, 0.08, 0.21 \rangle$	$\langle 0.05, 0.87, 0.06 \rangle$	$\langle 0.13, 0.75, 0.09 \rangle$

We use the $PFWG$, $PFOWG$ and $PFHG$ aggregation operators to calculate, respectively. As follows:

Step 1: *Since the four attributes are all benefit attributes, the decision matrix is the normalized decision matrix.*

Step 2: *Calculating the aggregation values of each alternative by using the PFWG, PFOWG and PFHG operators, respectively. The results in Table 8.*

Table 8. Aggregation values.

	$PFWG$	$PFOWG$	$PFHG$
A_1	$\langle 0.4336, 0.4912, 0.0752 \rangle$	$\langle 0.5102, 0.4253, 0.0645 \rangle$	$\langle 0.4145, 0.5166, 0.0689 \rangle$
A_2	$\langle 0.5545, 0.3024, 0.1093 \rangle$	$\langle 0.5102, 0.4253, 0.0645 \rangle$	$\langle 0.4494, 0.3832, 0.1289 \rangle$
A_3	$\langle 0.6159, 0.2904, 0.0937 \rangle$	$\langle 0.4693, 0.3619, 0.1688 \rangle$	$\langle 0.5445, 0.3128, 0.1427 \rangle$
A_4	$\langle 0.4251, 0.4949, 0.0800 \rangle$	$\langle 0.4214, 0.4976, 0.0810 \rangle$	$\langle 0.3515, 0.5622, 0.0863 \rangle$
A_5	$\langle 0.4756, 0.4288, 0.0690 \rangle$	$\langle 0.4710, 0.4372, 0.0663 \rangle$	$\langle 0.3838, 0.5076, 0.0806 \rangle$

Step 3: *Calculating the score function values of each aggregation alternative according to the Definition 6, the results in Table 9.*

Table 9. Score function values.

	$PFWG$	$PFOWG$	$PFHG$
$S(A_1)$	0.3584	0.4457	0.3456
$S(A_2)$	0.4452	0.2232	0.3205
$S(A_3)$	0.5222	0.3005	0.4018
$S(A_4)$	0.3451	0.3404	0.2652
$S(A_5)$	0.4066	0.4047	0.3032

Step 4: *Ranking the alternatives according to the Definition 7, the greatest score value is the best alternative. The results in Table 10.*

Table 10. Ranking of the alternatives.

	Ranking
PFWG	$A_3 > A_2 > A_5 > A_1 > A_4$
PFOWG	$A_1 > A_5 > A_4 > A_3 > A_2$
PFHG	$A_3 > A_1 > A_2 > A_5 > A_4$

The most desirable ERP system is A_3, A_1 and A_3 by using the PFWG, PFOWG and PFHG aggregation operators, respectively.

The proposed aggregation operators are compared with some existing ones, the results are summarized in Table 11.

Table 11. Comparison analysis of results.

References	*PFWG*	*PFOWG*	*PFHG*
Jana et al. [21]	$A_3 > A_5 > A_4 > A_2 > A_1$	$A_1 > A_5 > A_3 > A_4 > A_2$	$A_1 > A_3 > A_5 > A_4 > A_2$
Wei [22]	$A_3 > A_1 > A_2 > A_5 > A_4$	$A_1 > A_5 > A_3 > A_4 > A_2$	$A_3 > A_2 > A_1 > A_5 > A_4$
Ashraf et al. [25]	$A_2 > A_1 > A_3 > A_5 > A_4$	$A_1 > A_5 > A_3 > A_4 > A_2$	$A_3 > A_1 > A_2 > A_4 > A_5$
Wang et al. [26]	$A_2 > A_3 > A_5 > A_1 > A_4$	$A_5 > A_1 > A_4 > A_3 > A_2$	$A_3 > A_1 > A_4 > A_2 > A_5$
Wei [31]	$A_3 > A_1 > A_2 > A_5 > A_4$	$A_1 > A_5 > A_3 > A_4 > A_2$	$A_3 > A_1 > A_2 > A_4 > A_5$
The proposed	$A_3 > A_2 > A_5 > A_1 > A_4$	$A_1 > A_5 > A_4 > A_3 > A_2$	$A_3 > A_1 > A_2 > A_5 > A_4$

From the Table 11, we can see that: (1) Using the PFWG aggregation operators in [21,22,31], the most desirable ERP system are A_3, using the PFWG aggregation operators in [25,26], the most desirable ERP system are A_2, and using the proposed PFWG aggregation operator, the most desirable ERP system is A_3. (2) Using the PFOWG aggregation operators in [21,22,25,31], the most desirable ERP system are A_1, using the PFOWG aggregation operator in [26], the most desirable ERP system is A_5, and using the proposed PFOWG aggregation operator, the most desirable ERP system is A_1. (3) Using the PFHG aggregation operators in [22,26,31], the most desirable ERP system are A_3, using the PFHG aggregation operator in [21], the most desirable ERP system is A_1, and using the proposed PFHG aggregation operator, the most desirable ERP system is A_3. We can get the results of the proposed aggregation operators decision making are consistent with most of the results in the Table 11. It shows that the proposed aggregation operators are effective and it provides decision-makers with different options.

5.3. Comparative Analysis the Conditions of Using Some Aggregation Operators

In order to provide the decision-maker with accurate choice, in this subsection, we analyze the conditions of using some aggregation operators:

Condition 1. *All the degree of positive memberships are not equal to 0 and all the degree of neutral memberships are not equal to zero, i.e., $\mu_j \neq 0$ ($j = 1, 2, \ldots, n$) and $\eta_j \neq 0$ ($j = 1, 2, \ldots, n$).*

Condition 2. *At least one of the positive membership degrees is equal to zero, i.e., $\mu_i = 0$ (i = 1 or 2 or... or n), or at least one of the neutral membership degrees is equal to zero, i.e., $\eta_i = 0$ (i = 1 or 2 or... or n).*

Condition 3. *At least consider the relationship between the two degrees of membership, i.e., the relationship between degrees of positive membership and degrees of neutrality membership $\{\mu, \eta\}$, the relationship between degrees of positive membership and degrees of non-membership*

$\{\mu, \nu\}$, *the relationship between degrees of neutrality membership and degrees of non-membership* $\{\eta, \nu\}$.

Condition 4. *Consider the relationship among the three degrees of membership* $\{\mu, \eta, \nu\}$.

Based on conditions 1–4, we analyze some existing aggregation operators, and the conclusions are summarized in Table 12.

Table 12. Whether applies to the conditions.

References	Condition 1 $\mu_j \neq 0, \eta_j \neq 0$	Condition 2 $\mu_i = 0$	$\eta_i = 0$	Condition 3 $\{\mu, \eta\}$	$\{\mu, \nu\}$	$\{\eta, \nu\}$	Condition 4 $\{\mu, \eta, \nu\}$
Jana et al. [21]	Yes	No	Yes	No	No	No	No
Wei [22]	Yes	No	Yes	No	No	No	No
Ashraf et al. [25]	Yes	No	No	No	No	No	No
Wang et al. [26]	Yes	Yes	No	Yes	No	No	No
Wei [31]	Yes	No	Yes	No	No	No	No
The proposed operators	Yes	Yes	Yes	Yes	Yes	Yes	Yes

In the case of condition 1, all the above aggregation operators are effective. In the case of Condition 2, when at least one of the positive membership degrees is equal to zero, the aggregation operators in [26] is effective. When at least one of the neutral membership degrees is equal to zero, the aggregation operators in [21,22,31], are effective. For Condition 3, only the aggregation operators in [26] consider the relationship between degrees of positive membership and degrees of neutrality membership. For Condition 4, the aggregation operators in [21,22,25,26,31], are not satisfied.

However, we proposed aggregation operators satisfy the conditions 1–4. The aggregate result of positive membership degree or neutral membership degree is not zero, when at least one of the positive membership degree or neutral membership degree is equal to zero, and take into account the interaction relationships of three membership degrees. Therefor, the aggregate results of the proposed aggregation operators are more reliable and more widely used than existing aggregation operators.

6. Application to Pattern Recognition

6.1. Algorithm for Pattern Recognition

Suppose $P_j = \{\langle x_i, \mu_{p_j}(x_i), \eta_{p_j}(x_i), \nu_{p_j}(x_i)\rangle | x_i \in X\} (j = 1, 2, \ldots, m)$ be m known patterns and an unknown pattern $S = \{\langle x_i, \mu_s(x_i), \eta_s(x_i), \nu_s(x_i)\rangle | x_i \in X\}$ in the universal set $X = \{x_1, x_2, \ldots, x_n\}$. Which known pattern does unknown pattern S belong to? In the following, we give the pattern recognition algorithm which is shown in Figure 5.

Step 1: Calculating the aggregation values $\langle \mu_{P_j}, \eta_{P_j}, \nu_{P_j}\rangle$ of each known pattern $P_j = \{\langle x_i, \mu_{p_j}(x_i), \eta_{p_j}(x_i), \nu_{p_j}(x_i)\rangle | x_i \in X\} (j = 1, 2, \ldots, m)$. Calculating the aggregation values $\langle \mu_S, \eta_S, \nu_S\rangle$ of unknown pattern $S = \{\langle x_i, \mu_s(x_i), \eta_s(x_i), \nu_s(x_i)\rangle | x_i \in X\}$ by using the proposed aggregation operator *PFWG* (or *PFOWG*, or *PFHG*), where $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

Step 2: Calculating the distance $d(\langle \mu_{P_j}, \eta_{P_j}, \nu_{P_j}\rangle, \langle \mu_S, \eta_S, \nu_S\rangle)(j = 1, 2, ..., m)$ between P_j and S after aggregation by using the distance [15]:

$$d(\langle \mu_{P_j}, \eta_{P_j}, \nu_{P_j}\rangle, \langle \mu_S, \eta_S, \nu_S\rangle) = \frac{1}{3}[|\mu_{P_j} - \mu_S| + |\eta_{P_j} - \eta_S| + |\nu_{P_j} - \nu_S|]$$

Step 3: Select the minimum one $d(\langle \mu_{P_{j_0}}, \eta_{P_{j_0}}, \nu_{P_{j_0}}\rangle, \langle \mu_S, \eta_S, \nu_S\rangle)$ from $d(\langle \mu_{P_j}, \eta_{P_j}, \nu_{P_j}\rangle, \langle \mu_S, \eta_S, \nu_S\rangle)(j = 1, 2, \ldots, m)$. Then the unknown pattern S belongs to the known pattern P_{j_0}.

6.2. Application to Pattern Recognition

Example 4 ([40]). *Let P_1 and P_2 be two known patterns, S be an unknown pattern in the universal set $X = \{x_1, x_2, x_3\}$. Assume that there are two patterns P_1, P_2 and a sample S in Table 13, which pattern does sample S belong to?*

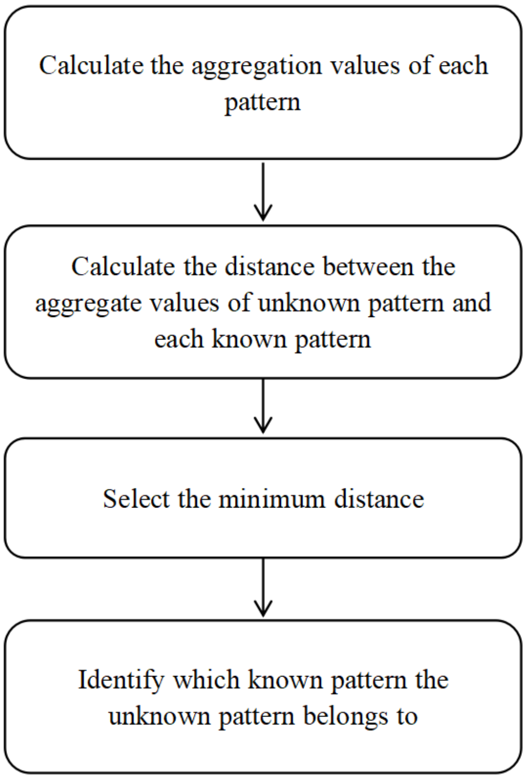

Figure 5. Pattern recognition algorithm.

Table 13. Known patterns and unknown pattern.

Patterns	x_1	x_2	x_3
P_1	$\langle 0.3, 0.2, 0.1\rangle$	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.6, 0.1, 0.3\rangle$
P_2	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.1, 0.2, 0.5\rangle$	$\langle 0.6, 0.3, 0.1\rangle$
S	$\langle 0.5, 0.3, 0.2\rangle$	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.4, 0.3, 0.2\rangle$

Step 1: *Calculating the aggregation values of each pattern by using the proposed aggregation operator PFWG. Where $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The aggregation values in Table 14.*

Table 14. Aggregation values.

Patterns	Aggregation Values
P_1	$\langle 0.4808, 0.1278, 0.3915\rangle$
P_2	$\langle 0.4759, 0.2483, 0.2759\rangle$
S	$\langle 0.4056, 0.3323, 0.2621\rangle$

Step 2: *Calculating the distances of the aggregate values:* $d(P_1, S) = 0.1364$, $d(P_2, S) = 0.0560$.

Step 3: *Select the minimum distance: since $d(P_1, S) > d(P_2, S)$, so sample S belongs to known pattern P_2.*

The unknown pattern S belongs to class P_2 *in [40]. Our recognition result is consistent with that of [40]. It shows that the proposed aggregation operators are effective. Thus, the proposed aggregation operators can be applied to solve the pattern recognition problem.*

7. Conclusions

In this paper, we have done the following. (1) We give a method of transform between picture fuzzy number and trapezoidal fuzzy number, and propose a multiplication operation and a power operation for picture fuzzy set based on trapezoidal fuzzy number. (2) We develop the $PFWG$ aggregation operator, $PFOWG$ aggregation operator and $PFHG$ aggregation operator. (3) We apply the proposed picture fuzzy aggregation operators to solve the multi-attribute decision and pattern recognition problems. The results show that the proposed aggregation operators overcome the defects of some existing aggregation operators. Moreover, the proposed aggregation operators have the advantage of taking the interaction relationships among three degrees of membership into account, and can be more effective and reliable. Next, we study some construction methods of aggregation operators, and apply them to solve some practical problems.

Author Contributions: M.L. initiated the research and provide the framework of this paper. H.L. wrote and complete this paper with M.L. validity and helpful suggeations. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Natural Science Foundation of China (Grant No. 61773019).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy Sets. *Inf. Control.* **1965**, *8*, 338–353. [CrossRef]
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Atanassov, K.T. New operations defined over the intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1994**, *61*, 137–142. [CrossRef]
4. Luo, M.X.; Zhao, R.R. A distance between intuitionistic fuzzy sets and its application in medical diagnosis. *Artif. Intell. Med.* **2018**, *89*, 34–39. [CrossRef]
5. Ngan, R.T.; Son, L.H.; Cuong, B.C.; Ali, M. H-max distance measure of intuitionistic fuzzy sets in decision making. *Appl. Soft Comput.* **2018**, *69*, 393–425. [CrossRef]
6. Iancu, I. Intuitionistic fuzzy similarity measures based on Frank t-norms family. *Pattern. Recognit. Lett.* **2014**, *42*, 128–136. [CrossRef]
7. Singh, P.; Mishra, N.K.; Kumar, M.; Saxena, S.; Singh, V. Risk analysis of flood disaster based on similarity measures in picture fuzzy environment. *Afr. Mat.* **2018**, *29*, 1019–1038. [CrossRef]
8. Song, Y.F.; Wang, X.D.; Quan, W.; Huang, W.L. A new approach to construct similarity measure for intuitionistic fuzzy sets. *Soft Comput.* **2019**, *23*, 1985–1998. [CrossRef]
9. Rahman, K.; Abdullah, S.; Jamil, M.; Khan, M.Y. Some generalized intuitionistic fuzzy einstein hybrid aggregation operators and their application to multiple attribute group decision making. *Int. J. Fuzzy Syst.* **2018**, *20*, 1–9. [CrossRef]
10. Xu, Z.S. Additive intuitionistic fuzzy aggregation operators based on fuzzy measure. *Int. J. Uncertain. Fuzz.* **2016**, *24*, 1–12. [CrossRef]
11. Ye, J. Intuitionistic fuzzy hybrid arithmetic and geometric aggregation operators for the decision-making of mechanical design schemes. *Appl. Intell.* **2017**, *47*, 743–751. [CrossRef]
12. Xu, Z.S.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. General Syst.* **2006**, *35*, 417–433. [CrossRef]
13. Cuong, B.C.; Kreinovich, V. Picture fuzzy sets-a new concept for computational intelligence problems. *Inf. Commun. Technol.* **2013**, 1–6. [CrossRef]
14. Son, L.H. Generalized picture distance measure and applications to picture fuzzy clustering. *Appl. Soft Comput.* **2016**, *46*, 284–295. [CrossRef]
15. Dinh, N.V.; Thao, N.X. Some measures of picture fuzzy sets and their application in multi-attribute decision making. *Int. J. Math. Sci. Comput.* **2018**, *4*, 23–41. [CrossRef]
16. Guleria, A.; Bajaj, R.K. A novel probabilistic distance measure for picture fuzzy sets with its application in classification problems. *Hacet. J. Math. Stat.* **2020**, *49*, 2134–2153. [CrossRef]
17. Wei, G.W. Some similarity measures for picture fuzzy sets and their applications. *Iran. J. Fuzzy Syst.* **2018**, *15*, 77–89. [CrossRef]

18. Wei, G.W. picture fizzy cross-entropy for multiple attribute desision making problems. *J. Bus. Econ. Manag.* **2016**, *17*, 491–502. [CrossRef]
19. Wei, G.W.; Alsaadi, F.E.; Hayat, T.; Alsaed, A. Picture 2-tuple linguistic aggregation operators in multiple attribute decision making. *Soft Comput.* **2018**, *22*, 989–1002. [CrossRef]
20. Joshi, R. A novel decision-making method using R-Norm concept and VIKOR approach under picture fuzzy environment. *Expert Syst. Appl.* **2020**, *147*, 113228. [CrossRef]
21. Jana, C.; Senapati, T.; Pal, M.; Yager, R.R. Picture fuzzy Dombi aggregation operators: Application to MADM process. *Appl. Soft Comput.* **2019**, *74*, 99–109. [CrossRef]
22. Wei, G.W. Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *Fundam. Inform.* **2018**, *157*, 271–320. [CrossRef]
23. Jana, C.; Pal, M. Assessment of enterprise performance based on picture fuzzy hamacher aggregation operators. *Symmetry* **2019**, *11*, 75. [CrossRef]
24. Harish, G. Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. *Arab. J. Sci. Eng.* **2017**, *42*, 5275–5290. [CrossRef]
25. Shahzaib, A.; Tahir, M.; Saleem, A.; Qaisar, K. Different approaches to multi-criteria group decision making problems for picture fuzzy environment. *Bull. New Ser. Am. Math Soc.* **2018**, 1–25. [CrossRef]
26. Wang, C.Y.; Zhou, X.Q.; Tu, H.N.; Tao, S.D. Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making. *Ital. J. Pure Appl. Math.* **2017**, *37*, 477–492.
27. Yanbing, J.; Dawei, J.; Santibanez, G.E.D.R.; Giannakis, M.; Wang, A. Study of site selection of electric vehicle charging station based on extended GRP method under picture fuzzy environment. *Comput. Ind. Eng.* **2019**, *135*, 1271–1285. [CrossRef]
28. Tian, C.; Peng, J.J.; Zhang, S.; Zhang, W.Y.; Wang, J.Q. Weighted picture fuzzy aggregation operators and their applications to multi-criteria decision-making problems. *Comput. Ind. Eng.* **2019**, *137*, 106037. [CrossRef]
29. Wang, R.; Wang, J.; Gao, H.; Wei, G.W. Methods for MADM with picture fuzzy Muirhead mean operators and their application for evaluating the financial investment risk. *Symmetry* **2019**, *11*, 6. [CrossRef]
30. Xu, Y.; Shang, X.P.; Wang, J.; Zhang, R.T.; Li, W.Z.; Xing, Y.P. A method to multi-attribute decision making with picture fuzzy information based on Muirhead mean. *J. Intell. Fuzzy Syst.* **2018**, 1–17. [CrossRef]
31. Wei, G.W. Picture fuzzy aggregation operators and their application to multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2017**, *33*, 713–724. [CrossRef]
32. Klement, E.P.; Mesiar, R.; Stupnanova, A. Picture fuzzy sets and 3-fuzzy sets. In Proceedings of the 2018 IEEE International Conference Fuzzy Syst, Rio de Janeiro, Brazil, 8–13 July 2018; pp. 1–7. [CrossRef]
33. Kaufmann, A.; Gupta, M.M. Introduction to fuzzy arithmetic: Theory and applications. *Math. Comput.* **1986**, *47*, 141–143. [CrossRef]
34. Dubois, D.; Prade, H. *Fuzzy Sets and Systems: Theory and Applications*; Academic Press: New York, NY, USA, 1980.
35. Deschrijiver, G.; Kerre, E.E. Implicators based on binary aggregation operators in interval-valued fuzzy set theory. *Fuzzy Sets Syst.* **2005**, *153*, 229–248. [CrossRef]
36. Kolesárová, A. Limit properties of quasi-arithmetic means. *Fuzzy Sets Syst.* **2001**, *124*, 65–71. [CrossRef]
37. Chen, S.M.; Tan, J.M. Handling multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy Sets Syst.* **1994**, *67*, 163–172. [CrossRef]
38. Chen, S.M.; Chang, C.H. Fuzzy multiattribute decision making based on transformation techniques of intuitionistic fuzzy values and intuitionistic fuzzy geometric averaging operators. *Inform. Sci.* **2016**, 133–149. [CrossRef]
39. Xu, Z.S. An overview of methods for determining OWA weights. *Int. J. Intell. Syst.* **2005**, *20*, 843–865. [CrossRef]
40. Khan, M.J.B.; Kumam, P.; Deebani, W.; Kumam, W.; Shah, Z. Bi-parametric distance and similarity measures of picture fuzzy sets and their applications in medical diagnosis. *Egypt. Inform. J.* **2020**. [CrossRef]