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# $m$-Polar Generalization of Fuzzy $T$-Ordering Relations: An Approach to Group Decision Making

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**Abstract:** Recently, $T$-orderings, defined based on a $t$-norm $T$ and infimum operator (for infinite case) or minimum operator (for finite case), have been applied as a generalization of the notion of crisp orderings to fuzzy setting. When this concept is extending to $m$-polar fuzzy data, it is questioned whether the generalized definition can be expanded for any aggregation function, not necessarily the minimum operator, or not. To answer this question, the present study focuses on constructing $m$-polar $T$-orderings based on aggregation functions $A$, in particular, $m$-polar $T$-preorderings (which are reflexive and transitive $m$-polar fuzzy relations w.r.t $T$ and $A$) and $m$-polar $T$-equivalences (which are symmetric $m$-polar $T$-preorderings). Moreover, the construction results for generating crisp preference relations based on $m$-polar $T$-orderings are obtained. Two algorithms for solving ranking problem in decision-making are proposed and validated by an illustrative example.

**Keywords:** $m$-polar fuzzy relations; $T$-orderings; $m$-polar $T$-preorder; $m$-polar $T$-equivalence; group decision making

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## 1. Introduction

In any decision situation, the pairwise comparison of alternatives for expressing the preferences is the essential part of extending an ordering model between objects. In real-world problems, this comparison information is usually expressed by linguistic variables or fuzzy preferences as they are known from their first appearance in 1971 (c.f. [1]). By adding the concept of membership degree to the binary relations, the class of fuzzy relations, introduced by Zadeh [1], provides more realistic environments for expressing of preferences over the set of alternatives that can be qualitative (linguistic) or quantitative (numeric).

Studies on fuzzy relations properties and fuzzy orderings have been received increasing attention [2–9]. Different researchers have attempted to generalize the basic concepts, such as reflexivity, symmetry, and transitivity for fuzzy relations; however, there is not a unique way for such development in fuzzy logic. While the notions of fuzzy reflexivity, fuzzy symmetry, and fuzzy antisymmetry depend only on the degrees of relations, the concept of fuzzy transitivity was defined by means of a binary operation $* : [0,1]^2 \to [0,1]$, especially where $*$ is a $t$-norm $T$. Accordingly, different variants of fuzzy orderings, such as $T$-preorder, $T$-partial order, and $T$-equivalence relations, were introduced for fuzzy binary relations. Later, Bodenhofer [10,11] discussed the axioms of fuzzy reflexivity and fuzzy antisymmetry being too strong conditions for fuzzification of the crisp case. The new concepts of $E$-reflexivity, $T$-antisymmetry, and $T-E$-antisymmetry, where $T$ is a $t$-norm and $E$ is a $T$-equivalence relation, with less requirements were then developed.

In Boolean logic, there is a close relationship between implication and ordering. If $P$ and $Q$ are two statements, then $P \preceq Q$ iff $P \to Q$ is a tautology. Accordingly, an equivalence relation can be defined over the set of statements where $P \sim Q$ iff $P \leftrightarrow Q$.

By using $t$-norms, the implication relations can be defined as operators associated with a $t$-norm $T$ over the two-valued and multi-valued statements or in a general case for fuzzy sets. It was also proved that, for any implication operator, associated with the $t$-norm $T$, a fuzzy $T$-preordering and a fuzzy $T$-equivalence can be formulated based on $T$. This helps researchers to construct fuzzy orderings from degrees of inclusion and equality of fuzzy sets (c.f. [11–14] for more information). However, the relation of fuzzy sets inclusion used in literature, characterized based on the operators of fuzzy joint and fuzzy implication, represents only a single way of different possible ways to define fuzzy orderings based on $t$-norms. In particular, when we restrict ourselves to the case dealing with finite records of fuzzy information, the fuzzy joint is represented as a minimum operator that can be considered as a conjunctive aggregation function.

In most real-world group decision-making problems, we deal with multi-polar or multi-index information that arises from multi-source or multi-parameter data. In 1994, Zhang [15] initiated the concept of bipolar fuzzy sets whose membership degrees are in $[-1,1]$ instead of $[0,1]$ to model situations where objects can be considered to have a certain property and its counter. This framework is successfully applied when both positive and negative sides of information are given or when objects have positive or negative influence on each other. By extending the range of fuzzy sets from $[0,1]$ into the $[0,1]^m$ that is $[0,1] \times \cdots \times [0,1]$ $m$-times, the concept of $m$-polar fuzzy set was introduced in [16,17] to cope with the problem of multi-polarity, where objects may have a relationship with each other in different directions based on various features of a given property. As a result, it is a very natural question regarding how the concept of $T$-orderings can be developed to $m$-polar fuzzy sets.

This paper contributes to generalizing the approach of construction fuzzy $T$-orderings for $m$-polar fuzzy data by extending the fuzzy joint operator to any aggregation functions. This generalized approach shows the close link between domination relationship of aggregation operators over $t$-norm $T$ and the existence of $m$-polar $T$-orderings. The obtained results do not only present the construction methods, but they are also illustrated to show the efficiency of this new class of $m$-polar $T$-orderings for solving decision-making problems. In this regard, the rest of the present paper is organized as the following: Section 2 gives basic information that is needed to get the main results of the paper. Next, in Section 3, the implication operator is developed to $m$-polar fuzzy sets in order to generate new classes of $m$-polar $T$-preorderings and $m$-polar $T$-equivalences. By using $\mathbf{a}$-cut relations, in Section 4, we discuss some construction methods to create crisp orderings from the proposed $m$-polar $T$-orderings. Two score function-based algorithms are designed for solving the problem of ranking in decision-making and then illustrated by a numerical example in Section 5. Lastly, the Conclusions section is presented.

## 2. Basic Definitions and Properties

In this section, we recall some theoretical background needed to develop the main results of this paper. Note that, throughout this paper, we use the following notations: $\mathbb{I} \subset \mathbb{R}$ as the closed unit interval $[0,1]$ and $\mathbb{I}^m = [0,1]^m = [0,1] \times \cdots \times [0,1]$, $m$-times, as the set of all $m$-dimensional real vectors whose components are in the interval $\mathbb{I}$.

### *2.1. Aggregation Functions*

In literature (c.f. [18–20]), an aggregation function of dimension $n \in \mathbb{N}$ is an $n$-ary function $A^{(n)} : [0,1]^n \to [0,1]$ satisfying:

**A1.** $A(x) = x$, for $n = 1$ and any $x \in [0,1]$;

**A2.** $A^{(n)}(x_1, \cdots, x_n) \leq A^{(n)}(y_1, \cdots, y_n)$ if $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$;

**A3.** $A^{(n)}(0, 0, \cdots, 0) = 0$ and $A^{(n)}(1, 1, \cdots, 1) = 1$.

An extended aggregation function is the function $A : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ whose restriction $A|_{\mathbb{I}^n} := A^{(n)}$ to $\mathbb{I}^n$ is the $n$-ary aggregation function $A^{(n)}$ for any $n \in \mathbb{N}$.

The aggregation function $A$ has a neutral element $e \in [0,1]$ if $A(x_1, \cdots, x_{i-1}, e, x_{i+1}, \cdots, x_n) = A(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. An element $a \in [0,1]$ is called an annihilator element (absorbing element or zero element) of $A$ if $A(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) = a$. Moreover, $A$ has no zero divisors if it has the zero element $a$ and $A(x_1, \cdots, x_n) = a$ implies that $x_s = a$ for some $1 \leq s \leq n$.

The aggregation function $A$ is called conjunctive if $A(x_1, \cdots, x_n) \leq \min(x_1, \cdots, x_n)$, called disjunctive if $A(x_1, \cdots, x_n) \geq \max(x_1, \cdots, x_n)$ and called average (idempotent) whenever $\min(x_1, \cdots, x_n) \leq A(x_1, \cdots, x_n) \leq \max(x_1, \cdots, x_n)$ for every $(x_1, \cdots, x_n) \in \mathbb{I}^n$. In the general case, the conjunctive and disjunctive aggregation functions do not need to have neutral elements. However, if they have them, these elements are, respectively, $e = 1$ and $e = 0$. Moreover, if an aggregation function $A$ has the neutral element $e = 1$, then it is necessarily conjunctive and, if it has the neutral element $e = 0$, then it is necessarily disjunctive.

**Definition 1** ([21])**.** *Let $A : [0,1]^m \to [0,1]$ and $B : [0,1]^n \to [0,1]$ be two aggregation functions where $m, n \in \mathbb{N}$. We say function $A$ dominates function $B$, denoted by $A \gg B$, if*

$$A(B(x_{11}, \cdots, x_{1n}), \cdots, B(x_{m1}, \cdots, a_{mn})) \geq B(A(x_{11}, \cdots, x_{m1}), \cdots, A(x_{1n}, \cdots, a_{mn})), \tag{1}$$

*where $x_{ij} \in [0,1]$.*

Note that, if $A \ll B$, then $B^d \ll A^d$.

Triangular Norms and Conorms

Triangular norms $T : [0,1]^2 \to [0,1]$ and conorms $S : [0,1]^2 \to [0,1]$, or $t$-norms and $t$-conorms in brief, are well-known examples of conjunctive and disjunctive aggregation operators which are associative and commutative with the neutral elements $e = 1$ and $e = 0$, respectively (c.f. [22]).

There are four basic $t$-norms: $T_D(x_1, x_2) = \min\{x_1, x_2\}$ if $x_1 = 1$ or $x_2 = 1$ and otherwise is zero, $T_M(x_1, x_2) = \min(x_1, x_2)$, $T_P(x_1, x_2) = x_1 x_2$ and $T_L(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$ such that $T_D \leq T_L \leq T_P \leq T_M$. In fact, the drastic product $T_D$ and the minimum operator $T_M$ are the smallest and the greatest $t$-norms, respectively, and $T_M$ is the only $t$-norm that is idempotent at any $x \in [0,1]$. On the other hand, the dual of these operators, i.e., $S_D(x_1, x_2) = \max\{x_1, x_2\}$ if $x_1 = 0$ or $x_2 = 0$ and otherwise is one, $S_M(x_1, x_2) = \max(x_1, x_2)$, $S_P(x_1, x_2) = x_1 + x_2 - x_1 x_2$ and $S_L(x_1, x_2) = \min(x_1 + x_2, 1)$ are $t$-conorms such that $S_M \leq S_P \leq S_L \leq S_D$.

It is shown in literature (c.f. [22], Section 6.3) that the domination relation $\gg$ on the set of all $t$-norms is reflexive (i.e., $T \gg T$) and antisymmetric (i.e., $T_1 \gg T_2, T_2 \gg T_1 \Rightarrow T_1 = T_2$); however, the transitivity property is still an open problem. It is also proved that, for any two $t$-norms $T_1$ and $T_2$, if $T_1 \gg T_2$, then $T_1 \geq T_2$, while the converse does not hold in general.

The $t$-norms $T$ and the $t$-conorms $S$ are, in fact, different membership functions from model conjunction (i.e., the logical AND) and disjunction (i.e., the logical OR) in fuzzy logic. However, besides these, it is needed to interpret the concept of implication in fuzzy logic. Therefore, by using $t$-norms and $t$-conorms, a particular class of implication operators was formulated based on the classical concept of implication in set theory as follows.

**Definition 2** ([12,13])**.** *Let $T$ be a left-continuous $t$-norm. The residuum or implication operation $\overrightarrow{T} : [0,1]^2 \to [0,1]$ with respect to the $t$-norm $T$ is defined as*

$$\overrightarrow{T}(x, y) = \sup\{\alpha \in \mathbb{I} : T(\alpha, x) \leq y\} \tag{2}$$

*for any $x, y \in [0,1]$.*

**Proposition 1** ([12,13])**.** *For any left-continuous $t$-norm $T$, the following is held:*

1. $x \leq y$ *if and only if* $\overrightarrow{T}(x,y) = 1$, *so* $\overrightarrow{T}(x,x) = 1$;
2. $T(x,y) \leq z$ *if and only if* $\overrightarrow{T}(y,z) \geq x$;
3. $T(\overrightarrow{T}(x,y), \overrightarrow{T}(y,z)) \leq \overrightarrow{T}(x,z)$;
4. $\overrightarrow{T}(1,x) = x$;
5. $T(x, \overrightarrow{T}(x,y)) \leq y$;
6. $y \leq \overrightarrow{T}(x, T(x,y))$.

The concept of logical equivalence was also extended from the Boolean case to the fuzzy case as below.

**Definition 3** ([11]). *Let T be a left-continuous t-norm. The biimplication relation of T, known as biimplication operator* $\overleftrightarrow{T} : [0,1]^2 \to [0,1]$, *is defined by*

$$\overleftrightarrow{T}(x,y) = T(\overrightarrow{T}(x,y), \overrightarrow{T}(y,x)) \tag{3}$$

*for any* $x, y \in [0,1]$.

**Proposition 2** ([11]). *For any left-continuous t-norm T the following is held.*

1. $x = y$ *if and only if* $\overleftrightarrow{T}(x,y) = 1$;
2. $\overleftrightarrow{T}(x,y) = \overleftrightarrow{T}(y,x)$;
3. $\overleftrightarrow{T}(x,y) = \min(\overrightarrow{T}(x,y), \overrightarrow{T}(y,x))$;
4. $\overleftrightarrow{T}(x,y) = \overrightarrow{T}(\max(x,y), \min(x,y))$;
5. $T(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(y,z)) \leq \overleftrightarrow{T}(x,z)$;
6. $\overleftrightarrow{T} : [0,1]^2 \to [0,1]$ *is a fuzzy T-equivalence relation.*

### 2.2. Fuzzy Orderings

Crisp binary relations, especially orders, are applied to explain the relationships between objects or to compare different objects. However, in real decision situations, we usually deal with the degree of preference not the simple case of yes or no comparison. By adding the membership degree to this Boolean information, binary relations are presented in the fuzzy logic-based framework that are called fuzzy binary relations (or FR in brief). In the sense of Zadeh [1], a fuzzy binary relation $R$ from $X$ to $Y$ is a fuzzy subset of $X \times Y$ characterized by the membership function $R : X \times Y \to [0,1]$, where, for each pair $(x,y) \in X \times Y$, the value $R(x,y)$ shows the strength of the relationship between $x$ and $y$.

In the case of $X = Y$, various properties of fuzzy relations including reflexivity (i.e., $R(x,x) = 1; \forall x \in X$), symmetry (i.e., $R(x,y) = R(y,x); \forall x,y \in X$), antisymmetry (i.e., $\min\{R(x,y), R(y,x)\} = 0; \forall x,y \in X$ such that $x \neq y$), transitivity (i.e., $R(x,z) \geq \max_y \left[\min(R(x,y), R(y,z)); y \in X\right]; \forall x,y,z \in X$) and completeness (i.e., $\max\{R(x,y), R(y,x)\} = 1; \forall x,y \in X$) were also introduced in [1]. Accordingly, the following fuzzy orderings were defined.

The fuzzy relation $R$ is:

- fuzzy ordering if it is reflexive;
- fuzzy preordering if it is reflexive and transitive;
- fuzzy total or linear preordering if it is strongly complete and transitive;
- fuzzy partial ordering or fuzzy weak preference ordering if it is reflexive, antisymmetric and transitive;
- fuzzy strict preference ordering if it is antisymmetric and transitive;
- similarity relation or fuzzy equivalence relation if it is reflexive, symmetric, and transitive.

Since in fuzzy logic there is not a unique way to express any concept, in contrast to the crisp case, by generalization, the $t$-norm $T_M$ and the $t$-conorm $S_M$ into any $t$-norm $T$ and

$t$-conorm $S$ or maybe into any binary operation $*$, the above definitions were developed as below.

**Definition 4.** *Let $*: [0,1]^2 \to [0,1]$ be a binary operation. The fuzzy relation $R$ is*

- *$\alpha$-reflexive if $\forall x \in X : R(x,x) \geq \alpha$, where $\alpha \in (0,1]$ ([12]);*
- *totally $*$-connected or $*$-complete if $\forall x, y \in X : *(R(x,y), R(y,x)) = 1$ (If $*$ is any t-conorm $S$, it is called $S$-connected) ([8,23,24]);*
- *$*$-antisymmetric if $\forall x, y \in X$ such that $x \neq y$: $*(R(x,y), R(y,x)) = 0$ (If $*$ is any t-norm $T$, it is called $T$-antisymmetric) ([8,23,24]);*
- *$*$-transitive if $\forall x, y, z \in X : *(R(x,y), R(y,z) \leq R(x,z)$ (([2,8,23,24])).*

Accordingly, fuzzy $T$-orderings or in general case fuzzy $*$-orderings can be defined as below.

**Definition 5** ([8])**.** *Let $*$ be a binary operation. A reflexive and $*$-transitive fuzzy relation is called fuzzy preordering w.r.t $*$ or fuzzy $*$-preordering. A fuzzy $*$-preorder relation which is also symmetric is called fuzzy $*$-equivalence relation.*

In the direction of generalization of the fuzzy orderings, some researchers pointed out that the above definitions are a straightforward extension of the associated concepts in a crisp case without taking the deeper algebraic background into account. Since the concept of equality in the above definitions is indeed the crisp concept and not a fuzzy equality, the class of $*$-orderings (or $T$-orderings where $T$ is a $t$-norm) are just half-way fuzzification of crisp orderings. Therefore, the following definitions were suggested.

**Definition 6** ([10,11])**.** *Let $T$ be a t-norm and $E$ be a fuzzy $T$-equivalence relation. The fuzzy binary relation $R$ is called $T$-$E$-ordering if and only if it is*

1. *$E$-reflexive, i.e., $E(x,y) \leq R(x,y)$ for all $x, y \in X$;*
2. *$T$-$E$-antisymmetric, i.e., $T(R(x,y), R(y,x)) \leq E(x,y)$ for all $x, y \in X$;*
3. *$T$-transitive.*

Valverde [14] showed that there is a link between the implication and biimplication operators and fuzzy $T$-orderings which can help us to construct a fuzzy $T$-preorder $R$ and a fuzzy $T$-equivalence $E$ from fuzzy data. The background idea of constructing these $T$-orderings is followed from measuring the degrees of $T$-inclusion and $T$-similarity of fuzzy sets.

**Theorem 1** ([14])**.** *For an arbitrary left-continuous t-norm $T$, the fuzzy relation $R$ on $X$ is $T$-preorder if and only if there exists a family of $\{\mu_j\}_{j \in J}$ of fuzzy subsets of $X$ such that*

$$R(x,y) = \inf_j \overrightarrow{T}(\mu_j(x), \mu_j(y)) \tag{4}$$

*for all $x, y \in X$.*

**Theorem 2** ([14])**.** *For an arbitrary left-continuous t-norm $T$, the fuzzy relation $R$ on $X$ is $T$-equivalence if and only if there exists a family of $\{\mu_j\}_{j \in J}$ of fuzzy subsets of $X$ such that*

$$E(x,y) = \inf_j \overrightarrow{T}(\max(\mu_j(x), \mu_j(y)), \min(\mu_j(x), \mu_j(y))) \tag{5}$$

*for all $x, y \in X$.*

*m*-Polar Fuzzy Relations

By expanding the range of membership function from the unit interval $[0,1]$ into the $m$th power of $[0,1]$, i.e., $[0,1]^m = [0,1] \times \cdots \times [0,1]$ where $\mathbf{0} = (0,0,\cdots,0)$ and

$\mathbf{1} = (1, 1, \cdots, 1)$ are, respectively, the least and greatest elements, the traditional fuzzy set, dealing with uni-polar data, is extended into the new concept of $m$-polar fuzzy set which can deal with multi-polar information.

**Definition 7** ([16]). *An $m$-polar fuzzy set $\mu$ on the universe $X$ is a mapping $\mu : X \to [0,1]^m$ such that $\mu(x) = (\pi_1 \circ \mu(x), \cdots, \pi_m \circ \mu(x))$ for any $x \in X$.*

Note that $\pi_s \circ \mu : X \to [0,1]$ is the $s$th degree of membership function $\mu$ where $\pi_s : [0,1]^m \to [0,1]$ is the $s$th projection mapping. The set $[0,1]^m$ is considered as a poset with order "$\leq$" such that for any $\mathbf{x}, \mathbf{y} \in [0,1]^m$ where $\mathbf{x} = (x_1, \cdots, x_m)$ and $\mathbf{y} = (y_1, \cdots, y_m)$, $\mathbf{x} \leq \mathbf{y}$ if $\pi_s(\mathbf{x}) \leq \pi_s(\mathbf{y})$ for each $s \in S = \{1, \cdots, m\}$.

For any two $m$-polar fuzzy subsets $\mu$ and $\nu$ of the universe $X$, the union and intersection are computed as the following:

- $(\mu \vee \nu)(x) = (\max(\pi_1 \circ \mu(x), \pi_1 \circ \nu(x)), \cdots, \max(\pi_m \circ \mu(x), \pi_m \circ \nu(x)))$
- $(\mu \wedge \nu)(x) = (\min(\pi_1 \circ \mu(x), \pi_1 \circ \nu(x)), \cdots, \min(\pi_m \circ \mu(x), \pi_m \circ \nu(x)))$

where $x \in X$.

The concept of $m$-polar fuzzy relation is also developed as below.

**Definition 8** ([25]). *An $m$-polar fuzzy relation $R$ on the universe $X$ is defined by a mapping $R : X \times X \to [0,1]^m$ such that $R(x,y) = (\pi_1 \circ R(x,y), \cdots, \pi_m \circ R(x,y))$ for any $x, y \in X$ where for each $1 \leq s \leq m$, the value $\pi_s \circ R(x,y)$ shows the sth degree of relationship between $x$ and $y$.*

Let $R$ be an $m$-polar fuzzy relation on $X$. Analogously to the fuzzy relations, if the $m$-tuple $\mathbf{a} = (a_1, \cdots, a_m) \in (0,1]^m$ is a given threshold vector on membership degrees, then $R_{\mathbf{a}} = \{(x,y) \in X^2 : \pi_s \circ R(x,y) \geq a_s : s = 1, 2, \cdots, m\}$ is a non-fuzzy (crisp) binary relation on $X$ that is called $\mathbf{a}$-level relation generated by $R$. Clearly, for any two given threshold vectors $\mathbf{a}, \mathbf{b} \in (0,1]^m$ such that $\mathbf{a} = (a_1, \cdots, a_m)$, $\mathbf{b} = (b_1, \cdots, b_m)$ and $\mathbf{a} \geq \mathbf{b}$, we have $R_{\mathbf{a}} \subseteq R_{\mathbf{b}}$.

The aggregating process of $n$ fuzzy relations $R_1, \cdots, R_n$ involves an $n$-ary aggregation function $F$ that assigns to the given fuzzy relations $R_1, \cdots, R_n$ a new fuzzy relation $R_F = F(R_1, \cdots, R_n)$, called an aggregated fuzzy relation. Aggregation functions on products of lattices/posets and $m$-polar fuzzy relations have been also studied in [26,27], where the aggregating over a profile $(R_1, \cdots, R_n)$ of $m$-polar fuzzy relations by aggregation function $F$ is defined by

$$F((\pi_1 \circ R_1, \cdots, \pi_m \circ R_1), \cdots, (\pi_1 \circ R_n, \cdots, \pi_m \circ R_n)) = (F(\pi_1 \circ R_1, \cdots, \pi_1 \circ R_n), \cdots, F(\pi_m \circ R_1, \cdots, \pi_m \circ R_n)) \quad (6)$$

## 3. Generating $m$-Polar $T$-Orders from $m$-Polar Fuzzy Data

This section discusses a method to construct an $m$-polar fuzzy $T$-preorder, where $T$ is a left continuous $t$-norm, based on an aggregation function $A$ and then provides an $m$-polar fuzzy $T$-equivalence. Let us first recall Theorem 1 which says that, if a family of fuzzy subsets of a universe $X$ is given, then it is always possible to derive a $T$-preorder on $X$ with respect to the $t$-norm $T$ as below:

$$R(x,y) = \inf_j \overrightarrow{T}(\mu_j(x), \mu_j(y))$$

where the fuzzy inclusion relation $INCl_T(\mu, \nu) = \inf_{x \in X} \overrightarrow{T}(\mu(x), \nu(x))$ has the fundamental role to define it. In real decision-making applications, the numbers of decision makers and decision parameters/criteria are usually finite. This means that we have mostly finite records of fuzzy information. In this case, the fuzzy $T$-preordering $R$, defined in Theorem 1, is computed by

$$R(x,y) = \min_{j=1}^{K} \overrightarrow{T}(\mu_j(x), \mu_j(y))$$

where the inf operator is replaced by the min operator that can be considered as a logic AND or conjunction belonging to a bigger class known as aggregation functions. It is, therefore, a very natural question whether the aggregating of $\overrightarrow{T}(\mu_1(x), \mu_1(y)), \cdots, \overrightarrow{T}(\mu_K(x), \mu_K(y))$ by any aggregation function $A$ still provides a $T$-preordering on $X$ or not?

Adding multi-polarity to the uni-polar scale of fuzzy sets that is known as $m$-polar fuzzy sets allows experts to deal with inputs from different categories or sources in decision situations. Since the two operators $\overrightarrow{T}$ and $\overleftrightarrow{T}$ play a basic role to construct $T$-preorderings and $T$-equivalences on the set $X$, to cope with the above question in the $m$-polar fuzzy framework, we need first to generalize these operators to obtain an $m$-polar-based representation of them.

**Definition 9.** *Let $T$ be a left-continuous t-norm. The m-polar implication operation $\overrightarrow{T} : ([0,1]^m)^2 \to [0,1]^m$ with respect to $T$ is defined as*

$$\overrightarrow{T}(\mathbf{x}, \mathbf{y}) = (\overrightarrow{T}(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \cdots, \overrightarrow{T}(\pi_m(\mathbf{x}), \pi_m(\mathbf{y}))) \tag{7}$$

*for any $\mathbf{x}, \mathbf{y} \in [0,1]^m$ such that $\mathbf{x} = (x_1, \cdots, x_m)$, $\mathbf{y} = (y_1, \cdots, y_m)$ and $\overrightarrow{T}(\pi_s(\mathbf{x}), \pi_s(\mathbf{y})) = \sup\{\alpha \in \mathbb{I} : T(\alpha, x_s) \leq y_s\}$ for any $1 \leq s \leq m$.*

By adopting the properties of implication operator, discussed in [12,13], for the $m$-polar case, the following properties are immediately obtained for $m$-polar implication operation given in Definition 9.

**Lemma 1.** *Consider the left-continuous t-norm $T$ and an m-polar fuzzy subset $\mu : X \to [0,1]^m$ of $X$. For any $x, y, z \in X$ and $1 \leq s \leq m$, the following assertions hold:*

1. $\pi_s \circ \mu(x) \leq \pi_s \circ \mu(y) \iff \overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(y)) = 1$;
2. $T(\pi_s \circ \mu(x), \pi_s \circ \mu(y)) \leq \pi_s \circ \mu(z) \iff \pi_s \circ \mu(x) \leq \overrightarrow{T}(\pi_s \circ \mu(y), \pi_s \circ \mu(z))$;
3. $T(\overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(y)), \overrightarrow{T}(\pi_s \circ \mu(y), \pi_s \circ \mu(z)) \leq \overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(z))$;
4. $\overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(y)) \leq \overrightarrow{T}(T(\pi_s \circ \mu(x), \pi_s \circ \mu(z)), T(\pi_s \circ \mu(y), \pi_s \circ \mu(z)))$;
5. $\overrightarrow{T}(1, \pi_s \circ \mu(x)) = \pi_s \circ \mu(x)$ *and* $\overrightarrow{T}(\pi_s \circ \mu(x), 1) = 1$;
6. $\overrightarrow{T}(0, \pi_s \circ \mu(x)) = 1$ *and* $\overrightarrow{T}(\pi_s \circ \mu(x), 0) = 0$ *if* $\pi_s \circ \mu(x) \neq 0$ *and $T$ without zero divisors, and otherwise is one;*
7. $T(\pi_s \circ \mu(x), \overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(y))) \leq \pi_s \circ \mu(y)$;
8. $\pi_s \circ \mu(y) \leq \overrightarrow{T}(\pi_s \circ \mu(x), T(\pi_s \circ \mu(x), \pi_s \circ \mu(y)))$.

**Proof.**

- To prove 1, first let $\overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(y)) = 1$ which means $T(\alpha, \pi_s \circ \mu(x)) \leq \pi_s \circ \mu(y)$ for all $\alpha < 1$. Since $T$ is a left-continuous $t$-norm, then $\pi_s \circ \mu(x) = T(1, \pi_s \circ \mu(x)) = T(\sup\{\alpha : \alpha < 1\}, \pi_s \circ \mu(x)) = \sup\{T(\alpha, \pi_s \circ \mu(x)) : \alpha < 1\} \leq \pi_s \circ \mu(y)$. The other side is immediately obtained because, for any $\alpha \in [0,1]$, $T(\alpha, \pi_s \circ \mu(x)) \leq \pi_s \circ \mu(x) \leq \pi_s \circ \mu(y)$.
- For 2, first suppose that $T(\pi_s \circ \mu(x), \pi_s \circ \mu(y)) \leq \pi_s \circ \mu(z)$. Thus, $\pi_s \circ \mu(x) \leq \sup\{\alpha \in [0,1] : T(\alpha, \pi_s \circ \mu(y)) \leq \pi_s \circ \mu(z)\} = \overrightarrow{T}(\pi_s \circ \mu(y), \pi_s \circ \mu(z))$. Conversely, let $\pi_s \circ \mu(x) \leq \overrightarrow{T}(\pi_s \circ \mu(y), \pi_s \circ \mu(z)) = w$. Then, by Definition 9, $T(w, \pi_s \circ \mu(y)) \leq \pi_s \circ \mu(z)$. Thus, $T(\pi_s \circ \mu(x), \pi_s \circ \mu(y)) \leq T(w, \pi_s \circ \mu(y)) \leq \pi_s \circ \mu(z)$.
- In 3, we have: $T(\overrightarrow{T}(\pi_s \circ \mu(x), \pi_s \circ \mu(y)), \overrightarrow{T}(\pi_s \circ \mu(y), \pi_s \circ \mu(z)) = T(\sup\{\alpha \in [0,1] : T(\alpha, \pi_s \circ \mu(x)) \leq \pi_s \circ \mu(y)\}, \sup\{\beta \in [0,1] : T(\beta, \pi_s \circ \mu(y)) \leq \pi_s \circ \mu(z)\}) =$

$\sup\{T(\alpha,\beta) : T(\alpha,\pi_s\circ\mu(x)) \leq \pi_s\circ\mu(y)\&T(\beta,\pi_s\circ\mu(y)) \leq \pi_s\circ\mu(z)\} \leq \overrightarrow{T}(\pi_s\circ\mu(x),\pi_s\circ\mu(z))$ since, for such $\alpha,\beta\in[0,1]$, we have: $T(T(\alpha,\beta),\pi_s\circ\mu(x)) = T(\beta,T(9\alpha,\pi_s\circ\mu(x))) \leq T(\beta,\pi_s\circ\mu(y)) \leq \pi_s\circ\mu(z)$ which implies $T(\alpha,\beta) \leq \sup\{\gamma\in[0,1] : T(\gamma,\pi_s\circ\mu(x)) \leq \pi_s\circ\mu(z)\}$.

- To show 4, let $\alpha\in[0,1]$ such that $T(\alpha,\pi_s\circ\mu(x)) \leq \pi_s\circ\mu(y)$. Then, because of the non-decreasing and associativity of $T$, we have: $T(T(\alpha,\pi_s\circ\mu(x)),\pi_s\circ\mu(z)) \leq T(\pi_s\circ\mu(y),\pi_s\circ\mu(z))$ or equivalently $T(\alpha,T(\pi_s\circ\mu(x),\pi_s\circ\mu(z))) \leq T(\pi_s\circ\mu(y),\pi_s\circ\mu(z))$. Thus, $\sup\{\alpha\in[0,1] : T(\alpha,\pi_s\circ\mu(x)) \leq \pi_s\circ\mu(y)\} \leq \sup\{\beta\in[0,1] : T(\beta,T(\pi_s\circ\mu(x),\pi_s\circ\mu(z))) \leq T(\pi_s\circ\mu(y),\pi_s\circ\mu(z))\}$. Therefore, $\overrightarrow{T}(\pi_s\circ\mu(x),\pi_s\circ\mu(y)) \leq \overrightarrow{T}(T(\pi_s\circ\mu(x),\pi_s\circ\mu(z)),T(\pi_s\circ\mu(y),\pi_s\circ\mu(z)))$.
- The items 5, 6, 7, and 8 are clearly obtained based on Definition 9.

□

**Corollary 1.** *Let $T$ be a left-continuous t-norm and $\mu : X \to [0,1]^m$ be an m-polar fuzzy subset of $X$. Then, for any $x,y,z\in X$:*

1. $\mu(x) \leq \mu(y) \iff \overrightarrow{T}(\mu(x),\mu(y)) = \mathbf{1}$;
2. $T(\mu(x),\mu(y)) \leq \mu(z) \iff \mu(x) \leq \overrightarrow{T}(\mu(y),\mu(z))$;
3. $T(\overrightarrow{T}(\mu(x),\mu(y)),\overrightarrow{T}(\mu(y),\mu(z)) \leq \overrightarrow{T}(\circ\mu(x),\mu(z))$;
4. $\overrightarrow{T}(\mu(x),\mu(y)) \leq \overrightarrow{T}(T(\mu(x),\mu(z)),T(\mu(z),\mu(y)))$;
5. $\overrightarrow{T}(\mathbf{1},\mu(x)) = \mu(x)$ *and* $\overrightarrow{T}(\mu(x),\mathbf{1}) = \mathbf{1}$;
6. $\overrightarrow{T}(\mathbf{0},\mu(x)) = \mathbf{1}$. $\overrightarrow{T}(\mu(x),\mathbf{0}) = \mathbf{0}$ *if* $\mu(x) \neq \mathbf{0}$, *otherwise it is* $\mathbf{1}$;
7. $T(\mu(x),\overrightarrow{T}(\mu(x),\mu(y))) \leq \mu(y)$;
8. $\mu(y) \leq \overrightarrow{T}(\mu(x),T(\mu(x),\mu(y)))$.

Motivated by Theorem 1, the next result gives a method to generate an $m$-polar fuzzy $T$-preorder relation on $X$ with respect to the $t$-norm $T$ and the aggregation function $A$. This change enables us to measure the strength of fuzzy relationship between each pair of elements $x$ and $y$ at any direction $s$; $1\leq s\leq m$, based on all implication degrees from $x$ to $y$, not necessarily the minimum one.

**Theorem 3.** *Consider a left-continuous t-norm $T$ and an aggregation function $A$ such that $A \gg T$. Let $X$ be a universal set and $\{\mu_j : 1\leq j\leq K\}$ be a finite family of m-polar fuzzy subsets of $X$. The m-polar fuzzy relation $R : X\times X \to [0,1]^m$ such that, for any $x,y\in X$, the degree $R(x,y) = (\pi_1\circ R(x,y),\cdots,\pi_m\circ R(x,y))$ is defined by*

$$\pi_s\circ R(x,y) = A[\overrightarrow{T}(\pi_s\circ\mu_1(x),\pi_s\circ\mu_1(y)),\cdots,\overrightarrow{T}(\pi_s\circ\mu_K(x),\pi_s\circ\mu_K(y))] : s = 1,\cdots,m \tag{8}$$

*is an m-polar T-preorder on X.*

**Proof.** Let $\{\mu_j : 1\leq j\leq K\}$ be a finite family of $m$-polar fuzzy sets on $X$. Suppose that $T$ is left continuous $t$-norm and $A$ is an aggregation function such that $A \gg T$. For any $1\leq j\leq K$; $1\leq s\leq m$ and $x\in X$, the reflexivity of $m$-polar fuzzy relation $R$ follows from boundary property $A(1,\cdots,1) = 1$ and the fact that $T(1,\pi_s\circ\mu_j(x)) \leq \pi_s\circ\mu_j(x)$. To prove $T$-transitivity for $R$, take $x,y,z\in X$ and $1\leq s\leq m$. Then, by Lemma 1, item (3), we have

$$T[\pi_s \circ R(x,y), \pi_s \circ R(y,z)] = T[A_{1\leq j\leq K}(\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y))), A_{1\leq j\leq K}(\overrightarrow{T}(\pi_s \circ \mu_j(y), \pi_s \circ \mu_j(z)))]$$
$$\leq A_{1\leq j\leq K}[T(\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y)), \overrightarrow{T}(\pi_s \circ \mu_j(y), \pi_s \circ \mu_j(z)))]$$
$$= A_{1\leq j\leq K}[\sup\{T(\alpha_j, \beta_j) : T(\alpha_j, \pi_s \circ \mu_j(x)) \leq \pi_s \circ \mu_j(y), T(\beta_j, \pi_s \circ \mu_j(y)) \leq \pi_s \circ \mu_j(z)\}]$$
$$\leq A_{1\leq j\leq K}(\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(z))) = R(x,z)$$

since $A$ is monotone and, for any $1 \leq j \leq K$, we have $T(T(\alpha_j, \beta_j), \pi_s \circ \mu_j(x)) \leq \pi_s \circ \mu_j(z)$, or equivalently $T(\alpha_j, \beta_j) \leq \overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(z))$. □

**Example 1.** *Let $T$ be the t-norms $T_L$, $T_P$, and $T_M$. For any $x, y \in X$ and $1 \leq s \leq m$, the following are the associated m-polar fuzzy T-preorderings generated by Equation (8):*

1. *If $T := T_L$ and $A :=$ WAM (weighted arithmetic mean) with weighting vector $w = (w_1, \cdots, w_K)$ where for any $1 \leq j \leq K$: $w_j \in [0,1]$ and $\sum_{j=1}^{K} w_j = 1$. Then, $\pi_s \circ R(x,y) = \sum_{j=1}^{K} w_j \cdot \min(1 - \pi_s \circ \mu_j(x) + \pi_s \circ \mu_j(y), 1)$.*
2. *If $T := T_P$ and $A :=$ WGM (weighted geometric mean) with weighting vector $w = (w_1, \cdots, w_K)$ where for any $1 \leq j \leq K$: $w_j \in [0,1]$, and $\sum_{j=1}^{K} w_j = 1$. Then, $\pi_s \circ R(x,y) = \prod_j (\frac{\pi_s \circ \mu_j(y)}{\pi_s \circ \mu_j(x)})^{w_j}$, where $\pi_s \circ \mu_j(x) \neq 0$ and $\pi_s \circ \mu_j(x) > \pi_s \circ \mu_j(y)$; otherwise, it is one.*
3. *If $T := T_M$ and $A :=$ Min. Then, $\pi_s \circ R(x,y) = 1$, if $\pi_s \circ \mu_j(x) \leq \pi_s \circ \mu_j(y)$ for all $1 \leq j \leq K$; otherwise, $\pi_s \circ R(x,y) = \min_j\{\pi_s \circ \mu_j(y) : \pi_s \circ \mu_j(y) < \pi_s \circ \mu_j(x)\}$.*

**Proposition 3.** *Consider the left-continuous t-norms $T_1, T_2$ and $T$; and the aggregation functions $A_1, A_2$ and $A$ such that $A_1 \gg T_1$, $A_2 \gg T_2$, and $A \gg T$. For the m-polar fuzzy preorder relations $R_1, R_2$ and $R$ with respect to $T_1, T_2$ and $T$, respectively, the following assertions hold.*

1. *If $A_1 \leq A_2$, then $R_1 \subseteq R_2$.*
2. *If $T_1 \leq T_2$, then $R_1 \supseteq R_2$.*
3. *If for any $1 \leq j \leq K$ and $1 \leq s \leq m$ we have $\pi_s \circ \mu_j(x) \leq \pi_s \circ \mu_j(y)$, then $R(x,y) = 1$. The converse will be held if $A$ is conjunction.*
4. *For any crisp point $x \in X$ such that $\pi_s \circ \mu_j(x) = 1$ for any $1 \leq j \leq K$ and $1 \leq s \leq m$, the $\pi_s \circ R(y,x) = 1$ and $\pi_s \circ R(x,y) = A(\pi_s \circ \mu_1(y), \cdots, \pi_s \circ \mu_K(y))$ for any $y \in X$.*
5. *If $\pi_s \circ \mu_j(x) \leq \pi_s \circ \mu_j(y)$ for any $1 \leq j \leq K$ and $1 \leq s \leq m$, then $\pi_s \circ R(z,x) \leq \pi_s \circ R(z,y)$ and $\pi_s \circ R(y,z) \leq \pi_s \circ R(x,z)$ for all $z \in X$.*
6. *If $A$ is disjunction, then $T(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y)) \leq \pi_s \circ \mu_j(z) \Rightarrow \pi_s \circ \mu_j(x) \leq \pi_s \circ R(y,z)$ for any $1 \leq j \leq K$, $1 \leq s \leq m$ and $x, y, z \in X$.*

**Proof.** Items (1) and (2) are immediately followed from Theorem 3. Items (3) and (4) are obvious by Lemma 1.
To prove item (5), let $\pi_s \circ \mu_j(x) \leq \pi_s \circ \mu_j(y)$ for any $1 \leq j \leq K$ and $1 \leq s \leq m$. Then, $A[\overrightarrow{T}(\pi_s \circ \mu_j(z), \pi_s \circ \mu_j(x))] \leq A[\overrightarrow{T}(\pi_s \circ \mu_j(z), \pi_s \circ \mu_j(y))]$ and $A[\overrightarrow{T}(\pi_s \circ \mu_j(y), \pi_s \circ \mu_j(z))] \leq A[\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(z))]$ are followed from the fact that $\{\beta \in [0,1] : T(\beta, \pi_s \circ \mu_j(z)) \leq \pi_s \circ \mu_j(x)\} \subseteq \{\alpha \in [0,1] : T(\alpha, \pi_s \circ \mu_j(z)) \leq \pi_s \circ \mu_j(y)\}$ and monotonicity of $A$. Therefore, respectively, $\pi_s \circ R(z,x) \leq \pi_s \circ R(z,y)$ and $\pi_s \circ R(y,z) \leq \pi_s \circ R(x,z)$.
For item (6), let $T(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y)) \leq \pi_s \circ \mu_j(z)$ for all $1 \leq j \leq K$ and $1 \leq s \leq m$. Then, by Lemma 1, we have $\pi_s \circ \mu_j(x) \leq \overrightarrow{T}(\pi_s \circ \mu_j(y), \pi_s \circ \mu_j(z))$. Therefore, $\pi_s \circ R(y,z) = A_{1\leq j\leq K}[\overrightarrow{T}(\pi_s \circ \mu_j(y), \pi_s \circ \mu_j(z))] \geq A_{1\leq j\leq K}[\pi_s \circ \mu_j(x)] \geq \pi_s \circ \mu_j(x)$ since $A$ is a disjunction aggregation function. □

**Theorem 4.** *Let $T$ be a left continuous t-norm and $A$ be a conjunctive aggregation function such that $A \gg T$. Consider the finite set $X$ with cardinality $K$. An m-polar fuzzy relation $R'$ on $X$ is reflexive if and only if there exists a family $\{\mu_j : 1 \leq j \leq K\}$ of m-polar fuzzy subsets of $X$*

*generating the m-polar fuzzy relation R by* $\pi_s \circ R(x,y) = A_j[\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y))]$ *such that* $R \subseteq R'$.

**Proof.** Let the left continuous $t$-norm $T$ be given. Let us suppose first that the $m$-polar fuzzy relation $R'$ is reflexive. Take the conjunction aggregation function $A \gg T$. For any $x, y \in X$ and $1 \leq s \leq m$, we have

$$\begin{aligned} A_{z\in X}[\overrightarrow{T}(\pi_s \circ R'(z,y), \pi_s \circ R'(z,x))] &\leq \min_{z\in X}[\overrightarrow{T}(\pi_s \circ R'(z,y), \pi_s \circ R'(z,x))] \\ &\leq \overrightarrow{T}(\pi_s \circ R'(y,y), \pi_s \circ R'(y,x)) \\ &= \overrightarrow{T}(1, \pi_s \circ R'(y,x)) = \pi_s \circ R'(y,x) \end{aligned}$$

Then, $\pi_s \circ R(y,x) = A_{z\in X}[\overrightarrow{T}(\pi_s \circ \mu_z(y), \pi_s \circ \mu_z(x))] \leq \pi_s \circ R'(y,x)$ if we put $X = J = \{1, 2, \cdots, K\}$, $\pi_s \circ R'(z,x) = \pi_s \circ \mu_z(x)$ and $\pi_s \circ R'(z,y) = \pi_s \circ \mu_z(y)$. Therefore, $R \subseteq R'$.

Conversely, let $R, R' : X \times X \to [0,1]^m$ be two $m$-polar fuzzy relations on $X$ such that $R \subseteq R'$ and $R$ is the $T$-preordering defined in Theorem 3 by a family $\{\mu_j : 1 \leq j \leq K\}$ of $m$-polar fuzzy subsets of $X$. The reflexivity of $R'$ is implied immediately by reflexivity of $R$. □

**Theorem 5.** *Let T be left continuous t-norm and A be a disjunctive aggregation function such that* $A \gg T$. *Consider the finite set X with cardinality K. An m-polar fuzzy relation* $R''$ *on X is T-transitive if and only if there exists a family* $\{\mu_j : 1 \leq j \leq K\}$ *of m-polar fuzzy subsets of X generating the m-polar fuzzy relation R by* $\pi_s \circ R(x,y) = A_j[\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y))]$ *such that* $R'' \subseteq R$.

**Proof.** Let the $m$-polar fuzzy relation $R''$ be $T$-transitive. For any $1 \leq s \leq m$ and $x, y, z \in X$, we have $T\big(\pi_s \circ R''(z,x), \pi_s \circ R''(x,y)\big) \leq \pi_s \circ R''(z,y)$, thus $\pi_s \circ R''(x,y) \leq \overrightarrow{T}(\pi_s \circ R''(z,x), \pi_s \circ R''(z,y))$. This implies that $\pi_s \circ R''(x,y) \leq A_{z\in X}[\overrightarrow{T}(\pi_s \circ R''(z,x), \pi_s \circ R''(z,y))]$ since $A \geq Max$. Now, it is sufficient to put $X = J = \{1, 2, \cdots, K\}$ and take $\pi_s \circ R''(z,x) = \pi_s \circ \mu_z(x)$, $\pi_s \circ R''(z,y) = \pi_s \circ \mu_z(y)$. □

**Proposition 4.** *Consider conjunction A and t-norm T, both of them without zero divisors. The relation R defined in Theorem 3 by the family* $\{\mu_j : 1 \leq j \leq K\}$ *of m-polar fuzzy sets is m-polar T-antisymmetric if and only if for any* $1 \leq s \leq m$ *there exists* $\mu_i$ *such that, for any* $x, y \in X$, *either* $\pi_s \circ \mu_i(x) = 0$ *or* $\pi_s \circ \mu_i(y) = 0$, *but not both.*

**Proof.** Take $x, y \in X$, and, for any $1 \leq s \leq m$, there exists $1 \leq i \leq K$ such that either $\pi_s \circ \mu_i(x) = 0$ or $\pi_s \circ \mu_i(y) = 0$, but not both. Therefore, either $\overrightarrow{T}(\pi_s \circ \mu_i(y), \pi_s \circ \mu_i(x)) = 0$ or $\overrightarrow{T}(\pi_s \circ \mu_i(x), \pi_s \circ \mu_i(y)) = 0$, respectively. Thus, either $\pi_s \circ R(y,x) = 0$ or $\pi_s \circ R(x,y) = 0$ since $A$ is a conjunction. Thus, $T(R(x,y), R(y,x)) = (T(\pi_1 \circ R(x,y), \pi_1 \circ R(y,x)), \cdots, T(\pi_m \circ R(x,y), \pi_m \circ R(y,x))) = (0, \cdots, 0) \in [0,1]^m$ since $T$ has no zero divisors. This means that $R$ is an $m$-polar fuzzy $T$-antisymmetric. Conversely, let $T(R(x,y), R(y,x)) = (0, \cdots, 0)$. Then, for any $1 \leq s \leq m$, we have $T(\pi_s \circ R(x,y), \pi_s \circ R(y,x)) = 0$, which implies that $\pi_s \circ R(x,y) = 0$ or $\pi_s \circ R(y,x) = 0$ since $T$ has no zero divisors. Without loss of generality, let $\pi_s \circ R(x,y) = 0$. Thus, there exists $1 \leq i \leq K$ such that $\overrightarrow{T}(\pi_s \circ \mu_i(x), \pi_s \circ \mu_i(y)) = 0$ and then $\pi_s \circ \mu_i(y) = 0$ since both conjunction $A$ and $t$-norm $T$ have no zero divisors. □

**Remark 1.** *Note that, under the conditions of Proposition 4, the property of T-antisymmetry is equivalent to A-antisymmetry and* min*-antisymmetry.*

**Theorem 6.** *Let R be the m-polar fuzzy T-preordering on X defined in Theorem 3. Then, the aggregated fuzzy relation* $R_F$ *over the components of R, where* $R_F(x,y) = F(\pi_1 \circ R(x,y), \cdots, \pi_m \circ R(x,y))$ *for any* $x, y \in X$, *by the aggregation function F is a fuzzy T-preorder on X if* $F \gg T$.

**Proof.** Take $x \in X$. The reflexivity of $R_F$ is followed from $R_F(x,x) = F(\pi_1 \circ R(x,x), \cdots, \pi_m \circ R(x,x)) = F(1, \cdots, 1) = 1$. To show $T$-transitivity, let $x, y, z \in X$. Then,

$$\begin{aligned} T(R_F(x,y), R_F(y,z)) &= T(F(\pi_1 \circ R(x,y), \cdots, \pi_m \circ R(x,y)), F(\pi_1 \circ R(y,z), \cdots, \pi_m \circ R(y,z))) \\ &\leq F(T(\pi_1 \circ R(x,y), \pi_1 \circ R(y,z)), \cdots, T(\pi_m \circ R(x,y), \pi_m \circ R(y,z))) \\ &\leq F(\pi_1 \circ R(x,z), \cdots, \pi_m \circ R(x,z)) = R_F(x,z) \end{aligned}$$

because of the monotonicity of $F$. This completes the proof. □

Clearly, if $R$ is an $m$-polar fuzzy $T$-preordering on $X$ w.r.t aggregation function $A$, as defined in Theorem 3, then $R_A$ is a fuzzy $T$-preorder on $X$. In particular, if $T := T_M$, then the aggregation functions $A$ and $F$ must be the minimum operator where the aggregated fuzzy relation $R_{\min}$ defined by $R_{\min}(x,y) = \min_{s=1}^{m} \big(\inf_j [\overrightarrow{T}(\pi_s \circ \mu_j(x), \pi_s \circ \mu_j(y))]\big)$ for $x, y \in X$ is min-transitive.

*m-Polar T-Equivalences*

In literature, there exists a fundamental representation theorem for T-equivalences with respect to left-continuous t-norms (c.f. Theorem 2), which shows that such relations can be generated from families of fuzzy sets by means of biimplications (see also Proposition 2, items 3, 4, and 6). This section is dedicated to developing an analogous result for constructing $m$-polar fuzzy equivalences by means of $B$-composition of $m$-polar fuzzy $T$-preorders introduced in Theorem 3, where $B$ is an aggregation function.

**Theorem 7.** *Consider the left-continuous t-norm $T$ and aggregation function $A \gg T$. Let $X$ be a universal set and $\{\mu_j : 1 \leq j \leq K\}$ be a finite family of $m$-polar fuzzy subsets of $X$ generating the $m$-polar $T$-preorder relation $R$ w.r.t $A$. If $B$ is a symmetric aggregation function such that $B \gg T$, then an $m$-polar fuzzy relation $E : X \times X \to [0,1]^m$ defined by $E(x,y) = (\pi_1 \circ E(x,y), \cdots, \pi_m \circ E(x,y))$ for any $x, y \in X$ such that, for each $1 \leq s \leq m$,*

$$\pi_s \circ E(x,y) = B\big(\pi_s \circ R(x,y), \pi_s \circ R(y,x)\big) \tag{9}$$

*is an $m$-polar $T$-equivalence relation on $X$.*

**Proof.** $m$-polar fuzzy reflexivity and symmetry of $E$ are clear. The $m$-polar fuzzy $T$-transitivity is followed from $T$-transitivity of $m$-polar fuzzy relation $R$. Take $x, y, z \in X$ and $1 \leq s \leq m$, then

$$\begin{aligned} \pi_s \circ E(x,z) &= B(\pi_s \circ R(x,z), \pi_s \circ R(z,x)) \\ &\geq B(T(\pi_s \circ R(x,y), \pi_s \circ R(y,z)), T(\pi_s \circ R(z,y), \pi_s \circ R(y,x))) \\ &\geq T(B(\pi_s \circ R(x,y), \pi_s \circ R(y,x)), B(\pi_s \circ R(y,z), \pi_s \circ R(z,y))) \\ &= T(\pi_s \circ E(x,y), \pi_s \circ E(y,z)) \end{aligned}$$

since $B \gg T$. □

**Example 2.** *Let $T$ be the t-norms $T_L$, $T_P$, and $T_M$. For any $x, y \in X$ and $1 \leq s \leq m$, the following are the associated $m$-polar fuzzy $T$-equivalences generated by Equation (9):*

1. *If $T := T_L$, $A := WAM$ and $B := AM$, then $\pi_s \circ E(x,y) = \frac{1}{2}\sum_{j=1}^{K} w_j \cdot \big[\min(1 - \pi_s \circ \mu_j(x) + \pi_s \circ \mu_j(y), 1) + \min(1 - \pi_s \circ \mu_j(y) + \pi_s \circ \mu_j(x), 1)\big]$ where, for $1 \leq j \leq K$: $w_j \in [0,1]$ and $\sum_{j=1}^{K} w_j = 1$.*
2. *If $T := T_P$ and $A := WGM$ with weighting vector $w = (w_1, \cdots, w_m)$ where for any $1 \leq j \leq K$: $w_j \in [0,1]$ and $\sum_{j=1}^{K} w_j = 1$ and $B := GM$. Then, $\pi_s \circ E(x,y) =$*

$\sqrt{\prod_j[\min(\frac{\pi_s\circ\mu_j(y)}{\pi_s\circ\mu_j(x)},1)]^{w_j}\cdot\min(\frac{\pi_s\circ\mu_j(x)}{\pi_s\circ\mu_j(y)},1)]^{w_j}}$, *where* $\pi_s\circ\mu_j(x)\neq 0$ *and* $\pi_s\circ\mu_j(y)\neq 0$; *otherwise, it is one.*

3. *If* $T := T_M$ *and* $A, B := Min$. *Then,* $\pi_s\circ E(x,y) = \min_{j=1}^{K}\{\pi_s\circ\mu_j(x)\}$ *if* $\pi_s\circ\mu_j(x)\leq\pi_s\circ\mu_j(y)$ *for all* $1\leq j\leq K$; $\pi_s\circ E(x,y) = \min_{j=1}^{K}\{\pi_s\circ\mu_j(y)\}$ *if* $\pi_s\circ\mu_j(y)\leq\pi_s\circ\mu_j(x)$ *for all* $1\leq j\leq K$; *otherwise,* $\pi_s\circ R(x,y) = \min(\min_j\{\pi_s\circ\mu_j(y):\pi_s\circ\mu_j(y)<\pi_s\circ\mu_j(x)\},\min_j\{\pi_s\circ\mu_j(x):\pi_s\circ\mu_j(x)<\pi_s\circ\mu_j(y)\})$.

**Proposition 5.** *Consider the left-continuous t-norms* $T_1, T_2$ *and* $T$; *and the aggregation functions* $A_1, A_2, A$ *and symmetric ones* $B_1, B_2, B$ *such that* $A_1, B_1 \gg T_1$, $A_2, B_2 \gg T_2$ *and* $A, B \gg T$. *For the m-polar fuzzy equivalences* $E_1, E_2$ *and* $E$ *with respect to* $T_1, A_1, B_1$; $T_2, A_2, B_2$; *and* $T, A, B$, *respectively, the following assertions hold.*

1. *If* $B_1 \leq B_2$, *then* $E_1 \subseteq E_2$.
2. *If* $A_1 \leq A_2$, *then* $E_1 \subseteq E_2$.
3. *If* $T_1 \leq T_2$, *then* $E_1 \supseteq E_2$.
4. *If for any* $1\leq i\leq K$, $\pi_s\circ\mu_i(x) = \pi_s\circ\mu_i(y)$ *for all* $s = 1,\cdots,m$; *then,* $\pi_s\circ E(x,y) = 1$. *The converse holds if B is a conjunction.*

**Proof.** It is obtained easily by Theorem 7 and Proposition 3. □

**Theorem 8.** *Suppose T be a left-continuous t-norm. Consider the m-polar fuzzy T-equivalence E w.r.t B. Then, the aggregated fuzzy relation* $E_F$ *by an aggregation function F where* $E_F(x,y) = F(\pi_1\circ E(x,y),\cdots,\pi_m\circ E(x,y))$ *for any* $x, y\in X$ *is a fuzzy T-equivalence on X.*

**Proof.** Analogous to the proof of Theorem 6. □

**Theorem 9.** *The m-polar fuzzy relation R given in Theorem 3 is a T-E-ordering where E is the T-equivalence relation defined by Equation (9) and B is a conjunction with the neutral element* $e = 1$.

**Proof.** Trivially, $R$ is supposed to be $T$-transitive by Theorem 3. For any $1\leq s\leq m$, we have $\pi_s\circ E(x,y) = B(\pi_s\circ R(x,y),\pi_s\circ R(y,x))\leq\pi_s\circ R(x,y)$ since $B$ is conjunction. This means that $R$ is an $m$-polar $E$-reflexive relation. To prove $T$-$E$-antisymmetry of $R$, we first show that $B\gg T$ implies $B\geq T$ by putting $x = y = 1$ in the following inequality $T(B(u,y),B(x,v))\leq B(T(u,x),T(y,v))$: $x,y,u,v\in[0,1]$. Then, it is immediately seen that $R$ is an $m$-polar $T$-$E$-antisymmetry. Thus, $R$ is an $m$-polar $T$-$E$-ordering. □

## 4. Constructing Crisp Orderings of the *m*-Polar *T*-Orderings

In decision sciences, the problem of ranking, dealing with definitions of preorder, and preference relations on the set of alternatives is the main task. Decision makers define a preorder (called partial ranking), which is a reflexive and transitive binary relation, or, in an ideal case, a total preorder (known as complete ranking) that is a complete transitive binary relation, on the set of alternatives/objects in order to compare the preference of objects and then choose the optimum one.

Dealing with $m$-polar fuzzy data, the first step is to derive a crisp preorder based on the $m$-polar fuzzy relation (called Defuzzification step) and then to develop a preference relation of alternatives. In this section, we provide a crisp ordering over $X$ by using the $m$-polar fuzzy $T$-preorder relation $R$ and the $m$-polar fuzzy $T$-equivalence relation $E$ defined in the previous section.

It is easy to check that, for the $m$-polar fuzzy $T$-preordering $R$ and the $m$-polar fuzzy $T$-equivalence $E$, the crisp relations "$\lhd_R^s$" and "$\sim_R^s$" on $X$ given by

$$\pi_s\circ R(x,y) = 1 \iff x\lhd_R^s y \tag{10}$$

and

$$\pi_s \circ E(x,y) = 1 \iff x \sim_R^s y \tag{11}$$

where $s = 1, \cdots, m$; are considered as preorder and equivalence relations, respectively. Moreover, if $x \lhd_R^s y$ and $y \lhd_R^s x$, then $\pi_s \circ R(x,y) = 1$ and $\pi_s \circ R(y,x) = 1$ that implies $x \sim_R^s y$ meaning that $\lhd_R^s$ is a preference relation on $X$ compatible with equivalence $\sim_R^s$. Note also that the intersection relation $\lhd_R = \bigcap_{s=1}^m \lhd_R^s$ is a preorder relation on $X$ compatible with equivalence $\sim_R = \bigcap_{s=1}^m \sim_R^s$.

However, the condition of equality to 1 makes the derived relations $\lhd_R^s$ and $\sim_R^s$ too strong. In many real-world examples, there exists a certain tolerance for computing the $m$-polar fuzzy relations. For example, $\pi_s \circ R(x,y) = 0.95$ can be considered as almost 1 which means the relation $x \lhd_R^s y$ may be also taken into account for such case. Motivated by Equations (10) and (11) and using the concept of **a**-cut for $m$-polar fuzzy set, where $\mathbf{a} = (a_1, \cdots, a_m) \in [0,1]^m$, a way to define crisp relations on $X$, known as **a**-cut relations, associated with the values $a_s \in [0,1]$ where $1 \leq s \leq m$, is considered as below.

**Theorem 10.** *Let $\mathbf{a} = (a_1, \cdots, a_m) \in [0,1]^m$ be a given m-tuple threshold vector. Consider the m-polar T-preordering R w.r.t A and the m-polar T-equivalence E w.r.t B over the X. The crisp relation "$\preceq_R^{s,\mathbf{a}}$" on X defined by*

$$x \preceq_R^{s,\mathbf{a}} y \iff \pi_s \circ R(x,y) \geq a_s \colon a_s \in (0,1] \tag{12}$$

*is a preference relation compatible with equivalence relation "$\sim_R^{s,\mathbf{a}}$", where $x \sim_R^{s,\mathbf{a}} y \iff \pi_s \circ E(x,y) \geq a_s \colon a_s \in (0,1]$; if and only if $T := T_M$.*

**Proof.** It is clear since the only continuous idempotent $t$-norm $T$ is $T_M$. □

The following results are obtained easily from Theorem 10.

**Proposition 6.** *Let $T_1$ and $T_2$ be some left-continuous t-norms. Consider the m-polar $T_1$-preordering $R_1$ and $T_2$-preordering $R_2$ w.r.t $A_1$ and $A_2$ and the m-polar $T_1$-equivalence $E_1$ and $T_2$-equivalence $E_2$ w.r.t $B_1$ and $B_2$, respectively. Suppose that R is the m-polar $T_M$-preordering and $\mathbf{b} = (b_1, \cdots, b_m)$ and $\mathbf{c} = (c_1, \cdots, c_m)$ are the given m-tuple threshold vectors.*
*For any $1 \leq s \leq m$, the following hold.*

1. *If $A_1 \leq A_2$, then $\lhd_{R_1}^s \subseteq \lhd_{R_2}^s$ and $\sim_{R_1}^s \subseteq \sim_{R_2}^s$.*
2. *If $T_1 \leq T_2$, then $\lhd_{R_1}^s \supseteq \lhd_{R_2}^s$ and $\sim_{R_1}^s \supseteq \sim_{R_2}^s$.*
3. *If $\mathbf{b} \leq \mathbf{c}$, then $\preceq_R^{s,\mathbf{b}} \supseteq \preceq_R^{s,\mathbf{c}}$ and $\sim_R^{s,\mathbf{b}} \supseteq \sim_R^{s,\mathbf{c}}$.*

**Proof.** It is straightforward. □

**Theorem 11.** *Suppose that the m-tuple threshold vector $\mathbf{a} = (a_1, \cdots, a_m) \in [0,1]^m$ is given such that for some $1 \leq s \leq m$; $a_s = b$. Let T be a left-continuous t-norm and F be an aggregation function. Consider the m-polar T-preordering R w.r.t A and the m-polar T-equivalence E w.r.t B. Then, the following assertions hold.*

1. *For all $1 \leq s \leq m$: if $x \lhd_R^s y$ then $x \lhd_{R_F} y$, moreover, if $x \sim_R^s y$ then $x \sim_{R_F} y$.*
2. *Let F be a disjunction. If for some $1 \leq s \leq m$: $x \lhd_R^s y$, then $x \lhd_{R_F} y$. Similarly, if for some $1 \leq s \leq m$: $x \sim_R^s y$, then $x \sim_{R_F} y$.*
3. *Let F be a conjunction. If for some $1 \leq s \leq m$: $x \not\lhd_R^s y$, then $x \not\lhd_{R_F} y$. Similarly, if for some $1 \leq s \leq m$: $x \nsim_R^s y$, then $x \nsim_{R_F} y$.*
4. *Let F have an annihilator element at b. If $x \preceq_R^{s,\mathbf{a}} y$, then $x \preceq_{R_F}^b y$. Similarly, if $x \sim_R^{s,\mathbf{a}} y$, then $x \sim_{R_F}^b y$.*
5. *Let F be a conjunction. If $\min(\pi_1 \circ R(x,y), \cdots, \pi_m \circ R(x,y)) = b_*$, then $x \preceq_{R_F}^{b_*} y$. Similarly, if $\min(\pi_1 \circ E(x,y), \cdots, \pi_m \circ E(x,y)) = c_*$, then $x \sim_{R_F}^{c_*} y$.*

6. *Let F be a disjunction. If* $\max(\pi_1 \circ R(x,y), \cdots, \pi_m \circ R(x,y)) = b^*$, *then* $x \preceq^{b^*}_{R_F} y$. *Similarly, if* $\max(\pi_1 \circ E(x,y), \cdots, \pi_m \circ E(x,y)) = c^*$, *then* $x \sim^{c^*}_{R_F} y$.

**Proof.** It is straightforward. □

## 5. Application in Decision-Making

Since the $m$-polar $T$-preorder $R$, given in the previous section, is not always complete (i.e., the equality $\max(\pi_s \circ R(x,y), \pi_s \circ R(y,x)) = 1$ is not necessarily true for all $x, y \in X$ and any $1 \leq s \leq m$), the results from the previous section do not necessarily define a complete or linear ranking over the alternatives. One way to provide a complete ranking, associated with the $m$-polar $T$-preorder relation $R$, is to use the score functions. When we deal with multi-polar data, two procedures may be offered to obtain a ranking of alternatives: (I) aggregating first (using Theorems 6 and 8) and then ranking with the help of score function, and (II) aggregating and ranking at once.

If the first way is used, we start with deriving the aggregated fuzzy relation $R_F$ of the profile $(\pi_1 \circ R, \cdots, \pi_m \circ R)$ based on an aggregation function $F$. After that, by using the crisp preorderings $\lhd_{R_F}$ or $\preceq^b_{R_F}$, the score functions $S(., \lhd_{R_F}) : X \to \mathbb{R}$ where

$$S(y, \lhd_{R_F}) = \mid \{x_i : x_i \lhd_{R_F} y\} \mid - \mid \{x_i : y \lhd_{R_F} x_i\} \mid \tag{13}$$

or $S(., \preceq^b_{R_F}) : X \to \mathbb{R}$ such that

$$S(y, \preceq^b_{R_F}) = \mid \{x_i : x_i \preceq^b_{R_F} y\} \mid - \mid \{x_i : y \preceq^b_{R_F} x_i\} \mid \tag{14}$$

for any $y \in X$ can be applied to obtain the rank of given objects (as proposed in Theorem 12). The idea behind these rules is based on the entering and leaving flows to each alternative in the crisp directed graphs $(X, \lhd_{R_F})$ or $(X, \preceq^b_{R_F})$, respectively.

**Theorem 12.** *Consider the crisp relations* $\lhd_{R_F}$ *and* $\preceq^b_{R_F}$ *on* $X$.

1. *The score function* $S(., \lhd_{R_F})$ *provides a complete preference relation on* $X$ *as*

$$y \preceq (S)x \iff S(y, \lhd_{R_F}) \leq S(x, \lhd_{R_F}) \tag{15}$$

2. *The score function* $S(., \preceq^b_{R_F})$ *provides a complete preference relation on* $X$ *defined by*

$$y \preceq (S)x \iff S(y, \preceq^b_{R_F}) \leq S(x, \preceq^b_{R_F}) \tag{16}$$

In Algorithm 1, we apply the proposed procedure for solving the problem of ranking in group decision-making with $m$-polar fuzzy inputs.

**Remark 2.** *In order to rank alternatives based on the m-polar fuzzy T-orderings, Algorithm 1 starts with computing the value of fuzzy relation* $\pi_s \circ R(x_i, x_j)$ *for* $1 \leq s \leq m$ *and any* $x_i, x_j \in X$ *which shows the strength of the relationship between* $x_i$ *and* $x_j$ *at direction s. By repeating this loop for* $s = 1, 2, 3, \cdots, m$, *a list of fuzzy relations* $\pi_1 \circ R, \pi_2 \circ R, \cdots, \pi_m \circ R$ *is provided. This helps us to construct the m-polar fuzzy relation* $R = (\pi_1 \circ R, \pi_2 \circ R, \cdots, \pi_m \circ R)$. *Then, at Step 2, the aggregated matrix* $R_F$ *is derived by applying the given aggregation function F on the profile* $R = (\pi_1 \circ R, \pi_2 \circ R, \cdots, \pi_m \circ R)$. *Each entry* $R_{Fij}$ *of matrix* $R_F$ *shows the consensus fuzzy relation between* $x_i$ *and* $x_j$. *This information is then converted to the preference matrix* $P(R_F) = [p_{ij}]_{n \times n}$ *in Step 3 to provide comparison results over the set of candidates. Finally, in Steps 4 and 5, the score of each alternative is calculated to rank candidates from the best to the worst and find the optimum choice.*

If the second way is applied, the score function $S(., R) : X \to \mathbb{R}$ defined by

$$\begin{aligned} S(y,R) = F\Big[&F(\pi_1 \circ R(x_1,y), \cdots, \pi_1 \circ R(x_n,y), 1-\pi_1 \circ R(y,x_1), \cdots, 1-\pi_1 \circ R(y,x_n)), \cdots, \\ &F(\pi_m \circ R(x_1,y), \cdots, \pi_m \circ R(x_n,y), 1-\pi_m \circ R(y,x_1), \cdots, 1-\pi_m \circ R(y,x_n))\Big] : x_i \neq y \end{aligned} \tag{17}$$

is computed based on the aggregating of $m$-polar $T$-preorder relation $R$ and its negation by function $F$. Analogues to Theorem 12, here we have the following Theorem 13.

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**Algorithm 1:** Ranking Alternatives by $m$-Polar Fuzzy $T$-Orderings

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**Input** : $m$-polar fuzzy sets $\mu_1, \cdots, \mu_K$ over the set $X$ such that $|X| = n$.
Left-continuous $t$-norm $T$.
Aggregation functions $A \gg T$ (for constructing $T$-preordering) and $F \gg T$ (for computing aggregated relation).
Threshold value $b \in [0, 1]$ (if $T$ is the minimum operator).
**Output:** Optimum solution.
**begin**
  Step 1. **for** $s = 1, 2, \ldots, m$ **do**
    **for** $i = 1, 2, \ldots, n$ **do**
      **for** $j = 1, 2, \ldots, n$ **do**
        Compute the fuzzy relation $\pi_s \circ R(x_i, x_j)$ by Equation (8).
      **end**
    **end**
  **end**
  Step 2. **for** $i = 1, 2, \ldots, n$ **do**
    **for** $j = 1, 2, \ldots, n$ **do**
      Using Theorem 6 to derive the aggregated relation $R_F(x_i, x_j) = F(\pi_1 \circ R(x_i, x_j), \cdots, \pi_m \circ R(x_i, x_j))$.
    **end**
  **end**
  Step 3. Utilize Equation (10) (or Equation (12)) for relation $R_F$ to construct an $n \times n$ preference matrix $P(R_F) = [p_{ij}]_{n\times n}$ related to the crisp relation $\lhd_{R_F}$ (relation $\preceq^b_{R_F}$) such that $p_{ij} = 1$ if $R_F(x_i, x_j) = 1$ (if $R_F(x_i, x_j) \geq b$), otherwise zero.
  Step 4. **for** $i = 1, 2, \ldots, n$ **do**
    Calculate the score value $S(x_i, \lhd_{R_F})$ (or the score value $S(x_i, \preceq^b_{R_F})$) based on the resultant matrix from Step 3 and Equation (13) (or Equation (14)).
  **end**
  Step 5. Rank the alternatives $x_i$ based on $\preceq (S)$ and then select the best one(s) (see Theorem 12).
**end**

---

**Theorem 13.** *Consider the $m$-polar $T$-preordering relation $R$ on $X$. The score function $S(., R)$ provides a complete preference relation on $X$ as*

$$y \preceq (S)x \iff S(y, R) \leq S(x, R) \tag{18}$$

Algorithm 2 shows how the second method can solve the ranking problem in $m$-polar fuzzy group decision-making.

**Algorithm 2:** Ranking Alternatives by $m$-Polar Fuzzy $T$-Orderings

**Input** : $m$-polar fuzzy sets $\mu_1, \cdots, \mu_K$ over the set $X$ such that $|X| = n$.
Left-continuous $t$-norm $T$.
Aggregation functions $A \gg T$ and $F$.
**Output:** Optimum solution.
**begin**
 Step 1. **for** $s = 1, 2, \ldots, m$ **do**
  **for** $i = 1, 2, \ldots, n$ **do**
   **for** $j = 1, 2, \ldots, n$ **do**
    Compute the fuzzy relation $\pi_s \circ R(x_i, x_j)$ by Equation (8).
   **end**
  **end**
 **end**
 Step 2. **for** $i = 1, 2, \ldots, n$ **do**
  **for** $s = 1, 2, \ldots, m$ **do**
   Derive $F(\pi_s \circ R(x_1, x_i), \cdots, \pi_s \circ R(x_{i-1}, x_i), \pi_s \circ R(x_{i+1}, x_i), \cdots, \pi_s \circ R(x_n, x_i), 1 - \pi_s \circ R(x_i, x_1), \cdots, 1 - \pi_s \circ R(x_i, x_{i-1}), 1 - \pi_s \circ R(x_i, x_{i+1}), \cdots, 1 - \pi_s \circ R(x_i, x_n))$.
  **end**
 **end**
 Step 3. **for** $i = 1, 2, \ldots, n$ **do**
  Utilize Equation (17) to compute the score value $S(x_i, R)$.
 **end**
 Step 4. Rank the alternatives $x_i$ based on $\preceq (S)$ and then select the best one(s) (see Theorem 13).
**end**

*Illustrative Example*

In any trip, the problem of accommodation is one of the most important issues. The best option is always selected after comparing different residences based on some parameters, such as facilities and location of hotels and the guest's budget. In this section, we discuss the problem of hotel booking, which is about selecting the best hotel to stay regarding a list of criteria, to provide a real-life example which shows the application of our method in decision-making problems. We apply some data given in [28], obtained from the "www.agoda.com" website, as our input (see Table 1).

**Example 3.** *Let us suppose that a travel agency wants to book a four-star hotel in Kuala Lumpur, Malaysia, for a customer. Let $H = \{h_1, \cdots, h_{10}\}$ be a short list of ten four-star hotels in Kuala Lumpur which are selected for consideration. To choose the optimum option, these hotels are compared with each other based on the following two parameters $P = \{p_1$ = Services and Facilities, $p_2$ = Food$\}$ that are the most important criteria for the customer. The comments of five guests of these hotels, which are shown by $\mu_1, \cdots, \mu_5$, who filled up the online questioners from five different categories "Families with Young Children", "Families with Elder Children", "Couples", "Solo Travelers", and "Group of Friends" are taken into consideration as the initial evaluation of these hotels by using 2-polar fuzzy sets (see Table 1).*

*For t-norm $T := T_L$ and by using Equation (8) (see Theorem 3), the 2-polar $T_L$-preordering $R_1$ w.r.t Min operator and the 2-polar $T_L$-equivalence $E_1$ w.r.t Min operator are obtained as shown in Tables 2 and 3.*

*If we change the t-norm $T$ to $T_M$, consequently, $A := F :=$ Min, the 2-polar $T_M$-preordering $R_2$ w.r.t Min operator and the 2-polar $T_M$-equivalence $E_2$ w.r.t Min operator are obtained as shown in Tables 4 and 5.*

*By using the proposed method in Algorithm 1 for aggregation functions $F := AM$, the score values $S(x_i, \lhd_{R_{AM}})$ for $1 \leq i \leq 10$ are computed as can be seen in Table 6.*

*Thus, the new score values $S(x_i, \lhd_{R_{Min}})$ will be obtained. Moreover, in this case, by using the cut-relation for $b = 0.5$ and $b = 0.7$, the scores $S(x_i, \preceq^{0.5}_{R_{Min}})$ and $S(x_i, \preceq^{0.7}_{R_{Min}})$ are computed (see Table 6).*

*Now, let the given procedure in Algorithm 2 be applied to find the score values $S(x_i, R_1)$ and $S(x_i, R_2)$, for $1 \leq i \leq 10$, where $F := AM$ and $T := T_L$ and $T_M$. Then, the new rankings of objects are obtained as can be seen in Table 7. However, in all methods, $h_{10}$ is the optimum selection.*

**Table 1.** Tabular representation of hotels evaluation by 2-polar fuzzy data.

| $H$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\mu_5$ |
|---|---|---|---|---|---|
| $h_1$ | (0.7,0.58) | (0.72,0.59) | (0.74,0.64) | (0.6,0.68) | (0.84,0.6) |
| $h_2$ | (0.55,0.71) | (0.6,0.69) | (0.7,0.66) | (0.4,0.64) | (0.74,0.64) |
| $h_3$ | (0.6,0.6) | (0.73,0.57) | (0.73,0.57) | (0.8,0.62) | (0.8,0.62) |
| $h_4$ | (0.88,0.55) | (0.87,0.54) | (0.73,0.61) | (0.74,0.69) | (0.77,0.59) |
| $h_5$ | (0.64,0.69) | (0.8,0.66) | (0.69,0.65) | (0.64,0.61) | (0.76,0.61) |
| $h_6$ | (0.66,0.67) | (0.68,0.69) | (0.65,0.69) | (0.76,0.66) | (0.72,0.66) |
| $h_7$ | (0.73,0.66) | (0.8,0.66) | (0.8,0.65) | (0.53,0.66) | (0.76,0.64) |
| $h_8$ | (0.73,0.68) | (0.75,0.64) | (0.8,0.67) | (0.7,0.64) | (0.76,0.67) |
| $h_9$ | (0.8,0.64) | (0.85,0.49) | (0.75,0.57) | (0.6,0.71) | (0.69,0.44) |
| $h_{10}$ | (0.86,0.83) | (0.86,0.85) | (0.8,0.77) | (1,0.75) | (0.82,0.77) |

**Table 2.** Tabular representation of 2-polar $T_L$-preorder $R_1$ w.r.t Min.

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ | $h_7$ | $h_8$ | $h_9$ | $h_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_1$ | (1,1) | (0.8,0.96) | (0.9,0.93) | (0.93,0.95) | (0.92,0.93) | (0.88,0.98) | (0.92,0.98) | (0.92,0.96) | (0.85,0.84) | (0.98,1) |
| $h_2$ | (1,0.87) | (1,1) | (1,0.88) | (1,0.84) | (0.99,0.97) | (0.95,0.96) | (1,0.95) | (1,0.95) | (0.95,0.8) | (1,1) |
| $h_3$ | (0.8,0.98) | (0.6,1) | (1,1) | (0.94,0.95) | (0.84,0.99) | (0.92,1) | (0.73,1) | (0.9,1) | (0.8,0.82) | (1,1) |
| $h_4$ | (0.82,0.99) | (0.66,0.95) | (0.72,0.93) | (1,1) | (0.76,0.92) | (0.78,0.97) | (0.79,0.97) | (0.85,0.95) | (0.86,0.85) | (0.98,1) |
| $h_5$ | (0.92,0.89) | (0.76,1) | (0.93,0.91) | (1,0.86) | (1,1) | (0.88,0.98) | (0.89,0.97) | (0.95,0.98) | (0.93,0.83) | (1,1) |
| $h_6$ | (0.84,0.9) | (0.64,0.97) | (0.94,0.88) | (0.98,0.85) | (0.88,0.95) | (1,1) | (0.77,0.96) | (0.94,0.95) | (0.84,0.78) | (1,1) |
| $h_7$ | (0.92,0.92) | (0.8,0.98) | (0.87,0.91) | (0.93,0.88) | (0.89,0.95) | (0.85,1) | (1,1) | (0.95,0.98) | (0.93,0.8) | (1,1) |
| $h_8$ | (0.9,0.9) | (0.7,0.97) | (0.87,0.9) | (0.93,0.87) | (0.89,0.94) | (0.85,0.99) | (0.83,0.97) | (1,1) | (0.9,0.77) | (1,1) |
| $h_9$ | (0.87,0.94) | (0.75,0.93) | (0.8,0.91) | (0.98,0.91) | (0.84,0.9) | (0.83,0.95) | (0.93,0.95) | (0.9,0.93) | (1,1) | (1,1) |
| $h_{10}$ | (0.6,0.74) | (0.4,0.84) | (0.74,0.72) | (0.74,0.69) | (0.64,0.81) | (0.76,0.84) | (0.53,0.81) | (0.7,0.79) | (0.6,0.64) | (1,1) |

**Table 3.** Tabular representation of 2-polar $T_L$-equivalence $E_1$ w.r.t Min.

| $H$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ | $h_7$ | $h_8$ | $h_9$ | $h_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_1$ | (1,1) | (0.8,0.87) | (0.8,0.93) | (0.82,0.95) | (0.92,0.89) | (0.84,0.9) | (0.92,0.92) | (0.9,0.9) | (0.85,0.84) | (0.6,0.74) |
| $h_2$ | (0.8,0.87) | (1,1) | (0.6,0.88) | (0.66,0.84) | (0.76,0.97) | (0.64,0.96) | (0.8,0.95) | (0.7,0.95) | (0.75,0.8) | (0.4,0.84) |
| $h_3$ | (0.8,0.93) | (0.6,0.88) | (1,1) | (0.72,0.93) | (0.84,0.91) | (0.92,0.88) | (0.73,0.91) | (0.87,0.9) | (0.8,0.82) | (0.74,0.72) |
| $h_4$ | (0.82,0.95) | (0.66,0.84) | (0.72,0.93) | (1,1) | (0.76,0.86) | (0.78,0.85) | (0.79,0.88) | (0.85,0.87) | (0.86,0.85) | (0.74,0.69) |
| $h_5$ | (0.92,0.89) | (0.76,0.97) | (0.84,0.91) | (0.76,0.86) | (1,1) | (0.88,0.95) | (0.89,0.95) | (0.89,0.94) | (0.84,0.83) | (0.64,0.81) |
| $h_6$ | (0.84,0.9) | (0.64,0.96) | (0.92,0.88) | (0.78,0.85) | (0.88,0.95) | (1,1) | (0.77,0.96) | (0.85,0.95) | (0.83,0.78) | (0.76,0.84) |
| $h_7$ | (0.92,0.92) | (0.8,0.95) | (0.73,0.91) | (0.79,0.88) | (0.89,0.95) | (0.77,0.96) | (1,1) | (0.83,0.97) | (0.93,0.8) | (0.53,0.81) |
| $h_8$ | (0.9,0.9) | (0.7,0.95) | (0.87,0.9) | (0.85,0.87) | (0.89,0.94) | (0.85,0.95) | (0.83,0.97) | (1,1) | (0.9,0.77) | (0.7,0.79) |
| $h_9$ | (0.85,0.84) | (0.75,0.8) | (0.8,0.82) | (0.86,0.85) | (0.84,0.83) | (0.83,0.78) | (0.93,0.8) | (0.9,0.77) | (1,1) | (0.6,0.64) |
| $h_{10}$ | (0.6,0.74) | (0.4,0.84) | (0.74,0.72) | (0.74,0.69) | (0.64,0.81) | (0.76,0.84) | (0.53,0.81) | (0.7,0.79) | (0.6,0.64) | (1,1) |

**Table 4.** Tabular representation of 2-polar $T_M$-preorder $R_2$ w.r.t. Min.

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ | $h_7$ | $h_8$ | $h_9$ | $h_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_1$ | (1,1) | (0.4,0.64) | (0.6,0.57) | (0.73,0.54) | (0.64,0.61) | (0.65,0.66) | (0.53,0.66) | (0.76,0.64) | (0.69,0.44) | (0.82,1) |
| $h_2$ | (1,0.58) | (1,1) | (1,0.57) | (1,0.54) | (0.69,0.61) | (0.65,0.67) | (1,0.65) | (1,0.64) | (0.69,0.44) | (1,1) |
| $h_3$ | (0.6,0.58) | (0.4,1) | (1,1) | (0.74,0.54) | (0.64,0.61) | (0.65,1) | (0.53,1) | (0.7,1) | (0.6,0.44) | (1,1) |
| $h_4$ | (0.6,0.68) | (0.4,0.64) | (0.6,0.57) | (1,1) | (0.64,0.61) | (0.65,0.66) | (0.53,0.66) | (0.7,0.64) | (0.6,0.44) | (0.86,1) |
| $h_5$ | (0.6,0.58) | (0.4,1) | (0.6,0.57) | (1,0.54) | (1,1) | (0.65,0.67) | (0.53,0.66) | (0.75,0.64) | (0.60.44) | (1,1) |
| $h_6$ | (0.6,0.58) | (0.4,0.64) | (0.6,0.57) | (0.74,0.54) | (0.64,0.61) | (1,1) | (0.53,0.64) | (0.7,0.64) | (0.6,0.44) | (1,1) |
| $h_7$ | (0.7,0.58) | (0.4,0.64) | (0.6,0.57) | (0.73,0.54) | (0.64,0.61) | (0.65,1) | (1,1) | (0.75,0.64) | (0.69,0.44) | (1,1) |
| $h_8$ | (0.6,0.58) | (0.4,0.64) | (0.6,0.57) | (0.73,0.54) | (0.64,0.61) | (0.65,0.66) | (0.53,0.64) | (1,1) | (0.6,0.44) | (1,1) |
| $h_9$ | (0.7,0.58) | (0.4,0.64) | (0.6,0.6) | (0.73,0.55) | (0.64,0.61) | (0.65,0.66) | (0.53,0.66) | (0.73,0.64) | (1,1) | (1,1) |
| $h_{10}$ | (0.6,0.58) | (0.4,0.64) | (0.6,0.57) | (0.73,0.54) | (0.64,0.61) | (0.65,0.66) | (0.53,0.64) | (0.7,0.64) | (0.6,0.44) | (1,1) |

**Table 5.** Tabular representation of 2-polar $T_M$-preorder $E_2$ w.r.t Min.

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ | $h_7$ | $h_8$ | $h_9$ | $h_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_1$ | (1,1) | (0.4,0.58) | (0.6,0.57) | (0.6,0.54) | (0.6,0.58) | (0.6,0.58) | (0.53,0.58) | (0.6,0.58) | (0.69,0.44) | (0.6,0.58) |
| $h_2$ | (0.4,0.58) | (1,1) | (0.4,0.57) | (0.4,0.54) | (0.4,0.61) | (0.4,0.64) | (0.4,0.64) | (0.4,0.64) | (0.4,0.44) | (0.4,0.64) |
| $h_3$ | (0.6,0.57) | (0.4,0.57) | (1,1) | (0.6,0.54) | (0.6,0.57) | (0.6,0.57) | (0.53,0.57) | (0.6,0.57) | (0.6,0.44) | (0.6,0.57) |
| $h_4$ | (0.6,0.54) | (0.4,0.54) | (0.6,0.54) | (1,1) | (0.64,0.54) | (0.65,0.54) | (0.53,0.54) | (0.7,0.54) | (0.6,0.44) | (0.73,0.54) |
| $h_5$ | (0.6,0.58) | (0.4,0.61) | (0.6,0.57) | (0.64,0.54) | (1,1) | (0.64,0.61) | (0.53,0.61) | (0.64,0.61) | (0.6,0.44) | (0.64,0.61) |
| $h_6$ | (0.6,0.58) | (0.4,0.64) | (0.6,0.57) | (0.65,0.54) | (0.64,0.61) | (1,1) | (0.53,0.64) | (0.65,0.64) | (0.6,0.44) | (0.65,0.66) |
| $h_7$ | (0.53,0.58) | (0.4,0.64) | (0.53,0.57) | (0.53,0.54) | (0.53,0.61) | (0.53,0.64) | (1,1) | (0.53,0.64) | (0.53,0.44) | (0.53,0.64) |
| $h_8$ | (0.6,0.58) | (0.4,0.64) | (0.6,0.57) | (0.7,0.54) | (0.64,0.61) | (0.65,0.64) | (0.53,0.64) | (1,1) | (0.6,0.44) | (0.7,0.64) |
| $h_9$ | (0.69,0.44) | (0.4,0.44) | (0.6,0.44) | (0.6,0.44) | (0.6,0.44) | (0.6,0.44) | (0.53,0.44) | (0.6,0.44) | (1,1) | (0.6,0.44) |
| $h_{10}$ | (0.6,0.58) | (0.4,0.64) | (0.6,0.57) | (0.73,0.54) | (0.64,0.61) | (0.65,0.66) | (0.53,0.64) | (0.7,0.64) | (0.6,0.44) | (1,1) |

**Table 6.** Ranking of Hotels by Algorithm 1.

| $H$ | $F := AM$<br>$T := T_L, A := Min$<br>$S(x_i, \lhd_{R_{AM}})$ | $S(x_i, \lhd_{R_{Min}})$ | $F := Min$<br>$T := T_M, A := Min$<br>$S(x_i, \preceq^{0.5}_{R_{Min}})$ | $S(x_i, \preceq^{0.7}_{R_{Min}})$ |
|---|---|---|---|---|
| $h_1$ | 0 | 0 | 2 | −1 |
| $h_2$ | −1 | −1 | −8 | −1 |
| $h_3$ | −1 | −1 | 2 | −2 |
| $h_4$ | 0 | 0 | 2 | −1 |
| $h_5$ | −1 | −1 | 2 | −1 |
| $h_6$ | −1 | −1 | 2 | −1 |
| $h_7$ | −1 | −1 | 2 | −1 |
| $h_8$ | −1 | −1 | 2 | 0 |
| $h_9$ | −1 | −1 | −8 | −1 |
| $h_{10}$ | 7 | 7 | 2 | 9 |
| Preference Order | $h_2, h_3, h_5, h_6, h_7, h_8,$<br>$h_9 \preceq (S)h_1, h_4 \preceq$<br>$(S)h_{10}$ | $h_2, h_3, h_5, h_6, h_7, h_8,$<br>$h_9 \preceq (S)h_1, h_4 \preceq$<br>$(S)h_{10}$ | $h_2, h_9 \preceq$<br>$(S)h_1, h_3, h_4, h_5,$<br>$h_6, h_7, h_8, h_{10}$ | $h_3 \preceq$<br>$(S)h_1, h_2, h_4, h_5, h_6,$<br>$h_7, h_9 \preceq (S)h_8 \preceq$<br>$(S)h_{10}$ |

**Table 7.** Ranking of Hotels by Algorithm 2.

| $H$ | $T := T_L, A := Min$ $S(x_i, R_1)$ | $F := AM$ | $T := T_M, A := Min$ $S(x_i, R_2)$ |
|---|---|---|---|
| $h_1$ | 0.479 | | 0.495 |
| $h_2$ | 0.44 | | 0.399 |
| $h_3$ | 0.487 | | 0.443 |
| $h_4$ | 0.512 | | 0.514 |
| $h_5$ | 0.483 | | 0.474 |
| $h_6$ | 0.508 | | 0.528 |
| $h_7$ | 0.485 | | 0.480 |
| $h_8$ | 0.511 | | 0.541 |
| $h_9$ | 0.462 | | 0.436 |
| $h_{10}$ | 0.634 | | 0.692 |
| Preference Order | $h_2 \preceq (S)h_9 \preceq (S)h_1 \preceq (S)h_5 \preceq (S)h_7 \preceq (S) h_3 \preceq (S)h_6 \preceq (S)h_8 \preceq (S)h_4 \preceq (S)h_{10}$ | $h_2 \preceq (S)h_9 \preceq (S)h_3 \preceq (S)h_5 \preceq (S)h_7 \preceq (S) h_1 \preceq (S)h_4 \preceq (S)h_6 \preceq (S)h_8 \preceq (S)h_{10}$ | |

## 6. Conclusions

The concept of $T$-orderings that is defined based on the infimum of implications depends only on the choice of $t$-norm $T$. By restricting the study to the case dealing with finite records of fuzzy information, the infimum is represented as minimum operator, belonging to the family of aggregation functions. In this case, a very natural extension is to generalize $T$-orderings based on any aggregation functions, such as arithmetic mean or geometric mean.

This paper has considered the generalized $T$-preorderings and $T$-equivalences for $m$-polar fuzzy sets based on $m$-polar implication operators and aggregation functions. The interesting observation is the domination of an aggregation operator over the $t$-norm $T$. In contrast to literature, which usually talked about fuzzification of crisp orderings, this paper has discussed how crisp orderings can be obtained by direct defuzzification (or cut-relations) of $m$-polar fuzzy $T$-orderings. As a result, two alternative ways of ranking the $m$-polar fuzzy data are formulated based on new score value functions, computed by generalized $m$-polar $T$-orderings, and illustrated by a numerical example. Some other problems that may be handled in the future are a deeper analysis of the proposed $m$-polar $T$-orderings and the search for necessary and sufficient conditions for the characterization of such relations based on pre-aggregation functions.

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