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Asymptotic Distributions for Power Variations of the Solutions to Linearized Kuramoto–Sivashinsky SPDEs in One-to-Three Dimensions

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Abstract: We study the realized power variations for the fourth order linearized Kuramoto–Sivashinsky (LKS) SPDEs and their gradient, driven by the space–time white noise in one-to-three dimensional spaces, in time, have infinite quadratic variation and dimension-dependent Gaussian asymptotic distributions. This class was introduced-with Brownian-time-type kernel formulations by Allouba in a series of articles starting in 2006. He proved the existence, uniqueness, and sharp spatio-temporal Hölder regularity for the above class of equations in $d = 1, 2, 3$. We use the relationship between LKS-SPDEs and the Houdré–Villaa bifractional Brownian motion (BBM), yielding temporal central limit theorems for LKS-SPDEs and their gradient. We use the underlying explicit kernels and spectral/harmonic analysis to prove our results. On one hand, this work builds on the recent works on the delicate analysis of variations of general Gaussian processes and stochastic heat equation driven by the space–time white noise. On the other hand, it builds on and complements Allouba's earlier works on the LKS-SPDEs and their gradient.

Keywords: quadratic variation; power variation; linearized Kuramoto–Sivashinsky SPDEs; space–time white noise; weak convergence

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1. Introduction

The fourth order linearized Kuramoto–Sivashinsky (LKS) SPDEs are related to the model of pattern formation phenomena accompanying the appearance of turbulence (see [1–4] for the LKS class and for its connection to many classical and new examples of deterministic and stochastic pattern formation PDEs, and see [5,6] for classical examples of deterministic and stochastic pattern formation PDEs).

The fundamental kernel associated with the deterministic version of this class is built on the Brownian-time process in [3,7,8]. In this article, we give exact dimension-dependent asymptotic distributions of the realized power variations in time, for the important class of stochastic equation:

$$\begin{cases} \dfrac{\partial U}{\partial t} = -\dfrac{\varepsilon}{8}(\mathcal{L} + 2\vartheta)^2 U + \dfrac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $\mathcal{L}$ is the d-dimensional Laplacian operator, $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ is a pair of parameters, the noise term $\partial^{d+1} W / \partial t \partial x$ is the space–time white noise corresponding to the real-valued Brownian sheet W on $\mathbb{R}_+ \times \mathbb{R}^d$, $d = 1, 2, 3$. The initial data u_0 here is assumed Borel measurable, deterministic, and 2-continuously differentiable on $\mathbb{R}^d$ whose 2-derivative is locally Hölder continuous with some exponent $0 < \gamma \leq 1$.

Of course, Equation (1) is the formal (and nonrigorous) equation. Its rigorous formulation, which we work with in this paper, is given in mild form as kernel stochastic integral equation (SIE). This SIE was first introduced and studied by [1–3,7–10]. We give it below in Section 3, along with some relevant details.

The existence/uniqueness as well as sharp dimension-dependent L^p and Hölder regularity of the linear and nonlinear noise version of (1) were investigated in [1,2,9,10]. It was studied in [4] that exact uniform and local moduli of continuity for the LKS-SPDE in the time variable t and space variable x, separately. In fact, it was established in [4] that exact, dimension-dependent, spatio-temporal, uniform and local moduli of continuity for the fourth order the LKS-SPDEs and their gradient. It was studied in [11] that the solution to a stochastic heat equation with the space–time white noise in time has infinite quadratic variation and is not a semimartingale, and also investigated temporal central limit theorems for modifications of the quadratic variation of the stochastic heat equation with space–time white noise in time.

The analysis of the asymptotic behavior of the realized variations is motivated by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by a Brownian motion B (see, e.g., [11,12]), besides, of course, the traditional applications of the realized variations to parameter estimation problems (see, e.g., [13–19] in which asymptotic distributions for power variations of fractional Brownian motion (FBM) and related Gaussian processes were investigated).

In this paper we show that the realized power variation of the process U and its gradient in time, have infinite quadratic variation and dimension-dependent Gaussian asymptotic distributions. It builds on and complements Allouba and Xiao's earlier works on the LKS-SPDEs and builds on the recent works on delicate analysis of variations of Gaussian processes and stochastic heat equations with space–time white noise. Our proof is based on the approach method in [11]. We make use of the product-moments of various orders of the normal correlation surface of two variates in [20] to establish exact convergence rates of variances of the realized power variation of the process U and its gradient in time. On one hand, this work builds on the recent works on delicate analysis of variations of general Gaussian processes and stochastic heat equation driven by the space–time white noise. Moreover, it builds on and complements Allouba's earlier works on the LKS-SPDEs and their gradient.

The rest of the paper is organized as follows. Some notations and main results of this paper are stated in Section 2. In Section 3, we discuss the rigorous LKS-SPDE kernel SIE (mild) formulation and estimate the temporal increments of LKS-SPDEs and their gradient by using the LKS-SPDE kernel SIE formulation and spectral/harmonic analysis. As a consequence of the result obtained, both LKS-SPDEs and their gradient in time have infinite quadratic variation. In Section 4, we prove Theorems 1 and 2 by using the product-moments of various orders of the normal correlation surface of two variates in [20] and the approach method in [11], respectively. In the final section, the results are summarized and discussed.

2. Statement of Results

2.1. Exact Convergence Rates of Variances and Temporal CLTs for the Realized Power Variations of LKS-SPDEs

In order to establish our main results we first introduce some notation. We consider discrete Riemann sums over a uniformly spaced time partition $t_j = j\Delta t$, where $\Delta t = n^{-1}$. Fix $x \in \mathbb{R}^d$. Let $\Delta U_{x;j} = U(t_j, x) - U(t_{j-1}, x)$ and $\sigma_{x;j} = (\mathbb{E}[\Delta U^2_{x;j}])^{1/2}$. For any $p \in \mathbb{N}_+$ and $n \in \mathbb{N}_+$, we define

$$\Xi^n_p(U(\cdot, x))_t = \sum_{j=1}^{\lfloor nt \rfloor} \Delta U^p_{x;j}.$$

Here and in the sequel, $\lfloor a \rfloor$ denotes an integer satisfying $a - 1 < \lfloor a \rfloor \le a$ for $a \in \mathbb{R}_+$.

Let μ_p denote the p-moment of a standard Gaussian random variable following an $\mathcal{N}(0,1)$ law, that is, $\mu_{2p-1} = 0$ and $\mu_{2p} = (2p-1)!! = (2p)!/(p!2^p)$ for all $p \in \mathbb{N}_+$. For $j \in \mathbb{N}_+$, let $\phi_{d;j} = 2j^{1-d/4} - (j-1)^{1-d/4} - (j+1)^{1-d/4}$. For real number $r \ge 1$, define $J_{d,r} = \sum_{j=1}^{\infty} \phi_{d;j}^r$. It follows from (49) below that $J_{d,r}$ is a positive and finite constant depending only on r. For any $p \in \mathbb{N}_+$, we define $\kappa_{d,p} = K_d^p \lambda_{d,p}$, where

$$K_d = \frac{1}{2^d(2-d/2)\pi^{d/2}\Gamma(d/2)} \left(\frac{8}{\varepsilon}\right)^{d/4} \int_0^{\infty} y^{d/2-1} e^{-y^2} dy, \tag{2}$$

and

$$\lambda_{d,p} = \begin{cases} \mu_{2p} - \mu_p^2 + \dfrac{p!p!}{2^{p-1}} \displaystyle\sum_{u=1}^{\lfloor p/2 \rfloor} \frac{2^{2u} J_{d,2u}}{(\lfloor p/2 \rfloor - u)!(\lfloor p/2 \rfloor - u)!(2u)!}, & \text{if } p \text{ is even}, \\ \mu_{2p} - \dfrac{p!p!}{2^{p-2}} \displaystyle\sum_{u=0}^{\lfloor p/2 \rfloor} \frac{2^{2u} J_{d,2u+1}}{(\lfloor p/2 \rfloor - u)!(\lfloor p/2 \rfloor - u)!(2u+1)!}, & \text{if } p \text{ is odd}. \end{cases} \tag{3}$$

Here $\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} du$, $s > 0$, is the Gamma function.

We will first show the exact convergence rates of variance for the realized power variation of processes U.

Theorem 1. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}^d$, and assume $d \in \{1,2,3\}$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then for each fixed $t > 0$ and any $p \in \mathbb{N}_+$,*

$$n^{-1+p(1-d/4)}\, Var(\Xi_p^n(U(\cdot, x))_t) \;\to\; \kappa_{d,p}\, t \tag{4}$$

as n tends to infinity.

By (4), we have the following convergence in probability for the realized power variation of the process U.

Corollary 1. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}^d$, and assume $d \in \{1,2,3\}$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then for each fixed $t > 0$ and any $p \in \mathbb{N}_+$,*

$$n^{-1+p(1-d/4)/2} \Xi_p^n(U(\cdot, x))_t \;\to\; K_d^{p/2} \mu_p\, t \tag{5}$$

in L^2 and in probability as n tends to infinity.

Remark 1. *Since $\Xi_{2p}^n(U(\cdot, x))_t$ is monotone, (5) implies that $n^{-1+p(1-d/4)}\ \Xi_{2p}^n(U(\cdot, x))_t \to K_d^p \mu_{2p}\, t$ uniform convergence in probability in the time interval $[0, T]$ with some $T > 0$. Moreover, (5) implies that for a fixed point in space, the process $U(\cdot, x)$ has infinite quadratic variation.*

Temporal central limit theorems (CLTs) for the realized power variation of processes U is as follows.

Theorem 2. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}^d$, and assume $d \in \{1,2,3\}$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then for any $p \in \mathbb{N}_+$,*

$$\left(U(t,x), \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} (n^{p(1-d/4)/2} \Delta U_{x;j}^p - K_d^{p/2} \mu_p) \right) \xrightarrow{\mathcal{L}} (U(t,x), \kappa_{d,p}^{1/2} B(t)) \tag{6}$$

as n tends to infinity, where $B = \{B(t), t \in [0, T]\}$ is a Brownian motion independent of the process U, and the convergence is in the space $D([0, T])^2$ equipped with the Skorohod topology.

Remark 2. *By (2) and (3), both K_d and $\kappa_{d,p}$ in (4)–(6) are dependent on spatial dimension but independent of x.*

2.2. Exact Convergence Rates of Variances and Temporal CLTs for the Realized Power Variations of LKS-SPDE Gradient

Fix $x \in \mathbb{R}$. Let $\partial_x \Delta U_{x;j} = \partial_x U(t_j, x) - \partial_x U(t_{j-1}, x)$ and $\partial_x \sigma_{x;j} = (\mathbb{E}[\partial_x \Delta U_{x;j}^2])^{1/2}$. For any $p \in \mathbb{N}_+$ and $n \in \mathbb{N}_+$, we define

$$\partial_x \Xi_p^n(U(\cdot, x))_t = \sum_{j=1}^{\lfloor nt \rfloor} \partial_x \Delta U_{x;j}^p.$$

For any $p \in \mathbb{N}_+$, we define $\chi_{d,p} = D_0^p \lambda_{d,p}$, where $\lambda_{d,p}$ is given in (2) and

$$D_0 = (2\pi)^{-1} \left(\frac{8}{\varepsilon}\right)^{3/4} \int_0^\infty y^{-1/4} e^{-y} dy. \tag{7}$$

We will first show the exact convergence rates of variance for the realized power variation of the gradient processes $\partial_x U(t, x)$.

Theorem 3. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}$, and assume $d = 1$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then for each fixed $t > 0$ and any $p \in \mathbb{N}_+$,*

$$n^{-1+p(1-d/4)}\, Var(\partial_x \Xi_p^n(U(\cdot, x))_t) \;\to\; \chi_{d,p}\, t \tag{8}$$

as n tends to infinity.

By (8), we have the following convergence in probability for the realized power variation of the gradient process $\partial_x U(t, x)$.

Corollary 2. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}$, and assume $d = 1$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then for each fixed $t > 0$ and any $p \in \mathbb{N}_+$,*

$$n^{-1+p(1-d/4)/2} \partial_x \Xi_p^n(U(\cdot, x))_t \;\to\; D_0^{p/2} \mu_p\, t \tag{9}$$

in L^2 and in probability as n tends to infinity.

Remark 3. *Since $\partial_x \Xi_{2p}^n(U(\cdot, x))_t$ is monotone, (9) implies that $n^{-1+p(1-d/4)}\, \partial_x \Xi_{2p}^n(U(\cdot, x))_t \to D_0^p \mu_{2p}\, t$ uniform convergence in probability in the time interval $[0, T]$ with some $T > 0$. Moreover, (9) implies that for a fixed point in space, the gradient process $\partial_x U(\cdot, x)$ has infinite quadratic variation.*

Temperal central limit theorems for the realized power variation of the gradient processes $\partial_x U(t, x)$ is as follows.

Theorem 4. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}$, and assume $d = 1$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then for any $p \in \mathbb{N}_+$,*

$$\left(\partial_x U(t, x), \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} (n^{p(1-d/4)/2} \partial_x \Delta U_{x;j}^p - D_0^{p/2} \mu_p)\right) \xrightarrow{\mathcal{L}} (\partial_x U(t, x), \chi_{d,p}^{1/2} B(t)) \tag{10}$$

as n tends to infinity, where $B = \{B(t), t \in [0, T]\}$ is a Brownian motion independent of the process U, and the convergence is in the space $D([0, T])^2$ equipped with the Skorohod topology.

Remark 4. *It is natural to expect that (6) and (10) hold for $x \mapsto U(t, x)$ in $d = 1, 2, 3$. However, substantial extra work is needed for proving these statements. In particular, in order to apply the method in [11], one will have to establish the property of the increments for $U(t, \cdot)$. Unfortunately the method in [11] does not seem useful anymore and some new ideas may be needed.*

Remark 5. *By using Lemma 3 below, following the same lines as the proof of Theorem 1, we get Theorem 3. Similarly, following the same lines as the proof of Theorem 2, we get Theorem 4. Therefore, only Theorems 1 and 2 are proved and Theorems 3 and 4 are omitted.*

3. Methodology

3.1. Rigorous Kernel Stochastic Integral Equations Formulations

As in [4], for the LKS-SPDE, we use the LKS kernel to define their rigorous mild SIE formulation. This LKS kernel, as shown in as in [1–3], is the fundamental solution to the deterministic version of (12) ($a \equiv 0$ and $b \equiv 0$) below, and is given by:

$$\begin{aligned}\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t;x,y} &= \int_{-\infty}^{0} \frac{e^{\mathbf{i}\vartheta s} e^{-|x-y|^2/(2\mathbf{i}s)}}{(2\pi \mathbf{i}s)^{d/2}} K^{\mathrm{BM}}_{\varepsilon t;s} ds + \int_{0}^{\infty} \frac{e^{\mathbf{i}\vartheta s} e^{-|x-y|^2/(2\mathbf{i}s)}}{(2\pi \mathbf{i}s)^{d/2}} K^{\mathrm{BM}}_{\varepsilon t;s} ds \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2} e^{\mathbf{i}\langle \xi, x-y\rangle} d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2} \cos(\langle \xi, x-y\rangle) d\xi, \quad (\varepsilon,\vartheta) \in \mathbb{R}_+ \times \mathbb{R},\end{aligned} \tag{11}$$

where $\mathbf{i} = \sqrt{-1}$ and $K^{\mathrm{BM}}_{t;s} = \frac{e^{-s^2/(2t)}}{\sqrt{2\pi t}}$. Let $b : \mathbb{R} \to \mathbb{R}$ be Borel measurable. The nonlinear drift-diffusion LKS-SPDE is

$$\begin{cases} \dfrac{\partial U}{\partial t} = -\dfrac{\varepsilon}{8}(\mathcal{L} + 2\vartheta)^2 U + b(U) + a(U)\dfrac{\partial^{d+1} W}{\partial t \partial x}, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\ U(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{12}$$

Then, the rigorous LKS kernel SIE (mild) formulation is the stochastic integral equation

$$\begin{aligned} U(t,x) = &\int_{\mathbb{R}^d} \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t;x,y} u_0(y) dy \\ &\int_{\mathbb{R}^d} \int_0^t \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t-s;x,y} [b(U(s,y)) ds dy + a(U(s,y)) W(ds \times dy)] \end{aligned} \tag{13}$$

(see p. 530 in [5] and Definition 1.1 and Equation (1.11) in [1]). Of course, the mild formulation of (1.1) is then obtained by setting $a \equiv 1$ and $b \equiv 0$ in (13).

Notation 1. *Positive and finite constants (independent of x) in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.*

We conclude this section by citing the following spatial Fourier transform of the (ε,ϑ) LKS kernels from Lemma 2.1 in [4].

Lemma 1. *Let $\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t;x}$ be the (ε,ϑ) LKS kernel. The spatial Fourier transform of the (ε,ϑ) LKS kernel in (11) is given by*

$$\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t;\xi} = (2\pi)^{-d/2} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2}; \quad (\varepsilon,\vartheta) \in \mathbb{R}_+ \times \mathbb{R}. \tag{14}$$

Here, the following symmetric form of the spatial Fourier transform has been used: $\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(u)\, e^{-\mathbf{i}\xi\cdot u} du$.

3.2. Estimates on the Temporal Increments of LKS-SPDEs and Their Gradient

Since $U(\cdot,x)$ is a centered Gaussian process, its law is determined by its covariance function, which is given in the following lemma. We also derive some needed estimates on the covariance function and the increment of $U(\cdot,x)$.

Lemma 2. *Fix $(\varepsilon,\vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}^d$, and assume $d \in \{1,2,3\}$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). For all $s,t \in (0,T]$, we have*

$$\mathbb{E}[U(t,x)U(s,x)] = K_d[(t+s)^{1-d/4} - |t-s|^{1-d/4}], \tag{15}$$

$$c_{4,1}|t-s|^{1-d/4} \le \mathbb{E}[(U(t,x)-U(s,x))^2] \le c_{4,2}|t-s|^{1-d/4}, \tag{16}$$

and

$$|\mathbb{E}[(U(t,x)-U(s,x))^2] - K_d|t-s|^{1-d/4}| \le \frac{c_{4,3}}{s^{d/4+1}}|t-s|^2, \tag{17}$$

where K_d is given in (3).

Proof. To show (15), we use Parseval's identity to get

$$\begin{aligned} \mathbb{E}[U(t,x)U(s,x)] &= \int_{\mathbb{R}^d}\int_0^s \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,y}\,\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,y}drdy \\ &= \int_0^s\int_{\mathbb{R}^d} \hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi}\,\overline{\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,\xi}}d\xi dr \\ &= (2\pi)^{-d}\int_0^s\int_{\mathbb{R}^d} e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4-\frac{\varepsilon(s-r)}{8}|\xi|^4}d\xi dr \\ &= (2\pi)^{-d}\int_0^s\int_{\mathbb{R}^d} e^{-\frac{\varepsilon(t+s-2r)}{8}|\xi|^4}d\xi dr. \end{aligned} \tag{18}$$

Thus, by using the following integral formula (see Corollary on page 23 in [21]):

$$\int_{\mathbb{R}^d} f\Big(\sum_{i=1}^d u_i^2\Big)du_1\cdots du_d = \frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty y^{d/2-1}f(y)dy, \tag{19}$$

(15) becomes

$$\mathbb{E}[U(t,x)U(s,x)] = (2\pi)^{-d}\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty y^{d/2-1}\int_0^s e^{-\frac{\varepsilon(t+s-2r)}{8}y^2}drdy. \tag{20}$$

This yields (15).

To verify (16), by (15), one has, up to a constant, the mean zero Gaussian process $\{U(t,x), t\ge 0\}$ is a BBM with indices $H=1/2$ and $K=1-d/4$. Thus, by the covariance function of BBM in [22], (15) holds.

To show (17), we introduce the following auxiliary Gaussian random field $\{G(t,x), t\in \mathbb{R}_+, x\in\mathbb{R}^d\}$:

$$G(t,x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}}\Big(\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{(t-r)_+;x,y} - \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{(-r)_+;x,y}\Big)W(dr\times dy). \tag{21}$$

where $a_+=\max\{a,0\}$ for all $a\in\mathbb{R}$. Then the LKS-SPDE solution U may be decomposed as $U(t,x)=G(t,x)-V(t,x)$, where

$$V(t,x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_-}\Big(\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{(t-r)_+;x,y} - \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{(-r)_+;x,y}\Big)W(dr\times dy). \tag{22}$$

This idea of decomposition originated in [23] in the second order SPDEs setting; and it has been applied in [24,25], also in the second order heat SPDE setting. Fix $x\in\mathbb{R}^d$. By Theorem 3.1 in [4], one has for any $0<s<t$,

$$\mathbb{E}[|G(t,x)-G(s,x)|^2] = K_d|t-s|^{1-d/4}. \tag{23}$$

Fix $x\in\mathbb{R}^d$. We apply Parseval's identity to the integral in y to get that for any $0<s<t$:

$$\begin{aligned} \mathbb{E}[|V(t,x)-V(s,x)|^2] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}}\Big|\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,y}\mathbb{I}_{\{0>r\}} - \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,y}\mathbb{I}_{\{0>r\}}\Big|^2 drdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big|\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi}\mathbb{I}_{\{0>r\}} - \hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,\xi}\mathbb{I}_{\{0>r\}}\Big|^2 d\xi dr. \end{aligned} \tag{24}$$

Since

$$\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi} = (2\pi)^{-d/2}e^{-\mathbf{i}\langle x,\xi\rangle - \frac{\varepsilon(t-r)}{8}|\xi|^4}, \tag{25}$$

Equation (24) becomes

$$\begin{aligned}&\mathbb{E}[|V(t,x)-V(s,x)|^2]\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{\left|e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}}-e^{-\frac{\varepsilon(s-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}}\right|^2}{(2\pi)^d}drd\xi.\end{aligned} \tag{26}$$

Now, we apply Parseval's identity to the inner integral in r. To this end, let

$$\phi(r,\xi)=e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}}-e^{-\frac{\varepsilon(s-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}}$$

Its Fourier transform in r is

$$\hat{\phi}(\tau,\xi)=\frac{1}{\mathbf{i}\tau+\frac{\varepsilon}{8}|\xi|^4}\Big(-e^{-\frac{\varepsilon t}{8}|\xi|^4}+e^{-\frac{\varepsilon s}{8}|\xi|^4}\Big).$$

Hence, by Parseval's identity, we see that for each $0<s<t$ Equation (26) becomes

$$\begin{aligned}\mathbb{E}[|V(t,x)-V(s,x)|^2]&=(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}}|\hat{\phi}(\tau,\xi)|^2d\tau d\xi\\&=(2\pi)^{-d}\int_{\mathbb{R}^d}|e^{-\frac{\varepsilon t}{8}|\xi|^4}-e^{-\frac{\varepsilon s}{8}|\xi|^4}|^2\int_{\mathbb{R}}\frac{1}{\tau^2+\frac{\varepsilon^2}{64}|\xi|^8}d\tau d\xi\\&\le c_{4,4}\int_{\mathbb{R}^d}|\xi|^{-4}e^{-\frac{\varepsilon s}{4}|\xi|^4}|1-e^{-\frac{\varepsilon(t-s)}{8}|\xi|^4}|^2d\xi.\end{aligned} \tag{27}$$

Since $|1-e^{-u}|\le 2u$ for all $u\ge 0$, one has that for each $0<s<t$ Equation (27) becomes

$$\begin{aligned}\mathbb{E}[|V(t,x)-V(s,x)|^2]&\le c_{4,5}(t-s)^2\int_{\mathbb{R}^d}|\xi|^4e^{-\frac{\varepsilon s}{4}|\xi|^4}d\xi\\&=\frac{c_{4,5}\pi^{d/2}}{\Gamma(d/2)}(t-s)^2\int_0^\infty y^{d/2+1}e^{-\frac{\varepsilon s}{4}y^2}dy\\&\le\frac{c_{4,6}}{s^{d/4+1}}(t-s)^2\int_0^\infty y^{d/2+1}e^{-y^2}dy.\end{aligned} \tag{28}$$

Fix $x\in\mathbb{R}^d$. Since U and V are independent, one has

$$\mathbb{E}[|G(t,x)-G(s,x)|^2]=\mathbb{E}[|U(t,x)-U(s,x)|^2]+\mathbb{E}[|V(t,x)-V(s,x)|^2].$$

This yields (17). The proof of Lemma 2 is completed. □

Since $\partial_x U(\cdot,x)$ is a centered Gaussian process, its law is determined by its covariance function, which is given in the following lemma. We also derive some needed estimates on the increment of $\partial_x U(\cdot,x)$.

Lemma 3. *Fix $(\varepsilon,\vartheta)\in\mathbb{R}_+\times\mathbb{R}$ and $x\in\mathbb{R}$, and assume $d=1$. Assume that $u_0\equiv 0$ and $\vartheta=0$ in (1). For all $s,t\in(0,T]$, we have*

$$\mathbb{E}[\partial_x U(t,x)\partial_x U(s,x)]=D_0[(t+s)^{1/4}-|t-s|^{1/4}], \tag{29}$$

$$c_{4,7}|t-s|^{1/4}\le\mathbb{E}[(\partial_x U(t,x)-\partial_x U(s,x))^2]\le c_{4,8}|t-s|^{1/4}, \tag{30}$$

and

$$|\mathbb{E}[(\partial_x U(t,x)-\partial_x U(s,x))^2]-D_0|t-s|^{1/4}|\le\frac{c_{4,9}}{s^{7/4}}|t-s|^2, \tag{31}$$

where D_0 is given in (7).

Proof. To show (29), we use Parseval's identity to get

$$\begin{aligned}\mathbb{E}[\partial_x U(t,x)\partial_x U(s,x)] &= \int_{\mathbb{R}}\int_0^s \partial_x \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,y}\, \partial_x \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,y} drdy \\ &= \int_0^s \int_{\mathbb{R}} \xi^2 \hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi}\, \overline{\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,\xi}} d\xi dr \\ &= (2\pi)^{-1}\int_0^s\int_{\mathbb{R}} \xi^2 e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4 - \frac{\varepsilon(s-r)}{8}|\xi|^4} d\xi dr \\ &= (2\pi)^{-1}\int_0^s\int_{\mathbb{R}} \xi^2 e^{-\frac{\varepsilon(t+s-2r)}{8}|\xi|^4} d\xi dr.\end{aligned} \tag{32}$$

Thus, (32) becomes

$$\mathbb{E}[\partial_x U(t,x)\partial_x U(s,x)] = (2\pi)^{-1}\left(\frac{8}{\varepsilon}\right)^{3/4}\left((t+s)^{1/4} - (t-s)^{1/4}\right)\int_0^\infty y^{-1/4}e^{-y}dy. \tag{33}$$

This yields (29).

To verify (30), by (29), one has, up to a constant, the mean zero Gaussian process $\{\partial_x U(t,x), t \ge 0\}$ is a BBM with indices $H = 1/2$ and $K = 1/4$. Thus, by the estimates on the increments of BBM in [22], (30) holds.

Fix $x \in \mathbb{R}$. We apply Parseval's identity to the integral in y to get that for any $0 < s < t$:

$$\begin{aligned}\mathbb{E}[|\partial_x V(t,x) - \partial_x V(s,x)|^2] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\partial_x \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,y}\mathbb{I}_{\{0>r\}} - \partial_x \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,y}\mathbb{I}_{\{0>r\}}\right|^2 drdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 \left|\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi}\mathbb{I}_{\{0>r\}} - \hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,\xi}\mathbb{I}_{\{0>r\}}\right|^2 d\xi dr.\end{aligned} \tag{34}$$

Since

$$\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi} = (2\pi)^{-1/2}e^{-\mathbf{i}\langle x,\xi\rangle - \frac{\varepsilon(t-r)}{8}|\xi|^4}, \tag{35}$$

Equation (34) becomes

$$\begin{aligned}&\mathbb{E}[|\partial_x V(t,x) - \partial_x V(s,x)|^2] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi^2\left|e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}} - e^{-\frac{\varepsilon(s-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}}\right|^2}{(2\pi)^d} drd\xi.\end{aligned} \tag{36}$$

Now, we apply Parseval's identity to the inner integral in r. To this end, let

$$\phi(r,\xi) = e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}} - e^{-\frac{\varepsilon(s-r)}{8}|\xi|^4}\mathbb{I}_{\{0>r\}}$$

Its Fourier transform in r is

$$\hat{\phi}(\tau,\xi) = \frac{1}{\mathbf{i}\tau + \frac{\varepsilon}{8}|\xi|^4}\left(-e^{-\frac{\varepsilon t}{8}|\xi|^4} + e^{-\frac{\varepsilon s}{8}|\xi|^4}\right).$$

Hence, by Parseval's identity, we see that for each $0 < s < t$ Equation (36) becomes

$$\begin{aligned}\mathbb{E}[|\partial_x V(t,x) - \partial_x V(s,x)|^2] &= (2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2|\hat{\phi}(\tau,\xi)|^2 d\tau d\xi \\ &= (2\pi)^{-1}\int_{\mathbb{R}} |e^{-\frac{\varepsilon t}{8}|\xi|^4} - e^{-\frac{\varepsilon s}{8}|\xi|^4}|^2 \int_{\mathbb{R}} \frac{\xi^2}{\tau^2 + \frac{\varepsilon^2}{64}|\xi|^8} d\tau d\xi \\ &\le c_{4,10}\int_{\mathbb{R}} |\xi|^{-2}e^{-\frac{\varepsilon s}{4}|\xi|^4}|1 - e^{-\frac{\varepsilon(t-s)}{8}|\xi|^4}|^2 d\xi.\end{aligned} \tag{37}$$

Since $|1 - e^{-x}| \le 2x$ for all $x \ge 0$, one has that for each $0 < s < t$ Equation (37) becomes

$$\begin{aligned}\mathbb{E}[|\partial_x V(t,x) - \partial_x V(s,x)|^2] &\le c_{4,11}(t-s)^2\int_{\mathbb{R}} |\xi|^6 e^{-\frac{\varepsilon s}{4}|\xi|^4} d\xi \\ &\le \frac{c_{4,12}}{s^{7/4}}(t-s)^2\int_0^\infty y^{7/4}e^{-y}dy.\end{aligned} \tag{38}$$

Thus, by using similar argument of the proof of (17), (31) holds. The proof of Lemma 3 is completed. □

4. Results

4.1. Exact Convergence Rates of Variances for LKS-SPDEs

We need the following product-moment of various orders of the normal correlation surface of two variate, which are Equations (viii) and (ix) in [20].

Lemma 4. *Suppose that* $(X,Y) \sim \mathcal{N}(0, \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix})$*, where* $\rho = (\sigma_1\sigma_2)^{-1}\mathbb{E}[XY]$*. Then,*

$$\mathbb{E}[X^pY^p] = \begin{cases} \dfrac{p!p!}{2^p}\sigma_1^p\sigma_2^p \displaystyle\sum_{j=1}^{p/2} \frac{(2\rho)^{2j}}{(p/2-j)!(p/2-j)!(2j)!}, & \text{if } p \text{ is even}, \\ \dfrac{\rho p!p!}{2^{p-1}}\sigma_1^p\sigma_2^p \displaystyle\sum_{j=0}^{\lfloor p/2\rfloor} \frac{(2\rho)^{2j}}{(\lfloor p/2\rfloor-j)!(\lfloor p/2\rfloor-j)!(2j+1)!}, & \text{if } p \text{ is odd}. \end{cases} \tag{39}$$

Proof of Theorem 1. It is sufficient to prove (4) for the even p case since the odd p case can be proved similarly. For $1 \le i < j \le \lfloor nt \rfloor$, define $\rho_{x;ij} = (\sigma_{x;i}\sigma_{x;j})^{-1}\mathbb{E}[\Delta U_{x;i}\Delta U_{x;j}]$. Note that for a random variable X following an $\mathcal{N}(0,\sigma^2)$ law,

$$\mathbb{E}[X^p] = \mu_p\sigma^p, \quad \forall p \in \mathbb{N}_+. \tag{40}$$

By (39) and (40), one has

$$\begin{aligned} \mathrm{Var}(\Xi_p^n(U(\cdot,x))_t) &= \mathbb{E}\Big[\Big|\sum_{j=1}^{\lfloor nt\rfloor}(\Delta U_{x;j}^p - \mu_p\sigma_{x;j}^p)\Big|^2\Big] \\ &= \sum_{j=1}^{\lfloor nt\rfloor}\mathbb{E}[(\Delta U_{x;j}^p - \mu_p\sigma_{x;j}^p)^2] + 2\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\mathbb{E}[(\Delta U_{x;i}^p - \mu_p\sigma_{x;i}^p)(\Delta U_{x;j}^p - \mu_p\sigma_{x;j}^p)] \\ &= \sum_{j=1}^{\lfloor nt\rfloor}(\mathbb{E}[\Delta U_{x;j}^{2p}] - \mu_p^2\sigma_{x;j}^{2p}) + 2\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}(\mathbb{E}[\Delta U_{x;i}^p\Delta U_{x;j}^p] - \mu_p^2\sigma_{x;i}^p\sigma_{x;j}^p) \\ &= (\mu_{2p} - \mu_p^2)\sum_{j=1}^{\lfloor nt\rfloor}\sigma_{x;j}^{2p} + \frac{p!p!}{2^{p-1}}\sum_{u=1}^{p/2}\frac{2^{2u}}{(p/2-u)!(p/2-u)!(2u)!}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\sigma_{x;i}^p\sigma_{x;j}^p\rho_{x;ij}^{2u}. \end{aligned} \tag{41}$$

It follows from (16) that

$$c_{5,1}^{-1}n^{-1+d/4} \le \sigma_{x;j}^2 \le c_{5,1}n^{-1+d/4} \ \text{ for all } 1 \le j \le \lfloor nt\rfloor. \tag{42}$$

By (17), (42) and Lagrange mean value theorem, it holds that for any real number $r > 0$ and $1 < j \le \lfloor nt \rfloor$,

$$\begin{aligned} |\sigma_{x;j}^r - (K_dn^{-1+d/4})^{r/2}| &\le c_{5,2}(\sigma_{x;j}^{r-2} + (K_dn^{-1+d/4})^{(r-2)/2})|\sigma_{x;j}^2 - K_dn^{-1+d/4}| \\ &\le c_{5,3}n^{-2+(-1+d/4)(r-2)/2}t_{j-1}^{-(d/4+1)}. \end{aligned} \tag{43}$$

Note that since $\alpha + 1 \le d < \alpha + 2$, one has $1/2 \le d/4 < 1$. Thus

$$\frac{1}{n}\sum_{j=2}^{\lfloor nt\rfloor}t_{j-1}^{-(d/4+1)/2} \to \int_0^t u^{-(d/4+1)/2}du = \frac{2}{1-d/4}t^{(1-d/4)/2}. \tag{44}$$

It follows from (43) (with $r = 2p$) and (44) that

$$n^{-1+p(1-d/4)}\sum_{j=1}^{\lfloor nt\rfloor}|\sigma_{x;j}^{2p} - (K_dn^{-1+d/4})^p| \to 0. \tag{45}$$

Hence

$$n^{-1+p(1-d/4)}\sum_{j=1}^{\lfloor nt\rfloor}\sigma_{x;j}^{2p}$$
$$= n^{-1+p(1-d/4)}\sum_{j=1}^{\lfloor nt\rfloor}(\sigma_{x;j}^{2p}-(K_d n^{-1+d/4})^p)+n^{-1+p(1-d/4)}(K_d n^{-1+d/4})^p\lfloor nt\rfloor \to K_d^p t. \tag{46}$$

It follows from (15) that

$$\begin{aligned}&\mathbb{E}[\Delta U_{x;i}\Delta U_{x;j}]\\&=K_d n^{-1+d/4}((j+i)^{1-d/4}-(j-i)^{1-d/4}-(j+i-1)^{1-d/4}+(j-i+1)^{1-d/4}\\&\quad -(j+i-1)^{1-d/4}+(j-i-1)^{1-d/4}+(j+i-2)^{1-d/4}-(j-i)^{1-d/4}),\end{aligned}$$

which simplifies to

$$\mathbb{E}[\Delta U_{x;i}\Delta U_{x;j}] = -K_d(n^{-1+d/4}\phi_{d;j+i-1}+n^{-1+d/4}\phi_{d;j-i}), \tag{47}$$

where $\phi_{d;j}=2j^{1-d/4}-(j-1)^{1-d/4}-(j+1)^{1-d/4}$. Thus, by binomial expansion, for every $1\le u\le p/2$ and $1\le i<j\le\lfloor nt\rfloor$,

$$\begin{aligned}\sigma_{x;i}^p\sigma_{x;j}^p\rho_{x;ij}^{2u} &= \sigma_{x;i}^{p-2u}\sigma_{x;j}^{p-2u}(\mathbb{E}[\Delta U_{x;i}\Delta U_{x;j}])^{2u}\\&=K_d^{2u}\sigma_{x;i}^{p-2u}\sigma_{x;j}^{p-2u}(n^{-1+d/4}\phi_{d;j+i-1}+n^{-1+d/4}\phi_{d;j-i})^{2u}\\&=K_d^{2u}\sum_{v=0}^{2u}\binom{2u}{v}\sigma_{x;i}^{p-2u}\sigma_{x;j}^{p-2u}(n^{-1+d/4}\phi_{d;j+i-1})^{v}(n^{-1+d/4}\phi_{d;j-i})^{2u-v}.\end{aligned} \tag{48}$$

If we write $\phi_{d;k}=g(k-1)-g(k)$, where $g(s)=(s+1)^{1-d/4}-s^{1-d/4}$, then for each $k\ge 2$, the Lagrange mean value theorem gives $\phi_{d;k}=|g'(k-\zeta_1)|=(d/4)(1-d/4)(k-\zeta_1+\zeta_2)^{-d/4-1}$ for some $\zeta_1,\zeta_2\in[0,1]$. This yields that for all $k\in\mathbb{N}_+$,

$$0<\phi_{d;k}\le\frac{c_{5,4}}{k^{d/4+1}}, \tag{49}$$

and hence, for any $r\ge 1$,

$$\sum_{k=1}^{M}\phi_{d;k}^r\to J_{d,r} \tag{50}$$

with some $J_{d,r}>0$ as $M\to\infty$.

Note that since $j+i-1\ge(j+i)/2$, one has

$$n^{-1+d/4}\phi_{d;j+i-1}\le\frac{c_{5,5}}{n^2}\frac{1}{(t_i+t_j)^{d/4+1}}. \tag{51}$$

Note that (49) gives $n^{-1+d/4}\phi_{d;j-i}\le c_{5,6}n^{-1+d/4}$ and $n^{-1+d/4}\phi_{d;j+i-1}\le c_{5,7}n^{-1+d/4}$ for all $1\le i<j\le\lfloor nt\rfloor$. Thus, by (42) and (51), for every $1\le u\le p/2$ and $1\le v\le 2u$,

$$\begin{aligned}&n^{-1+p(1-d/4)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\sigma_{x;i}^{p-2u}\sigma_{x;j}^{p-2u}(n^{-1+d/4}\phi_{d;j+i-1})^{v}(n^{-1+d/4}\phi_{d;j-i})^{2u-v}\\&\le c_{5,8}n^{-d/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}(n^{-1+d/4}\phi_{d;j+i-1})\\&\le c_{5,9}n^{-2-d/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\frac{1}{(t_i+t_j)^{d/4+1}},\end{aligned} \tag{52}$$

which tends to zero as $n\to\infty$ since $\int_0^t\int_0^t(u+v)^{-(d/4+1)}dudv<\infty$.

We now consider the term $v = 0$ in (48). Let $B^H = \{B^H(t),\, t \in \mathbb{R}_+\}$ be a FBM with index $H \in (0,1)$, which is a centered Gaussian process with $\mathbb{E}[(B^H(t) - B^H(s))^2] = |s-t|^{2H}$ for $s, t \in \mathbb{R}_+$. Then, for $H_0 = (1-d/4)/2$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(B^{H_0}\Big(\frac{j+1}{n}\Big) - B^{H_0}\Big(\frac{j}{n}\Big)\Big)\Big(B^{H_0}\Big(\frac{i+1}{n}\Big) - B^{H_0}\Big(\frac{i}{n}\Big)\Big)\Big] \\
&= -\frac{1}{2}\Big[2\Big(\frac{j-i}{n}\Big)^{1-d/4} - \Big(\frac{j-i-1}{n}\Big)^{1-d/4} - \Big(\frac{j-i+1}{n}\Big)^{1-d/4}\Big] \\
&= -\frac{1}{2} n^{-1+d/4}\phi_{d;j-i}.
\end{aligned} \tag{53}$$

Thus,

$$\begin{aligned}
n^{-1+d/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\phi_{d;j-i} &= n^{-1+d/4}\sum_{i=1}^{\lfloor nt\rfloor-1}\sum_{j=i+1}^{\lfloor nt\rfloor}\phi_{d;j-i} \\
&= -2\sum_{i=1}^{\lfloor nt\rfloor-1}\sum_{j=i+1}^{\lfloor nt\rfloor}\mathbb{E}\Big[\Big(B^{H_0}\Big(\frac{j+1}{n}\Big) - B^{H_0}\Big(\frac{j}{n}\Big)\Big)\Big(B^{H_0}\Big(\frac{i+1}{n}\Big) - B^{H_0}\Big(\frac{i}{n}\Big)\Big)\Big] \\
&= -2\sum_{i=1}^{\lfloor nt\rfloor-1}\mathbb{E}\Big[\Big(B^{H_0}\Big(\frac{\lfloor nt\rfloor+1}{n}\Big) - B^{H_0}\Big(\frac{i+1}{n}\Big)\Big)\Big(B^{H_0}\Big(\frac{i+1}{n}\Big) - B^{H_0}\Big(\frac{i}{n}\Big)\Big)\Big] \\
&= -\sum_{i=1}^{\lfloor nt\rfloor-1}\Big[-\Big(\frac{\lfloor nt\rfloor-i}{n}\Big)^{1-d/4} + \Big(\frac{\lfloor nt\rfloor+1-i}{n}\Big)^{1-d/4} - \Big(\frac{1}{n}\Big)^{1-d/4}\Big] \\
&= -\Big(\frac{\lfloor nt\rfloor}{n}\Big)^{1-d/4} + \Big(\frac{1}{n}\Big)^{1-d/4} + \lfloor nt\rfloor n^{-1+d/4}.
\end{aligned} \tag{54}$$

This yields

$$n^{-d/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}(n^{-1+d/4}\phi_{d;j-i}) \to t. \tag{55}$$

By (42) and (49), one has for every $1 \le u \le p/2$ and any $M > 0$,

$$\begin{aligned}
&n^{-1+p(1-d/4)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\sigma_{x;i}^{p-2u}\sigma_{x;j}^{p-2u}(n^{-1+d/4}\phi_{d;j-i})^{2u} \\
&\le c_{5,10}M^{-(d/4+1)(2u-1)}n^{-d/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}(n^{-1+d/4}\phi_{d;j-i}) \\
&\le c_{5,11}M^{-(d/4+1)(2u-1)}n^{-d/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}(n^{-1+d/4}\phi_{d;j-i}).
\end{aligned} \tag{56}$$

This, together with (50), yields

$$n^{-1+p(1-d/4)}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\sigma_{x;i}^{p-2u}\sigma_{x;j}^{p-2u}(n^{-1+d/4}\phi_{d;j-i})^{2u} \le c_{5,12}M^{-(d/4+1)(2u-1)}t, \tag{57}$$

which tends to zero by letting $M \to \infty$.

By (43) (with $r = p - 2u$), (42) and (53), one has for every $1 \le u \le p/2$,

$$
\begin{aligned}
& n^{-1+p(1-d/4)} \sum_{i=2}^{\lfloor nt \rfloor} \sum_{j=i+1}^{\lfloor nt \rfloor} |\sigma_{x;i}^{p-2u} - (K_d n^{-1+d/4})^{(p-2u)/2} | \sigma_{x;j}^{p-2u} (n^{-1+d/4} \phi_{d;j-i})^{2u} \\
& \leq c_{5,13} n^{-1-d/2} \sum_{i=2}^{\lfloor nt \rfloor} \frac{1}{t_{i-1}^{d/4+1}} \sum_{j=i+1}^{\lfloor nt \rfloor} (n^{-1+d/4} \phi_{d;j-i}) \\
& = -2c_{5,13} n^{-1-d/2} \sum_{i=2}^{\lfloor nt \rfloor} \frac{1}{t_{i-1}^{d/4+1}} \Big[-\Big(\frac{\lfloor nt \rfloor - i}{n} \Big)^{1-d/4} + \Big(\frac{\lfloor nt \rfloor + 1 - i}{n} \Big)^{1-d/4} - \Big(\frac{1}{n} \Big)^{1-d/4} \Big] \\
& \leq c_{5,14} n^{-d/4} \sum_{i=2}^{\lfloor nt \rfloor} \Big[-\Big(\frac{\lfloor nt \rfloor - i}{n} \Big)^{1-d/4} + \Big(\frac{\lfloor nt \rfloor + 1 - i}{n} \Big)^{1-d/4} \Big] + c_{5,15} n^{-2-d/4} \sum_{i=2}^{\lfloor nt \rfloor} \frac{1}{t_{i-1}^{d/4+1}} \\
& \leq c_{5,16} n^{-d/4} \Big[\Big(\frac{1}{n} \Big)^{1-d/4} + \Big(\frac{\lfloor nt \rfloor - 1}{n} \Big)^{1-d/4} \Big] + c_{5,17} n^{-3/2-d/4/2} \sum_{i=2}^{\lfloor nt \rfloor} t_{i-1}^{-(d/4+1)/2},
\end{aligned}
\tag{58}
$$

which tends to zero as $n \to \infty$ since $\int_0^t s^{-(d/4+1)/2} ds < \infty$. Hence, one has for every $1 \leq u \leq p/2$,

$$
n^{-1+p(1-d/4)} \sum_{i=2}^{\lfloor nt \rfloor} \sum_{j=i+1}^{\lfloor nt \rfloor} (\sigma_{x;i}^{p-2u} - (K_d n^{-1+d/4})^{(p-2u)/2}) \sigma_{x;j}^{p-2u} (n^{-1+d/4} \phi_{d;j-i})^{2u} \to 0. \tag{59}
$$

Similarly, one has for every $1 \leq u \leq p/2$,

$$
n^{-1+p(1-d/4)} \sum_{i=2}^{\lfloor nt \rfloor} \sum_{j=i+1}^{\lfloor nt \rfloor} (K_d n^{-1+d/4})^{(p-2u)/2} (\sigma_{x;j}^{p-2u} - (K_d n^{-1+d/4})^{(p-2u)/2}) (n^{-1+d/4} \phi_{d;j-i})^{2u} \to 0. \tag{60}
$$

For every $1 \leq u \leq p/2$ and any $M > 0$,

$$
\begin{aligned}
& n^{-1+p(1-d/4)} \sum_{i=2}^{\lfloor nt \rfloor} \sum_{j=i+1}^{i+M} (K_d n^{-1+d/4})^{p-2u} (n^{-1+d/4} \phi_{d;j-i})^{2u} \\
& = K_d^{p-2u} \frac{\lfloor nt \rfloor - 1}{n} \sum_{j=1}^{M} \phi_{d;j}^{2u} \to K_d^{p-2u} J_{d,2u} t
\end{aligned}
\tag{61}
$$

as $n \to \infty$ and $M \to \infty$.

Note that for every $1 \leq u \leq p/2$ and $1 \leq i < j \leq \lfloor nt \rfloor$,

$$
\begin{aligned}
\sigma_{x;i}^{p-2u} \sigma_{x;j}^{p-2u} & = (\sigma_{x;i}^{p-2u} - (K_d n^{-1+d/4})^{(p-2u)/2}) \sigma_{x;j}^{p-2u} \\
& \quad + (K_d n^{-1+d/4})^{(p-2u)/2} (\sigma_{x;j}^{p-2u} - (K_d n^{-1+d/4})^{(p-2u)/2}) + (K_d n^{-1+d/4})^{p-2u}.
\end{aligned}
\tag{62}
$$

Hence, by (59)–(62), one has for every $1 \leq u \leq p/2$,

$$
n^{-1+p(1-d/4)} \sum_{i=2}^{\lfloor nt \rfloor} \sum_{j=i+1}^{i+M} \sigma_{x;i}^{p-2u} \sigma_{x;j}^{p-2u} (n^{-1+d/4} \phi_{d;j-i})^{2u} \to K_d^{p-2u} J_{d,2u} t \tag{63}
$$

as $n \to \infty$ and $M \to \infty$. It follows from (42) that

$$
n^{-1+p(1-d/4)} \sum_{j=2}^{1+M} \sigma_{x;i}^{p-2u} \sigma_{x;j}^{p-2u} (n^{-1+d/4} \phi_{d;j-1})^{2u} \to 0. \tag{64}
$$

This, together with (48), (52) and (63), yields for every $1 \leq u \leq p/2$,

$$
n^{-1+p(1-d/4)} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=i+1}^{\lfloor nt \rfloor} \sigma_{x;i}^{p} \sigma_{x;j}^{p} \rho_{x;ij}^{2u} \to K_d^p J_{d,2u} t \tag{65}
$$

Therefore, by (41), (46) and (65), one has

$$\begin{aligned} & n^{-1+p(1-d/4)}\,\mathrm{Var}(\Xi_p^n(U(\cdot,x))_t) \\ & \to K_d^p\Big(\mu_{2p}-\mu_p^2+\frac{p!p!}{2^{p-1}}\sum_{u=1}^{p/2}\frac{2^{2u}J_{d,2u}}{(p/2-u)!(p/2-u)!(2u)!}\Big)t=\kappa_{d,p}\,t. \end{aligned} \tag{66}$$

This proves (4). The proof of Theorem 1 is completed. □

Proof of Corollary 1. Write

$$\begin{aligned} & n^{-1+p(1-d/4)/2}\Xi_p^n(U(\cdot,x))_t-K_d^{p/2}\mu_p t \\ & = n^{-1+p(1-d/4)/2}(\Xi_p^n(U(\cdot,x))_t-\mathbb{E}[\Xi_p^n(U(\cdot,x))_t]) \\ & \quad +\mu_p n^{-1+p(1-d/4)/2}\sum_{j=1}^{\lfloor nt\rfloor}(\sigma_{x;j}^p-(K_d n^{-1+d/4})^{p/2})+K_d^{p/2}\mu_p\Big(\frac{\lfloor nt\rfloor}{n}-t\Big). \end{aligned} \tag{67}$$

Obviously, the third term of (67) tends to zero as $n\to\infty$. It follows from (43) (with $r=p$) and (45) that the second term of (67) tends to zero as $n\to\infty$. Thus, by (4), one has

$$\mathbb{E}[|n^{-1+p(1-d/4)/2}\Xi_p^n(U(\cdot,x))_t-K_d^{p/2}\mu_p t|^2]\to 0.$$

This proves (5). □

4.2. Temporal CLTs for LKS-SPDEs

The following lemma is needed to prove Theorem 2.

Lemma 5. *Let $X_1,\ldots,X_4$ be mean zero, jointly normal random variables, such that $\mathbb{E}[X_j^2]=1$ and $\rho_{ij}=\mathbb{E}[X_iX_j]$. Put $Z_j=X_j^p-E[X_j^p]$. Then, for any $p\in\mathbb{N}_+$,*

$$\Big|\mathbb{E}\Big[\prod_{j=1}^4 Z_j\Big]\Big|\le c_{6,1}\Big(|\rho_{12}\,\rho_{34}|+\frac{1}{\sqrt{1-\rho_{12}^2}}\max_{i\le 2<j}|\rho_{ij}|\Big) \tag{68}$$

whenever $|\rho_{12}|<1$. Moreover,

$$\Big|\mathbb{E}\Big[\prod_{j=1}^4 Z_j\Big]\Big|\le c_{6,2}\max_{2\le j\le 4}|\rho_{1j}|. \tag{69}$$

Furthermore, there exists $\varepsilon>0$ such that

$$\Big|\mathbb{E}\Big[\prod_{j=1}^4 Z_j\Big]\Big|\le c_{6,3}\max_{1\le i\ne j\le 4}\rho_{ij}^2 \tag{70}$$

whenever $|\rho_{ij}|<\varepsilon$ for all $1\le i\ne j\le 4$.

Proof. Following the same lines as the proof of Lemma 3.3 in [11] with $h_j(X_j)=Z_j$, $1\le j\le 4$, we get Lemma 5 immediately. □

Proposition 1. *Fix $(\varepsilon,\vartheta)\in\mathbb{R}_+\times\mathbb{R}$ and $x\in\mathbb{R}^d$, and assume $d\in\{1,2,3\}$. Assume that $u_0\equiv 0$ and $\vartheta=0$ in (1). Fix $r\in\mathbb{N}_+$. Put*

$$\Theta_r^n(U(\cdot,x))_t=n^{-1/2+r(1-d/4)/2}\sum_{i=1}^{\lfloor nt\rfloor}(\Delta U_{x;i}^r-\mu_r\sigma_{x;i}^r).$$

Then, for all $0 \le s < t$ *and all* $n \in \mathbb{N}_+$,

$$\mathbb{E}[|\Theta_r^n(U(\cdot,x))_t - \Theta_r^n(U(\cdot,x))_s|^4] \le c_{6,4}\Big(\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}\Big)^2. \tag{71}$$

The sequence $\{\Theta_r^n(U(\cdot,x))\}$ *is therefore relatively compact in the Skorohod space* $D_{\mathbb{R}}[0,\infty)$.

Proof. We follow the method of Proposition 3.5 in [11] to prove (71). Let $\mathcal{S} = \{j \in \mathbb{N}_+^4 : \lfloor ns\rfloor + 1 \le j_1 \le \cdots \le j_4 \le \lfloor nt\rfloor\}$. For $j \in \mathcal{S}$ and $k \in \{1,2,3\}$, define $h_k = j_{k+1} - j_k$ and let $\mathcal{S}_k = \{j \in \mathcal{S} : h_k = \max\{h_1,h_2,h_3\}\}$. Define $N = \lfloor nt\rfloor - (\lfloor ns\rfloor + 1)$ and for $i \in \{0,1,...,N\}$, let $\mathcal{S}_k^i = \{j \in \mathcal{S}_k : \max\{h_1,h_2,h_3\} = i\}$. Further define $\mathcal{T}_k^\ell = \mathcal{T}_k^{i,\ell} = \{j \in \mathcal{S}_k^i : \min\{h_1,h_2,h_3\} = \ell\}$ and $\mathcal{V}_k^v = \mathcal{V}_k^{i,\ell,v} = \{j \in \mathcal{T}_k^\ell : \mathrm{med}\{h_1,h_2,h_3\} = v\}$, where "med" denotes the median function. For $j \in \mathcal{S}$, define

$$\Lambda_{x;j} = \prod_{k=1}^4 (\Delta U_{x;j_k}^r - \mu_r \sigma_{x;j_k}^r).$$

Observe that

$$\begin{aligned} \mathbb{E}[|\Theta_r^n(U(\cdot,x))_t - \Theta_r^n(U(\cdot,x))_s|^4] &= n^{-2+2r(1-d/4)}\mathbb{E}\Big[\Big|\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\Delta U_{x;i}^r - \mu_r\sigma_{x;i}^r)\Big|^4\Big] \\ &\le 4! n^{-2+2r(1-d/4)}\sum_{j\in\mathcal{S}}|\mathbb{E}[\Lambda_{x;j}]| \\ &\le 4! n^{-2+2r(1-d/4)}\sum_{k=1}^3\sum_{j\in\mathcal{S}_k}|\mathbb{E}[\Lambda_{x;j}]|, \end{aligned} \tag{72}$$

and that

$$\begin{aligned} \sum_{j\in\mathcal{S}_k}|\mathbb{E}[\Lambda_{x;j}]| &= \sum_{i=0}^N\sum_{j\in\mathcal{S}_k^i}|\mathbb{E}[\Lambda_{x;j}]| \\ &= \sum_{i=0}^N\sum_{\ell=0}^{\lfloor i^{d/4}\rfloor}\sum_{j\in\mathcal{T}_k^\ell}|\mathbb{E}[\Lambda_{x;j}]| + \sum_{i=0}^N\sum_{\ell=\lfloor i^{d/4}\rfloor+1}^{i}\sum_{j\in\mathcal{T}_k^\ell}|\mathbb{E}[\Lambda_{x;j}]| \\ &= \sum_{i=0}^N\sum_{\ell=0}^{\lfloor i^{d/4}\rfloor}\sum_{v=\ell}^{i}\sum_{j\in\mathcal{V}_k^v}|\mathbb{E}[\Lambda_{x;j}]| + \sum_{i=0}^N\sum_{\ell=\lfloor i^{d/4}\rfloor+1}^{i}\sum_{v=\ell}^{i}\sum_{j\in\mathcal{V}_k^v}|\mathbb{E}[\Lambda_{x;j}]|. \end{aligned} \tag{73}$$

Let $Z_{x;k} = \sigma_{x;j_k}^{-1}\Delta U_{x;j_k}$ and

$$\xi_{x;k} = Z_{x;k}^r - \mathbb{E}[Z_{x;k}^r] = \sigma_{x;j_k}^{-r}(\Delta U_{x;j_k}^r - \mu_r\sigma_{x;j_k}^r).$$

Then

$$|\mathbb{E}[\Lambda_{x;j}]| = \Big(\prod_{k=1}^4\sigma_{x;j_k}^r\Big)\Big|\mathbb{E}\Big[\prod_{k=1}^4\xi_{x;k}\Big]\Big|. \tag{74}$$

By (47) and (49), one has for all $k \ne l \in \mathbb{N}_+$,

$$|\mathbb{E}[\Delta U_{x;k}\Delta U_{x;l}]| \le \frac{c_{6,5}n^{-1+d/4}}{|k-l|^{d/4+1}}. \tag{75}$$

It follows from (42) and (75) that

$$|\rho_{x;kl}| = |\mathbb{E}[Z_{x;k}Z_{x;l}| = \sigma_{x;j_k}^{-1}\sigma_{x;j_l}^{-1}|\mathbb{E}[\Delta U_{x;j_k}\Delta U_{x;j_l}]| \le \frac{c_{6,6}}{|j_k - j_l|^{d/4+1}}. \tag{76}$$

Suppose $0 \le \ell \le \lfloor i^{d/4} \rfloor$. Fix v and let $j \in \mathcal{V}_k^v$ be arbitrary. If $k = 1$, then $i = \max\{h_1, h_2, h_3\} = h_1 = j_2 - j_1$. If $k = 3$, then $i = \max\{h_1, h_2, h_3\} = h_3 = j_4 - j_3$. In either case, by (69), (42), (74) and (76), one has

$$|\mathbb{E}[\Lambda_{x;j}]| \le \frac{c_{6,7} n^{-2r(1-d/4)}}{i^{d/4+1}} \le c_{6,7} \Big(\frac{1}{(\ell v)^{d/4+1}} + \frac{1}{i^{d/4+1}} \Big) n^{-2r(1-d/4)}.$$

If $k = 2$, then $i = \max\{h_1, h_2, h_3\} = h_2 = j_3 - j_2$ and $\ell v = h_3 h_1 = (j_4 - j_3)(j_2 - j_1)$. Hence, by (68), (42), (74) and and (76),

$$|\mathbb{E}[\Lambda_{x;j}]| \le c_{6,8} \Big(\frac{1}{(\ell v)^{d/4+1}} + \frac{1}{i^{d/4+1}} \Big) n^{-2r(1-d/4)}.$$

Now choose $k' \neq k$ such that $h_{k'} = \ell$. With k' given, j is determined by j_k. Since there are two possibilities for k' and $N+1$ possibilities for j_k, $|\mathcal{V}_k^v| \le 2(N+1)$. Therefore,

$$\begin{aligned} \sum_{\ell=0}^{\lfloor i^{d/4} \rfloor} \sum_{v=\ell}^{i} \sum_{j \in \mathcal{V}_k^v} |\mathbb{E}[\Lambda_{x;j}]| &\le c_{6,9}(N+1) \sum_{\ell=0}^{\lfloor i^{d/4} \rfloor} \sum_{v=\ell}^{i} \Big(\frac{1}{(\ell v)^{d/4+1}} + \frac{1}{i^{d/4+1}} \Big) n^{-2r(1-d/4)} \\ &\le c_{6,10}(N+1) \sum_{\ell=0}^{\lfloor i^{d/4} \rfloor} \Big(\frac{1}{\ell^{d/4+1}} + \frac{1}{i^{d/4}} \Big) n^{-2r(1-d/4)} \\ &\le c_{6,11}(N+1) n^{-2r(1-d/4)}. \end{aligned} \tag{77}$$

For the second summation, suppose $\lfloor i^{d/4} \rfloor + 1 \le \ell \le i$. In this case, if $j \in \mathcal{T}_k^\ell$, then $\ell = \min\{h_1, h_2, h_3\}$, so that by (42), (70), (74) and (76),

$$|\mathbb{E}[\Lambda_{x;j}]| \le \frac{c_{6,12} n^{-2r(1-d/4)}}{\ell^{2(d/4+1)}}.$$

Since $\sum_{v=\ell}^{i} |\mathcal{V}_k^v| \le 2(N+1)i$ and $1/2 \le d/4 < 1$, one has

$$\begin{aligned} \sum_{\ell=\lfloor i^{d/4} \rfloor+1}^{i} \sum_{v=\ell}^{i} \sum_{j \in \mathcal{V}_k^v} |\mathbb{E}[\Lambda_{x;j}]| &\le c_{6,13}(N+1) i \sum_{\lfloor i^{d/4} \rfloor+1}^{i} \frac{n^{-2r(1-d/4)}}{\ell^{2(d/4+1)}} \\ &\le c_{6,14}(N+1) i \Big(\int_{\lfloor i^{d/4} \rfloor}^{\infty} \frac{1}{u^{2(d/4+1)}} du \Big) n^{-2r(1-d/4)} \\ &\le c_{6,15}(N+1) n^{-2r(1-d/4)}. \end{aligned} \tag{78}$$

Thus, using (72), (73), (77) and (78), one has

$$n^{-2+2r(1-d/4)} \mathbb{E}\Big[\Big| \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} (\Delta U_{x;j}^r - \mu_r \sigma_{x;j}^r) \Big|^4 \Big] \le c_{6,16} \sum_{i=0}^{N} (N+1) n^{-2} = c_{6,16} \Big(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} \Big)^2,$$

which is (71).

To show that a sequence of cadlag processes $\{F_n\}$ is relatively compact, it suffices to show that for each $T > 1$, there exist constants $\beta > 0$, $C > 0$, and $q > 1$ such that

$$R_{F_n}(t,h) = \mathbb{E}[|F_n(t+h) - F_n(t)|^\beta |F_n(t) - F_n(t-h)|^\beta] \le Ch^q$$

for all $n \in \mathbb{N}$, all $t \in [0,T]$ and all $h \in [0,t]$. (See, e.g., Theorem 3.8.8 in [26].) Taking $\beta = 2$ and using (71) together with Hölder inequality gives

$$R_{\Theta_r^n(U(\cdot,x))}(t,h) \le c_{6,17} \Big(\frac{\lfloor nt + nh \rfloor - \lfloor nt \rfloor}{n} \Big) \Big(\frac{\lfloor nt \rfloor - \lfloor nt - nh \rfloor}{n} \Big).$$

If $nh < 1/2$, then the right-hand side of this inequality is zero. Assume $nh \geq 1/2$. Then

$$\frac{\lfloor nt + nh \rfloor - \lfloor nt \rfloor}{n} \leq \frac{nh+1}{n} \leq 3h.$$

The other factor is similarly bounded, so that $R_{\Theta^n_r(U(\cdot,x))}(t,h) \leq c_{6,18}h^2$. □

Proposition 2. *Fix $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}^d$, and assume $d \in \{1,2,3\}$. Assume that $u_0 \equiv 0$ and $\vartheta = 0$ in (1). Then, for any $0 \leq s < t$ and $r \in \mathbb{N}_+$,*

$$\Theta^n_r(U(\cdot,x))_t - \Theta^n_r(U(\cdot,x))_s \xrightarrow{\mathcal{L}} \kappa^{1/2}_{d,r}|t-s|^{1/2}\mathcal{N}$$

as $n \to \infty$, where $\mathcal{N}$ is a standard normal random variable.

Proof. Let $\{n(j)\}_{j=1}^{\infty}$ be any sequence of natural numbers. We will prove that there exists a subsequence $\{n(j_m)\}$ such that $\Theta^{n(j_m)}_r(U(\cdot,x))_t - \Theta^{n(j_m)}_r(U(\cdot,x))_s$ converges in law to the given random variable.

For each $m \in \mathbb{N}_+$, choose $n(j_m) \in \{n(j)\}$ such that $n(j_m) > n(j_{m-1})$ and $n(j_m) \geq m^{2/d/4}(t-s)^{-1}$. Let $b = b(m) = n(j_m)(t-s)/m$. For $0 \leq k \leq m$, define $u_k = n(j_m)s + kb$, so that

$$\begin{aligned}\Theta^{n(j_m)}_r(U(\cdot,x))_t - \Theta^{n(j_m)}_r(U(\cdot,x))_s &= n(j_m)^{-1/2+r(1-d/4)/2} \sum_{i=\lfloor n(j_m)s \rfloor+1}^{\lfloor n(j_m)t \rfloor} (\Delta U^r_{x;i} - \mu_r \sigma^r_{x;i}) \\ &= n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \sum_{i=u_{k-1}+1}^{u_k} (\Delta U^r_{x;i} - \mu_r \sigma^r_{x;i}).\end{aligned} \quad (79)$$

Let us now introduce the filtration

$$\mathcal{F}_t = \sigma\{W(A) : A \subset [0,t] \times \mathbb{R}^d, \lambda(A) < \infty\},$$

where λ denotes Lebesgue measure on $\mathbb{R}^{d+1}$. Let $\tau_k = n(j_m)^{-1}u_{k-1}$. For each pair (i,k) such that $u_{k-1} < i \leq u_k$, define

$$\xi_{x;i,k} = \Delta U_{x;i} - \mathbb{E}[\Delta U_{x;i}|\mathcal{F}_{\tau_k}].$$

Note that $\xi_{x;i,k}$ is $\mathcal{F}_{\tau_{k+1}}$-measurable and independent of $\mathcal{F}_{\tau_k}$. Recall that

$$U(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t-s;x,y} W(ds \times dy). \quad (80)$$

Moreover, given constants $0 \leq \tau \leq s \leq t$, one has

$$\mathbb{E}[U(t,x)|\mathcal{F}_\tau] = \int_0^\tau \int_{\mathbb{R}^d} \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t-s;x,y} W(ds \times dy). \quad (81)$$

It follows from (80) and (81) that

$$U(t+\tau_k,x) - \mathbb{E}[U(t+\tau_k,x)|\mathcal{F}_{\tau_k}] = \int_{\tau_k}^{t+\tau_k} \int_{\mathbb{R}^d} \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}_{t+\tau_k-s;x,y} W(ds \times dy).$$

This yields that $\{\xi_{x;i,k}\}$ has the same law as $\{\Delta U_{x;i-u_{k-1}}\}$.
Now define $\sigma^2_{x;i,k} = \mathbb{E}[\xi^2_{x;i,k}] = \sigma^2_{x;i-u_{k-1}}$ and

$$\zeta_{x;m,k} = \sum_{i=u_{k-1}+1}^{u_k} (\xi^r_{x;i,k} - \mu_r \sigma^r_{x;i,k}),$$

so that $\zeta_{x;m,k}$, $1 \le k \le m$, are independent and

$$\Theta_r^{n(j_m)}(U(\cdot,x))_t - \Theta_r^{n(j_m)}(U(\cdot,x))_s = n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \zeta_{x;m,k} + \epsilon_{x;m}, \tag{82}$$

where

$$\epsilon_{x;m} = n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \sum_{i=u_{k-1}+1}^{u_k} ((\Delta U_{x;i}^r - \mu_r \sigma_{x;i}^r) - (\xi_{x;i,k}^r - \mu_r \sigma_{x;i,k}^r))$$

Since $\xi_{x;i,k}$ and $\Delta U_{x;i} - \xi_{x;i,k} = \mathbb{E}[\Delta U_{x;i} | \mathcal{F}_{\tau_k}]$ are independent, one has

$$\begin{aligned} \sigma_{x;i}^2 &= \mathbb{E}[\Delta U_{x;i}^2] = \mathbb{E}[\xi_{x;i,k}^2] + \mathbb{E}[|\Delta U_{x;i} - \xi_{x;i,k}|^2] \\ &= \sigma_{x;i-u_{k-1}}^2 + \mathbb{E}[|\Delta U_{x;i} - \xi_{x;i,k}|^2]. \end{aligned} \tag{83}$$

This, together with (17), gives

$$\mathbb{E}[|\Delta U_{x;i} - \xi_{x;i,k}|^2] = \sigma_{x;i}^2 - \sigma_{x;i-u_{k-1}}^2 \le \frac{c_{6,19} n(j_m)^{-1+d/4}}{(i-u_{k-1})^{d/4+1}}. \tag{84}$$

Thus, since $\Delta U_{x;i} - \xi_{x;i,k}$ is Gaussian, by (40) and (84), one has

$$\mathbb{E}[|\Delta U_{x;i} - \xi_{x;i,k}|^4] \le \frac{c_{6,20} n(j_m)^{-2+d/2}}{(i-u_{k-1})^{d/2+2}}. \tag{85}$$

Note that (40) and (42) give $\mathbb{E}[|\Delta U_{x;i}|^{4r-4}] \le c_{6,21} \sigma_{x;i}^{4r-4} \le c_{6,22} n(j_m)^{(-1+d/4)(2r-2)}$ and $\mathbb{E}[|\xi_{x;i,k}|^{4r-4}] \le c_{6,23} \sigma_{x;i-u_{k-1}}^{4r-4} \le c_{6,24} n(j_m)^{(-1+d/4)(2r-2)}$. By Lagrange mean value theorem,

$$|\Delta U_{x;i}^r - \xi_{x;i,k}^r| \le c_{6,25}(|\Delta U_{x;i}|^{r-1} + |\xi_{x;i,k}|^{r-1})|\Delta U_{x;i} - \xi_{x;i,k}|.$$

Thus, by (85) and Hölder inequality,

$$\begin{aligned} \mathbb{E}[|\Delta U_{x;i}^r - \xi_{x;i,k}^r|^2] &\le c_{6,26}(\mathbb{E}[|\Delta U_{x;i}|^{4r-4}] + \mathbb{E}[|\xi_{x;i,k}|^{4r-4}])^{1/2}(\mathbb{E}[|\Delta U_{x;i} - \xi_{x;i,k}|^4])^{1/2} \\ &\le \frac{c_{6,27} n(j_m)^{-r(1-d/4)}}{(i-u_{k-1})^{d/4+1}}. \end{aligned} \tag{86}$$

Similarly, by (84) and Lagrange mean value theorem,

$$|\sigma_{x;i}^r - \sigma_{x;i,k}^r| \le c_{6,28}(|\sigma_{x;i}|^{r-2} + |\sigma_{x;i,k}|^{r-2})|\sigma_{x;i}^2 - \sigma_{x;i,k}^2| \le \frac{c_{6,29} n(j_m)^{-r(1-d/4)/2}}{(i-u_{k-1})^{d/4+1}}. \tag{87}$$

Therefore, by (86), (87) and Hölder inequality,

$$\begin{aligned} \mathbb{E}[|\epsilon_{x;m}|] &\le n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \sum_{j=u_{k-1}+1}^{u_k} ((\mathbb{E}[|\Delta U_{x;i}^r - \xi_{x;i,k}^r|^2])^{1/2} + \mu_r |\sigma_{x;j}^r - \sigma_{x;j,k}^r|) \\ &\le c_{6,30} n(j_m)^{-1/2} \sum_{k=1}^{m} \sum_{i=u_{k-1}+1}^{u_k} (i-u_{k-1})^{-(d/4+1)/2} \\ &= c_{6,31} n(j_m)^{-1/2} \sum_{k=1}^{m} \sum_{i=1}^{u_k-u_{k-1}} i^{-(d/4+1)/2}. \end{aligned}$$

Since $u_k - u_{k-1} \le b$, this gives

$$\mathbb{E}[|\epsilon_{x;m}|] \le c_{6,32} n(j_m)^{-1/2} m b^{(1-d/4)/2} = c_{6,32} m^{(d/4+1)/2} n(j_m)^{-d/4/2} (t-s)^{(1-d/4)/2}.$$

But since $n(j_m)$ was chosen so that $n(j_m) \geq m^{2/d/4}(t-s)^{-1}$, one has $E[|\epsilon_{x;m}|] \leq c_{6,33} m^{-(1-d/4)/2}\, |t-s|^{1/2}$ and $\epsilon_{x;m} \to 0$ in L^1 and in probability. Therefore, by (82), we need only to show that

$$n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \zeta_{x;m,k} \xrightarrow{\mathcal{L}} \kappa_{d,r}^{1/2} |t-s|^{1/2} \mathcal{N}$$

in order to complete the proof.

For this, we will use the Lindeberg-Feller theorem (see, e.g., Theorem 2.4.5 in [27]), which states the following: for each m, let $\zeta_{x;m,k}, 1 \leq k \leq m$, be independent random variables with $\mathbb{E}[\zeta_{x;m,k}] = 0$. Suppose:

(a) $n(j_m)^{-1+r(1-d/4)} \sum_{k=1}^{m} \mathbb{E}[\zeta_{x;m,k}^2] \to \nu^2$, and

(b) for all $\delta > 0$, $\lim_{m\to\infty} n(j_m)^{-1+r(1-d/4)} \sum_{k=1}^{m} \mathbb{E}[|\zeta_{x;m,k}|^2 \mathbb{I}_{\{n(j_m)^{-1/2+r(1-d/4)/2} |\zeta_{x;m,k}| > \delta\}}] \to 0$.

Then $n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \zeta_{x;m,k} \xrightarrow{\mathcal{L}} \nu\mathcal{N}$ as $n \to \infty$.

To verify these conditions, recall that $\{\xi_{x;i,k}\}$ and $\{\Delta U_{x;i-u_{k-1}}\}$ have the same law, so that

$$\mathbb{E}[|\zeta_{x;m,k}|^4] = n(j_m)^{-2+2r(1-d/4)} \mathbb{E}\Big[\Big| \sum_{i=1}^{u_k-u_{k-1}} (\Delta U_{x;i}^r - \mu_r \sigma_{x;i}^r) \Big|^4\Big].$$

Hence, by (71),

$$n(j_m)^{-2+2r(1-d/4)} \mathbb{E}[|\zeta_{x;m,k}|^4] \leq c_{6,34} (u_k - u_{k-1})^2 n(j_m)^{-2}.$$

Jensen inequality now gives $m^{-1+r(1-d/4)} \sum_{k=1}^{m} \mathbb{E}[|\zeta_{x;m,k}|^2] \leq c_{6,35} mbn(j_m)^{-1} = c_{6,35} (t-s)$, so that by passing to a subsequence, we may assume that (a) holds for some $\nu \geq 0$.

For (b), let $\delta > 0$ be arbitrary. Then

$$\begin{aligned}
&n(j_m)^{-1+r(1-d/4)} \sum_{k=1}^{m} \mathbb{E}[|\zeta_{x;m,k}|^2 \mathbb{I}_{\{n(j_m)^{-1/2+r(1-d/4)/2} |\zeta_{x;m,k}| > \delta\}}] \\
&\leq \delta^{-2} n(j_m)^{-2+2r(1-d/4)} \sum_{k=1}^{m} \mathbb{E}[|\zeta_{x;m,k}|^4] \\
&\leq c_{6,36} \delta^{-2} m b^2 n(j_m)^{-2} \\
&= c_{6,36} \delta^{-2} m^{-1} (t-s)^2,
\end{aligned}$$

which tends to zero as $m \to \infty$.

It therefore follows that $n(j_m)^{-1/2+r(1-d/4)/2} \sum_{k=1}^{m} \zeta_{x;m,k} \xrightarrow{\mathcal{L}} \nu\mathcal{N}$ as $n \to \infty$ and it remains only to show that $\nu = \kappa_{d,r}^{1/2} |t-s|^{1/2}$. For this, observe that the continuous mapping theorem implies that $|\Theta_r^m(U(\cdot,x))_t - \Theta_r^m(U(\cdot,x))_s|^2 \xrightarrow{\mathcal{L}} \nu^2 \mathcal{N}^2$. By the Skorohod representation theorem, we may assume that the convergence is a.s. By Proposition 1, the family $|\Theta_r^m(U(\cdot,x))_t - \Theta_r^m(U(\cdot,x))_s|^2$ is uniformly integrable. Hence, $|\Theta_r^m(U(\cdot,x))_t - \Theta_r^m(U(\cdot,x))_s|^2 \to \nu^2 \mathcal{N}^2$ in L^1, which implies $\mathbb{E}[|\Theta_r^m(U(\cdot,x))_t - \Theta_r^m(U(\cdot,x))_s|^2] \to \nu^2$. But by Theorem 1, $\mathbb{E}[|\Theta_r^m(U(\cdot,x))_t - \Theta_r^m(U(\cdot,x))_s|^2] \to \kappa_{d,r}|t-s|$, so $\nu = \kappa_{d,r}^{1/2} |t-s|^{1/2}$ and the proof is complete. □

Proof of Theorem 2. It is sufficient to prove (6) for the even p case since the odd p case can be proved similarly. Let $\{n(j)\}_{j=1}^{\infty}$ be any sequence of natural numbers. By Proposition 1, the sequence $\{(U(\cdot,x), \Theta_p^{n(j)}(U(\cdot,x)))\}$ is relatively compact. Therefore, there exists a subsequence $\{n(j_k)\}$ and a cadlag process Y such that $(U(\cdot,x), \Theta_p^{n(j_k)}(U(\cdot,x))) \xrightarrow{\mathcal{L}} (U,Y)$. Fix $0 < s_1 < s_2 < \cdots < s_\ell < s < t$. With notation as in Proposition 2, let

$$\zeta_{x;n(j_k)} = n(j_k)^{-1/2+p(1-d/4)/2} \sum_{i=\lfloor n(j_k)s \rfloor + 2}^{\lfloor n(j_k)t \rfloor} (\xi_{x;i,k}^p - \mu_p \sigma_{x;i,k}^p),$$

and define

$$\eta_{x;n(j_k)} = \Theta_p^{n(j_k)}(U(\cdot,x))_t - \Theta_p^{n(j_k)}(U(\cdot,x))_s - \zeta_{x;n(j_k)}.$$

As in the proof of Proposition 2, $\eta_{x;n(j_k)} \to 0$ in probability. It therefore follows that

$$(\Theta_p^{n(j_k)}(U(\cdot,x))_{s_1}, ..., \Theta_p^{n(j_k)}(U(\cdot,x))_{s_\ell}, \zeta_{x;n(j_k)}) \xrightarrow{\mathcal{L}} (Y(s_1), ..., Y(s_\ell), Y(t) - Y(s)).$$

Note that $\mathcal{F}_{(\lfloor n(j_k)s\rfloor+1)n(j_k)^{-1}}$ and $\zeta_{x;n(j_k)}$ are independent. Hence, $(\Theta_p^{n(j_k)}(U(\cdot,x))_{s_1}, ...,$ $\Theta_p^{n(j_k)}(U(\cdot,x))_{s_\ell})$ and $\zeta_{x;n(j_k)}$ are independent, which implies $Y(t) - Y(s)$ and $(Y(s_1), ...,$ $Y(s_\ell))$ are independent. This yields that the process Y has independent increments.

By Proposition 2, the increment $Y(t) - Y(s)$ is normally distributed with mean zero and variance $\kappa_{d,p}|t-s|$. Moreover, $U(0,x) = 0$ since $\Theta_p^n(U(\cdot,x))_0 = 0$ for all n. Hence, Y is equal in law to $\kappa_{d,p}^{1/2}B$, where B is a standard Brownian motion. It remains only to show that U and B are independent.

Fix $0 < s_1 < s_2 < \cdots < s_\ell \le T$ and $x \in \mathbb{R}^d$. Let $Z_x = (U(s_1,x), ..., U(s_\ell,x))^T$ and $\Sigma_x = \mathbb{E}[Z_x Z_x^T]$. It is easy to see that Σ_x is invertible. Hence, we may define the vectors $v_{x;j} \in \mathbb{R}^\ell$ by $v_{x;j} = \mathbb{E}[Z_x \Delta U_{x;j}]$, and $w_{x;j} = \Sigma_x^{-1} v_{x;j}$. Let $\xi_{x;j} = \Delta U_{x;j} - w_{x;j}^T Z_x$, so that $\xi_{x;j}$ and Z_x are independent.

Define

$$\widetilde{\Theta}_p^n(U(\cdot,x))_t = n^{-1/2+p(1-d/4)/2} \sum_{j=1}^{\lfloor nt\rfloor} (\xi_{x;j}^p - \mu_p \sigma_{x;j}^p).$$

Then

$$|\Theta_p^n(U(\cdot,x))_t - \widetilde{\Theta}_p^n(U(\cdot,x))_t| \le n^{-1/2+p(1-d/4)/2} \Big| \sum_{j=1}^{\lfloor nt\rfloor} (\Delta U_{x;j}^p - \xi_{x;j}^p) \Big|.$$

By (40), binomial expansion and Hölder inequality,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} |\Theta_p^n(U(\cdot,x))_t - \widetilde{\Theta}_p^n(U(\cdot,x))_t|\Big] \\
&\le c_{6,37} n^{-1/2+p(1-d/4)/2} \sum_{\nu=1}^{p} \sum_{j=1}^{\lfloor nT\rfloor} (\mathbb{E}[\Delta U_{x;j}^{2p-2\nu}])^{1/2} (\mathbb{E}[(w_{x;j}^T Z_x)^{2\nu}])^{1/2} \\
&\le c_{6,38} \sum_{\nu=1}^{p} n^{-1/2+\nu(1-d/4)/2} \sum_{j=1}^{\lfloor nT\rfloor} (\mathbb{E}[(w_{x;j}^T Z_x)^{2\nu}])^{1/2} \\
&\le c_{6,39} \max_{1\le i\le \ell} \sum_{\nu=1}^{p} n^{-1/2+\nu(1-d/4)/2} \sum_{j=1}^{\lfloor nT\rfloor} |\mathbb{E}[U(s_i,x)\Delta U_{x;j}]|^\nu.
\end{aligned}$$

Note that by (42) and Hölder inequality, one has $|\mathbb{E}[U(s_i,x)\Delta U_{x;j}]| \le c_{6,40}\sigma_{x;j} \le c_{6,41}$ $n^{-(1-d/4)/2}$ for all $1 \le i \le \ell$ and $1 \le j \le \lfloor nt\rfloor$, and that by (15) and Lagrange mean value theorem, for any $1 \le i \le \ell$ and $1 \le j \le \lfloor nt\rfloor$,

$$\begin{aligned}
\mathbb{E}[U(s_i,x)\Delta U_{x;j}] &= K_d((s_i+t_j)^{1-d/4} - (s_i+t_{j-1})^{1-d/4} - (s_i-t_j)^{1-d/4} + (s_i-t_{j-1})^{1-d/4}) \\
&= \frac{K_d(1-d/4)}{n}((s_i+(j-\zeta_1)/n)^{-d/4} + (s_i-(j-\zeta_2)/n)^{-d/4}) \\
&\le \frac{2K_d(1-d/4)}{n}(s_i-(j-\zeta_2)/n)^{-d/4},
\end{aligned}$$

where $\zeta_1, \zeta_2 \in (0,1)$. Then, for any $1 \le i \le \ell$ and $1 \le \nu \le 2p$,

$$n^{-1/2+\nu(1-d/4)/2} \sum_{j=1}^{\lfloor nT \rfloor} |\mathbb{E}[U(s_i,x)\Delta U_{x;j}]|^{\nu}$$
$$\le c_{6,42} n^{1/2-\nu(1+d/4)/2} \frac{1}{n} \sum_{j=1}^{\lfloor nT \rfloor} (s_i - (j-\zeta_2)/n)^{-d/4},$$

which tends to zero as $n \to \infty$ since $\int_0^T (s_i - u)^{-d/4} du < \infty$. Thus, $(Z_x, \widetilde{\Theta}_p^n(U(\cdot,x))_{s_1}, ..., \widetilde{\Theta}_p^n(U(\cdot,x))_{s_d}) \xrightarrow{\mathcal{L}} (Z_x, \kappa_{d,p}^{1/2} B(s_1), ..., \kappa_{d,p}^{1/2} B(s_\ell))$. Since Z_x and $\widetilde{\Theta}_p^n(U(\cdot,x))$ are independent, this gives that U and B are independent

We now can complete the proof. Note that by (43) and (44),

$$\max_{0 \le t \le T} \left| \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} (n^{p(1-d/4)/2} \Delta U_{x;j}^p - K_d^{p/2} \mu_p) - \Theta_p^n(U(\cdot,x))_t \right|$$
$$\le \mu_p n^{-1/2+p(1-d/4)/2} \sum_{j=1}^{\lfloor nT \rfloor} |\sigma_{x;j}^p - (K_d n^{-1+d/4})^{p/2}|$$
$$\to 0.$$

This finish the proof. □

5. Conclusions

In this paper, we have presented that the realized power variations for the fourth order LKS-SPDEs and their gradient, driven by the space–time white noise in one-to-three dimensional spaces, in time, have infinite quadratic variation and dimension-dependent Gaussian asymptotic distributions. We are concerned with the fluctuation behavior, with delicate analysis of the realized variations, of the sample paths for the above class of equations and their gradient, and complement Allouba's earlier works on the spatio-temporal Hölder regularity of LKS-SPDEs and their gradient. These asymptotic distributions are expressed in terms of the parameters of the problem, and may be used to analyze how the fluctuation behavior depends on those parameters.

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Abbreviations

The following abbreviations are used in this manuscript:

SPDE	Stochastic partial differential equation
LKS	Linearized Kuramoto–Sivashinsky
SIE	Stochastic integral equation
FBM	fractional Brownian motion
BBM	bifractional Brownian motion

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