symmetry

MDPI

Article

Clausen's Series ${}_3F_2(1)$ with Integral Parameter Differences

Kwang-Wu Chen †

Department of Mathematics, University of Taipei, Taipei 10048, Taiwan; kwchen@uTaipei.edu.tw
† Current address: No. 1, Ai-Guo West Road, Taipei 10048, Taiwan.

Abstract: Ebisu and Iwassaki proved that there are three-term relations for ${}_3F_2(1)$ with a group symmetry of order 72. In this paper, we apply some specific three-term relations for ${}_3F_2(1)$ to partially answer the open problem raised by Miller and Paris in 2012. Given a known value ${}_3F_2((a,b,x),(c,x+1),1)$, if $f-x$ is an integer, then we construct an algorithm to obtain ${}_3F_2((a,b,f),(c,f+n),1)$ in an explicit closed form, where n is a positive integer and a,b,c and f are arbitrary complex numbers. We also extend our results to evaluate some specific forms of ${}_{p+1}F_p(1)$, for any positive integer $p \geq 2$.

Keywords: Clausen's hypergeometric functions; summation formulas; Catalan constant

MSC: 33C20; 33C75

check for updates

Citation: Chen, K.-W. Clausen's Series ${}_3F_2(1)$ with Integral Parameter Differences. *Symmetry* **2021**, *13*, 1783. https://doi.org/10.3390/sym13101783

Academic Editor: Paolo Emilio Ricci

Received: 28 August 2021
Accepted: 20 September 2021
Published: 25 September 2021

1. Introduction

The Clausen's hypergeometric function with unit argument [1] is defined to be the complex number

$$ {}_3F_2\left[\begin{matrix} a_1,a_2,a_3 \\ b_1,b_4 \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n n!}, $$

where $\Re(b_1+b_2-a_1-a_2-a_3)>0$,

$$ (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1) $$

denotes the Pochhammer symbol, $\Gamma(s)$ is the Euler's Γ-function, and the complex numbers a_i, b_j are called the numerator and denominator parameters, respectively. The denominator parameters are not allowed to be zero or negative integers ($b_j \notin \mathbb{Z}_{\leq 0}$).

The classical hypergeometric function, which is often called the Gauss hypergeometric function, is defined by

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix} \middle| z\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad \Re(c-a-b)>0, |z|\leq 1, c\notin \mathbb{Z}_{\leq 0}. $$

The most famous formula is the Gauss formula:

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix} \middle| 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b)>0. \tag{1} $$

Another interesting formula is the Ramanujan's formula ([2] (Page 39)):

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2},\frac{1}{2},n \\ 1,n+1 \end{matrix} \middle| 1\right] = \frac{16^n}{\pi n\binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k}, \tag{2} $$

where n is a positive integer. When Ramanujan's collected papers were published in 1927, Equation (2) attracted great attention. There are numerous hypergeometric series identities in the mathematical literature (see [3,4]). The evaluation of the hypergeometric sum ${}_3F_2(1)$ (the Clausenian hypergeometric function with unit argument) is of ongoing interest, since it appears ubiquitously in many physics and statistics problems [5–7].

There is a formula worth noting in ([8] Equation 3.13-(37)):

$$ {}_3F_2\left[\begin{matrix} a, b, f+1 \\ c, f \end{matrix} \middle| 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 - \frac{ab}{f(a+b+1-c)}\right], \quad \Re(c-a-b) > 1. \tag{3} $$

In 1971, Karlsson [9] has deduced the summation formula

$$ {}_3F_2\left[\begin{matrix} a, b, f+m \\ c, f \end{matrix} \middle| 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k (a)_k (b)_k}{(f)_k (1+a+b-c)_k}. \tag{4} $$

For the sake of simplicity, in this paper we assume that the parameters of the hypergeometric series ${}_3F_2(1)$ are such that it converges and the summation formula for it makes sense. Miller and Srivastava [10] and Miller and Paris [11] have obtained the following generalized form. The value of

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| 1\right] $$

is equal to

$$ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \frac{(a)_k (b)_k}{(1+a+b-c)_k k!} {}_{r+1}F_r\left[\begin{matrix} -k, (f_r + m_r) \\ (f_r) \end{matrix} \middle| 1\right]. \tag{5} $$

Thus, one may ask for a similar formula for ${}_3F_2(1)$ where at least one pair of numeratorial and denominatorial parameters differs by a negative integer. Miller and Paris [12] gave the following formula for positive integers n and p:

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} a, b, n \\ c, n+p \end{matrix} \middle| 1\right] &= (p)_n \sum_{k=0}^{p-1} (-1)^k (n)_k \binom{p-1}{k} \frac{(1-c)_{k+n}}{(1-a)_{k+n}(1-b)_{k+n}} \\ &\quad + (n)_p \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{n-1} (-1)^k (p)_k \binom{n-1}{k} \frac{(c-a-b)_{k+p}}{(1-a)_{k+p}(1-b)_{k+p}}. \end{aligned} \tag{6} $$

Shpot and Srivastava [6] derived an elegant summation formula for Clausen's series ${}_3F_2$ with the unit argument, with which they determined that:

$$ \frac{1}{(b)_{n+1}(c)_{n+1}} {}_3F_2\left[\begin{matrix} a, b, c \\ b+1+m, c+1+n \end{matrix} \middle| 1\right] $$

equals

$$ \frac{B(1-a,b)}{(c-b)_{n+1}} {}_3F_2\left[\begin{matrix} -n, b, b-c-n \\ 1+b-a, 1+b-c \end{matrix} \middle| 1\right] + \frac{B(1-a,c)}{(b-c)_{n+1}} {}_3F_2\left[\begin{matrix} -n, c, c-b-n \\ 1+c-a, 1+c-b \end{matrix} \middle| 1\right], \tag{7} $$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler beta function. Miller and Paris [12] indicated that the problem remains of deriving a summation formula for the series:

$$ {}_3F_2\left[\begin{matrix} a, b, f \\ c, f+n \end{matrix} \middle| 1\right], \tag{8} $$

where n is a positive integer and a, b, c and f are arbitrary complex numbers.

In this paper, we try to answer this question. In fact, we evaluate this Clausen's series ${}_3F_2(1)$ under some additional conditions. That is, if we can find a known value ${}_3F_2\left[\begin{matrix}a,b,x\\c,x+1\end{matrix}\,\middle|\,1\right]$ and $f-x\in\mathbb{Z}$, then we can follow our algorithm to give the explicit value of ${}_3F_2\left[\begin{matrix}a,b,f\\c,f+n\end{matrix}\,\middle|\,1\right]$, for any positive integer n. We state the Algorithm 1 in Section 4.

We will use this algorithm to obtain equivalent forms of Equations (6) and (7), and some other formulas as our examples.

Let $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathbb{C}^5$. Ebisu and Iwasaki [13] proved that there exist unique rational functions $u(\mathbf{a}),v(\mathbf{a})\in\mathbb{Q}(\mathbf{a})$ such that

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}a_1,a_2,a_3\\a_4,a_5\end{matrix}\,\middle|\,1\right]&=u(\mathbf{a})\,{}_3F_2\left[\begin{matrix}a_1+p_1,a_2+p_2,a_3+p_3\\a_4+p_4,a_5+p_5\end{matrix}\,\middle|\,1\right]\\
&\quad+v(\mathbf{a})\,{}_3F_2\left[\begin{matrix}a_1+q_1,a_2+q_2,a_3+q_3\\a_4+q_4,a_5+q_5\end{matrix}\,\middle|\,1\right],
\end{aligned}
\tag{9}
$$

where $\mathbf{p}=(p_1,p_2,p_3,p_4,p_5),\mathbf{q}=(q_1,q_2,q_3,q_4,q_5)\in\mathbb{Z}^5$ are distinct shift vectors ([13] Theorem 1.1). They presented a systematic recipe to determine $u(\mathbf{a})$ and $v(\mathbf{a})$ in finite steps. The three-term relation (9) admits an $S_2\ltimes(S_3\times S_3)$-symmetry of order 72. Karp and Prilepkina [14] also gave an alternative method for computing $u(\mathbf{a}),v(\mathbf{a})$ for given shifts. Some of our results could be viewed as particular cases of the results (9). However, our algorithm aims to answer the open problem raised by Miller and Paris. The method of proof in our work is also different from theirs.

Here, we present one more formula obtained by our algorithm as the last example in this introductory section. For any non-negative integer n, we know that:

$$
{}_3F_2\left[\begin{matrix}\frac{1}{3},\frac{2}{3},\frac{1}{2}-n\\1,\frac{3}{2}\end{matrix}\,\middle|\,1\right]
$$

is equal to

$$
\frac{\sqrt{3}\binom{2n}{n}}{\pi 4^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{6k-1}\frac{\binom{6k}{3k}\binom{3k}{2k}}{27^k\binom{2k}{k}^2}\left\{3\log(2)+\sum_{j=1}^{k}\frac{6j-1}{(2j-1)^2}\frac{27^j\binom{2j}{j}^2}{\binom{6j}{3j}\binom{3j}{2j}}\right\}. \tag{10}
$$

We will list some known ${}_3F_2(1)$, which we could use as an initial value in our algorithm in Section 6. In the final section, we extend our results to evaluate some specific forms of ${}_{p+1}F_p(1)$, for $p\geq 2$.

2. Preliminaries

For the sake of brevity, we sometimes denote ${}_3F_2\left[\begin{matrix}a,b,x\\c,x+1\end{matrix}\,\middle|\,1\right]$ as $F(x)$. We cite an explicit formula in ([8] Equation 3.13-(41)) which we need to use later.

$$
{}_3F_2\left[\begin{matrix}a,b,1\\c,2\end{matrix}\,\middle|\,1\right]=\frac{(c-1)}{(a-1)(b-1)}\left[\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right], \tag{11}
$$

where $a\neq1$, $b\neq1$, and $\Re(c-a-b+1)>0$.

Lemma 1. *Let a, b, c, x be complex numbers with $x+1, c \notin \mathbb{Z}_{\leq 0}$, $x+1 \neq c$, and $\Re(c-a-b) > -1$. Then,*

$$
{}_3F_2\left[\begin{matrix} a, b, x \\ c, x+1 \end{matrix} \,\middle|\, 1\right] = \frac{(x+1-a)(x+1-b)}{(x+1-c)(x+1)} {}_3F_2\left[\begin{matrix} a, b, x+1 \\ c, x+2 \end{matrix} \,\middle|\, 1\right] - \frac{\Gamma(c)\Gamma(c+1-a-b)}{(x+1-c)\Gamma(c-a)\Gamma(c-b)}. \tag{12}
$$

Proof. We rewrite this hypergeometric series $F(x)$ as:

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a, b, x \\ c, x+1 \end{matrix} \,\middle|\, 1\right] &= 1 + \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}x}{(c)_{n+1}(1)_{n+1}(n+1+x)} \\
&= 1 + \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \frac{(a+n)(b+n)x}{(c+n)(n+1)(n+1+x)}.
\end{aligned}
$$

If $c \neq 1$ and $c \neq x+1$, we regard the inner summand:

$$
\frac{(a+n)(b+n)x}{(c+n)(n+1)(n+1+x)},
$$

as a rational function of n. Then its partial fraction decomposition is the following decomposition:

$$
\frac{x(b-c)(a-c)}{(c-1)(c-1-x)(c+n)} + \frac{(b-1)(a-1)}{(c-1)(n+1)} - \frac{(b-x-1)(a-x-1)}{(c-x-1)(n+1+x)},
$$

if $c \neq 1$ and $c \neq x+1$. We substitute this representation into the above formula of $F(x)$, then we have:

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a, b, x \\ c, x+1 \end{matrix} \,\middle|\, 1\right] &= 1 + \frac{x(b-c)(a-c)}{c(c-1)(c-1-x)} {}_3F_2\left[\begin{matrix} a, b, c \\ c, c+1 \end{matrix} \,\middle|\, 1\right] \\
&+ \frac{(b-1)(a-1)}{(c-1)} {}_3F_2\left[\begin{matrix} a, b, 1 \\ c, 2 \end{matrix} \,\middle|\, 1\right] - \frac{(b-x-1)(a-x-1)}{(c-x-1)(x+1)} {}_3F_2\left[\begin{matrix} a, b, x+1 \\ c, x+2 \end{matrix} \,\middle|\, 1\right].
\end{aligned}
$$

We know that

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ c, c+1 \end{matrix} \,\middle|\, 1\right] = {}_2F_1\left[\begin{matrix} a, b \\ c+1 \end{matrix} \,\middle|\, 1\right].
$$

Using Equation (11) and the Gauss formula Equation (1), we can evaluate the first two hypergeometric series in the right-hand side of the above equation. Therefore, we get Equation (12), if $c \neq 1$ and $c \neq x+1$:

$$
{}_3F_2\left[\begin{matrix} a, b, x \\ c, x+1 \end{matrix} \,\middle|\, 1\right] = \frac{(x+1-a)(x+1-b)}{(x+1-c)(x+1)} {}_3F_2\left[\begin{matrix} a, b, x+1 \\ c, x+2 \end{matrix} \,\middle|\, 1\right] - \frac{\Gamma(c)\Gamma(c+1-a-b)}{(x+1-c)\Gamma(c-a)\Gamma(c-b)}.
$$

On the other hand, if $c = 1$ and $x \neq 0$, we regard the inner summand

$$
\frac{(a+n)(b+n)x}{(n+1)^2(n+1+x)}
$$

as a rational function of n. Then its partial fraction decomposition is the following decomposition

$$\frac{(b-1)(a-1)}{(n+1)^2}+\frac{(b-x-1)(a-x-1)}{x(n+1+x)}-\frac{ab-ax-bx-a-b+2x+1}{(n+1)x}.$$

We substitute this representation into the above formula of $F(x)$, then we have

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a,b,x\\c,x+1\end{matrix}\middle|\,1\right]&=1-\frac{(ab-ax-bx-a-b+2x+1)}{x}{}_3F_2\left[\begin{matrix}a,b,1\\1,2\end{matrix}\middle|\,1\right]\\&+(a-1)(b-1){}_4F_3\left[\begin{matrix}a,b,1,1\\1,2,2\end{matrix}\middle|\,1\right]+\frac{(a-x-1)(b-x-1)}{x(1+x)}{}_3F_2\left[\begin{matrix}a,b,x+1\\1,x+2\end{matrix}\middle|\,1\right].\end{aligned}$$

We simplify the first three terms on the right-hand side of the above equation, then the resulting identity matches Equation (12) with $c=1$. □

We reverse Equation (12) and get another recurrence relation for our $F(x)$.

Lemma 2. *Let a,b,c,x be complex numbers with $x,c\notin\mathbb{Z}_{\le 0}$, $x\ne a$, $x\ne b$, and $\Re(c-a-b)>-1$. Then,*

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a,b,x\\c,x+1\end{matrix}\middle|\,1\right]&=\frac{x(x-c)}{(x-a)(x-b)}{}_3F_2\left[\begin{matrix}a,b,x-1\\c,x\end{matrix}\middle|\,1\right]\\&+\frac{x}{(x-a)(x-b)}\frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c-a)\Gamma(c-b)}.\end{aligned}\tag{13}$$

3. Main Results

We will solve the recurrence relations in Lemmas 1 and 2 as explicit formulas with a specific known value $F(f)$.

Theorem 1. *Let m be a non-negative integer, a,b,c,f be complex numbers with $f+m,c\notin\mathbb{Z}_{\le 0}$, $m+f\ne a$, $m+f\ne b$, and $\Re(c-a-b)>-1$. Then,*

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a,b,f+m\\c,f+m+1\end{matrix}\middle|\,1\right]&=\frac{(f+1)_m(f+1-c)_m}{(f+1-a)_m(f+1-b)_m}\left\{{}_3F_2\left[\begin{matrix}a,b,f\\c,f+1\end{matrix}\middle|\,1\right]\right.\\&\left.+\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-b)\Gamma(c-a)}\sum_{k=0}^{m-1}\frac{(f+1-a)_k(f+1-b)_k}{(f+1)_k(f+1-c)_{k+1}}\right\}.\end{aligned}$$

Proof. Applying the recurrence relation in Lemma 2, a positive integer ℓ times, we get

$${}_3F_2\left[\begin{matrix}a,b,x\\c,x+1\end{matrix}\middle|\,1\right]=T(\ell)\,{}_3F_2\left[\begin{matrix}a,b,x-\ell\\c,x+1-\ell\end{matrix}\middle|\,1\right]+A\sum_{k=0}^{\ell-1}\frac{T(k+1)}{x-k-c},$$

where

$$A=\frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \text{and}\ T(\ell)=\prod_{j=1}^{\ell}\frac{(x+1-j)(x+1-c-j)}{(x+1-a-j)(x+1-b-j)}.$$

Using the mathematical induction on the integer ℓ, we know that the above formula is correct. Let $x=f+m$ and we use the Pochhammer symbols to rewrite the function T. Then we have

$$T(m-k)=\frac{(f+1)_m(f+1-c)_m}{(f+1-a)_m(f+1-b)_m}\frac{(f+1-a)_k(f+1-b)_k}{(f+1-c)_k(f+1)_k}.$$

Therefore,

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a,b,f+m \\ c,f+m+1 \end{matrix} \middle| 1\right] &= T(m)\,{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right] + A\sum_{k=0}^{m-1}\frac{T(k+1)}{x-k-c} \\
&= T(m)\,{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right] + A\sum_{k=0}^{m-1}\frac{T(m-k)}{f+1+k-c} \\
&= \frac{(f+1)_m(f+1-c)_m}{(f+1-a)_m(f+1-b)_m}\left\{{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right]\right. \\
&\quad \left. +\sum_{k=0}^{m-1}\frac{A}{f+1+k-c}\frac{(f+1-a)_k(f+1-b)_k}{(f+1)_k(f+1-c)_k}\right\}.
\end{aligned}
$$

This result is what we want. □

Similarly, if we apply the recurrence relation in Lemma 1 a positive integer ℓ times, then we get

$$
{}_3F_2\left[\begin{matrix} a,b,x \\ c,x+1 \end{matrix} \middle| 1\right] = T(\ell)\,{}_3F_2\left[\begin{matrix} a,b,x+\ell \\ c,x+1+\ell \end{matrix} \middle| 1\right] + A\sum_{k=0}^{\ell-1}\frac{T(k)}{x+1+k-c}, \tag{14}
$$

where

$$
A = \frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \text{ and } T(\ell) = \prod_{j=1}^{\ell}\frac{(x-a+j)(x-b+j)}{(x-c+j)(x+j)}.
$$

Using the mathematical induction on the integer ℓ, we prove that the above formula is correct. Now we apply this formula to get the following theorem.

Theorem 2. *Let m be a non-negative integer, a,b,c,f be complex numbers with $f-m+1, c \notin \mathbb{Z}_{\leq 0}$, $f-m+1 \neq c$, and $\Re(c-a-b) > -1$. Then,*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a,b,f-m \\ c,f-m+1 \end{matrix} \middle| 1\right] &= \frac{(f+1)_{-m}(f+1-c)_{-m}}{(f+1-a)_{-m}(f+1-b)_{-m}}\left\{{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right]\right. \\
&\quad \left. -\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-b)\Gamma(c-a)}\sum_{k=1}^{m}\frac{(f+1-a)_{-k}(f+1-b)_{-k}}{(f+1)_{-k}(f+1-c)_{-k+1}}\right\},
\end{aligned}
$$

where $(a)_{-k}$ is defined by

$$
(a)_{-k} = \frac{\Gamma(a-k)}{\Gamma(a)} = \frac{1}{(a-k)(a-k+1)\cdots(a-1)} = \frac{1}{(a-k)_k}.
$$

Proof. We set $x = f-m$ in Equation (14) and we use the Pochhammer symbols to rewrite the function T. Then we have

$$
T(m-k) = \frac{(f+1)_{-m}(f+1-c)_{-m}}{(f+1-a)_{-m}(f+1-b)_{-m}}\frac{(f+1-a)_{-k}(f+1-b)_{-k}}{(f+1-c)_{-k}(f+1)_{-k}}.
$$

Therefore,

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a,b,f-m \\ c,f-m+1 \end{matrix} \middle| 1\right] &= T(m)\,{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right] - A\sum_{k=0}^{m-1}\frac{T(k)}{x+1+k-c} \\
&= T(m)\,{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right] - A\sum_{k=1}^{m}\frac{T(m-k)}{f+1-k-c}
\end{aligned}
$$

$$= \frac{(f+1)_{-m}(f+1-c)_{-m}}{(f+1-a)_{-m}(f+1-b)_{-m}} \left\{ {}_3F_2\left[\begin{matrix} a,b,f \\ c,f+1 \end{matrix} \middle| 1\right] - \sum_{k=1}^{m} \frac{A}{f+1-k-c} \frac{(f+1-a)_{-k}(f+1-b)_{-k}}{(f+1)_{-k}(f+1-c)_{-k}} \right\}.$$

We obtain the desired result. □

For any non-negative integer n, we know that

$$\frac{(-n)_k}{k!} = (-1)^k \binom{n}{k}.$$

Thus, we apply the Gauss formula (i.e., Equation (1)) to get

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (x)_k}{(y)_k} = {}_2F_1\left[\begin{matrix} -n, x \\ y \end{matrix} \middle| 1\right] = \frac{\Gamma(y)\Gamma(y+n-x)}{\Gamma(y+n)\Gamma(y-x)}.$$

We state this result as the following lemma.

Lemma 3 (Chu-Vandermonde identity)**.** *For any complex number x, y with $\Re(y-x) > 0$, and any non-negative integer n, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (x)_k}{(y)_k} = \frac{(y-x)_n}{(y)_n}. \tag{15}$$

If we set $y = x+1$, then Equation (15) will become

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{n!}{(x)_{n+1}}. \tag{16}$$

This formula is a special case in ([15] Equation (14)).

Theorem 3. *Let m be a positive integer, a, b, c, f be complex numbers with $f, c \notin \mathbb{Z}_{\leq 0}$. Then,*

$${}_3F_2\left[\begin{matrix} a,b,f \\ c,f+m \end{matrix} \middle| 1\right] = \frac{(f)_m}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{f+k} {}_3F_2\left[\begin{matrix} a,b,f+k \\ c,f+k+1 \end{matrix} \middle| 1\right]. \tag{17}$$

Proof. First, we know that

$$\frac{(f)_m}{(f+n)_m} = \frac{\Gamma(f+n)\Gamma(f+m)}{\Gamma(f+n+m)\Gamma(f)} = \frac{(f)_n}{(f+m)_n}.$$

Therefore, we have

$${}_3F_2\left[\begin{matrix} a,b,f \\ c,f+m \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \frac{(f)_n}{(f+m)_n} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \frac{(f)_m}{(f+n)_m}.$$

Applying Equation (16) with $n = m-1$ and $x = n+f$, we have

$$\sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{n+f+k} = \frac{(m-1)!}{(f+n)_m}.$$

Thus,

$$
{}_3F_2\left[\begin{matrix} a,b,f \\ c,f+m \end{matrix}\middle|\, 1\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \frac{(f)_m}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{n+f+k}
$$
$$
= \frac{(f)_m}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{f+k} {}_3F_2\left[\begin{matrix} a,b,f+k \\ c,f+k+1 \end{matrix}\middle|\, 1\right].
$$

This completes the proof. □

4. Algorithm

In this section, we will use Theorems 1–3 to give an algorithm. Under an additional condition that $F(f_0)$ is known, we can use Algorithm 1 to express ${}_3F_2\left[\begin{matrix} a,b,f_0+n \\ c,f_0+n+m \end{matrix}\middle|\, 1\right]$ in explicit form.

Algorithm 1: Express ${}_3F_2\left[\begin{matrix} a,b,f_0+n \\ c,f_0+n+m \end{matrix}\middle|\, 1\right]$ in an explicit form.

1: Given ${}_3F_2\left[\begin{matrix} a,b,f_0 \\ c,f_0+1 \end{matrix}\middle|\, 1\right]$ as an explicit value. Let m be a positive integer, n be a integer, and $f = f_0 + n$.

2: Using Theorem 3 results in ${}_3F_2\left[\begin{matrix} a,b,f \\ c,f+m \end{matrix}\middle|\, 1\right]$ a linear combination of ${}_3F_2\left[\begin{matrix} a,b,f+k \\ c,f+k+1 \end{matrix}\middle|\, 1\right]$, where $0 \le k \le m-1$.

3: **for** $k \leftarrow 0$ **to** $m-1$.

4: **do**

5: **if** $n+k<0$ **then**

6: Applying Theorem 2 yields ${}_3F_2\left[\begin{matrix} a,b,f+k \\ c,f+k+1 \end{matrix}\middle|\, 1\right]$ as a finite sum and add a constant multiple of ${}_3F_2\left[\begin{matrix} a,b,f_0 \\ c,f_0+1 \end{matrix}\middle|\, 1\right]$.

7: **else**

8: Using Theorem 1 results in ${}_3F_2\left[\begin{matrix} a,b,f+k \\ c,f+k+1 \end{matrix}\middle|\, 1\right]$ as a finite sum and add a constant multiple of ${}_3F_2\left[\begin{matrix} a,b,f_0 \\ c,f_0+1 \end{matrix}\middle|\, 1\right]$.

9: **end if**

10: We get ${}_3F_2\left[\begin{matrix} a,b,f_0+n \\ c,f_0+n+m \end{matrix}\middle|\, 1\right]$ as an explicit value.

Therefore, we have partially solved the open problem of Miller and Paris. Now we will use our algorithm to evaluate some Clausenian hypergeometric functions with unit argument. The first example is concerned with a kind of hypergeometric series,

$$R(r) = {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, r \\ 1, r+1\end{matrix} \middle| 1\right], \tag{18}$$

which is generated by elliptic integrals. Ramanujan stated without proof in his first letter to Hardy a particular case of $R(r)$ with $r \in \mathbb{N}$ ([16] (Equation (2))). Ramanujan [16] gave the explicit formulas for $R(r)$ and $R(r+\frac{1}{2})$ with $r \in \mathbb{N}$, and also the values at $r \in \{1, \frac{1}{2}, \frac{1}{4}\}$.

Example 1. *Prove that*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{11}{4} \\ 1, \frac{9}{4}\end{matrix} \middle| 1\right] = \frac{5}{88{,}704\pi^2}\left(4096\pi + 711 \cdot \Gamma\left(\frac{1}{4}\right)^4\right). \tag{19}$$

Proof. Let us run an exact value by using our algorithm. Since $-\frac{11}{4} - \frac{1}{4} = -3$ is an integer, the known value used in this example is [16]:

$$R(\frac{1}{4}) = {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{4} \\ 1, \frac{5}{4}\end{matrix} \middle| 1\right] = \frac{\Gamma(\frac{1}{4})^4}{16\pi^2}.$$

First, we let $a = b = \frac{1}{2}$, $c = 1$, $f_0 = \frac{1}{4}$, $n = -3$, and $m = 5$. So $f = \frac{1}{4} + (-3) = -\frac{11}{4}$. Second, we use Theorem 3 to yield:

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{11}{4} \\ 1, \frac{9}{4}\end{matrix} \middle| 1\right] = \frac{(-\frac{11}{4})_5}{4!}\sum_{k=0}^{4}\binom{4}{k}\frac{(-1)^k}{-\frac{11}{4}+k}\, {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{11}{4}+k \\ 1, -\frac{7}{4}+k\end{matrix} \middle| 1\right].$$

Third, we need to calculate

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{11}{4}+k \\ 1, -\frac{7}{4}+k\end{matrix} \middle| 1\right] \tag{20}$$

for $0 \le k \le 4$.

Fourth, for $0 \le k \le 2$, we have $-3 \le n+k \le -1 < 0$. We apply Theorem 2 to evaluate Equation (20). We obtain the following three values.

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{11}{4} \\ 1, -\frac{7}{4}\end{matrix} \middle| 1\right] = \frac{5}{8624\pi^2}\left(4544\pi - 45\,\Gamma\left(\frac{1}{4}\right)^4\right), \tag{21}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{7}{4} \\ 1, -\frac{3}{4}\end{matrix} \middle| 1\right] = \frac{2176\pi - 25\,\Gamma\left(\frac{1}{4}\right)^4}{1008\pi^2}, \tag{22}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{3}{4} \\ 1, \frac{1}{4}\end{matrix} \middle| 1\right] = \frac{64\pi - \Gamma\left(\frac{1}{4}\right)^4}{48\pi^2}. \tag{23}$$

Fifth, for $3 \le k \le 4$, we have $0 \le n+k \le 1$. The case $k=3$, Equation (20) is $R(\frac{1}{4})$. For $k=4$, we apply Theorem 3 to evaluate Equation (20).

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{5}{4} \\ 1, \frac{9}{4}\end{matrix} \middle| 1\right] = \frac{5}{144\pi^2}\left(64\pi + \Gamma\left(\frac{1}{4}\right)^4\right). \tag{24}$$

For the final step, if we combine all these together, we will have

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -\frac{11}{4} \\ 1, \frac{9}{4}\end{matrix} \middle| 1\right] = \frac{5}{88{,}704\pi^2}\left(4096\pi + 711\,\Gamma\left(\frac{1}{4}\right)^4\right). \tag{25}$$

This is what we want. □

Reference [17] gave many explicit formulas for

$$ {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, q+n \\ 1, q+n+1 \end{matrix} \middle| 1 \right], $$

where n is an integer and $q = \frac{1}{2}, \frac{i}{3}, \frac{j}{4}$, with $i \in \{1,2\}$, and $j \in \{1,2,3\}$. It is clear that all the formulas appearing in [17] are special cases for Theorems 1 and 2 with $a = 1/6, b = 5/6, c = 1$. We present a more general formula which could not be obtained in [17] as our second example.

Example 2. *For any non-negative integers n, m with $m > 0$, we can get that the value of*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{2}+n \\ 1, \frac{1}{2}+n+m \end{matrix} \middle| 1 \right] $$

is equal to

$$ \frac{\left(\frac{1}{2}+n\right)_m}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k 27^{n+k} \binom{2n+2k}{n+k}}{\pi(3n+3k+1)16^{n+k}\binom{3n+3k}{2n+2k}} \times \left\{ \begin{matrix} 3\sqrt{3}\log(2+\sqrt{3}) \\ +2\displaystyle\sum_{j=0}^{n+k-1} \frac{(3j+1)16^j \binom{3j}{2j}}{(2j+1)^2 27^j \binom{2j}{j}} \end{matrix} \right\}. $$

Solution. Let $a = \frac{1}{6}$, $b = \frac{5}{6}$, $c = 1$, and $f = \frac{1}{2}+n$. Applying Theorem 3 yields

$$ {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{2}+n \\ 1, \frac{1}{2}+n+m \end{matrix} \middle| 1 \right] = \frac{(\frac{1}{2}+n)_m}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{\frac{1}{2}+n+k} {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{2}+n+k \\ 1, \frac{3}{2}+n+k \end{matrix} \middle| 1 \right]. $$

Since $n+k \geq 0$ and we have obtained ([17] (Equation (17))) as follows.

$$ {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{2}+n \\ 1, \frac{3}{2}+n \end{matrix} \middle| 1 \right] = \frac{(2n+1)}{2\pi(3n+1)} \frac{27^n}{16^n} \frac{\binom{2n}{n}}{\binom{3n}{2n}} \times \left\{ \begin{matrix} 3\sqrt{3}\log(2+\sqrt{3}) \\ +2\displaystyle\sum_{k=0}^{n-1} \frac{(3k+1)}{(2k+1)^2} \frac{16^k}{27^k} \frac{\binom{3k}{2k}}{\binom{2k}{k}} \end{matrix} \right\}, \tag{26} $$

where n is a non-negative integer. By replacing the parameter n in the above formula with our new parameter $n+k$, we get the required formula. ■

5. More Examples

We will use our algorithm to obtain equivalent forms of Equations (6), (7) and (10) in this section.

Example 3. *Let a, b, c be complex numbers with $a \neq 1$, $b \neq 1$, $\Re(c-a-b+1) > 0$. For any positive integres n and p, we have*

$$ {}_3F_2\left[\begin{matrix} a, b, n \\ c, n+p \end{matrix} \middle| 1 \right] = \frac{(n)_p}{(p-1)!} \sum_{k=0}^{p-1} \binom{p-1}{k} \frac{(-1)^k}{n+k} \frac{(2)_{n+k-1}(2-c)_{n+k-1}}{(2-a)_{n+k-1}(2-b)_{n+k-1}} \times \left\{ {}_3F_2\left[\begin{matrix} a, b, 1 \\ c, 2 \end{matrix} \middle| 1 \right] + \frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-b)\Gamma(c-a)} \sum_{\ell=0}^{n+k-2} \frac{(2-a)_\ell(2-b)_\ell}{(2)_\ell(2-c)_{\ell+1}} \right\}. \tag{27} $$

Solution. Since n is a positive integer, we use Equation (11) as our known value. First, we set $f_0 = 1, f = n, m = p$. Second, we use Theorem 3 to yield

$$ {}_3F_2\left[\begin{matrix} a,b,n \\ c,n+p \end{matrix} \middle| 1 \right] = \frac{(n)_p}{(p-1)!} \sum_{k=0}^{p-1} \binom{p-1}{k} \frac{(-1)^k}{n+k} {}_3F_2\left[\begin{matrix} a,b,n+k \\ c,n+k+1 \end{matrix} \middle| 1 \right]. $$

Since $n + k - 1 \geq 0$, for $0 \leq k \leq p - 1$, we use Theorem 1 to evaluate

$$ {}_3F_2\left[\begin{matrix} a,b,n+k \\ c,n+k+1 \end{matrix} \middle| 1 \right] = \frac{(2)_{n+k-1}(2-c)_{n+k-1}}{(2-a)_{n+k-1}(2-b)_{n+k-1}} \left\{ {}_3F_2\left[\begin{matrix} a,b,1 \\ c,2 \end{matrix} \middle| 1 \right] + \frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-b)\Gamma(c-a)} \sum_{\ell=0}^{n+k-2} \frac{(2-a)_\ell(2-b)_\ell}{(2)_\ell(2-c)_{\ell+1}} \right\}. $$

Therefore, we get our conclusion. ■

It should be noted that our formula is different from ([12] (Equation (1.7))), but it is an equivalent form.

In 2018, Paris [18] gave two proofs to evaluate a special type of ${}_3F_2(1)$:

$$ {}_3F_2\left[\begin{matrix} 1,1,c \\ d,n+2 \end{matrix} \middle| 1 \right] = \frac{(n+1)\Gamma(d)}{(1-c)_{n+1}} \left\{ \frac{(d-c)_n}{\Gamma(d-1)} [\psi(d-c+n) - \psi(d-1)] - \sum_{k=1}^{n} \binom{n}{k} \binom{n+1-c}{k} \frac{k![\psi(n+1) - \psi(n+1-k)]}{\Gamma(d-n-1+k)} \right\}. $$

This type of hypergeometric series could be obtained by letting $b \to 1$ in the above example (Ref. [18] (Method 1)). Therefore, the known value we need in our algorithm in this case is the limit case of Equation (11) (see [8] (Equation 3.13-(42))):

$$ {}_3F_2\left[\begin{matrix} c,1,1 \\ d,2 \end{matrix} \middle| 1 \right] = \frac{d-1}{c-1} [\psi(d-1) - \psi(d-c)]. \tag{28} $$

The following is the formula which is obtained by our algorithm.

Example 4. *Let c, d be complex numbers with $\Re(d - c + n) > 0$. Then*

$$ {}_3F_2\left[\begin{matrix} c,1,1 \\ d,n+2 \end{matrix} \middle| 1 \right] = (n+1) \frac{(1-d)}{(1-c)} [\psi(d-1) - \psi(d-c)] \frac{(d-c)_n}{(2-c)_n} + \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k (k+1) \frac{(2-d)_k}{(2-c)_k} \sum_{\ell=0}^{k-1} \frac{(d-1)\cdot(2-c)_\ell}{(\ell+1)\cdot(2-d)_{\ell+1}}. \tag{29} $$

Solution. We use Theorem 3 with $f = 1, m = n + 1$ to this hypergeometric series,

$$ {}_3F_2\left[\begin{matrix} c,1,1 \\ d,n+2 \end{matrix} \middle| 1 \right] = \frac{(1)_{n+1}}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} {}_3F_2\left[\begin{matrix} c,1,k+1 \\ d,k+2 \end{matrix} \middle| 1 \right]. $$

For $0 \leq k \leq n$, we use Theorem 1 with $f = 1, m = k$ to the above identity

$$ {}_3F_2\left[\begin{matrix} c,1,1 \\ d,n+2 \end{matrix} \middle| 1 \right] = (n+1) \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} \left[\frac{(2)_k(2-d)_k}{(1)_k(2-c)_k} \left\{ {}_3F_2\left[\begin{matrix} c,1,1 \\ d,2 \end{matrix} \middle| 1 \right] \right.\right. $$

$$+\frac{\Gamma(d)\Gamma(d-c)}{\Gamma(d-c)\Gamma(d)}\sum_{\ell=0}^{k-1}\frac{(d-1)\cdot(1)_\ell(2-c)_\ell}{(2)_\ell(2-d)_{\ell+1}}\Bigg\}\Bigg].$$

We substitute Equation (28) into the above equation and simplify them.

$$\begin{aligned}{}_3F_2\left[\begin{matrix}c,1,1\\d,n+2\end{matrix}\,\middle|\,1\right] &= \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k(k+1)\frac{(1-d)_{k+1}}{(1-c)_{k+1}}[\psi(d-1)-\psi(d-c)]\\ &\quad+\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k(k+1)\frac{(2-d)_k}{(2-c)_k}\sum_{\ell=0}^{k-1}\frac{(d-1)\cdot(2-c)_\ell}{(\ell+1)\cdot(2-d)_{\ell+1}}.\end{aligned}$$

We apply Equation (15) with $x=2-d, y=2-c$, then

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k(k+1)\frac{(1-d)_{k+1}}{(1-c)_{k+1}} &= (n+1)\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(2-d)_k}{(2-c)_k}\frac{(1-d)}{(2-c)}\\ &= (n+1)\frac{(1-d)\cdot(d-c)_n}{(1-c)\cdot(1-c)_n}.\end{aligned}$$

Hence, we have

$$\begin{aligned}{}_3F_2\left[\begin{matrix}c,1,1\\d,n+2\end{matrix}\,\middle|\,1\right] &= (n+1)\frac{(1-d)}{(1-c)}[\psi(d-1)-\psi(d-c)]\frac{(d-c)_n}{(2-c)_n}\\ &\quad+\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k(k+1)\frac{(2-d)_k}{(2-c)_k}\sum_{\ell=0}^{k-1}\frac{(d-1)\cdot(2-c)_\ell}{(\ell+1)\cdot(2-d)_{\ell+1}}.\end{aligned}$$

This formula is then obtained. ■

Example 5. *Shpot and Srivastava [6] derived the elegant formula for Clausen's series ${}_3F_2$ with unit argument:*

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,b,c\\b+1+m,c+1+n\end{matrix}\,\middle|\,1\right]\frac{1}{(b)_{m+1}(c)_{n+1}}\\ &=\frac{B(1-a,b)}{(c-b)_{n+1}m!}{}_3F_2\left[\begin{matrix}-m,b,b-c-n\\1+b-a,1+b-c\end{matrix}\,\middle|\,1\right]+\frac{B(1-a,c)}{(b-c)_{n+1}n!}{}_3F_2\left[\begin{matrix}-n,c,c-b-m\\1+c-a,1+c-b\end{matrix}\,\middle|\,1\right].\end{aligned}$$

We still can use our algorithm to get any specific values of a,b,c,m,n, but the formula is very complicated. Therefore, we just state the process of how we apply our algorithm to get the exact value.

First, the known initial value is from ([8] (Equation 3.13.(38)))

$$ {}_3F_2\left[\begin{matrix}a,b,c\\d+1,c+1\end{matrix}\,\middle|\,z\right]=\frac{c}{c-d}{}_2F_1\left[\begin{matrix}a,b\\d+1\end{matrix}\,\middle|\,z\right]-\frac{d}{c-d}{}_3F_2\left[\begin{matrix}a,b,c\\d,c+1\end{matrix}\,\middle|\,z\right], \tag{30}$$

with $d=b$ and $z=1$. That is,

$$ {}_3F_2\left[\begin{matrix}a,b,c\\b+1,c+1\end{matrix}\,\middle|\,1\right]=\frac{bc}{c-b}\Gamma(1-a)\left[\frac{\Gamma(b)}{\Gamma(1-a+b)}-\frac{\Gamma(c)}{\Gamma(1-a+c)}\right]. \tag{31}$$

Second, the process we use is stated as below. For any non-negative integers p,q,

$$ {}_3F_2\left[\begin{matrix}a,b,c\\b+1+p,c+1+q\end{matrix}\,\middle|\,1\right]\xrightarrow[f=c,m=q+1]{\text{Theorem 3}}{}_3F_2\left[\begin{matrix}a,b,c+k\\b+1+p,c+1+k\end{matrix}\,\middle|\,1\right],0\le k\le q$$

$$\xrightarrow[f=c,m=k]{\text{Theorem 1}} {}_3F_2\left[\begin{matrix} a,b,c \\ b+1+p,c+1 \end{matrix}\middle|\, 1\right]$$
$$\xrightarrow[f=b,m=p+1]{\text{Theorem 3}} {}_3F_2\left[\begin{matrix} a,c,b+j \\ c+1,b+1+j \end{matrix}\middle|\, 1\right], \qquad 0 \le j \le p$$
$$\xrightarrow[f=b,m=j]{\text{Theorem 1}} {}_3F_2\left[\begin{matrix} a,b,c \\ b+1,c+1 \end{matrix}\middle|\, 1\right].$$

Thus, we can write this hypergeometric series as a finite sum and add a constant multiple of the known initial value (Equation (31)). ■

For $0 \le s < \frac{1}{2}$ and $0 \le k \le 1$, let

$$K^s(k) = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}-s, \frac{1}{2}+s \\ 1 \end{matrix}\middle|\, k^2\right]$$

be Ramanujan's generalized elliptic integral of the first kind of order s. The moment $K_{n,s}$ is defined by

$$K_{n,s} = \int_0^1 k^n K^s(k)\,dk,$$

where n is a real number. Borwein et al. ([19] (Theorem 2)) proved that, for $0 \le s < \frac{1}{2}$,

$$K_{n,s} = \frac{\pi}{2(n+1)}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}-s, \frac{1}{2}+s, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix}\middle|\, 1\right]. \tag{32}$$

They also defined the generalized Catalan constant

$$G_s = \frac{K_{0,s}}{2} = \frac{\pi}{4}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}-s, \frac{1}{2}+s, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix}\middle|\, 1\right]$$

and gave the formula ([19] (Theorem 6))

$$K_{0,s} = \frac{\cos(\pi s)}{4s}\left[\psi(\frac{s}{2}+\frac{1}{4}) + \psi(\frac{s}{2}+\frac{3}{4})\right] + \frac{\pi}{4s} = 2G_s, \tag{33}$$

for $s > 0$ and

$$K_{0,0} = \frac{1}{8}\psi'(\frac{1}{4}) - \frac{1}{8}\psi'(\frac{3}{4}) = 2G_0,$$

where $G_0 = G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is the Catalan constant. The function $R(r)$ (see Equation (18)) is related as follows:

$$R(r) = \frac{4r}{\pi} K_{2r-1,0}. \tag{34}$$

The odd moments of K^s can be evaluated by ([19] (Theorem 3)). If we assume that Equation (33) is evaluable, here we can use our algorithm to evaluate the even moments of K^s:

$$K_{2n,s} = \frac{(\frac{3}{2})_n(\frac{1}{2})_n}{(1+s)_n(1-s)_n}\left\{\frac{4G_s}{\pi} + \frac{1}{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)}\sum_{k=0}^{n-1}\frac{(1+s)_k(1-s)_k}{(\frac{3}{2})_k(\frac{1}{2})_{k+1}}\right\}. \tag{35}$$

Borwein et al. [19] said that the results for $s = 1/6$ are especially interesting. Therefore, we derive the following formula, relating $G_{1/6} = \frac{3\sqrt{3}}{4}\log(2)$ as our final example.

Example 6. *For a non-negative integer n, we have:*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2}-n \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right] = \frac{\sqrt{3}\binom{2n}{n}}{\pi 4^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{6k-1} \frac{\binom{6k}{3k}\binom{3k}{2k}}{27^k \binom{2k}{k}^2} \left\{ \begin{matrix} 3\log(2) \\ + \sum_{j=1}^{k} \frac{6j-1}{(2j-1)^2} \frac{27^j \binom{2j}{j}^2}{\binom{6j}{3j}\binom{3j}{2j}} \end{matrix} \right\}. $$

Solution. The initial value that we use is:

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right] = \frac{4}{\pi} G_{1/6} = \frac{3\sqrt{3}}{\pi} \log(2). $$

First, we let $a = \frac{1}{3}$, $b = \frac{2}{3}$, $c = 1$, $f_0 = \frac{1}{2}$, and $f = \frac{1}{2} - n$. Second, we use Theorem 3 to yield

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2}-n \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right] = \frac{(\frac{1}{2}-n)_{n+1}}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\frac{1}{2}-n+k} {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2}-n+k \\ 1, \frac{3}{2}-n+k \end{matrix} \middle| 1 \right]. \tag{36} $$

We set $n-k$ as the new index k in the summation and then we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2}-n \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right] = \frac{\binom{2n}{n}}{4^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1-2k} {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2}-k \\ 1, \frac{3}{2}-k \end{matrix} \middle| 1 \right]. \tag{37} $$

Third, we apply Theorem 2 to yield

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2}-k \\ 1, \frac{3}{2}-k \end{matrix} \middle| 1 \right] = \frac{(\frac{3}{2})_{-k}(\frac{1}{2})_{-k}}{(\frac{7}{6})_{-k}(\frac{5}{6})_{-k}} \left\{ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1 \right] - \frac{\Gamma(1)\Gamma(1)}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \sum_{j=1}^{k} \frac{1}{\frac{1}{2}-j} \frac{(\frac{7}{6})_{-j}(\frac{5}{6})_{-j}}{(\frac{3}{2})_{-j}(\frac{1}{2})_{-j}} \right\}. $$

Combing them all together, we get our conclusion. ■

6. Some Known Initial Values

An important part of our algorithm is to find the corresponding initial value $F(f_0)$ first. Therefore, we enclose a list of some known initial values in this section.

$$ {}_3F_2\left[\begin{matrix} a, b, 2a \\ a+1, 2a-b+1 \end{matrix} \middle| 1 \right] = \frac{\Gamma(a+1)^2\Gamma(2a-b+1)\Gamma(1-b)}{\Gamma(2a+1)\Gamma(a-b+1)^2}, \qquad \Re(b) < 1, \tag{38} $$

$$ {}_3F_2\left[\begin{matrix} a, b, a+b \\ a+1, b+1 \end{matrix} \middle| 1 \right] = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(1+\frac{a+b}{2})\Gamma(1-\frac{a+b}{2})}{\Gamma(a+b+1)\Gamma(1+\frac{a-b}{2})\Gamma(1+\frac{b-a}{2})}, \qquad \Re(a+b) < 2, \tag{39} $$

$$ {}_3F_2\left[\begin{matrix} 2a-1, a, b \\ 2a, a+\frac{b}{2} \end{matrix} \middle| 1 \right] = \frac{\sqrt{\pi}\Gamma(a+\frac{1}{2})\Gamma(a+\frac{b}{2})\Gamma(1-\frac{b}{2})}{\Gamma(a)\Gamma(\frac{b+1}{2})\Gamma(a+\frac{1-b}{2})}, \qquad \Re(b) < 1, \tag{40} $$

$$ {}_3F_2\left[\begin{matrix} a, 1-a, b \\ a+1, 2b-a \end{matrix} \middle| 1 \right] = \frac{\pi\Gamma(a+1)\Gamma(2b-a)}{2^{2c-1}\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b-a+\frac{1}{2})}, \qquad \Re(b) > 0, \tag{41} $$

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ a+1, b+1 \end{matrix} \middle| 1 \right] = \frac{ab}{b-a}\Gamma(1-c)\left(\frac{\Gamma(a)}{\Gamma(a-c+1)} - \frac{\Gamma(b)}{\Gamma(b-c+1)}\right), \qquad a \neq b, \tag{42} $$

$$ {}_3F_2\left[\begin{matrix} 1, a, b \\ 2, c \end{matrix} \middle| 1 \right] = \frac{c-1}{(a-1)(b-1)}\left(\frac{\Gamma(c-1)\Gamma(c+1-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right), \qquad a \neq 1, b \neq 1, \ \Re(c-a-b) > -1, \tag{43} $$

$$ {}_3F_2\left[\begin{matrix} 1, 1, a \\ 2, b \end{matrix} \middle| 1 \right] = \frac{b-1}{a-1}(\psi(b-1) - \psi(b-a)), \qquad a \neq 1, \Re(b-a) > 0, \tag{44} $$

$$ {}_3F_2\left[\begin{matrix}1,1,1\\2,a\end{matrix}\middle|\,1\right] = (a-1)\psi'(a-1), \qquad \Re(a) > 1, \tag{45} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},a,1-a\\\frac{3}{2},1\end{matrix}\middle|\,1\right] = \frac{\sin(\pi a)}{\pi(1-2a)}\left(\psi(\frac{a+1}{2}) - \psi(\frac{a}{2})\right) - \frac{1}{1-2a}, \qquad a \neq \frac{1}{2}, \tag{46} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2}-a,\frac{1}{2}+a\\\frac{3}{2},1\end{matrix}\middle|\,1\right] = \frac{\cos(\pi a)}{2\pi a}\left(\psi(\frac{a}{2}+\frac{1}{4}) - \psi(\frac{a}{2}+\frac{3}{4})\right) + \frac{1}{2a}, \qquad 0 < a < \frac{1}{2}, \tag{47} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\\frac{3}{2},\frac{3}{2}\end{matrix}\middle|\,1\right] = 2G = \frac{\pi}{2}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\\frac{3}{2},1\end{matrix}\middle|\,1\right], \tag{48} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},1,\frac{5}{4}\\\frac{3}{2},\frac{7}{4}\end{matrix}\middle|\,1\right] = 3 - \frac{6\pi^3}{\Gamma^4(\frac{1}{4})}, \tag{49} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},1,\frac{7}{4}\\\frac{3}{2},\frac{9}{4}\end{matrix}\middle|\,1\right] = \frac{10}{9} + \frac{5\Gamma^4(\frac{1}{4})}{288\pi}, \tag{50} $$

$$ {}_3F_2\left[\begin{matrix}\frac{3}{4},1,\frac{3}{2}\\\frac{7}{4},\frac{9}{4}\end{matrix}\middle|\,1\right] = \frac{15}{16}\left(4 - \sqrt{2}\pi + 2\sqrt{2}\log(1+\sqrt{2})\right), \tag{51} $$

$$ {}_3F_2\left[\begin{matrix}\frac{5}{6},1,\frac{4}{3}\\\frac{11}{6},\frac{3}{2}\end{matrix}\middle|\,1\right] = \frac{5\sqrt{3}}{2}\log(2+\sqrt{3}), \tag{52} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{1}{4},1\\\frac{5}{4},\frac{5}{4}\end{matrix}\middle|\,1\right] = \frac{\pi^2+8G}{16}, \tag{53} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{3}\\1,\frac{4}{3}\end{matrix}\middle|\,1\right] = \frac{\sqrt{3}\sqrt[3]{2}}{2\pi}A - \frac{\sqrt[3]{2}}{\pi}B, \tag{54} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{2}{3}\\1,\frac{5}{3}\end{matrix}\middle|\,1\right] = \frac{\sqrt{3}\sqrt[3]{4}}{3\pi}A + \frac{2\sqrt[3]{4}}{3\pi}B, \tag{55} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{4}\\1,\frac{5}{4}\end{matrix}\middle|\,1\right] = \frac{12^{3/4}(C-D)}{2\pi}, \tag{56} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{3}{4}\\1,\frac{7}{4}\end{matrix}\middle|\,1\right] = \frac{9\cdot 12^{3/4}(C+D)}{14\sqrt{3}\pi}, \tag{57} $$

The parameters A and B in Equations (54) and (55) are defined by:

$$ A = \log\left((1-2^{-\frac{2}{3}})^2 + (1+2^{-\frac{2}{3}}\sqrt{3})^2\right) - \log\left((1-2^{-\frac{2}{3}})^2 + (1-2^{-\frac{2}{3}}\sqrt{3})^2\right), \tag{58} $$

$$ B = \arctan\left(\frac{3}{3+\sqrt[3]{2}+3\sqrt[3]{4}}\right), \tag{59} $$

and the parameters C and D in Equations (56) and (57) are defined by:

$$ C = \frac{1}{2}\log\left(\frac{3^{5/4}-3^{3/4}+\sqrt{2}}{3^{5/4}-3^{3/4}-\sqrt{2}}\right), \quad \text{and} \quad D = \arccos\left(\frac{3^{5/4}+3^{3/4}}{2\sqrt{5+3\sqrt{3}}}\right). \tag{60} $$

Here, we list the references where all the known values mentioned above can be found: Equations (38) and (39) are derived from Dixon [1,8]; Equation (40) is derived from Watson [1,8]; Equation (41) is derived from Whipple [1,8]; Equation (42) is derived from ([8] Equation 3.13-(38)); Equation (43) is derived from ([8] Equation 3.13.(41)); Equation (44) is derived from ([8] Equation 3.13.(42)); Equation (45) is derived from ([8] Equation 3.13.(43)); Equation (46) is derived from ([19] Equation (29)); Equation (47) is derived from ([19] Equation (51)); Equation (48) is derived from ([19] Equation (82)); Equations (49) and (50) are derived from [20]; Equation (51) is derived from [21]; Equation (52) is derived from Watson [22]; Equation (53) is derived from [22]; Equations (54)–(57) are derived from [23].

The initial value we used in Example 1 is Equation (38) with $a = \frac{1}{4}$, $b = \frac{1}{2}$. In our Example 2, the initial value is obtained from Equation (47) with $a = \frac{1}{3}$. The initial values used in Example 3, Example 4, and Example 5 are Equation (43), Equation (44), and Equation (42), respectively. The initial value in Example 6 that we used can be obtained by Equation (46) with $a = \frac{1}{3}$, or Equation (47) with $a = \frac{1}{6}$.

7. Evaluations of Some Specific Forms of ${}_{p+1}F_p(1)$

Let r be a positive integer. If all elements of the vector $\mathbf{g} = (g_1, g_2, \ldots, g_r)$ are distinct, then we have the following partial fraction decomposition [24]:

$$\frac{1}{(g_1+x)(g_2+x)\cdots(g_m+x)} = \sum_{q=1}^{m} \frac{B_q^{-1}}{g_q+x}, \tag{61}$$

where $B_q = \prod_{\substack{v=1 \\ v\neq q}}^{m} (g_v - g_q)$. This will provide us with the following lemma so that we can extend our results to evaluate some specific forms of ${}_{p+1}F_p(1)$.

Lemma 4 ([24]). *Suppose r is a positive integer, $\mathbf{f} = (f_1, f_2, \ldots, f_r)$ is a complex vector, $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a vector of positive integers, $m = m_1 + m_2 + \cdots + m_r$, and all elements of the vector $\mathbf{g} = (f_1, f_1+1, \ldots, f_1+m_1-1, \ldots, f_r, f_r+1, \ldots, f_r+m_r-1) = (g_1, g_2, \ldots, g_m)$ are distinct. If $\Re(1+c-a-b) > 0$, then:*

$${}_{r+2}F_{r+1}\left[\begin{matrix} a, b, \mathbf{f} \\ c, \mathbf{f}+\mathbf{m} \end{matrix} \middle| 1\right] = \sum_{q=1}^{m} \frac{(\mathbf{f})_{\mathbf{m}}}{g_q B_q} {}_3F_2\left[\begin{matrix} a, b, g_q \\ c, g_q+1 \end{matrix} \middle| 1\right], \tag{62}$$

where $B_q = \prod_{\substack{v=1 \\ v\neq q}}^{m} (g_v - g_q)$, $\mathbf{f}+\mathbf{m} = (f_1+m_1, \ldots, f_r+m_r)$, and $(\mathbf{f})_{\mathbf{m}} = (f_1)_{m_1}\cdots(f_r)_{m_r}$.

Proof. Applying the partial fraction decomposition (61), we have:

$$\frac{(\mathbf{f})_n}{(\mathbf{f}+\mathbf{m})_n} = \frac{(\mathbf{f})_{\mathbf{m}}}{(\mathbf{f}+n)_{\mathbf{m}}} = \frac{(\mathbf{f})_{\mathbf{m}}}{\prod_{q=1}^{m}(g_q+n)} = (\mathbf{f})_{\mathbf{m}} \sum_{q=1}^{m} \frac{B_q^{-1}(g_q)_n}{g_q(g_q+1)_n},$$

where $(\mathbf{f})_n = (f_1)_n(f_2)_n\cdots(f_r)_n$. Therefore,

$${}_{r+2}F_{r+1}\left[\begin{matrix} a, b, \mathbf{f} \\ c, \mathbf{f}+\mathbf{m} \end{matrix} \middle| 1\right] = \sum_{q=1}^{m} \frac{(\mathbf{f})_{\mathbf{m}}}{g_q B_q} {}_3F_2\left[\begin{matrix} a, b, g_q \\ c, g_q+1 \end{matrix} \middle| 1\right].$$

This completes the proof. □

We combine the algorithm of ${}_3F_2(1)$ with the above lemma, and get the following results.

Theorem 4. *Suppose r is a positive integer, $\mathbf{f} = (f_1, f_2, \ldots, f_r)$ is a complex vector, $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ is a vector of integers, $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a vector of positive integers, $m = m_1 + m_2 + \cdots + m_r$, and all elements of the vector $\mathbf{g} = (f_1+n_1, f_1+n_1+1, \ldots, f_1+n_1+m_1-1, \ldots, f_r+n_r, f_r+n_r+1, \ldots, f_r+n_r+m_r-1) = (g_1, g_2, \ldots, g_m)$ are distinct. If $\Re(1+c-a-b) > 0$, and for $1 \le i \le r$, the values of ${}_3F_2\left[\begin{matrix} a, b, f_i \\ c, f_1+1 \end{matrix} \middle| 1\right]$ are known, then we can express ${}_{r+2}F_{r+1}\left[\begin{matrix} a, b, \mathbf{f}+\mathbf{n} \\ c, \mathbf{f}+\mathbf{n}+\mathbf{m} \end{matrix} \middle| 1\right]$ in explicit form.*

Here, we provide some examples.

Example 7. *Let* $a = \frac{1}{6}$, $b = \frac{5}{6}$, $c = 1$, $\mathbf{f} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, $\mathbf{n} = (0,0,0)$, $\mathbf{m} = (1,1,1)$, $\mathbf{g} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, *and* $m = 3$. *We substitute these parameters into Theorem 4 to get:*

$$
\begin{aligned}
{}_5F_4\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\\ 1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}\end{matrix}\,\middle|\, 1\right] &= 2\,{}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{2}\\ 1, \frac{3}{2}\end{matrix}\,\middle|\, 1\right] - 9\,{}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{3}\\ 1, \frac{4}{3}\end{matrix}\,\middle|\, 1\right] + 8\,{}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{4}\\ 1, \frac{5}{4}\end{matrix}\,\middle|\, 1\right] \\
&= \frac{3\sqrt{3}}{\pi}\log(2+\sqrt{3}) - \frac{9\sqrt{3}\sqrt[3]{2}}{2\pi}A + \frac{9\sqrt[3]{2}}{\pi}B + \frac{4\cdot 12^{3/4}}{\pi}(C-D),
\end{aligned}
$$

where A, B, C, *and* D *are defined in (58)–(60).*

Example 8. *For any distinct numbers* a, b, c *with* $\Re(a+b+c) < 3$, *we have:*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}a, b, c, k\\ a+1, b+1, c+1\end{matrix}\,\middle|\, 1\right] \\
&= \frac{abc}{(a-b)(a-c)}B(a, 1-k) + \frac{abc}{(b-a)(b-c)}B(b, 1-k) + \frac{abc}{(c-a)(c-b)}B(c, 1-k),
\end{aligned}
\tag{63}
$$

where $k = a+b+c$ *and* $B(x,y)$ *is the Euler beta function.*

Proof. Applying Lemma 4, we have:

$$
{}_4F_3\left[\begin{matrix}a, b, c, k\\ a+1, b+1, c+1\end{matrix}\,\middle|\, 1\right] = \frac{b}{b-a}\,{}_3F_2\left[\begin{matrix}a, c, k\\ a+1, c+1\end{matrix}\,\middle|\, +\right] \frac{a}{a-b}\,{}_3F_2\left[\begin{matrix}b, c, k\\ b+1, c+1\end{matrix}\,\middle|\, .\right]
$$

Using Equation (42) in the above formula, we get our result. □

We use Equation (63) to evaluate the following hypergeometric series.

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}1, \frac{1}{2}, \frac{1}{4}, \frac{7}{4}\\ 2, \frac{3}{2}, \frac{5}{4}\end{matrix}\,\middle|\, 1\right] &= \frac{1}{3}B\left(1, \frac{-3}{4}\right) - B\left(\frac{1}{2}, \frac{-3}{4}\right) + \frac{2}{3}B\left(\frac{1}{4}, \frac{-3}{4}\right) \\
&= -\frac{4}{9} + \frac{4}{9\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2 - \frac{\sqrt{2}}{6\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2.
\end{aligned}
$$

Example 9. *Suppose* r *is a positive integer,* $\mathbf{f} = (f_1, f_2, \ldots, f_r)$ *and* $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ *are vectors of positive integers,* $m = m_1 + m_2 + \cdots + m_r$, *and all elements of the vector* $\mathbf{g} = (f_1, f_1+1, \ldots, f_1+m_1-1, \ldots, f_r, f_r+1, \ldots, f_r+m_r-1) = (g_1, g_2, \ldots, g_m)$ *are distinct. If* $a \neq 1$, $b \neq 1$, $\Re(1+c-a-b) > 0$, *then we can express* ${}_{r+2}F_{r+1}\left[\begin{matrix}a, b, \mathbf{f}\\ c, \mathbf{f}+\mathbf{m}\end{matrix}\,\middle|\, 1\right]$ *in the following explicit form:*

$$
\begin{aligned}
\sum_{q=1}^{m} \frac{(\mathbf{f})_{\mathbf{m}}}{g_q B_q}\Bigg\{ \frac{(2)_{g_q-1}(2-c)_{g_q-1}}{(2-a)_{g_q-1}(2-b)_{g_q-1}} \Bigg({}_3F_2\left[\begin{matrix}a, b, 1\\ c, 2\end{matrix}\,\middle|\, 1\right] \\
+ \frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{\ell=0}^{g_q-2} \frac{(2-a)_\ell(2-b)_\ell}{(2)_\ell(2-c)_{\ell+1}} \Bigg) \Bigg\}.
\end{aligned}
\tag{64}
$$

Proof. Example 3 gives the explicit formula of ${}_3F_2\left[\begin{matrix}a, b, n\\ c, n+p\end{matrix}\,\middle|\, 1\right]$, for any $n, p \in \mathbb{N}$. Therefore, we set $p = 1$ in that formula and get

$$
{}_3F_2\left[\begin{matrix} a,b,n \\ c,n+1 \end{matrix} \middle| 1\right] = \frac{(2)_{n-1}(2-c)_{n-1}}{(2-a)_{n-1}(2-b)_{n-1}} \left\{ {}_3F_2\left[\begin{matrix} a,b,1 \\ c,2 \end{matrix} \middle| 1\right] + \frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{\ell=0}^{n-2} \frac{(2-a)_\ell(2-b)_\ell}{(2)_\ell(2-c)_{\ell+1}} \right\}.
$$

From the above formula and Lemma 4, we complete the proof. □

Let us use Equation (64) to evaluate

$$
{}_5F_4\left[\begin{matrix} \frac{1}{2},\frac{1}{2},2,4,6 \\ 1,3,5,8 \end{matrix} \middle| 1\right].
$$

The parameters of this hypergeometric series are $a = \frac{1}{2} = b$, $c = 1$, $\mathbf{f} = (2,4,6)$, $\mathbf{m} = (1,1,2)$, $m = 4$, and $\mathbf{g} = (2,4,6,7)$. Applying Equation (64), we have

$$
\begin{aligned}
{}_5F_4\left[\begin{matrix} \frac{1}{2},\frac{1}{2},2,4,6 \\ 1,3,5,8 \end{matrix} \middle| 1\right] &= \sum_{q=1}^{4} \frac{(2)_1(4)_1(6)_2}{g_q B_q} \left\{ \frac{(2)_{g_q-1}(1)_{g_q-1}}{(\frac{3}{2})^2_{g_q-1}} \left(\frac{4}{\pi} + \frac{1}{\pi} \sum_{\ell=0}^{g_q-2} \frac{(\frac{3}{2})^2_\ell}{(2)^2_\ell} \right) \right\} \\
&= \frac{21}{5} \cdot \frac{40}{9\pi} - 7 \cdot \frac{6096}{1225\pi} + 7 \cdot \frac{851{,}240}{160{,}083\pi} - \frac{16}{5} \cdot \frac{7{,}023{,}364}{1{,}288{,}287\pi} \\
&= \frac{348{,}741{,}632}{96{,}621{,}525\pi}.
\end{aligned}
$$

In fact, if $a = b = \frac{1}{2}$ and $c = 1$, then we could use Equation (2) to get a more beautiful form:

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} \frac{1}{2},\frac{1}{2},\mathbf{f} \\ 1,\mathbf{f}+\mathbf{m} \end{matrix} \middle| 1\right] = \sum_{q=1}^{m} \frac{16^{g_q} \cdot (\mathbf{f})_{\mathbf{m}}}{\pi g_q^2 B_q \binom{2g_q}{g_q}^2} \sum_{k=0}^{g_q-1} \frac{\binom{2k}{k}^2}{16^k}. \tag{65}
$$

Funding: This research was funded by the Ministry of Science and Technology, Taiwan, R.O.C. grant number MOST 109-2115-M-845-001.

Acknowledgments: The author is indebted to the two anonymous referees and the editor of the journal for their helpful remarks.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bailey, W.N. *Generalized Hypergeometric Series*; Cambridge Tracts in Mathematics and Mathematical Physics, No. 32; Stechert-Hafner, Inc.: New York, NY, USA, 1964.
2. Berndt, B. *Ramanujan's Notebooks*; Springer: New York, NY, USA, 1989; Volume 2.
3. Li, N.N.; Chu, W. Non-terminating ${}_3F_2$-series with unit argument. *Integral Transforms Spec. Funct.* **2018**, *29*, 450–469. [CrossRef]
4. Masjed-Jamei, M.; Koepf, W. A new identity for generalized hypergeometric functions and applications. *Axioms* **2019**, *8*, 12. [CrossRef]
5. Milgram, M. On hypergeometric ${}_3F_2(1)$—A review. *arXiv* **2010**, arXiv:1011.4546v1[math.CA].
6. Shpot, M.A.; Srivastava, H.M. The Clausenian hypergeometric function ${}_3F_2$ with unit argument and negative integral parameter differences. *Appl. Math. Comput.* **2015**, *259*, 819–827. [CrossRef]
7. Srivastava, H.M.; Qureshi, M.I.; Jabee, S. Some general series identities and summation theorems for Clausen's hypergeometric function with negative integer numerator and denominator parameters. *J. Nonlinear Convex Anal.* **2020**, *21*, 805–819.
8. Luke, Y.L. *The Special Functions and Their Approximations*; Academic Press, Inc., Harcourt Brace Jovanovich, Publishers: San Diego, CA, USA, 1969; Volume 1.
9. Karlsson, P.W. Karlsson-Hypergeometric functions with integral parameter differences. *J. Math. Phys.* **1971**, *12*, 270–271. [CrossRef]

10. Miller, A.R.; Srivastava, H.M. Karlsson-Minton summation theorems for the generalized hypergeometric series of unit argument. *Integral Transforms Spec. Funct.* **2010**, *21*, 603–612. [CrossRef]
11. Miller, A.R.; Paris, R.B. Transformation formulas for the generalized hypergeometric function with integral parameter differences. *Rocky Mountain J. Math.* **2013**, *43*, 291–327. [CrossRef]
12. Miller, A.R.; Paris, R.B. Clausen's series ${}_3F_2(1)$ with integral parameter differences and transformations of the hypergeometric function ${}_2F_2(x)$. *Integral Transforms Spec. Funct.* **2012**, *23*, 21–33. [CrossRef]
13. Ebisu, A.; Iwasaki, K. Three-term relations for ${}_3F_2(1)$. *J. Math. Anal. Appl.* **2018**, *463*, 593–610. [CrossRef]
14. Karp, D.; Prilepkina, E. Transformations of the hypergeometric ${}_4F_3$ with one unit shift: A group theoretic study. *Mathematics* **2020**, *8*, 1966. [CrossRef]
15. Batir, N.; Chen, K.-W. Finite Hurwitz-Lerch functions. *Filomat* **2019**, *33*, 101–109. [CrossRef]
16. Adamchik, V.S. A certain series associated with Catalan's constant. *Z. Anal. Anwend.* **2002**, *21*, 817–826. [CrossRef]
17. Chen, K.-W. Explicit formulas for some infinite ${}_3F_2(1)$-series. *Axioms* **2021**, *10*, 125. [CrossRef]
18. Paris, R.B. A summation formula for a ${}_3F_2(1)$ hypergeometric series. *arXiv* **2018**, arXiv:1803.03012v1 [math.CA].
19. Borwein, D.; Borwein, J.M.; Glasser, M.L.; Wan, J.G. Moments of Ramanujan's generalized elliptic integrals and extensions of Catalan's constant. *J. Math. Anal. Appl.* **2011**, *384*, 478–496. [CrossRef]
20. Campbell, J.M.; Cantarini, M.; D'Aurizio, J. Symbolic computations via Fourier-Legendre expansions and fractional operators. *Integral Transforms Spec. Funct.* **2021**. [CrossRef]
21. Campbell, J.M.; D'Aurizio, J.; Sondow, J. Hypergeometry of the parbelos. *Am. Math. Mon.* **2020**, *127*, 23–32. [CrossRef]
22. Ebisu, A.; Iwasaki, K. Contiguous relations, Laplace's methods, and continued fractions for ${}_3F_2(1)$. *Ramanujan J.* **2019**, *49*, 159–213. [CrossRef]
23. Asakura, M.; Yabu, T. Explicit logarithmic formulas of special values of hypergeometric funciton ${}_3F_2$. *Commun. Contemp. Math.* **2020**, *22*, 1950040. [CrossRef]
24. Karp, D.B.; Prilepkina, E.G. Extensions of Karlsson-Minton summation theorem and some consequences of the first Miller-Paris transformation. *Integral Transforms Spec. Funct.* **2018**, *29*, 955–970. [CrossRef]