

Article

# Classification of Integrodifferential $C^*$ -Algebras

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**Abstract:** The infinite product of matrices with integer entries, known as a modified Glimm–Bratteli symbol  $\mathfrak{n}$ , is a new, sufficiently simple, and very powerful tool for the characterization of approximately finite-dimensional (AF) algebras. This symbol provides a convenient algebraic representation of the Bratteli diagram for AF algebras in the same way as was previously performed by J. Glimm for more simple uniformly hyperfinite (UHF) algebras. We apply this symbol to characterize integrodifferential algebras. The integrodifferential algebra  $\mathcal{F}_{N,M}$  is the  $C^*$ -algebra generated by the following operators acting on  $L^2([0, 1]^N \rightarrow \mathbb{C}^M)$ : (1) operators of multiplication by bounded matrix-valued functions, (2) finite-difference operators, and (3) integral operators. Most of the operators and their approximations studying in physics belong to these algebras. We give a complete characterization of  $\mathcal{F}_{N,M}$ . In particular, we show that  $\mathcal{F}_{N,M}$  does not depend on  $M$ , but depends on  $N$ . At the same time, it is known that differential algebras  $\mathcal{H}_{N,M}$ , generated by the operators (1) and (2) only, do not depend on both dimensions  $N$  and  $M$ ; they are all  $*$ -isomorphic to the universal UHF algebra. We explicitly compute the Glimm–Bratteli symbols (for  $\mathcal{H}_{N,M}$ , it was already computed earlier) which completely characterize the corresponding AF algebras. This symbol  $\mathfrak{n}$  is an infinite product of matrices with nonnegative integer entries. Roughly speaking, all the symmetries appearing in the approximation of complex infinite-dimensional integrodifferential and differential algebras by finite-dimensional ones are coded by a product of integer matrices.

**Keywords:** representation of integrodifferential operators; Glimm–Bratteli symbols; AF algebras



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## 1. Introduction

Discrete and continuous analogs of integrodifferential algebras are actively used in various applications, for example, in the development of computer algorithms for symbolic and numerical solving of integrodifferential equations, see, e.g., [1–4]. These studies do not concern the structure of  $C^*$ -algebras. On the other hand, differential algebras are closely related to the rotation  $C^*$ -algebras well studied in, e.g., [5–8]. In contrast to the rotation algebras, the integrodifferential algebras contain operators of multiplication by discontinuous functions and integral operators. Nevertheless, the integrodifferential algebras are AF algebras and, hence, they admit a classification in terms of, e.g., the Bratteli diagrams. We give an algebraic representation of the Bratteli diagrams based on the infinite products of matrices with nonnegative integer entries. These infinite products of matrices extend the infinite products of natural numbers that J. Glimm used for the characterization of uniformly hyper-finite algebras. This is a reason why we call the infinite products of matrices Glimm–Bratteli symbols. GB symbols are a fairly powerful tool because it allows one to use a simple matrix product technique to prove the presence or absence of isomorphism between algebras. In general, isomorphism is very important in understanding the differences and similarities between different classes of operators. Recently, in [9], it was proved that all the differential  $C^*$ -algebras are isomorphic, independently of their dimensions. A natural question arises: what will happen if we add integral operators to differential algebras? We see below that the resulting integrodifferential algebras are already nonisomorphic to each other, they depend on the dimension. Thus, while there are unified methods for analyzing finite-difference approximations of differential equations

for any dimension, the analysis of such approximations of integrodifferential equations may strongly depend on the dimensions.

The current research is devoted to the integrodifferential operators acting on multi-dimensional tori. In the future, we plan to adapt the analysis to more complex stochastic integrodifferential operators acting on non-compact and fractal domains.

The manuscript is organized as follows: Section 2 contains basic information about \*-homomorphisms of finite-dimensional C\*-algebras, a structure of AF-algebras, related graphical Bratteli diagrams along with their algebraic representations by the Glimm–Bratteli symbols, and some examples; Section 3 is devoted to the characterization of integrodifferential C\*-algebras in terms of the Glimm–Bratteli symbols; Section 4 contains the proofs of the results from Sections 2 and 3; we conclude in Section 5.

### 2. Characterization of AF-Algebras. Infinite Product of Matrices with Integer Entries

Let us recall some facts about Bratteli diagrams. It is well known that any finite-dimensional C\*-algebra is \*-isomorphic to the direct sum of simple matrix algebras. Up to the order of terms, this direct sum is determined uniquely. It is convenient to use the following notation for finite-dimensional C\*-algebras. Let  $\mathbf{p} = (p_j)_{j=1}^n \in \mathbb{N}^n$ , then

$$\mathcal{M}(\mathbf{p}) := \mathbb{C}^{p_1 \times p_1} \oplus \dots \oplus \mathbb{C}^{p_n \times p_n}. \tag{1}$$

Any \*-homomorphism from  $\mathcal{M}(\mathbf{p})$  to  $\mathcal{M}(\mathbf{q})$  with  $\mathbf{p} \in \mathbb{N}^n$ ,  $\mathbf{q} \in \mathbb{N}^m$  is internally (inside each  $\mathbb{C}^{q_j \times q_j}$ ) unitary equivalent to some canonical \*-homomorphism. Any canonical \*-homomorphism is completely and uniquely determined by the matrix of multiplicities of partial embeddings  $\mathbf{E} \in \mathbb{Z}_+^{m \times n}$  (E matrix), satisfying  $\mathbf{E}(p_j)_{j=1}^n = (\tilde{q}_j)_{j=1}^m$ , where  $\tilde{q}_j \leq q_j$ , and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . For example, the canonical \*-homomorphism

$$\varphi : \mathcal{M}(2, 2, 3) \rightarrow \mathcal{M}(4, 4), \quad \mathbf{A} \oplus \mathbf{B} \oplus \mathbf{C} \mapsto \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix} \oplus (0)$$

has the E matrix

$$\mathbf{E} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For simplicity, we can write

$$\mathcal{M}(2, 2, 3) \xrightarrow{\mathbf{E}} \mathcal{M}(4, 4).$$

If a canonical \*-homomorphism is unital then there are no zero rows in the E matrix, and we should replace the above-mentioned condition  $\tilde{q}_j \leq q_j$  with  $\tilde{q}_j = q_j$ . For example, the unital embedding

$$\mathcal{M}(2, 2, 3) \xrightarrow{\mathbf{E}} \mathcal{M}(4, 5), \quad \mathbf{E} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

has the form

$$\mathbf{A} \oplus \mathbf{B} \oplus \mathbf{C} \mapsto \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{pmatrix}.$$

The AF algebra is a separable C\*-algebra, any finite subset of which can be approximated by a finite-dimensional C\*-subalgebra. For convenience, we will consider unital AF algebras only. This is not a restriction because the unitalization of an AF algebra is obviously an AF algebra. It is well known that for any unital AF algebra  $\mathcal{A}$ , there is a family of nested finite-dimensional C\*-subalgebras  $\mathcal{A}_n \subseteq \mathcal{A}$ , satisfying

$$\mathbb{C}^{1 \times 1} \cong \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots, \quad \mathcal{A} = \overline{\bigcup_{n=0}^{\infty} \mathcal{A}_n}. \tag{2}$$

Since  $\mathcal{A}_n$  are nested finite-dimensional  $C^*$ -algebras, they are isomorphic to some canonical algebras  $\mathcal{M}(\mathbf{p}_n)$ , where  $\mathbf{p}_n \in \mathbb{N}^{M_n}$ ,  $M_n \in \mathbb{N}$ , and the inclusions (2) can be written as

$$\mathcal{M}(\mathbf{p}_0) \xrightarrow{E_0} \mathcal{M}(\mathbf{p}_1) \xrightarrow{E_1} \mathcal{M}(\mathbf{p}_2) \xrightarrow{E_2} \dots, \tag{3}$$

where  $\mathbf{p}_0 = 1$ ,  $M_0 = 1$ , and  $E_n \in \mathbb{Z}_+^{M_{n+1} \times M_n}$ . Moreover, due to the unital embeddings  $\mathcal{M}(\mathbf{p}_n) \subseteq \mathcal{M}(\mathbf{p}_{n+1})$ , all E matrices have no zero rows and columns and they satisfy  $E_n \mathbf{p}_n = \mathbf{p}_{n+1}$ . Because  $\mathbf{p}_0 = 1$ , we obtain

$$\mathbf{p}_{n+1} = E_n \dots E_1 E_0 = \prod_{i=0}^n E_i. \tag{4}$$

We will always assume the right-to-left order in the product  $\prod$ . Using (3) and (4), we conclude that the matrices  $\{E_n\}_{n=0}^{+\infty}$  completely determine the structure of the unital AF algebra  $\mathcal{A}$ . It is useful to note that the choice of E matrices is not unique. For example, E matrices  $\{E'_n\}_{n=0}^{+\infty}$ , where  $E'_n = E_{2n+1} E_{2n}$ , determine the same algebra  $\mathcal{A}$ . This is because the composition of two embeddings has the E-matrix equivalent to the product of E matrices corresponding to the embeddings. It is possible to describe the class of all E-matrices determining the same unital AF-algebra.

**Definition 1.** Let  $\mathfrak{E}$  be the set of sequences of matrices  $\{E_n\}_{n=0}^{\infty}$ , where  $E_n \in \mathbb{Z}_+^{M_{n+1} \times M_n}$  have no zero rows and columns, and  $M_0 = 1$ ,  $M_n \in \mathbb{N}$  are some positive integer numbers. Let us define the equivalence relation on  $\mathfrak{E}$ . Two sequences  $\{A_n\}_{n=0}^{\infty} \sim \{B_n\}_{n=0}^{\infty}$  are equivalent if there is  $\{C_n\}_{n=0}^{\infty} \in \mathfrak{E}$  such that

$$C_0 = \prod_{i=0}^{r_1-1} A_i, \quad C_{2n-1} C_{2n-2} = \prod_{i=m_{n-1}}^{m_n-1} B_i, \quad C_{2n} C_{2n-1} = \prod_{i=r_n}^{r_{n+1}-1} A_i, \quad n \geq 1, \tag{5}$$

where  $0 = r_0 < r_1 < r_2 < \dots$  and  $0 = m_0 < m_1 < m_2 < \dots$  are some monotonic sequences of integer numbers. The corresponding set of equivalence classes is denoted by  $\mathfrak{E} := \mathfrak{E} / \sim$ .

It is convenient to denote the equivalence classes as

$$\overline{\{E_n\}_{n=0}^{\infty}} = \prod_{n=0}^{\infty} E_n$$

because, see (5),

$$\prod_{n=0}^{\infty} A_n = \prod_{n=0}^{\infty} \prod_{i=r_n}^{r_{n+1}-1} A_i = \prod_{n=0}^{\infty} C_n = \prod_{n=0}^{\infty} \prod_{i=m_n}^{m_{n+1}-1} B_i = \prod_{n=0}^{\infty} B_n.$$

In other words, we can perform the standard manipulations in the product of matrices without leaving the equivalence class. Of course, the manipulations should not go beyond  $\mathfrak{E}$ , i.e., all the resulting matrices should have non-negative integer entries and should not have zero rows and zero columns.

Let  $\mathcal{A}$  be a unital AF algebra. Following (2)–(4), there is  $\{E_n\}_{n=0}^{\infty}$ , which represents a Bratteli diagram for  $\mathcal{A}$ . Let us define the mapping

$$\mathfrak{n} : \mathcal{A} \mapsto \prod_{n=0}^{\infty} E_n. \tag{6}$$

Because the Bratteli diagram is not unique, the correctness of the mapping  $\mathfrak{n}$  should be checked. This is already complete in the main structure theorem for Bratteli diagrams.

**Theorem 1.** (i) The relation  $\sim$  defined in Definition 1 is the equivalence relation. (ii) Let  $\mathfrak{A}$  be the set of classes of nonisomorphic unital  $A$ -algebras. Then,  $n : \mathfrak{A} \rightarrow \mathfrak{E}$  is 1–1 mapping.

Note that the inverse mapping  $n^{-1}$  has a more explicit form than  $n$ . For example,  $n^{-1}(\prod_{n=0}^{\infty} E_n)$  is the  $C^*$ -algebra  $\mathcal{A}$  given by the inductive limit (3).

The proof of Theorem 1 follows from the similar results formulated for the graphical representations of Bratteli diagrams, see, e.g., [10,11], and Theorem 3.4.4 in [12]. The equivalence relation  $\sim$  is the complete analogue of telescopic transformations of Bratteli diagrams defined in [12]. While the Bratteli diagram is not unique, it provides a kind of good classification tool. Other types of classification of AF algebras, including the efficient K-theoretic Elliott classification, are discussed in [13–17]. The infinite product  $n(\mathcal{A})$  representing the Bratteli diagram for AF-algebra  $\mathcal{A}$  can be called the Glimm–Bratteli symbol. The reason for using this name is the following. Working with supernatural symbols (numbers), when  $E_n$  are natural numbers in (6), J. Glimm provides the classification of UHF algebras in [18]. In turn, the UHF algebras are a partial case of AF algebras for which the supernatural numbers should be replaced with “supernatural matrices” connected with the corresponding Bratteli diagram. Let us consider some examples of AF algebras.

**Example 1 (Compact operators).** Let  $\mathcal{K}$  be the  $C^*$ -algebra of compact operators acting on a separable Hilbert space. Let  $\mathcal{K}_1 = \text{Alg}(\mathcal{K}, 1)$  be its unitalization. It is well known, see, e.g., [12], that the Bratteli diagram for  $\mathcal{K}_1$  can be presented as it is drawn in Figure 1.

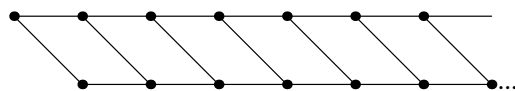


Figure 1. Bratteli diagram for the unitalized  $C^*$ -algebra of compact operators  $\mathcal{K}_1$ .

The nodes correspond to simple matrix sub-algebras, and the edges show the multiplicity of embedding: one line means the multiplicity is equal to 1, two lines means the multiplicity is 2, and so on. The first node is a node that has no incoming edges; it is always  $C^{1 \times 1}$ . The dimensions of nodes are determined by the dimensions of nodes connected on the left and by the multiplicities of embedding. The corresponding Glimm–Bratteli symbol is

$$n(\mathcal{K}_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\infty} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Combining terms in the infinite product, we can write another form of the Glimm–Bratteli symbol

$$n(\mathcal{K}_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\infty} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left( \prod_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left( \prod_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which leads to the labeled Bratteli diagram depicted in Figure 2.

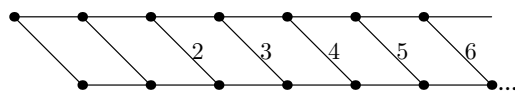


Figure 2. Another variant of the Bratteli diagram for the unitalized  $C^*$ -algebra of compact operators  $\mathcal{K}_1$ .

In the labeled Bratteli diagram, the edge numbers are the multiplicities of embeddings. The multiplicity 1 is usually omitted.

**Example 2 (CAR algebra).** For the CAR (canonical anticommutation relations) algebra  $\mathcal{C}$ , which is a UHF algebra, the Glimm–Bratteli symbol is  $\mathfrak{n}(\mathcal{C}) = 2^\infty$ . At the same time,

$$\mathfrak{n}(\mathcal{C}) = 2^\infty = \left( (1 \ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^\infty = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \ 1) \right)^\infty \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^\infty \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The corresponding Bratteli diagrams are depicted in Figure 3.

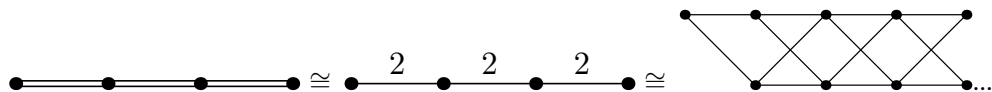


Figure 3. Different Bratteli diagrams represent the same  $C^*$ -algebra  $\mathfrak{n}^{-1}(2^\infty)$ .

Comparing this explanation with the similar one given in [12], it is seen how the symbol  $\mathfrak{n}$  simplifies the proof of equivalence of presented Bratteli diagrams.

**Example 3 (Direct sum of AF algebras).** Above, we already used the notation  $\oplus$  for the direct sum of matrix algebras and for their elements. We will use the same symbol in a slightly different context, namely for the direct sum of not necessarily square  $E$  matrices. Suppose that  $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$  is the standard direct sum of two AF algebras. If  $\mathfrak{n}(\mathcal{A}) = \prod_{n=0}^\infty \mathbf{A}_n$  and  $\mathfrak{n}(\mathcal{B}) = \prod_{n=0}^\infty \mathbf{B}_n$  then it can be shown that

$$\mathfrak{n}(\mathcal{C}) = \left( \prod_{n=0}^\infty \mathbf{C}_n \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ where } \mathbf{C}_n = \mathbf{A}_n \oplus \mathbf{B}_n = \begin{pmatrix} \mathbf{A}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_n \end{pmatrix}.$$

**Example 4 (Tensor product of AF-algebras).** It is useful to note the following property of the tensor product

$$\mathcal{M}(\mathbf{p}) \otimes \mathcal{M}(\mathbf{q}) = \mathcal{M}(\mathbf{p} \otimes \mathbf{q}),$$

where

$$\mathbf{p} = (p_i)_{i=1}^n \in \mathbb{N}^n, \quad \mathbf{q} = (q_j)_{j=1}^m \in \mathbb{N}^m, \quad \mathbf{p} \otimes \mathbf{q} = (p_i q_j)_{i,j=1}^{n,m} \in \mathbb{N}^{nm}.$$

Moreover, it is easy to check that if

$$\mathcal{M}(\mathbf{p}_1) \xrightarrow{\mathbf{E}_1} \mathcal{M}(\mathbf{q}_1), \quad \mathcal{M}(\mathbf{p}_2) \xrightarrow{\mathbf{E}_2} \mathcal{M}(\mathbf{q}_2)$$

then

$$\mathcal{M}(\mathbf{p}_1 \otimes \mathbf{p}_2) \xrightarrow{\mathbf{E}_1 \otimes \mathbf{E}_2} \mathcal{M}(\mathbf{q}_1 \otimes \mathbf{q}_2),$$

where the tensor product of matrices is defined in the standard way

$$(A_{i,j}) \otimes (B_{r,s}) = (C_{(i,r),(j,s)}), \quad C_{(i,r),(j,s)} = A_{i,j} B_{r,s}.$$

Hence, the standard tensor product  $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$  of two AF algebras is an AF algebra which satisfies

$$\mathfrak{n}(\mathcal{C}) = \prod_{n=0}^\infty \mathbf{A}_n \otimes \mathbf{B}_n,$$

where  $\mathfrak{n}(\mathcal{A}) = \prod_{n=0}^\infty \mathbf{A}_n$ ,  $\mathfrak{n}(\mathcal{B}) = \prod_{n=0}^\infty \mathbf{B}_n$ , and the tensor product of matrices is then

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} b_{11}\mathbf{A} & \dots & b_{1N}\mathbf{A} \\ \dots & \dots & \dots \\ b_{M1}\mathbf{A} & \dots & b_{MN}\mathbf{A} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1N} \\ \dots & \dots & \dots \\ b_{M1} & \dots & b_{MN} \end{pmatrix}.$$

We will also use the following result.

**Theorem 2.** Let  $\{\mathbf{A}_n\}_{n=1}^\infty$  be a commutative (multiplicative) semigroup of square matrices with non-negative integer entries and with non-zero determinants. Let  $\mathbf{A}_0$  be a matrix column with

positive integer entries such that  $\mathbf{A}_1 \mathbf{A}_0$  is defined. Let  $\{\mathbf{B}_n\}_{n=1}^\infty \subset \{\mathbf{A}_n\}_{n=1}^\infty$  be a subset consisting of not necessarily different matrices satisfying the condition  $(\sigma)$ : for any  $p \in \mathbb{N}$  there are  $r, s \in \mathbb{N}$  such that  $\mathbf{A}_p \mathbf{A}_r = \prod_{i=1}^s \mathbf{B}_i$ . Then,

$$n^{-1} \left( \left( \prod_{n=1}^\infty \mathbf{B}_n \right) \mathbf{A}_0 \right) \cong n^{-1} \left( \left( \prod_{n=1}^\infty \mathbf{A}_n \right) \mathbf{A}_0 \right). \tag{7}$$

Even if  $(\sigma)$  is not fulfilled, LHS in (7) is a sub-algebra of RHS.

**Remark 1.** The universal UHF algebra  $\mathcal{U}$  is the AF algebra generated by the multiplicative semigroup of natural numbers

$$\mathcal{U} = n^{-1} \left( \prod_{n=1}^{+\infty} n \right) = n^{-1} \left( \prod_{n=1}^{+\infty} p_n^\infty \right) = n^{-1} \left( \prod_{n=1}^{+\infty} (p_1 \dots p_n)^n \right) = n^{-1} \left( \prod_{n=1}^{+\infty} (p_1 \dots p_n) \right),$$

where  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  are the prime numbers. Any UHF algebra is a sub-algebra of  $\mathcal{U}$ . The CAR algebra is the UHF algebra generated by any of the following multiplicative semigroups  $\{2^n : n \in m\mathbb{N}\}$ , where  $m \in \mathbb{N}$ .

There is another useful proposition describing nonisomorphic classes of AF algebras.

**Theorem 3.** Let  $N, M \in \mathbb{N}$ . Let  $\{\mathbf{A}_n\} \subset \mathbb{Z}_+^{N \times N}, \{\mathbf{B}_n\} \subset \mathbb{Z}_+^{M \times M}$  be two sequences of matrices with non-zero determinants. Let  $\mathbf{A}_0 \in \mathbb{Z}_+^{N \times 1}, \mathbf{B}_0 \in \mathbb{Z}_+^{M \times 1}$  be two matrix columns without zero entries. If  $N \neq M$ , then  $n^{-1} \left( \prod_{n=0}^\infty \mathbf{A}_n \right) \not\cong n^{-1} \left( \prod_{n=0}^\infty \mathbf{B}_n \right)$  are nonisomorphic  $C^*$ -algebras.

### 3. Main Results

Let  $N, M \in \mathbb{N}$  be positive integers. Let  $L^2_{N,M} = L^2(\mathbb{T}^N \rightarrow \mathbb{C}^M)$  be the Hilbert space of periodic vector-valued functions defined on the multidimensional torus  $\mathbb{T}^N$ , where  $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0, 1)$ . Everywhere in the article, it is assumed the Lebesgue measure in the definition of Hilbert spaces of square-integrable functions. Let  $R^\infty_{N,M} = R^\infty(\mathbb{T}^N \rightarrow \mathbb{C}^{M \times M})$  be the  $C^*$ -algebra of matrix-valued regulated functions with rational discontinuities. The regulated functions with possible rational discontinuities are the functions that can be uniformly approximated by the step functions of the form

$$\mathbf{S}(\mathbf{x}) = \sum_{n=1}^P \chi_{J_n}(\mathbf{x}) \mathbf{S}_n, \tag{8}$$

where  $P \in \mathbb{N}, \mathbf{S}_n \in \mathbb{C}^{M \times M}$ , and  $\chi_{J_n}$  is the characteristic function of the parallelepiped  $J_n = \prod_{i=1}^N [p_{in}, q_{in})$  with rational end points  $p_{in}, q_{in} \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$ . In particular, continuous matrix-valued functions belong to  $R^\infty_{N,M}$ . Let us introduce generating operators for the integrodifferential algebras. These operators are operators of multiplication by a function  $\mathcal{M}$ , finite-difference operators  $\mathcal{D}$ , and integral operators  $\mathcal{I}$ , all of which act on  $L^2_{N,M}$ :

$$\begin{aligned} \mathcal{M}_{\mathbf{S}} \mathbf{u}(\mathbf{x}) &= \mathbf{S}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \\ \mathcal{D}_{i,h} \mathbf{u}(\mathbf{x}) &= h^{-1} (\mathbf{u}(\mathbf{x} + h \mathbf{e}_i) - \mathbf{u}(\mathbf{x})) \text{ , } \mathbf{u}(\mathbf{x}) \in L^2_{N,M}, \mathbf{x} \in \mathbb{T}^N, \\ \mathcal{I}_i \mathbf{u} &= \int_0^1 \mathbf{u}(\mathbf{x}) dx_i \end{aligned} \tag{9}$$

where the function  $\mathbf{S} \in R^\infty_{N,M}$ , the index  $i \in \mathbb{N}_N := \{1, \dots, N\}$ , the step of differentiation  $h \in \mathbb{Q}$ , the standard basis vector  $\mathbf{e}_i = (\delta_{ij})_{j=1}^N$ , and  $\delta_{ij}$  is the Kronecker symbol. The  $C^*$ -algebra of integrodifferential operators is generated by all the operators (9)

$$\mathcal{F}_{N,M} = \overline{\text{Alg}}^{\mathcal{B}} \{ \mathcal{M}_{\mathbf{S}}, \mathcal{D}_{i,h}, \mathcal{I}_i : \mathbf{S} \in R^\infty_{N,M}, i \in \mathbb{N}_N, h \in \mathbb{Q} \}, \tag{10}$$

where  $\mathcal{B} \equiv \mathcal{B}_{N,M} = \mathcal{B}(L^2_{N,M})$  is the  $C^*$ -algebra of all the bounded operators acting on  $L^2_{N,M}$ . The typical example of an operator  $\mathcal{A}$  from  $\mathcal{F}_{1,1}$  is

$$Au(x) = \sum_{n=1}^p A_n(x) \mathcal{D}_{1, \frac{n}{p}} u(x) + \int_0^1 K(x, y) u(y) dy, \quad u \in L^2_{1,1}, \quad x \in \mathbb{T},$$

where  $A_n \in R_{1,1}^\infty, K \in R_{2,1}^\infty$ , and  $p \in \mathbb{N}$ . Let us provide the characterization of  $\mathcal{F}_{N,M}$ .

**Theorem 4.** *The AF-algebra  $\mathcal{F}_{N,M}$  has the following Glimm–Bratteli symbol*

$$\mathfrak{n}(\mathcal{F}_{N,M}) = \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix}^{\otimes N} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes N}. \tag{11}$$

In particular,  $\mathcal{F}_{N,M}$  and  $\mathcal{F}_{N_1,M_1}$  are isomorphic if, and only if,  $N = N_1$ .

Integrodifferential algebras with a different number of variables are nonisomorphic. This fact indicates a significant difference between the integrodifferential algebras  $\mathcal{F}_{N,M}$  and differential algebras  $\mathcal{H}_{N,M}$  generated by  $\mathcal{M}_S$  and  $\mathcal{D}_{i,h}$ . The algebras  $\mathcal{H}_{N,M}$  are all isomorphic to the universal UHF algebra  $\mathcal{U} = \otimes_{n=1}^\infty \mathbb{C}^{n \times n}$  independently on the number of variables  $N$  and the number of functions  $M$ ; see [9].

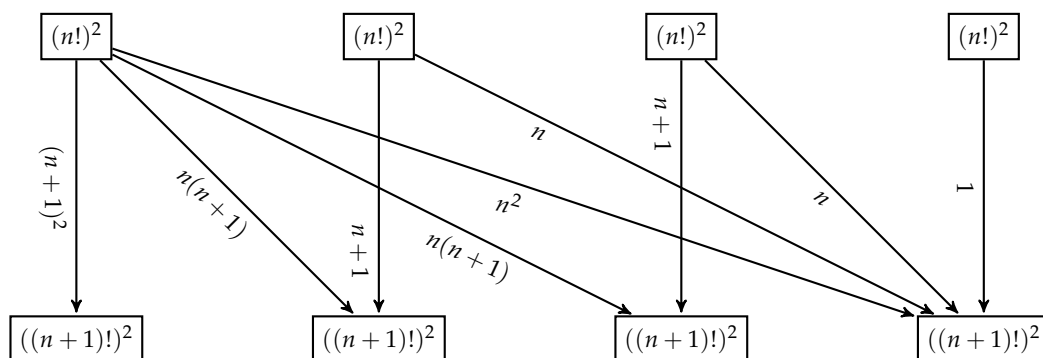
**Example 5.** *Let us consider the  $C^*$ -algebra of two-dimensional integrodifferential operators  $\mathcal{F}_{2,M}$ . We have*

$$\prod_{i=1}^n \begin{pmatrix} i & 0 \\ i-1 & 1 \end{pmatrix}^{\otimes 2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes 2} = \begin{pmatrix} n! \\ n! \end{pmatrix}^{\otimes 2} = (n!)^2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} n+1 & 0 \\ n & 1 \end{pmatrix}^{\otimes 2} = \begin{pmatrix} (n+1)^2 & 0 & 0 & 0 \\ n(n+1) & n+1 & 0 & 0 \\ n(n+1) & 0 & n+1 & 0 \\ n^2 & n & n & 1 \end{pmatrix}.$$

A fragment of the Bratteli diagram for  $\mathcal{F}_{2,M}$  is depicted in Figure 4. Here, the vertices in the row represent direct summands of the finite-dimensional sub-algebra, the edges represent partial embeddings into the next finite-dimensional sub-algebra appearing in the direct limit, and the edge labels are multiplicities of partial embeddings. Note that the orientation of the Bratteli diagram for  $\mathcal{F}_{2,M}$  is up→down, but the Bratteli diagrams for the examples in the previous section are oriented as left→right. The only reason for that is the convenience of the corresponding graphic illustration.



**Figure 4.** The fragment of Bratteli diagrams for  $C^*$ -algebra  $\mathcal{F}_{2,M}$ .



**Remark 2.** Let us consider the algebra of one-dimensional scalar integrodifferential operators  $\mathcal{F}_{1,1}$ . The E matrices for  $\mathcal{F}_{1,1}$  are given by Theorem 4

$$\mathbf{E}_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \mathbf{E}_n = \begin{pmatrix} n+1 & 0 \\ & 1 \end{pmatrix}.$$

It is clear that

$$\mathbf{E}_n = \begin{pmatrix} n+1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}^n.$$

Thus, there are arbitrary large elements  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n$ ,  $n \in \mathbb{N}$  in  $\mathfrak{n}(\mathcal{F}_{1,1})$ . Remembering that these elements correspond to the unitalized algebra of compact operators  $\mathbb{C}1 + \mathcal{K}(L^2_{1,1})$ , see above and, e.g., Example 3.3.1 in [12], we can expect that  $\mathcal{K}(L^2_{1,1}) \subset \mathcal{F}_{1,1}$ . This is true because any compact operator can be uniformly approximated by finite-dimensional operators in some orthonormal basis of  $L^2_{1,1}$ . Taking the Walsh basis  $f_n$ ,  $n \in \mathbb{N}$  consisting of step functions, we see that for any  $n, m \in \mathbb{N}$ , the one-rank operator  $\mathcal{C}_{n,m}$  given by  $\mathcal{C}_{n,m} : u(x) \rightarrow f_m(x) \int_0^1 f_n(t)u(t)dt$ , where  $u \in L^2_{1,1}$ , belongs to  $\mathcal{F}_{1,1}$ . Hence, any compact operator belongs to  $\mathcal{K}_{1,1}$ , since it can be uniformly approximated by linear combinations of  $\mathcal{C}_{n,m}$ .

Finally, note that  $\begin{pmatrix} n+1 & 0 \\ 0 & 1 \end{pmatrix} = (n+1) \oplus (1)$ . E matrices  $(n+1)$  for  $n \in \mathbb{N}$  correspond to the universal uniformly hyper-finite algebra  $\mathcal{U} = \otimes_{n=1}^{\infty} \mathbb{C}^{n \times n}$  which has the supernatural number  $\mathfrak{n}(\mathcal{U}) = \prod_{n=1}^{\infty} n$ . As is shown in [9], the algebra  $\mathcal{U}$ , generated by  $\mathcal{M}_S$  and  $\mathcal{D}_{i,h}$ , see (9), is a sub-algebra of  $\mathcal{F}_{1,1}$ . Thus, roughly speaking,  $\mathcal{F}_{1,1}$  is a combination of the universal UHF algebra  $\mathcal{U}$  and the algebra of compact operators  $\mathcal{K}$ .

The natural extension of  $\mathcal{F}_{1,1}$  (or  $\mathcal{F}_{1,M}$ ) is the AF algebra  $\mathcal{F}_1$  generated by the following commutative semigroup

$$\mathfrak{n}(\mathcal{F}_1) = \prod_{n=1}^{\infty} \prod_{m=1}^n \begin{pmatrix} n & 0 \\ n-m & m \end{pmatrix}.$$

This is the maximal commutative semigroup of  $2 \times 2$  matrices from  $\mathfrak{E}$  with the eigenvectors  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Perhaps it would be interesting to see the “physical meaning” of extended integrodifferential operators from  $\mathcal{F}_1$ .

#### 4. Proof of the Main Results

**Proof of Theorem 2.** The conditions of Definition 1 will be checked. We set  $\mathbf{C}_0 = \mathbf{A}_0$ ,  $\mathbf{C}_1 = \mathbf{B}_1$ , and  $r_1 = 1$ ,  $m_1 = 2$  correspondingly. Next,  $\mathbf{B}_1 = \mathbf{A}_{n_1}$  for some  $n_1 \geq 1$ . We take  $\mathbf{C}_2 = \prod_{i=0}^{n_1-1} \mathbf{A}_i$ , or  $\mathbf{C}_2 = \mathbf{A}_2$  if  $n_1 = 1$ . In the first case, we set  $r_2 = n_1 + 1$ ; in the second case we set  $r_2 = 3$ .

Anyway,  $\mathbf{C}_2 \in \{\mathbf{A}_n\}_{n=1}^{\infty}$ , since this is a semigroup. Hence, for some  $\mathbf{C}_3 \in \{\mathbf{A}_n\}_{n=1}^{\infty}$ , we have  $(\mathbf{B}_1 \mathbf{C}_2) \mathbf{C}_3 = \prod_{i=1}^{m_2-1} \mathbf{B}_i$  by the condition  $(\sigma)$ . Thus,  $\mathbf{C}_2 \mathbf{C}_3 = \prod_{i=m_1}^{m_2-1} \mathbf{B}_i$  because  $\mathbf{B}_1$  is invertible, and all the matrices are commute. In general, commutativity greatly simplifies the reasoning, since we do not need to worry about the order of the factors.

By induction, suppose that for some  $n > 1$ , we already found  $1 = r_1 < \dots < r_n$ , and  $2 = m_1 < \dots < m_n$ , and  $\mathbf{C}_i \in \{\mathbf{A}_n\}_{n=1}^{\infty}$ , satisfying

$$\mathbf{C}_{2j-1} \mathbf{C}_{2j-2} = \prod_{i=m_{j-1}}^{m_j-1} \mathbf{B}_i, \quad \mathbf{C}_{2j-2} \mathbf{C}_{2j-3} = \prod_{i=r_{j-1}}^{r_j-1} \mathbf{A}_i, \quad 2 \leq j \leq n. \tag{12}$$



Let  $\mu(\mathbf{A})$  be the maximal element of the matrix  $\mathbf{A}$ . It is true that

$$\mu(\mathbf{A}_n \mathbf{A}_m) \geq \max(\mu(\mathbf{A}_n), \mu(\mathbf{A}_m)), \tag{13}$$

since  $\mathbf{A}_n, \mathbf{A}_m$  are matrices with non-negative integer entries, without zero rows and columns. There are two possibilities: (a)  $\lim_{p \rightarrow \infty} \mu(\mathbf{C}_{2n-1}^p) = \infty$ , and (b)  $\mu(\mathbf{C}_{2n-1}^p)$  are bounded. In the case of (a), for some sufficiently large  $p > 1$ , we have  $\mathbf{C}_{2n-1}^p = \mathbf{A}_r$ , where  $r > r_n$ . We set  $r_{n+1} = r + 1, \mathbf{C}_{2n} = \mathbf{C}_{2n-1}^{p-1} \prod_{i=r_n}^{r-1} \mathbf{A}_i$ . Hence, we obtain

$$\mathbf{C}_{2n} \mathbf{C}_{2n-1} = \prod_{i=r_n}^{r_{n+1}-1} \mathbf{A}_i. \tag{14}$$

Note that  $\mathbf{C}_{2n} \in \{\mathbf{A}_n\}_{n=1}^\infty$ , since this is a semigroup. Another possibility: (b)  $\mu(\mathbf{C}_{2n-1}^p)$  are uniformly bounded for all  $p$ . Then,  $\mathbf{C}_{2n-1}^p = \mathbf{C}_{2n-1}^s$  for some  $p > s$  because  $\{\mathbf{C}_{2n-1}^p\}$  is a sequence of matrices with bounded non-negative integer entries. The existence of inverse matrix  $\mathbf{C}_{2n-1}^{-1}$  leads to  $\mathbf{C}_{2n-1}^{p-s} = \mathbf{I}$  being the identity matrix. We set  $r_{n+1} = r_n + 1, \mathbf{C}_{2n} = \mathbf{C}_{2n-1}^{p-s-1} \mathbf{A}_{r_n}$ . These values also satisfy (14). Note that  $\mathbf{E}$  matrices satisfying condition (b) correspond to a permutation of elements in the Bratteli diagrams.

Again, there are two possibilities: (a)  $\lim_{p \rightarrow \infty} \mu(\mathbf{C}_{2n-1}^p) = \infty$ , and (b)  $\mu(\mathbf{C}_{2n-1}^p)$  are bounded. Consider the first case (a), the second case (b) can be treated as above. There is  $p \geq 1$ , such that

$$\mu(\mathbf{C}_{2n}^p) > \mu(\prod_{i=1}^{m_n} \mathbf{B}_i). \tag{15}$$

Hence, by the condition  $(\sigma)$ , taking  $\mathbf{A}_p = (\prod_{i=1}^{m_n-1} \mathbf{B}_i) \mathbf{C}_{2n}^p$  (recall that the set  $\{\mathbf{A}_n\}_{n=1}^\infty$  is a semigroup), we have

$$((\prod_{i=1}^{m_n-1} \mathbf{B}_i) \mathbf{C}_{2n}^p) \mathbf{A}_r = \prod_{i=1}^{m_{n+1}-1} \mathbf{B}_i \tag{16}$$

for some,  $m_{n+1} > m_n$  because of (13) and (15). We set  $\mathbf{C}_{2n+1} = \mathbf{C}_{2n}^{p-1} \mathbf{A}_r$ . Using (16), we deduce that

$$\mathbf{C}_{2n+1} \mathbf{C}_{2n} = \prod_{i=m_n}^{m_{n+1}-1} \mathbf{B}_i. \tag{17}$$

Thus, by induction we prove that  $\prod_{n=0}^\infty \mathbf{A}_n$  and  $(\prod_{n=1}^\infty \mathbf{B}_n) \mathbf{A}_0$  are equivalent; see Definition 1. By Theorem 1, they represent the same algebra.

If  $\mu(\prod_{i=1}^p \mathbf{B}_i)$  are bounded for all  $p$ , then there is  $\mathbf{A}_r$  and  $1 \leq m_1 < m_2 < \dots$  such that  $\prod_{i=1}^{m_n} \mathbf{B}_i = \mathbf{A}_r$  for all  $n$ . Thus,  $n^{-1}((\prod_{n=1}^\infty \mathbf{B}_n) \mathbf{A}_0) \cong \mathcal{M}(\mathbf{A}_r \mathbf{A}_0)$  is a sub-algebra of  $\mathcal{M}(\prod_{n=0}^r \mathbf{A}_n)$ , which, in turn, is the sub-algebra of  $n^{-1}(\prod_{n=0}^\infty \mathbf{A}_n)$ . Now, suppose that  $\mu(\prod_{i=1}^p \mathbf{B}_i) \rightarrow \infty$ . Then, we can take  $1 = m_1 < m_2 < \dots$  such that

$$\prod_{i=m_n}^{m_{n+1}-1} \mathbf{B}_i = \mathbf{A}_{r_n} \quad n \geq 1,$$

where  $0 = r_0 < r_1 < r_2 < \dots$ , denoting

$$\mathbf{D}_n = \prod_{i=r_{n-1}+1}^{r_n} \mathbf{A}_i, \quad \mathbf{E}_n = \prod_{j=1}^n (\prod_{i=r_{j-1}+1}^{r_j-1} \mathbf{A}_i),$$

where  $E_n = I$  is the identity matrix if  $r_n - 2 < r_{n-1}$ . Then, the following infinite commutative diagrams

$$\begin{array}{ccccccc}
 \mathcal{M}(\mathbf{A}_0) & \xrightarrow{D_1} & \mathcal{M}(D_1\mathbf{A}_0) & \xrightarrow{D_2} & \mathcal{M}(D_2D_1\mathbf{A}_0) & \dots & \longrightarrow & n^{-1}(\prod_{n=0}^{\infty} \mathbf{A}_n) \\
 \uparrow I & & \uparrow E_1 & & \uparrow E_2 & & & \uparrow \\
 \mathcal{M}(\mathbf{A}_0) & \xrightarrow{A_{r_1}} & \mathcal{M}(A_{r_1}\mathbf{A}_0) & \xrightarrow{A_{r_2}} & \mathcal{M}(A_{r_2}A_{r_1}\mathbf{A}_0) & \dots & \longrightarrow & n^{-1}((\prod_{n=1}^{\infty} \mathbf{B}_n)\mathbf{A}_0)
 \end{array}$$

show that  $n^{-1}((\prod_{n=1}^{\infty} \mathbf{B}_n)\mathbf{A}_0)$  is the sub-algebra of  $n^{-1}(\prod_{n=0}^{\infty} \mathbf{A}_n)$ .  $\square$

**Proof of Theorem 3.** Suppose that  $N > M$ . If  $n^{-1}(\prod_{n=0}^{\infty} \mathbf{A}_n) \cong n^{-1}(\prod_{n=0}^{\infty} \mathbf{B}_n)$ ; then, there is a sequence of matrices  $\{C\}_{n=0}^{\infty}$  satisfying (5), namely

$$C_{2n-2}C_{2n-3} = \prod_{i=r_{n-1}}^{r_n-1} \mathbf{A}_i, \quad C_{2n-1}C_{2n-2} = \prod_{i=m_{n-1}}^{m_n-1} \mathbf{B}_i, \quad C_{2n}C_{2n-1} = \prod_{i=r_n}^{r_{n+1}-1} \mathbf{A}_i$$

for some  $n > 2$ . This yields to

$$\prod_{i=r_{n-1}}^{r_{n+1}-1} \mathbf{A}_i = C_{2n} \left( \prod_{i=m_{n-1}}^{m_n-1} \mathbf{B}_i \right) C_{2n-3}.$$

The matrix in LHS has the full rank  $N$ , while the matrix in RHS has a rank less or equal to  $M$ . This is the contradiction.  $\square$

**Proof of Theorem 4.** Let us start from the 1D case  $N = M = 1$ . For  $h \in \mathbb{Q}$ : define the shift operator  $S_h = 1 - hD_{1,h}$ . Define also the operators of multiplication by the characteristic functions of intervals

$$\mathcal{M}_{j,p} \equiv \mathcal{M}_{\chi_{I_j^p}}, \quad I_j^p = \left[ \frac{j}{p}, \frac{j+1}{p} \right), \quad j \in \mathbb{Z}_p := \{0, \dots, p-1\}, \quad p \in \mathbb{N}. \tag{18}$$

The operators satisfy some elementary properties that can be checked directly; see also [19],

$$\begin{aligned}
 \mathcal{M}_{i,p}\mathcal{M}_{j,p} &= \delta_{ij}\mathcal{M}_{i,p}, \quad S_{\frac{j}{p}}\mathcal{M}_{i,p} = \mathcal{M}_{i+j,p}S_{\frac{j}{p}}, \quad S_hS_t = S_{h+t}, \quad \mathcal{M}_{i,p}^* = \mathcal{M}_{i,p}, \\
 S_h^* &= S_{-h}, \quad S_h\mathcal{I}_1 = \mathcal{I}_1S_h = \mathcal{I}_1, \quad \mathcal{I}_1\mathcal{M}_{i,p}\mathcal{I}_1 = p^{-1}, \tag{19}
 \end{aligned}$$

where  $i, j \in \mathbb{Z}_p, h, t \in \mathbb{Q}$ , and  $p \in \mathbb{N}$ . For  $i, j \in \mathbb{Z}_p$ , define the basis operators

$$\mathcal{B}_{i,j}^p = p\mathcal{M}_{i,p}\mathcal{I}_1\mathcal{M}_{j,p}, \quad \mathcal{A}_{i,j}^p = \mathcal{M}_{i,p}S_{\frac{i-j}{p}} - \mathcal{B}_{i,j}^p. \tag{20}$$

Using (19), we can directly check the properties

$$\begin{aligned}
 \mathcal{B}_{i,j}^p\mathcal{B}_{n,m}^p &= \delta_{jn}\mathcal{B}_{i,m}^p, \quad (\mathcal{B}_{i,j}^p)^* = \mathcal{B}_{j,i}^p, \quad \mathcal{A}_{i,j}^p\mathcal{A}_{n,m}^p = \delta_{jn}\mathcal{A}_{i,m}^p, \\
 (\mathcal{A}_{i,j}^p)^* &= \mathcal{A}_{j,i}^p, \quad \mathcal{A}_{i,j}\mathcal{B}_{n,m} = 0. \tag{21}
 \end{aligned}$$

Identities (21) mean that

$$\mathcal{H}_p \equiv \text{Alg}\{\mathcal{A}_{i,j}^p, \mathcal{B}_{i,j}^p : i, j \in \mathbb{Z}_p\} \cong \mathcal{M}(p) \oplus \mathcal{M}(p) = \mathcal{M}(p, p) \tag{22}$$

with the  $*$ -isomorphism defined by

$$\mathcal{A}_{ij}^p \mapsto (\delta_{in}\delta_{jm})_{n,m=0}^{p-1} \oplus \mathbf{0}_p, \quad \mathcal{B}_{ij}^p \mapsto \mathbf{0}_p \oplus (\delta_{in}\delta_{jm})_{n,m=0}^{p-1} \tag{23}$$

where  $\mathbf{0}_p$  is the zero element in  $\mathcal{M}_p$ . Let  $q \in \mathbb{N}$  be some positive integer. Using (20) and the identity

$$\mathcal{M}_{i,p} = \sum_{n=iq}^{(i+1)q-1} \mathcal{M}_{n,pq},$$

we obtain

$$\begin{aligned} \mathcal{B}_{ij}^p &= \frac{1}{q} \sum_{n=iq}^{(i+1)q-1} \sum_{m=jq}^{(j+1)q-1} \mathcal{B}_{nm}^{pq}, \quad \mathcal{A}_{ij}^p = \left( \sum_{n=iq}^{(i+1)q-1} \mathcal{M}_{n,qp} \right) \mathcal{S}_{\frac{iq-pq-n+n}{pq}} - \mathcal{B}_{i,j}^p = \\ &= \sum_{n=iq}^{(i+1)q-1} (\mathcal{A}_{n+(j-i)q}^{pq} + \mathcal{B}_{n+(j-i)q}^{pq}) - \frac{1}{q} \sum_{n=iq}^{(i+1)q-1} \sum_{m=jq}^{(j+1)q-1} \mathcal{B}_{nm}^{pq}. \end{aligned} \quad (24)$$

Identity (24) shows how  $\mathcal{H}_p$  is embedded into  $\mathcal{H}_{pq}$ . Namely, the corresponding  $*$ -embedding is defined by

$$\mathbf{A} \oplus \mathbf{B} \mapsto (\mathbf{A} \otimes \mathbf{I}_q) \oplus (\mathbf{A} \otimes \mathbf{I}_q - \mathbf{A} \otimes (q^{-1}\mathbf{1}_q) + \mathbf{B} \otimes q^{-1}\mathbf{1}_q), \quad (25)$$

where  $\mathbf{I}_q$  is the identity matrix in  $\mathcal{M}_q$  and  $\mathbf{1}_q = (1) \in \mathcal{M}_q$  is the matrix, which has all entries equal to 1. The matrix  $q^{-1}\mathbf{1}_q$  is the rank-one matrix with the unit trace and, hence, it is unitarily equivalent to the matrix with one nonzero entry

$$q^{-1}\mathbf{1}_q \simeq \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \end{pmatrix}. \quad (26)$$

Using (26), we conclude that  $*$ -embedding (25) between  $\mathcal{H}_p \cong \mathcal{M}(p, p)$  and  $\mathcal{H}_{pq} \cong \mathcal{M}(pq, pq)$  has the E matrix

$$\mathbf{E} = \begin{pmatrix} q & 0 \\ q-1 & 1 \end{pmatrix}. \quad (27)$$

The integral operator  $\mathcal{I}_1$  belongs to all  $\mathcal{H}_p$ , since

$$\mathcal{I}_1 = \sum_{i \in \mathbb{Z}_p} \sum_{j \in \mathbb{Z}_p} \mathcal{B}_{ij}^p \in \mathcal{H}_p. \quad (28)$$

By definition, any  $S \in R_{1,1}^\infty$  can be uniformly approximated by step functions with rational discontinuities. Thus, the operator of multiplication by the function  $\mathcal{M}_S$  can be uniformly approximated by linear combinations of  $\mathcal{M}_{i,p}$ ,  $i \in \mathbb{Z}_p$ . On the other hand, using (20), we have

$$\mathcal{M}_{i,p} = \mathcal{A}_{i,i}^p + \mathcal{B}_{i,i}^p \in \mathcal{H}_p \quad (29)$$

Hence, for any  $S \in R_{1,1}^\infty$ , the operator  $\mathcal{M}_S$  can be uniformly approximated by the elements from  $\mathcal{H}_p$  with arbitrary precision when  $p \rightarrow \infty$ . The identity operator 1 belongs to all  $\mathcal{H}_p$ , since

$$1 = \sum_{j=0}^{p-1} \mathcal{M}_{j,p}. \quad (30)$$

The shift operators  $\mathcal{S}_h$  (with  $h = q/p \in \mathbb{Q}$ ) belongs to  $\mathcal{H}_p$ , since

$$\mathcal{S}_{\frac{1}{p}} = \sum_{i \in \mathbb{Z}_p} (\mathcal{A}_{i,i-1}^p + \mathcal{B}_{i,i-1}^p) \in \mathcal{H}_p, \quad \mathcal{S}_{\frac{q}{p}} = \mathcal{S}_{\frac{1}{p}}^q \in \mathcal{H}_p \quad (31)$$

by (19), (20) and (30). Hence, the finite differentials belong also to  $\mathcal{H}_p$ :

$$\mathcal{D}_{1,h} = h^{-1}(1 - \mathcal{S}_h) \in \mathcal{H}_p. \quad (32)$$

Using (28) and (32), and the above-mentioned fact about the approximation of  $\mathcal{M}_S$  (for any  $S \in R_{1,1}^\infty$ ) by the elements from  $\mathcal{H}_p$ , we conclude that  $\mathcal{F}_{1,1}$  is the inductive limit of  $\mathcal{H}_p$  for  $p \rightarrow \infty$ . In particular, taking  $p_n = n!$  and using (27) for  $\mathcal{H}_{p_n} \subset \mathcal{H}_{p_{n+1}}$ , we obtain (11) for  $N = M = 1$ .

The algebra  $\mathcal{F}_{1,M} = \mathcal{M}(M) \otimes \mathcal{F}_{1,1}$  has the same Glimm–Bratteli symbol as  $\mathcal{F}_{1,1}$ , since

$$\begin{aligned} n(\mathcal{F}_{1,M}) &= n(\mathcal{M}(M) \otimes \mathcal{F}_{1,1}) = \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} M = \\ & \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \begin{pmatrix} M \\ M \end{pmatrix} = \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \begin{pmatrix} M & 0 \\ M-1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \\ & \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = n(\mathcal{F}_{1,1}). \end{aligned} \tag{33}$$

Let us discuss why the first identity in the last string of (33) is true. The matrices (27) form a commutative (multiplicative) semigroup. Then, the infinite product with one duplicated term in the LHS of (33) obviously satisfies the condition ( $\sigma$ ) from Theorem 2. Hence,  $\mathcal{F}_{1,M}$  and  $\mathcal{F}_{1,1}$  are isomorphic. There is also a more intuitive similarity with supernatural numbers

$$\begin{aligned} \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right) \begin{pmatrix} M & 0 \\ M-1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \\ \left( \prod_{n=1}^\infty \begin{pmatrix} p_n & 0 \\ p_n-1 & 1 \end{pmatrix} \right)^\infty \left( \prod_{j=1}^K \begin{pmatrix} p_{n_j} & 0 \\ p_{n_j}-1 & 1 \end{pmatrix}^{R_j} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \\ \prod_{n=1}^\infty \begin{pmatrix} p_n & 0 \\ p_n-1 & 1 \end{pmatrix}^\infty \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

where  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  are prime numbers and  $M = \prod_{j=1}^K p_{n_j}^{R_j}$  is the prime factorization of  $M$ . This similarity with supernatural numbers is possible because all the matrices commute with each other. In this sense, the presented idea is somewhat similar to that one related to supernatural numbers, see [18].

Consider the case  $N > 1$ . Using the fact that  $L_{N,M}^2 = \bigoplus_{j=1}^M (L_{1,1}^2)^{\otimes N}$ , we deduce that  $\mathcal{F}_{N,M} = \mathcal{M}(M) \otimes \mathcal{F}_{1,1}^{\otimes N}$ . This means that

$$n(\mathcal{M}(M) \otimes \mathcal{F}_{1,1}^{\otimes N}) = n(\mathcal{F}_{1,M} \otimes \mathcal{F}_{1,1}^{\otimes N-1}) = n(\mathcal{F}_{1,1}^{\otimes N}) = \left( \prod_{n=2}^\infty \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix} \right)^{\otimes N} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes N},$$

which proves (11). Thus, the  $C^*$ -algebras  $\mathcal{F}_{N,M}$  and  $\mathcal{F}_{N_1,M_1}$  are isomorphic if, and only if,  $N = N_1$  by Theorem 3.  $\square$

### 5. Conclusions

We have considered the Glimm–Bratteli symbol  $n$  that is a convenient algebraic representation of the Bratteli diagrams for AF algebras in the form of infinite products of matrices with integer entries. We have applied this symbol for the classification of the algebra of integrodifferential operators. In particular, the Glimm–Bratteli symbol allowed us to prove easily that the algebras of integrodifferential operators acting on the tori of different dimensions are nonisomorphic. This is an essential addition to the result proved recently: all the algebras of differential operators are isomorphic to each other, independently to the dimensions of the tori that they act on, see [19]. In the future, we plan to characterize the algebras of operators acting on more complex non-compact domains, including fractal ones. Another interesting research area is the application of modified Glimm–Bratteli symbols to the algebras of stochastic integrodifferential operators.

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