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Power Aggregation Operators Based on t-Norm and t-Conorm under the Complex Intuitionistic Fuzzy Soft Settings and Their Application in Multi-Attribute Decision Making

Zeeshan Ali [1], Tahir Mahmood [1], Kifayat Ullah [2,*], Dragan Pamucar [3,*] and Goran Cirovic [4]

[1] Department of Mathematics and Statistics, International Islamic University, Islamabad 44000, Pakistan; zeeshanalinsr@gmail.com (Z.A.); tahirbakhat@iiu.edu.pk (T.M.)
[2] Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University Islamabad (Lahore Campus), Lahore 54000, Pakistan
[3] Department of Logistics, Military Academy, University of Defence in Belgrade, 11000 Belgrade, Serbia
[4] Faculty of Civil Engineering Management, University "UNION—Nikola Tesla", 11000 Belgrade, Serbia; goran.cirovic@uns.ac.rs or cirovic@sezampro.rs
* Correspondence: kifayat.khan.dr@gmail.com (K.U.); dragan.pamucar@va.mod.gov.rs (D.P.); Tel.: +92-334-909-0225 (K.U.)

Abstract: Multi-attribute decision-making (MADM) is commonly used to investigate fuzzy information effectively. However, selecting the best alternative information is not always symmetric because the alternatives do not have complete information, so asymmetric information is often involved. In this analysis, we use the massive dominant and more consistent principle of power aggregation operators (PAOs) based on general t-norm and t-conorm, which manage awkward and inconsistent data in real-world dilemmas such as medical diagnosis, pattern recognition, cleaner production evaluation in gold mines, the analysis of the cancer risk factor, etc. The principle of averaging, geometric, Einstein, and Hamacher aggregation operators are specific cases of generalized PAOs. We combine the principle of complex intuitionistic fuzzy soft (CIFS) information with PAOs to initiate CIFS power averaging (CIFSPA), CIFS weighted power averaging (CIFSWPA), CIFS ordered weighted power averaging (CIFSOWPA), CIFS power geometric (CIFSPG), CIFS weighted power geometric (CIFSWPG), and CIFS ordered weighted power geometric (CIFSOWPG), and their flexible laws are elaborated. Certain specific cases (such as averaging, Einstein, and Hamacher operators) of the explored operators are also illustrated with the help of different t-norm and t-conorm operators. A MADM process is presented under the developed operators based on the CIFS environment. Finally, to investigate the supremacy of the demonstrated works, we employed a sensitivity analysis and geometrical expressions of the initiated operators with numerous prevailing works to verify the efficiency of the proposed works. This manuscript shows how to make decisions when there is asymmetric information about enterprises.

Keywords: complex intuitionistic fuzzy soft sets; power aggregation operators; decision-making techniques

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1. Introduction

These days, decision-making (DM) is mostly utilized rapidly in our day-to-day lives when the object is to choose based on various known or obscure standards the best option out of those that are accessible. MADM is a massive dominant piece of the DM process and is credited as a psychological-based human activity. To deal with and total the data accumulated from a few assets, the main advance is information assortment. Multicriteria DM (MCDM) or multicriteria decision analysis (MCDA) is a sub-discipline of activities research that unequivocally assesses numerous clashing rules in DM (both in day-to-day existence and in settings such as business, government, and medication). Clashing rules are commonplace in assessing choices: cost versus cost is generally one of the primary rules, and some proportion of value is regularly another basis, effectively in a struggle with the expense. In buying a vehicle, cost, solace, wellbeing, and mileage are perhaps a portion of the primary measures we consider—surprisingly, the least expensive vehicle is the most agreeable and the most secure one. Generally, all the data are given as a crisp number. However, in several practical situations, testing is normally believed to show the functioning circumstances, utilizing the crisp number-based crude information when dealing with available strategies. These strategies lead the decision makers to obscure ends just as much as to unsure choices. Therefore, to manage such sorts of complicated situations, the principle of the fuzzy set (FS) was elaborated by Zadeh [1] by modifying the range of crisp set that is {0,1} into the range of FS that is $[0,1]$. Moreover, Atanassov [2] initiated the intuitionistic FS (IFS) as one of the massive dominant techniques for managing inconsistent data. IFS takes the truth grade (TG) $\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}})$ and falsity grade (FG) $\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}})$ with $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) \leq 1$. Numerous scholars have expended effort employing the principle of IFS under the circumstances of distinct fields—as illustrated by bipolar soft sets [3], weighted-based hybrid works [4], divergence measures [5], photovoltaic project [6], matrix games [7], solar site section dilemmas [8], and three-way decisions based on IFS [9].

By using the above prevailing examinations, it is seen that the MADM dilemmas have been employed for FS, IFS speculations, or their development. These models cannot deal with time-periodic dilemmas and two-dimensional data together in a single set. To operate with such a problematic circumstance, the principle of complex FS (CFS) [10] is more generalized than that of FS. CFS covers the complex-valued TG, whose real and unreal parts have described the support of the element into the set. The principle of CFS has been utilized in the distinct fields—for instance, in del-equality for CFSs [11], complex fuzzy logic [12], systematic view for CFSs [13], neuro architecture for CFSs [14], complex neuro-fuzzy sets [15], distance measures for CFSs [16], periodic factor predictions [17], linguistic variables employed in CFSs [18], cross-entropy measures for CFSs [19], and distance measures for interval-valued CFSs [20]. However, the principle of CFS has been neglected, when an individual has faced data in the shape of yes or no; then, the principle of CFS has failed. For this, the principle of complex IFS (CIFS), elaborated by Alkouri and Salleh [21], covers the TG and FG in the shape of complex-valued numbers. CIFS is one of the massive dominant techniques for managing inconsistent data. CIFS takes the TG $\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) = \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)}$ and FG $\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) = \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)}$, with $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) \leq 1$ and $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}) \leq 1$. Numerous scholars have expended effort employing the principle of CIFS in the circumstances of distinct fields—as illustrated by information measures [22], correlation measures [23], geometric aggregation operators (AOs) [24], robust AOs [25], generalized AOs [26], PAOs [27], and preference relations [28].

The concept of soft set (SS) was elaborated by Molodtsov [29], after which every scholar has employed it in the environment of distinct fields; for illustration, Maji et al. [30] initiated the fuzzy SS (FSS). Maintaining the benefit of the FSS, several scholars have employed it in the environment of distinct fields; for illustration, Maji et al. [31] again developed the intuitionistic FSS (IFSS) by modifying the theory of FSS by putting the FG in the circumstance of FSS. Furthermore, Jiang et al. [32] proposed several properties for interval-valued IFSS. Agarwal et al. [33] initiated the generalized form of CIFSSs. Jiang et al. [34] investigated the adjustable approach for IFSSs. Feng et al. [35] revised the idea of IFSSs. Muthukumar and Krishnan [36] proposed several measures for IFSSs. Garg and Arora [37] explored the group-based generalized IFSSs. Feng et al. [38] presented the PROMETHEE method for IFSSs. Khan et al. [39] employed the MADM technique for IFSSs. Babitha and Sunil [40] initiated the generalized IFSSs. Hayat et al. [41] developed the AOs for IFSSs.

Thinking about the significance of the relationship among the traits, focusing on the accumulation operators is one of the incredible meanings of the choice issue. A focus on the conglomeration operator permits the various pieces of information to inform each of the other pieces of information during the accumulation interaction. In our real-life circumstance, there consistently exists a circumstance wherein a connection between the various rules, such as prioritization, backing, and the effect each has on a prevailing job during an accumulation measure. For instance, when a decision-maker makes a judgment that depends on an assessment of hazard in addition to the costs involved in a task, he ought to relegate a higher need to addressing the hazard than the cost. To manage ambiguous and problematic data, Yager [42] developed the PAOs. Arora and Garg [43] demonstrated the prioritized AOs for IFSSs. Karaaslan [44] presented the parameterized IFSSs. Arora and Garg [45] initiated the robust AOs for IFSSs. Garg and Arora [46] explored the Bonferroni mean operators for IFSSs, and the decision-making procedures for IFSSs were developed by Mao et al. [47].

The notion of complex intuitionistic fuzzy soft sets and power aggregation operators based on t-norm and t-conorms are very closely related to the notion of symmetry. Based on symmetry, we can talk about the mixture of both theories. PAOs are more important and useful for determining the best optimal choice from the group of alternatives. PAOs are massively more powerful than the averaging, geometric, Einstein, and Hamacher aggregation operators for reaching decisions that involve inconsistent and complicated data in realistic dilemmas. Using the benefits of the PAOS and CIFSSs, the main goal of this analysis is discussed below:

To initiate CIFSPA, CIFSWPA, CIFSOWPA, CIFSPG, CIFSWPG, CIFSOWPG, and their flexible laws are elaborated.

(i) Certain specific cases of the explored operators, such as averaging, Einstein, and Hamacher operators, are also illustrated with the help of t-norm and t-conorm $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = -\log\left(\bar{\bar{\mathfrak{x}}}\right), \widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = \log\left(\frac{2-\bar{\bar{\mathfrak{x}}}}{\bar{\bar{\mathfrak{x}}}}\right), \bar{\bar{\mathfrak{x}}} \neq 0$, and $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = \log\left(\widetilde{\delta} + \frac{\left(1-\widetilde{\delta}\right)\bar{\bar{\mathfrak{x}}}}{\bar{\bar{\mathfrak{x}}}}\right)$, $\widetilde{\delta} \in (0,\infty), \bar{\bar{\mathfrak{x}}} \neq 0$.

(ii) A MADM process is presented under the developed operators based on the CIFS environment.

(iii) To investigate the supremacy of the demonstrated works, we employed a sensitivity analysis and geometrical expressions of the initiated operators with numerous prevailing works to verify the efficiency of the proposed works.

The main aim of this analysis is reviewed in the subsequent sections: Section 2 highlights some prevailing principles, such as CIFSs, SSs, CIFSSs, and their flexible operational laws. Moreover, the principle of PAOs and the generalized t-norm (TN) and t-conorm (TCN) are also highlighted in this study. The object $\bar{\bar{\mathfrak{X}}}$ stated the fixed sets. In Section 3, we combine the principle of CIFS information with PAOs to initiate CIFSPA, CIFSWPA, CIFSOWPA, CIFSPG, CIFSWPG, and CIFSOWPG and their flexible laws are elaborated. Certain specific cases (such as averaging, Einstein, and Hamacher operators) of the explored operators are also illustrated with the help of t-norm and t-conorm $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = -\log\left(\bar{\bar{\mathfrak{x}}}\right), \widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = \log\left(\frac{2-\bar{\bar{\mathfrak{x}}}}{\bar{\bar{\mathfrak{x}}}}\right), \bar{\bar{\mathfrak{x}}} \neq 0$, and $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = \log\left(\widetilde{\delta} + \frac{\left(1-\widetilde{\delta}\right)\bar{\bar{\mathfrak{x}}}}{\bar{\bar{\mathfrak{x}}}}\right), \widetilde{\delta} \in (0,\infty), \bar{\bar{\mathfrak{x}}} \neq 0$. In Section 4, a MADM process is presented under the developed operators based on the CIFS environment. Finally, to investigate the supremacy of the demonstrated works, we employed a sensitivity analysis and geometrical expressions of the initiated operators with numerous prevailing works to verify the efficiency of the proposed works. Section 5 finishes with closing statements.

2. Preliminaries

This analysis highlights some prevailing principles, such as CIFSs, SSs, CIFSSs, and their flexible operational laws. Moreover, the principle of PAOs and the generalized t-norm (TN) and t-conorm (TCN) are also highlighted in this study. The terms used in this analysis are illustrated in Table 1.

Table 1. Presentation of the symbols used and their meanings.

Symbol	Meaning	Symbol	Meaning
$\overline{\overline{\mathfrak{C}}}_{IF}$	Complex intuitionistic fuzzy sets	$\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_C}$	Complex-valued truth grade
$\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_C}$	Complex-valued falsity grade	$\overline{\overline{\mathfrak{X}}}$	Universal sets
$\overline{\overline{\mathfrak{x}}}$	Element of the universal set	$\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}$	A real part of truth grade
$\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}$	The imaginary part of truth grade	$\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}$	The real part of falsity grade
$\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}$	The imaginary part of falsity grade	$\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_C}$	Complex-valued refusal grade
$\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_R}$	The real part of the refusal grade	$\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_I}$	The imaginary part of refusal grade
$\overline{\overline{\mathcal{P}}}^{\overline{\overline{\mathfrak{x}}}}$	Power set	$\overline{\overline{\mathcal{E}}}$	Set of parameters
$\overline{\overline{\mathcal{F}}}$	Soft function	$\overline{\overline{\mathcal{S}}}$	Score function
$\overline{\overline{\mathcal{H}}}$	Accuracy function		

Definition 1 ([21]). *A CIFS $\overline{\overline{\mathfrak{C}}}_{IF}$ is stated by*

$$\overline{\overline{\mathfrak{C}}}_{CIF} = \left\{ \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}), \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) \right) : \overline{\overline{\mathfrak{x}}} \in \overline{\overline{\mathfrak{X}}} \right\} \tag{1}$$

where $\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) = \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)}$, *and* $\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) = \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)}$, *with* $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) \leq 1$ *and* $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}) \leq 1$, *where* $\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}), \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}})$, $\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}), \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}) \in [0,1]$. *Moreover, the term* $\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_C}(\overline{\overline{\mathfrak{x}}}) = \mathcal{R}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)}$ $= 1 - \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) \right) e^{i2\pi\left(1 - \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)\right)}$, *states the refusal grade. The object* $\overline{\overline{\mathfrak{C}}}_{IF-i} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_R}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_I}(\overline{\overline{\mathfrak{x}}})\right)} \right), i = 1, 2, \ldots, n$, *states the CIFNs.*

Definition 2 ([29]). *If* $\overline{\overline{\mathcal{F}}} : \overline{\overline{\mathcal{E}}} = \overline{\overline{\mathcal{P}}}^{\overline{\overline{\mathfrak{X}}}}$, *where* $\overline{\overline{\mathcal{P}}}^{\overline{\overline{\mathfrak{X}}}}$ *is the set containing subsets of* $\overline{\overline{\mathfrak{X}}}$, *and* $\overline{\overline{\mathcal{E}}}$ *is the set of parameters, then it is expressed as the SSs.*

Definition 3 ([48]). *If* $\overline{\overline{\mathcal{F}}} : \overline{\overline{\mathcal{E}}} \to CIF^{\overline{\overline{\mathfrak{X}}}}$, *the set covering the complex intuitionistic fuzzy subsets of* $\overline{\overline{\mathfrak{X}}}$. *Then the CIFSS* $\overline{\overline{\mathfrak{C}}}_{CIFS}$ *is stated by*

$$\overline{\overline{\mathfrak{C}}}_{CIFS-e_j} = \left\{ \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{C-j}}(\overline{\overline{\mathfrak{x}}}), \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{C-j}}(\overline{\overline{\mathfrak{x}}}) \right) : \overline{\overline{\mathfrak{x}}} \in \overline{\overline{\mathfrak{X}}} \right\} \tag{2}$$

where $\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{C-j}}(\overline{\overline{\mathfrak{x}}}) = \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}})\right)}$, *and* $\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{C-j}}(\overline{\overline{\mathfrak{x}}}) = \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}})\right)}$, *with* $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) \leq 1$ *and* $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}}) \leq 1$. *Moreover, the term* $\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_{C-j}}(\overline{\overline{\mathfrak{x}}}) = \mathcal{R}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{R}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}})\right)} = 1 - \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) \right)$ $\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-j}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(1 - \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-j}}(\overline{\overline{\mathfrak{x}}})\right)\right)}$, *which states the refusal grade. The object* $\overline{\overline{\mathfrak{C}}}_{CIFS-ij} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)} \right), i = 1, 2, \ldots, n$ *states the CIFSNs.*

Definition 4. ([49]). *Let* $\overline{\overline{\mathfrak{C}_{IF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)} \right)$ *be any CIFSN. The score value (SV) is stated by*

$$\overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) = \left| \mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) - \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}}) - \mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}}) \right| \tag{3}$$

where $\overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) \in [-1, 1]$.

Definition 5. ([49]). *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)} \right)$ *be any CIFSN. The accuracy value (AV) is stated by*

$$\overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) = \left| \mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}}) \right| \tag{4}$$

where $\overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) \in [0, 1]$.

For any two CIFSNs $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)} \right)$ and $\overline{\overline{\mathfrak{C}_{CIF-ij}}}^* = \left(\mathcal{M}^*_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}^*_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}^*_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}^*_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)} \right)$, then

1. If $\overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) > \overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}^*\right)$, then $\overline{\overline{\mathfrak{C}_{CIF-ij}}} > \overline{\overline{\mathfrak{C}_{CIF-ij}}}^*$;
2. If $\overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) < \overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}^*\right)$, then $\overline{\overline{\mathfrak{C}_{CIF-ij}}} < \overline{\overline{\mathfrak{C}_{CIF-ij}}}^*$;
3. If $\overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) = \overline{\overline{\mathcal{S}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}^*\right)$, then
    (i) If $\overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) > \overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}^*\right)$, then $\overline{\overline{\mathfrak{C}_{CIF-ij}}} > \overline{\overline{\mathfrak{C}_{CIF-ij}}}^*$;
    (ii) If $\overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) < \overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}^*\right)$, then $\overline{\overline{\mathfrak{C}_{CIF-ij}}} < \overline{\overline{\mathfrak{C}_{CIF-ij}}}^*$;
    (iii) If $\overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}\right) = \overline{\overline{\mathcal{H}}}\left(\overline{\overline{\mathfrak{C}_{CIF-ij}}}^*\right)$, then $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \overline{\overline{\mathfrak{C}_{CIF-ij}}}^*$.

Definition 6. ([42]). *Let* $\overline{\overline{\mathfrak{C}_{PI-i}}}, i = 1, 2, \ldots, n$ *be a group of attributes. The power averaging operator is stated by*

$$PA\left(\overline{\overline{\mathfrak{C}_{PI-1}}}, \overline{\overline{\mathfrak{C}_{PI-2}}}, \ldots, \overline{\overline{\mathfrak{C}_{PI-n}}}\right) = \sum_{i=1}^{n} \frac{\left(1 + \overline{\overline{\mathcal{T}}}\left(\overline{\overline{\mathfrak{C}_{PI-i}}}\right)\right)\overline{\overline{\mathfrak{C}_{PI-i}}}}{\sum_{i=1}^{n}\left(1 + \overline{\overline{\mathcal{T}}}\left(\overline{\overline{\mathfrak{C}_{PI-i}}}\right)\right)} \tag{5}$$

where $\overline{\overline{\mathcal{T}}}\left(\overline{\overline{\mathfrak{C}_{PI-i}}}\right) = \sum^{n}_{\substack{k=1,\\ k \neq i}} Sup\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right)$ *, states the support for* $\overline{\overline{\mathfrak{C}_{PI-i}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-k}}}$ *, is elaborated by*

$$Sup\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right) = 1 - d\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right) \tag{6}$$

where $d\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right)$*, states the hamming measure for* $\overline{\overline{\mathfrak{C}_{PI-i}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-k}}}$*, with a rule that*

1. $Sup\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right) \in [0, 1]$;
2. $Sup\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right) = Sup\left(\overline{\overline{\mathfrak{C}_{PI-k}}}.\overline{\overline{\mathfrak{C}_{PI-i}}}\right)$;

3. If $d\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right) \leq d\left(\overline{\overline{\mathfrak{C}_{PI-l}}}.\overline{\overline{\mathfrak{C}_{PI-q}}}\right)$ *then* $Sup\left(\overline{\overline{\mathfrak{C}_{PI-i}}}.\overline{\overline{\mathfrak{C}_{PI-k}}}\right) \geq Sup\left(\overline{\overline{\mathfrak{C}_{PI-l}}}.\overline{\overline{\mathfrak{C}_{PI-q}}}\right)$.

Definition 7. ([50]) *A structure* $\overline{\overline{\mathcal{T}}} : [0,1] \times [0,1] \to [0,1]$ *is stated by TN if* $\overline{\overline{\mathcal{T}}}$ *holds the boundary, monotonicity, commutative, and associativity conditions, where the TCN is stated by* $\overline{\overline{\mathfrak{S}}}\left(\overline{\overline{\mathfrak{x}}}, \overline{\overline{\mathfrak{y}}}\right) = 1 - \overline{\overline{\mathcal{T}}}\left(1 - \overline{\overline{\mathfrak{x}}}, 1 - \overline{\overline{\mathfrak{y}}}\right)$. *Moreover, the general form of the Archimedean TN and TCN is stated by* $\overline{\overline{\mathcal{T}}}\left(\overline{\overline{\mathfrak{x}}}, \overline{\overline{\mathfrak{y}}}\right) = \widehat{\mathfrak{f}}^{-1}\left(\widehat{\mathfrak{f}}\left(\overline{\overline{\mathfrak{x}}}\right) + \widehat{\mathfrak{f}}\left(\overline{\overline{\mathfrak{y}}}\right)\right)$ *and* $\overline{\overline{\mathfrak{S}}}\left(\overline{\overline{\mathfrak{x}}}, \overline{\overline{\mathfrak{y}}}\right) = \widehat{\mathfrak{g}}^{-1}\left(\widehat{\mathfrak{g}}\left(\overline{\overline{\mathfrak{x}}}\right) + \widehat{\mathfrak{g}}\left(\overline{\overline{\mathfrak{y}}}\right)\right)$, *under the continuous increasing (or decreasing) mapping with* $\widehat{\mathfrak{f}}(1) = 0, \widehat{\mathfrak{g}}(0) = 0$ *and* $\widehat{\mathfrak{f}}\left(\overline{\overline{\mathfrak{x}}}\right) = 1 - \widehat{\mathfrak{g}}\left(1 - \overline{\overline{\mathfrak{x}}}\right)$.

3. Complex Intuitionistic Fuzzy Soft Power Aggregation Operators

In this analysis, we use the massive dominant and more consistent principle of PAOs based on general t-norm and t-conorm, which manage awkward and inconsistent data in real-world dilemmas. The principle of averaging, geometric, Einstein, and Hamacher aggregation operators are the specific cases of the generalized PAOs. We combine the principle of CIFS information with PAOs to initiate CIFSPA, CIFSWPA, CIFSOWPA, CIFSPG, CIFSWPG, and CIFSOWPG, and their flexible laws are elaborated. Certain specific cases (such as averaging, Einstein, and Hamacher operators) of the explored operators are also illustrated with the help of t-norm and t-conorm $\widehat{\mathfrak{g}}\left(\overline{\overline{\mathfrak{x}}}\right) = -\log\left(\overline{\overline{\mathfrak{x}}}\right), \widehat{\mathfrak{g}}\left(\overline{\overline{\mathfrak{x}}}\right) = \log\left(\frac{2-\overline{\overline{\mathfrak{x}}}}{\overline{\overline{\mathfrak{x}}}}\right), \overline{\overline{\mathfrak{x}}} \neq 0$, and $\widehat{\mathfrak{g}}\left(\overline{\overline{\mathfrak{x}}}\right) = \log\left(\widehat{\delta} + \frac{\left(1-\widehat{\delta}\right)\overline{\overline{\mathfrak{x}}}}{\overline{\overline{\mathfrak{x}}}}\right), \widehat{\delta} \in (0,\infty), \overline{\overline{\mathfrak{x}}} \neq 0$.

3.1. Operational Laws for CIFSSs

In this study, we elaborated certain algebraic laws by using the CIFSSs.

Definition 8. *Let* $\overline{\overline{\mathfrak{C}_{CIF-1j}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)}\right)$, $j = 1, 2$, *be any two CIFSNs with* $\widehat{\delta} > 0$. *Then*

$$\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overline{\overline{\mathfrak{C}_{CIF-12}}} = \begin{pmatrix} \widehat{\mathfrak{f}}^{-1}\left(\widehat{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right) + \widehat{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)e^{i2\pi\left(\widehat{\mathfrak{f}}^{-1}\left(\widehat{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)+\widehat{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)\right)}, \\ \widehat{\mathfrak{g}}^{-1}\left(\widehat{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right) + \widehat{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)e^{i2\pi\left(\widehat{\mathfrak{g}}^{-1}\left(\widehat{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)+\widehat{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)\right)} \end{pmatrix} \quad (7)$$

$$\overline{\overline{\mathfrak{C}_{CIF-11}}} \otimes \overline{\overline{\mathfrak{C}_{CIF-12}}} = \begin{pmatrix} \widehat{\mathfrak{g}}^{-1}\left(\widehat{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right) + \widehat{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)e^{i2\pi\left(\widehat{\mathfrak{g}}^{-1}\left(\widehat{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)+\widehat{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)\right)}, \\ \widehat{\mathfrak{f}}^{-1}\left(\widehat{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right) + \widehat{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)e^{i2\pi\left(\widehat{\mathfrak{f}}^{-1}\left(\widehat{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)+\widehat{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-12}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)\right)} \end{pmatrix} \quad (8)$$

$$\widehat{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-1j}}} = \left(\widehat{\mathfrak{f}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)e^{i2\pi\left(\widehat{\mathfrak{f}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)}, \widehat{\mathfrak{g}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)e^{i2\pi\left(\widehat{\mathfrak{g}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)}\right) \quad (9)$$

$$\overline{\overline{\mathfrak{C}_{CIF-1j}}}^{\widehat{\delta}} = \left(\widehat{\mathfrak{g}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{g}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)e^{i2\pi\left(\widehat{\mathfrak{g}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{g}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)}, \widehat{\mathfrak{f}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{f}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)e^{i2\pi\left(\widehat{\mathfrak{f}}^{-1}\left(\widehat{\delta}\,\widehat{\mathfrak{f}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)}\right) \quad (10)$$

Theorem 1. *Let* $\overline{\overline{\mathfrak{C}_{CIF-1j}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1,2,$ *be any two CIFSNs with* $\overset{\frown}{\delta} > 0$*. Then we prove that* $\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overline{\overline{\mathfrak{C}_{CIF-12}}}$, $\overline{\overline{\mathfrak{C}_{CIF-11}}} \otimes \overline{\overline{\mathfrak{C}_{CIF-12}}}$, $\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-1j}}}$ *and* $\overline{\overline{\mathfrak{C}_{CIF-1j}}}^{\overset{\frown}{\delta}}$ *are also CIFSNs.*

See the Appendix A.

Theorem 2. *Let* $\overline{\overline{\mathfrak{C}_{CIF-1j}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1,2,$ *be any two CIFSNs with* $\overset{\frown}{\delta_i} > 0, i = 1,2$*. Then*

1. $\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overline{\overline{\mathfrak{C}_{CIF-12}}} = \overline{\overline{\mathfrak{C}_{CIF-12}}} \oplus \overline{\overline{\mathfrak{C}_{CIF-11}}}$;
2. $\overline{\overline{\mathfrak{C}_{CIF-11}}} \otimes \overline{\overline{\mathfrak{C}_{CIF-12}}} = \overline{\overline{\mathfrak{C}_{CIF-12}}} \otimes \overline{\overline{\mathfrak{C}_{CIF-11}}}$;
3. $\overset{\frown}{\delta}\left(\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overline{\overline{\mathfrak{C}_{CIF-12}}}\right) = \overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-12}}}$;
4. $\left(\overline{\overline{\mathfrak{C}_{CIF-11}}} \otimes \overline{\overline{\mathfrak{C}_{CIF-12}}}\right)^{\overset{\frown}{\delta}} = \overline{\overline{\mathfrak{C}_{CIF-11}}}^{\overset{\frown}{\delta}} \otimes \overline{\overline{\mathfrak{C}_{CIF-12}}}^{\overset{\frown}{\delta}}$;
5. $\overset{\frown}{\delta_1}\,\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overset{\frown}{\delta_2}\,\overline{\overline{\mathfrak{C}_{CIF-11}}} = \left(\overset{\frown}{\delta_1} + \overset{\frown}{\delta_2}\right)\overline{\overline{\mathfrak{C}_{CIF-11}}}$;
6. $\overline{\overline{\mathfrak{C}_{CIF-11}}}^{\overset{\frown}{\delta_1}} \otimes \overline{\overline{\mathfrak{C}_{CIF-11}}}^{\overset{\frown}{\delta_2}} = \overline{\overline{\mathfrak{C}_{CIF-11}}}^{\left(\overset{\frown}{\delta_1} + \overset{\frown}{\delta_2}\right)}$.

See the Appendix B.

3.2. Power Aggregation Operators for CIFSSs

In this analysis, we initiated CIFSPA, CIFSWPA, CIFSOWPA, CIFSPG, CIFSWPG, and CIFSOWPG, and their flexible laws are elaborated. Certain specific cases (such as averaging, Einstein, and Hamacher operators) of the explored operators are also illustrated with the help of t-norm and t-conorm.

Definition 9. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1,2,$ *be any group of CIFSNs with* $\overset{\frown}{\delta_i} > 0, i = 1,2,\ldots,n$*. If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} \in \overline{\overline{\Xi}}$*, CIFSPA operator is stated by*

$$CIFSPA : \overline{\overline{\Xi}}^n \to \overline{\overline{\Xi}}.$$

by

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \oplus_{j=1}^{m}\left(\breve{\mathbb{M}}_j \oplus_{i=1}^{n}\left(\breve{\mathbb{N}}_i \overline{\overline{\mathfrak{C}_{PI-ij}}}\right)\right) \tag{11}$$

where $\breve{\mathbb{M}}_j = \frac{\left(1+\overline{\overline{\mathcal{T}}}_j\right)}{\sum_{j=1}^{m}\left(1+\overline{\overline{\mathcal{T}}}_j\right)}, \breve{\mathbb{N}}_i = \frac{(1+\Re_i)}{\sum_{i=1}^{n}\left(1+\overline{\overline{\Re}}_i\right)}$*, and* $\overline{\overline{\Re}}_i = \sum_{\substack{k=1\\ i\neq k}}^{n} Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right), \overline{\overline{\mathcal{T}}}_j = \sum_{\substack{j=1\\ i\neq j}}^{m} Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}.\overline{\overline{\mathfrak{C}_{PI-i}}}\right)$*, and* $Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right)$*, state the support for* $\overline{\overline{\mathfrak{C}_{PI-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-kj}}}$*.*

Theorem 3. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1,2,$ *be any group of CIFSNs with* $\overset{\frown}{\delta_i} > 0, i = 1,2,\ldots,n$*. Then keeping Equation (11), we determine*

$$IFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\begin{pmatrix}\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)},\\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}\end{pmatrix}\quad(12)$$

See the Appendix C.

Property 1. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2,$ *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\overline{\overline{\mathfrak{C}_{CIF}}}$, *then*

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\overline{\overline{\mathfrak{C}_{CIF}}}.\quad(13)$$

See the Appendix D (Property 1).

Property 2. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2,$ *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{CIF}}}$ *are any two CIFSNs, then*

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overline{\overline{\mathfrak{C}_{CIF}}},\overline{\overline{\mathfrak{C}_{CIF-12}}}\oplus\overline{\overline{\mathfrak{C}_{CIF}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\oplus\overline{\overline{\mathfrak{C}_{CIF}}}\right)=CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)\oplus\overline{\overline{\mathfrak{C}_{CIF}}}.\quad(14)$$

See the Appendix E (Property 2).

Property 3. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2,$ *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *Then*

$$CIFSPA\left(\widetilde{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-11}}},\widetilde{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\widetilde{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\widetilde{\delta}\,CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right).\quad(15)$$

See the Appendix F (Property 3).

Property 4. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2,$ *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}^*_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right)$, *then*

$$\begin{gathered}CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overline{\overline{\mathfrak{C}^*_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}}\oplus\overline{\overline{\mathfrak{C}^*_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\oplus\overline{\overline{\mathfrak{C}^*_{CIF-nm}}}\right)=\\CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)\oplus CIFSPA\left(\overline{\overline{\mathfrak{C}^*_{CIF-11}}},\overline{\overline{\mathfrak{C}^*_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}^*_{CIF-nm}}}\right).\end{gathered}\quad(16)$$

See the Appendix G (Property 4).

For purposes of discussion, the massive important cases of Equation (12) are initiated below.

1. For $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = -\log\left(\bar{\bar{\mathfrak{x}}}\right)$, then Equation (12) is stated by

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} e^{i2\pi\left(1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}\right)}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} e^{i2\pi\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}\right)} \end{pmatrix} \tag{17}$$

The operator is called a complex intuitionistic fuzzy soft Archimedean weighted averaging (CIFSAWA) operator.

2. For $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = \log\left(\frac{2-\bar{\bar{\mathfrak{x}}}}{\bar{\bar{\mathfrak{x}}}}\right), \bar{\bar{\mathfrak{x}}} \neq 0$, then Equation (12) is stated by

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}} e^{i2\pi\left(\frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)}, \\ \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}} e^{i2\pi\left(\frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\bar{\bar{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)} \end{pmatrix}. \tag{18}$$

The operator is called a complex intuitionistic fuzzy soft Einstein weighted averaging (CIFSEWA) operator.

3. For $\widetilde{\mathfrak{g}}\left(\bar{\bar{\mathfrak{x}}}\right) = \log\left(\widetilde{\delta} + \frac{\left(1-\widetilde{\delta}\right)\bar{\bar{\mathfrak{x}}}}{\bar{\bar{\mathfrak{x}}}}\right), \widetilde{\delta} \in (0,\infty), \bar{\bar{\mathfrak{x}}} \neq 0$, then Equation (12) is stated by

$$CIFSPA\left(\overline{\overline{\mathfrak{C}}}_{CIF-11},\overline{\overline{\mathfrak{C}}}_{CIF-12},\ldots,\overline{\overline{\mathfrak{C}}}_{CIF-nm}\right)=\left(\begin{array}{c}\frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\\ e^{i2\pi\left(\frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)},\\ \frac{\overset{\frown}{\delta}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\\ e^{i2\pi\left(\frac{\overset{\frown}{\delta}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)}\end{array}\right) \quad (19)$$

The operator is called a complex intuitionistic fuzzy soft Hamacher weighted averaging (CIFSHWA) operator.

Definition 10. *Let* $\overline{\overline{\mathfrak{C}}}_{CIF-ij}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)}\right), j=1,2,$ *be any group of CIFSNs with* $\overset{\frown}{\delta_i}>0, i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}}}_{CIF-ij}\in\bar{\bar{\Xi}}$, *CIFSWPA operator is stated by*

$$CIFSWPA:\bar{\bar{\Xi}}^{n}\rightarrow\bar{\bar{\Xi}}.$$

by

$$CIFSWPA\left(\overline{\overline{\mathfrak{C}}}_{CIF-11},\overline{\overline{\mathfrak{C}}}_{CIF-12},\ldots,\overline{\overline{\mathfrak{C}}}_{CIF-nm}\right)=\oplus_{j=1}^{m}\left(\breve{\mathbb{M}}'_j\oplus_{i=1}^{n}\left(\breve{\mathbb{N}}'_i\overline{\overline{\mathfrak{C}}}_{PI-ij}\right)\right) \quad (20)$$

where $\breve{\mathbb{M}}'_j=\frac{\hat{\mu}_j\left(1+\bar{\bar{\mathcal{T}}}_j\right)}{\sum_{j=1}^{m}\hat{\mu}_j\left(1+\bar{\bar{\mathcal{T}}}_j\right)}$, $\breve{\mathbb{N}}'_i=\frac{\hat{\eta}_i(1+\Re_i)}{\sum_{i=1}^{n}\hat{\eta}_i\left(1+\bar{\bar{\Re}}_i\right)}$, *and* $\bar{\bar{\Re}}_i=\sum^{n}_{\substack{k=1\\ i\neq k}} Sup\left(\overline{\overline{\mathfrak{C}}}_{PI-ij}.\overline{\overline{\mathfrak{C}}}_{PI-kj}\right)$, $\bar{\bar{\mathcal{T}}}_j=\sum^{m}_{\substack{j=1\\ i\neq j}} Sup\left(\overline{\overline{\mathfrak{C}}}_{PI-j}.\overline{\overline{\mathfrak{C}}}_{PI-i}\right)$, *and* $Sup\left(\overline{\overline{\mathfrak{C}}}_{PI-ij}.\overline{\overline{\mathfrak{C}}}_{PI-kj}\right)$, *state the support for* $\overline{\overline{\mathfrak{C}}}_{PI-ij}$ *and* $\overline{\overline{\mathfrak{C}}}_{PI-kj}$, *where* $\hat{\mu}_j$ *and* $\hat{\eta}_i$, *express the weight vector with* $\sum_{j=1}^{m}\hat{\mu}_j=1$ and $\sum_{i=1}^{n}\hat{\eta}_i=1$.

Theorem 4. *Let* $\overline{\overline{\mathfrak{C}}}_{CIF-ij}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\bar{\bar{\mathfrak{x}}})\right)}\right), j=1,2,$ *be any group of CIFSNs with* $\overset{\frown}{\delta_i}>0, i=1,2,\ldots,n$. *Then keeping Equation (20), we determine*

$$CIFSWPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix} \quad (21)$$

Proof. Omitted. (The proof of this theorem is similar to the proof of Theorem 3). □

Definition 11. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1, 2,$ *be any group of CIFSNs with* $\widetilde{\delta_i} > 0, i = 1, 2, \ldots, n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} \in \overline{\overline{\Xi}}$, *CIFSOWPA operator is stated by*

$$CIFSOWPA : \overline{\overline{\Xi}}^n \to \overline{\overline{\Xi}}.$$

by

$$CIFSOWPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \oplus_{j=1}^{m}\left(\breve{\mathbb{M}}'_j \oplus_{i=1}^{n}\left(\breve{\mathbb{N}}'_i \overline{\overline{\mathfrak{C}_{PI-o(i)o(j)}}}\right)\right) \quad (22)$$

where $\breve{\mathbb{M}}'_j = \frac{\hat{\mu}_j\left(1+\overline{\overline{\mathcal{T}}}_j\right)}{\sum_{j=1}^{m} \hat{\mu}_j\left(1+\overline{\overline{\mathcal{T}}}_j\right)}, \breve{\mathbb{N}}'_i = \frac{\hat{\eta}_i(1+\Re_i)}{\sum_{i=1}^{n} \hat{\eta}_i\left(1+\overline{\overline{\Re}}_i\right)}$, *and* $\overline{\overline{\Re}}_i = \sum_{\substack{k=1 \\ i \neq k}}^{n} Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right), \overline{\overline{\mathcal{T}}}_j = \sum_{\substack{j=1 \\ i \neq j}}^{m} Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}.\overline{\overline{\mathfrak{C}_{PI-i}}}\right)$, *and* $Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}}}_{PI-kj}\right)$, *state the support for* $\overline{\overline{\mathfrak{C}_{PI-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-kj}}}$, *where* $\hat{\mu}_j$ *and* $\hat{\eta}_i$, *express the weight vector with* $\sum_{j=1}^{m} \hat{\mu}_j = 1$ *and* $\sum_{i=1}^{n} \hat{\eta}_i = 1$ *with* $o(i)j \geq o(i-1)j$ *and* $io(j) \geq io(j-1)$.

Theorem 5. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1, 2,$ *be any group of CIFSNs with* $\widetilde{\delta_i} > 0, i = 1, 2, \ldots, n$. *Then keeping Equation (22), we determine*

$$CIFSOWPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} \overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j\left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}. \quad (23)$$

Proof. Omitted. (The proof of this theorem is similar to the proof of the Theorem 3). □

Definition 12. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1, 2,$ *be any group of CIFSNs with* $\widetilde{\delta_i} > 0, i = 1, 2, \ldots, n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} \in \overline{\overline{\Xi}}$, *CIFSPG operator is stated by*

$$CIFSPG : \overline{\overline{\Xi}}^n \to \overline{\overline{\Xi}}.$$

by

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} \quad (24)$$

where $\breve{\mathbb{M}}_j=\frac{\left(1+\overline{\overline{\mathcal{T}}}_j\right)}{\sum_{j=1}^{m}\left(1+\overline{\overline{\mathcal{T}}}_j\right)},\breve{\mathbb{N}}_i=\frac{(1+\Re_i)}{\sum_{i=1}^{n}\left(1+\overline{\overline{\Re}}_i\right)}$, *and* $\overline{\overline{\Re}}_i=\sum^{n}_{\substack{k=1\\ i\neq k}} Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right),\overline{\overline{\mathcal{T}}}_j=\sum^{m}_{\substack{j=1\\ i\neq j}} Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}.\overline{\overline{\mathfrak{C}_{PI-i}}}\right)$, *and* $Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right)$, *state the support for* $\overline{\overline{\mathfrak{C}_{PI-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-kj}}}$.

Theorem 6. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2$, *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *Then keeping Equation (24), we determine*

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\begin{pmatrix}\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)},\\ \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}\end{pmatrix}. \quad (25)$$

Proof. Omitted. (The proof of this theorem is similar to the proof of the Theorem 3). □

Property 5. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2$, *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\overline{\overline{\mathfrak{C}_{CIF}}}$, *then*

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\overline{\overline{\mathfrak{C}_{CIF}}}\,. \quad (26)$$

Proof. Omitted. (The proof of this property is similar to the proof of the Property 1). □

Property 6. Let $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2$, *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{CIF}}}$ *be any two CIFSNs, then*

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}\otimes\overline{\overline{\mathfrak{C}_{CIF}}},\overline{\overline{\mathfrak{C}_{CIF-12}}}\otimes\overline{\overline{\mathfrak{C}_{CIF}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\otimes\overline{\overline{\mathfrak{C}_{CIF}}}\right)=CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)\otimes\overline{\overline{\mathfrak{C}_{CIF}}}\,. \quad (27)$$

Proof. Omitted. (The proof of this property is similar to the proof of the Property 2). □

Property 7. Let $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right),j=1,2$, *be any group of CIFSNs with* $\widetilde{\delta_i}>0,i=1,2,\ldots,n$. *Then*

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}^{\widetilde{\delta}},\overline{\overline{\mathfrak{C}_{CIF-12}}}^{\widetilde{\delta}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}^{\widetilde{\delta}}\right)=CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)^{\widetilde{\delta}}\,. \quad (28)$$

Proof. Omitted. (The proof of this property is similar to the proof of the Property 3). □

Property 8. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1, 2,$ *be any group of CIFSNs with* $\widetilde{\delta_i} > 0, i = 1, 2, \ldots, n$. *If* $\overline{\overline{\mathfrak{C}^*_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right)$, *then*

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}} \otimes \overline{\overline{\mathfrak{C}^*_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}} \otimes \overline{\overline{\mathfrak{C}^*_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}} \otimes \overline{\overline{\mathfrak{C}^*_{CIF-nm}}}\right) = CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) \otimes CIFSPG\left(\overline{\overline{\mathfrak{C}^*_{CIF-11}}}, \overline{\overline{\mathfrak{C}^*_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}^*_{CIF-nm}}}\right). \tag{29}$$

Proof. Omitted. (The proof of this property is similar to the proof of the Property 4). □

For purposes of discussion, the massive important cases of Equation (25) are initiated below.

1. For $\widetilde{\mathfrak{g}}(\overline{\overline{\mathfrak{x}}}) = -\log(\overline{\overline{\mathfrak{x}}})$, then Equation (25) is stated by

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} e^{i2\pi\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}\right)}, \\ 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j} e^{i2\pi\left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}\right)} \end{pmatrix}. \tag{30}$$

The operator is called a complex intuitionistic fuzzy soft Archimedean weighted geometric (CIFSAWG) operator.

2. For $\widetilde{\mathfrak{g}}(\overline{\overline{\mathfrak{x}}}) = \log\left(\frac{2-\overline{\overline{\mathfrak{x}}}}{\overline{\overline{\mathfrak{x}}}}\right), \overline{\overline{\mathfrak{x}}} \neq 0$, then Equation (25) is stated by

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} \dfrac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}} e^{i2\pi\left(\dfrac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)}, \\ \dfrac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}} e^{i2\pi\left(\dfrac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)} \end{pmatrix}. \tag{31}$$

The operator is called a complex intuitionistic fuzzy soft Einstein weighted geometric (CIFSEWG) operator.

3. For $\overset{\frown}{\mathfrak{g}}\left(\overline{\overline{\mathfrak{x}}}\right)=\log\left(\overset{\frown}{\delta}+\frac{\left(1-\overset{\frown}{\delta}\right)\overline{\overline{\mathfrak{x}}}}{\overline{\overline{\mathfrak{x}}}}\right),\overset{\frown}{\delta}\in(0,\infty),\overline{\overline{\mathfrak{x}}}\neq 0$, then Equation (25) is stated by

$$CIFSPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\left(\begin{array}{c}\frac{\overset{\frown}{\delta}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\\ e^{i2\pi\left(\frac{\overset{\frown}{\delta}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\left(1-\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)},\\ \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\\ e^{i2\pi\left(\frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\left(\overset{\frown}{\delta}-1\right)\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}+\left(\overset{\frown}{\delta}-1\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)^{\breve{\mathbb{N}}_i}\right)^{\breve{\mathbb{M}}_j}}\right)}\end{array}\right). \quad (32)$$

The operator is called a complex intuitionistic fuzzy soft Hamacher weighted geometric (CIFSHWG) operator.

Definition 13. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)}\right),j=1,2,$ *be any group of CIFSNs with* $\overset{\frown}{\delta_i}>0,i=1,2,\ldots,n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}\in\overline{\overline{\Xi}}$, *CIFSWPG operator is stated by*

$$CIFSWPG:\overline{\overline{\Xi}}^{n}\rightarrow\overline{\overline{\Xi}}.$$

by

$$CIFSWPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}},\overline{\overline{\mathfrak{C}_{CIF-12}}},\ldots,\overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)=\otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}\right)^{\breve{\mathbb{N}}'_i}\right)^{\breve{\mathbb{M}}'_j} \quad (33)$$

where $\breve{\mathbb{M}}'_j=\frac{\hat{\mu}_j\left(1+\overline{\overline{\mathcal{T}}}_j\right)}{\sum_{j=1}^{m}\hat{\mu}_j\left(1+\overline{\overline{\mathcal{T}}}_j\right)},\breve{\mathbb{N}}'_i=\frac{\hat{\eta}_i(1+\Re_i)}{\sum_{i=1}^{n}\hat{\eta}_i\left(1+\overline{\overline{\Re}}_i\right)}$, *and* $\overline{\overline{\Re}}_i=\sum_{\substack{k=1\\ i\neq k}}^{n}Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right),\overline{\overline{\mathcal{T}}}_j=\sum_{\substack{j=1\\ i\neq j}}^{m}Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}.\overline{\overline{\mathfrak{C}_{PI-i}}}\right)$, *and* $Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right)$, *state the support for* $\overline{\overline{\mathfrak{C}_{PI-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-kj}}}$, *where* $\hat{\mu}_j$ *and* $\hat{\eta}_i$, *express the weight vector with* $\sum_{j=1}^{m}\hat{\mu}_j=1$ *and* $\sum_{i=1}^{n}\hat{\eta}_i=1$.

Theorem 6. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}}=\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)},\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}\left(\overline{\overline{\mathfrak{x}}}\right)\right)}\right),j=1,2,$ *be any group of CIFSNs with* $\overset{\frown}{\delta_i}>0,i=1,2,\ldots,n$. *Then keeping Equation (33), we determine*

$$CIFSWPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \begin{pmatrix} \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}. \quad (34)$$

Proof. Omitted. (The proof of this theorem is similar to the proof of the Theorem 3). □

Definition 14. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right), j = 1, 2,$ *be any group of CIFSNs with* $\widetilde{\delta_i} > 0, i = 1, 2, \ldots, n$. *If* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} \in \overline{\overline{\Xi}}$, *CIFSOWPG operator is stated by*

$$CIFSOWPG : \overline{\overline{\Xi}}^n \to \overline{\overline{\Xi}}.$$

by

$$CIFSOWPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\overline{\overline{\mathfrak{C}_{PI-o(i)o(j)}}}\right)^{\breve{\mathbb{N}}_i'}\right)^{\breve{\mathbb{M}}_j'} \quad (35)$$

where $\breve{\mathbb{M}}_j' = \frac{\hat{\mu}_j\left(1+\overline{\overline{\mathcal{T}}}_j\right)}{\sum_{j=1}^{m} \hat{\mu}_j\left(1+\overline{\overline{\mathcal{T}}}_j\right)}, \breve{\mathbb{N}}_i' = \frac{\hat{\eta}_i(1+\Re_i)}{\sum_{i=1}^{n} \hat{\eta}_i\left(1+\overline{\overline{\Re}}_i\right)}$, and $\overline{\overline{\Re}}_i = \sum^{n}_{\substack{k=1 \\ i \neq k}} Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right)$, $\overline{\overline{\mathcal{T}}}_j = \sum^{m}_{\substack{j=1 \\ i \neq j}} Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}.\overline{\overline{\mathfrak{C}_{PI-i}}}\right)$, *and* $Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}.\overline{\overline{\mathfrak{C}_{PI-kj}}}\right)$, *state the support for* $\overline{\overline{\mathfrak{C}_{PI-ij}}}$ *and* $\overline{\overline{\mathfrak{C}_{PI-kj}}}$, *where* $\hat{\mu}_j$ *and* $\hat{\eta}_i$, *express the weight vector with* $\sum_{j=1}^{m} \hat{\mu}_j = 1$ *and* $\sum_{i=1}^{n} \hat{\eta}_i = 1$ *with* $o(i)j \geq o(i-1)j$ *and* $io(j) \geq io(j-1)$.

Theorem 7. *Let* $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}}))}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}}))}\right), j = 1, 2,$ *be any group of CIFSNs with* $\widetilde{\delta_i} > 0, i = 1, 2, \ldots, n$. *Then keeping Equation (35), we determine*

$$CIFSOWPG\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)$$

$$= \begin{pmatrix} \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-o(i)o(j)}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}. \quad (36)$$

Proof. Omitted. (The proof of this theorem is similar to the proof of the Theorem 3). □

4. MADM Processes under the Investigated Operators

In this analysis, we choose a MADM dilemma and resolve it by using the investigated works under the CIFS setting.

4.1. MADM Procedures

To handle ambiguous and problematic data, which occurs in real-world dilemmas, we take a group of m alternatives and n attributes such that $\overline{\overline{\mathfrak{C}_{AT}}} = \left\{\overline{\overline{\mathfrak{C}_{AT-1}}}, \overline{\overline{\mathfrak{C}_{AT-2}}}, \ldots, \overline{\overline{\mathfrak{C}_{AT-m}}}\right\}$ and $\overline{\overline{\mathfrak{C}_{AL}}} = \left\{\overline{\overline{\mathfrak{C}_{AL-1}}}, \overline{\overline{\mathfrak{C}_{AL-2}}}, \ldots, \overline{\overline{\mathfrak{C}_{AL-n}}}\right\}$. For this, we consider that $\sum_{j=1}^{m} \hat{\mu}_j = 1$ and $\sum_{i=1}^{n} \hat{\eta}_i = 1$ state the weight vector for attributes and parameters. To investigate the above dilemmas, the experts give their opinions in the shape of $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\bar{\bar{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\bar{\bar{\mathfrak{x}}})\right)}\right)$, stated the CIFSNs. Where $\mathcal{M}_{\overline{\overline{\mathfrak{C}_C}}}(\bar{\bar{\mathfrak{x}}}) = \mathcal{M}_{\overline{\overline{\mathfrak{C}_R}}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_I}}}(\bar{\bar{\mathfrak{x}}})\right)}$, and $\mathcal{N}_{\overline{\overline{\mathfrak{C}_C}}}(\bar{\bar{\mathfrak{x}}}) = \mathcal{N}_{\overline{\overline{\mathfrak{C}_R}}}(\bar{\bar{\mathfrak{x}}})e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_I}}}(\bar{\bar{\mathfrak{x}}})\right)}$, with $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}_R}}}(\bar{\bar{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}_R}}}(\bar{\bar{\mathfrak{x}}}) \leq 1$ and $0 \leq \mathcal{M}_{\overline{\overline{\mathfrak{C}_I}}}(\bar{\bar{\mathfrak{x}}}) + \mathcal{N}_{\overline{\overline{\mathfrak{C}_I}}}(\bar{\bar{\mathfrak{x}}}) \leq 1$, where $\mathcal{M}_{\overline{\overline{\mathfrak{C}_R}}}(\bar{\bar{\mathfrak{x}}}), \mathcal{N}_{\overline{\overline{\mathfrak{C}_R}}}(\bar{\bar{\mathfrak{x}}}), \mathcal{M}_{\overline{\overline{\mathfrak{C}_I}}}(\bar{\bar{\mathfrak{x}}}), \mathcal{N}_{\overline{\overline{\mathfrak{C}_I}}}(\bar{\bar{\mathfrak{x}}}) \in [0,1]$. By using the above data, the stages of the explored algorithm are stated below.

Stage 1: First of all, we collect the data in the shape of CIFSNs to develop the matrices $\overline{\overline{\mathfrak{C}_{CIF-ij}}}^{(b)}, b = 1, 2, \ldots, z$.

Stage 2: Then by using the data of the original matrix, we initiate the support $\Re_i$ for the experts, such that

$$\overline{\overline{\Re_i^{(b)}}} = \sum_{\substack{k=1 \\ i \neq k}}^{n} Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}^{(b)}.\overline{\overline{\mathfrak{C}_{PI-kj}}}^{(b)}\right)$$

where $Sup\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}^{(b)}.\overline{\overline{\mathfrak{C}_{PI-kj}}}^{(b)}\right) = 1 - d\left(\overline{\overline{\mathfrak{C}_{PI-ij}}}^{(b)}.\overline{\overline{\mathfrak{C}_{PI-kj}}}^{(b)}\right)$.

Stage 3: Under the data of the support vector, we investigate the weighted averaging/geometric operators. For this, we determine the support $\overline{\overline{\mathcal{T}}}_j$ for the experts, such that

$$\overline{\overline{\mathcal{T}}}_j^{(b)} = \sum_{\substack{j=1 \\ i \neq j}}^{m} Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}^{(b)}.\overline{\overline{\mathfrak{C}_{PI-i}}}^{(b)}\right)$$

where $Sup\left(\overline{\overline{\mathfrak{C}_{PI-j}}}^{(b)}.\overline{\overline{\mathfrak{C}_{PI-i}}}^{(b)}\right) = 1 - d\left(\overline{\overline{\mathfrak{C}_{PI-j}}}^{(b)}.\overline{\overline{\mathfrak{C}_{PI-i}}}^{(b)}\right)$.

Stage 4: By using the CIFSPA and CIFSPG operators, we accumulate the data, which are given in the shape of the matrix to initiate the exact values under the TN and TCN $\widetilde{\mathfrak{g}}(\bar{\bar{\mathfrak{x}}}) = -\log(\bar{\bar{\mathfrak{x}}})$.

Stage 5: By using the accumulated values we explore the SV.

Stage 6: By using the SVs, we initiate the ranking result to find the best optimal.

4.2. Illustrated Example

For this situation study, a model for the assessment of up-and-comers is utilized to outline the appropriateness of the proposed strategy. An insurance agency, HG, in Guangzhou, China, is locked in for the protection of items, charging insurance expenses and conferring on insurance, monetary issues, and different ways of approaching the administration of people and ventures. Consistently, this organization selects new staff for the post of insurance deals specialists and advisors. To keep up with the respected and highly esteemed reputation, the organization counsels specialists during their appraisals and concludes to enroll the up-and-comers. Moreover, the insurance business office and HR office are effectively occupied with enrollment measures.

Labor and products tax (LPT) is a circuitous duty that is intended to provide China with a coordinated normal market. While LPT promises the client a contract with a bound together wraparound expense system, incorporating China into a solitary homogenous market, it accompanies certain inconveniences acquired from the inherited charge system. With the public authority equipped to uphold the LPT in Beijing from 1 July, there is an issue in that dealers have restricted PC information and a poor network. To counter this, the state government wanted to prepare more than 2000 young people, proposing to help citizens in addressing the subtleties of external supplies, internal supplies and returns; documenting cases or discounts; recording some other applications; and so on in the LPT system. The state government intended to gather duty experts who would be accessible in the territory or at the doorstep of citizens, at reasonable expense, all through the territory of Beijing. To resolve the above dilemma, we considered the five attributes and their five parameters for four alternatives including, $\overline{\overline{\mathfrak{C}_{AL-1}}}$: Bharti Airtel, $\overline{\overline{\mathfrak{C}_{AL-2}}}$: Reliance Communications; $\overline{\overline{\mathfrak{C}_{AL-3}}}$: Vodafone China; and $\overline{\overline{\mathfrak{C}_{AT-4}}}$: Mahanagar Telecom Nigam. The data relating to parameters were followed as: e_1: Customer services, e_2: bandwidth, e_3: package deal, e_4: total cost, and e_5: Internet speed. For this, we chose the weight vectors of $(0.3, 0.2, 0.3, 0.1, 0.1)$ and $(0.3, 0.2, 0.25, 0.15, 0.1)$, given by expert u_i. By using the above data, the stages of the explored algorithm are stated below.

Stage 1: We developed the matrices $\overline{\overline{\mathfrak{C}_{CIF-ij}}}^{(b)}, b = 1, 2, \ldots, z$, in the form of Tables 2–5, which were given by distinct experts.

Table 2. Original decision matrix for Expert 1.

	e_1	e_2	e_3	e_4	e_5
u_1	$\left(0.5e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)$	$\left(0.51e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.21)}\right)$	$\left(0.52e^{i2\pi(0.62)}, 0.12e^{i2\pi(0.22)}\right)$	$\left(0.53e^{i2\pi(0.63)}, 0.13e^{i2\pi(0.23)}\right)$	$\left(0.54e^{i2\pi(0.64)}, 0.14e^{i2\pi(0.24)}\right)$
u_2	$\left(0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.3)}\right)$	$\left(0.41e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.31)}\right)$	$\left(0.42e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.32)}\right)$	$\left(0.43e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.33)}\right)$	$\left(0.44e^{i2\pi(0.34)}, 0.24e^{i2\pi(0.34)}\right)$
u_3	$\left(0.7e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.3)}\right)$	$\left(0.71e^{i2\pi(0.51)}, 0.11e^{i2\pi(0.31)}\right)$	$\left(0.72e^{i2\pi(0.52)}, 0.12e^{i2\pi(0.32)}\right)$	$\left(0.73e^{i2\pi(0.53)}, 0.13e^{i2\pi(0.33)}\right)$	$\left(0.74e^{i2\pi(0.54)}, 0.14e^{i2\pi(0.34)}\right)$
u_4	$\left(0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.3)}\right)$	$\left(0.31e^{i2\pi(0.41)}, 0.21e^{i2\pi(0.31)}\right)$	$\left(0.32e^{i2\pi(0.42)}, 0.22e^{i2\pi(0.32)}\right)$	$\left(0.33e^{i2\pi(0.43)}, 0.23e^{i2\pi(0.33)}\right)$	$\left(0.34e^{i2\pi(0.44)}, 0.24e^{i2\pi(0.34)}\right)$
u_5	$\left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)$	$\left(0.71e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.21)}\right)$	$\left(0.72e^{i2\pi(0.62)}, 0.12e^{i2\pi(0.22)}\right)$	$\left(0.73e^{i2\pi(0.63)}, 0.13e^{i2\pi(0.23)}\right)$	$\left(0.74e^{i2\pi(0.64)}, 0.14e^{i2\pi(0.24)}\right)$

Table 3. Original decision matrix for Expert 2.

	e_1	e_2	e_3	e_4	e_5
u_1	$\left(0.5e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.3)}\right)$	$\left(0.51e^{i2\pi(0.41)}, 0.21e^{i2\pi(0.31)}\right)$	$\left(0.52e^{i2\pi(0.42)}, 0.22e^{i2\pi(0.32)}\right)$	$\left(0.53e^{i2\pi(0.43)}, 0.23e^{i2\pi(0.33)}\right)$	$\left(0.54e^{i2\pi(0.44)}, 0.24e^{i2\pi(0.34)}\right)$
u_2	$\left(0.6e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.2)}\right)$	$\left(0.61e^{i2\pi(0.31)}, 0.11e^{i2\pi(0.21)}\right)$	$\left(0.62e^{i2\pi(0.32)}, 0.12e^{i2\pi(0.22)}\right)$	$\left(0.63e^{i2\pi(0.33)}, 0.13e^{i2\pi(0.23)}\right)$	$\left(0.64e^{i2\pi(0.34)}, 0.14e^{i2\pi(0.24)}\right)$
u_3	$\left(0.7e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.2)}\right)$	$\left(0.71e^{i2\pi(0.51)}, 0.11e^{i2\pi(0.21)}\right)$	$\left(0.72e^{i2\pi(0.52)}, 0.12e^{i2\pi(0.22)}\right)$	$\left(0.73e^{i2\pi(0.53)}, 0.13e^{i2\pi(0.23)}\right)$	$\left(0.74e^{i2\pi(0.54)}, 0.14e^{i2\pi(0.24)}\right)$
u_4	$\left(0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.2)}\right)$	$\left(0.41e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.21)}\right)$	$\left(0.42e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.22)}\right)$	$\left(0.43e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.23)}\right)$	$\left(0.44e^{i2\pi(0.34)}, 0.24e^{i2\pi(0.24)}\right)$
u_5	$\left(0.3e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.1)}\right)$	$\left(0.31e^{i2\pi(0.21)}, 0.21e^{i2\pi(0.11)}\right)$	$\left(0.32e^{i2\pi(0.22)}, 0.22e^{i2\pi(0.12)}\right)$	$\left(0.33e^{i2\pi(0.23)}, 0.23e^{i2\pi(0.13)}\right)$	$\left(0.34e^{i2\pi(0.24)}, 0.24e^{i2\pi(0.14)}\right)$

Table 4. Original decision matrix for Expert 3.

	e_1	e_2	e_3	e_4	e_5
u_1	$\begin{pmatrix} 0.7e^{i2\pi(0.5)}, \\ 0.1e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.71e^{i2\pi(0.51)}, \\ 0.11e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.52)}, \\ 0.12e^{i2\pi(0.32)} \end{pmatrix}$	$\begin{pmatrix} 0.73e^{i2\pi(0.53)}, \\ 0.13e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.74e^{i2\pi(0.54)}, \\ 0.14e^{i2\pi(0.34)} \end{pmatrix}$
u_2	$\begin{pmatrix} 0.3e^{i2\pi(0.4)}, \\ 0.2e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.31e^{i2\pi(0.41)}, \\ 0.21e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.32e^{i2\pi(0.42)}, \\ 0.22e^{i2\pi(0.32)} \end{pmatrix}$	$\begin{pmatrix} 0.33e^{i2\pi(0.43)}, \\ 0.23e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.34e^{i2\pi(0.44)}, \\ 0.24e^{i2\pi(0.34)} \end{pmatrix}$
u_3	$\begin{pmatrix} 0.7e^{i2\pi(0.6)}, \\ 0.1e^{i2\pi(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.71e^{i2\pi(0.61)}, \\ 0.11e^{i2\pi(0.21)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.62)}, \\ 0.12e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.73e^{i2\pi(0.63)}, \\ 0.13e^{i2\pi(0.23)} \end{pmatrix}$	$\begin{pmatrix} 0.74e^{i2\pi(0.64)}, \\ 0.14e^{i2\pi(0.24)} \end{pmatrix}$
u_4	$\begin{pmatrix} 0.5e^{i2\pi(0.4)}, \\ 0.2e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.51e^{i2\pi(0.41)}, \\ 0.21e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.52e^{i2\pi(0.42)}, \\ 0.22e^{i2\pi(0.32)} \end{pmatrix}$	$\begin{pmatrix} 0.53e^{i2\pi(0.43)}, \\ 0.23e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.54e^{i2\pi(0.44)}, \\ 0.24e^{i2\pi(0.34)} \end{pmatrix}$
u_5	$\begin{pmatrix} 0.6e^{i2\pi(0.3)}, \\ 0.1e^{i2\pi(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.61e^{i2\pi(0.31)}, \\ 0.11e^{i2\pi(0.21)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.32)}, \\ 0.12e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.33)}, \\ 0.13e^{i2\pi(0.23)} \end{pmatrix}$	$\begin{pmatrix} 0.64e^{i2\pi(0.34)}, \\ 0.14e^{i2\pi(0.24)} \end{pmatrix}$

Table 5. Original decision matrix for Expert 4.

	e_1	e_2	e_3	e_4	e_5
u_1	$\begin{pmatrix} 0.5e^{i2\pi(0.6)}, \\ 0.1e^{i2\pi(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.51e^{i2\pi(0.61)}, \\ 0.11e^{i2\pi(0.21)} \end{pmatrix}$	$\begin{pmatrix} 0.52e^{i2\pi(0.62)}, \\ 0.12e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.53e^{i2\pi(0.63)}, \\ 0.13e^{i2\pi(0.23)} \end{pmatrix}$	$\begin{pmatrix} 0.54e^{i2\pi(0.64)}, \\ 0.14e^{i2\pi(0.24)} \end{pmatrix}$
u_2	$\begin{pmatrix} 0.4e^{i2\pi(0.3)}, \\ 0.2e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.41e^{i2\pi(0.31)}, \\ 0.21e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.42e^{i2\pi(0.32)}, \\ 0.22e^{i2\pi(0.32)} \end{pmatrix}$	$\begin{pmatrix} 0.43e^{i2\pi(0.33)}, \\ 0.23e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.44e^{i2\pi(0.34)}, \\ 0.24e^{i2\pi(0.34)} \end{pmatrix}$
u_3	$\begin{pmatrix} 0.7e^{i2\pi(0.5)}, \\ 0.1e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.71e^{i2\pi(0.51)}, \\ 0.11e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.52)}, \\ 0.12e^{i2\pi(0.32)} \end{pmatrix}$	$\begin{pmatrix} 0.73e^{i2\pi(0.53)}, \\ 0.13e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.74e^{i2\pi(0.54)}, \\ 0.14e^{i2\pi(0.34)} \end{pmatrix}$
u_4	$\begin{pmatrix} 0.3e^{i2\pi(0.4)}, \\ 0.2e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.31e^{i2\pi(0.41)}, \\ 0.21e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.32e^{i2\pi(0.42)}, \\ 0.22e^{i2\pi(0.32)} \end{pmatrix}$	$\begin{pmatrix} 0.33e^{i2\pi(0.43)}, \\ 0.23e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.34e^{i2\pi(0.44)}, \\ 0.24e^{i2\pi(0.34)} \end{pmatrix}$
u_5	$\begin{pmatrix} 0.7e^{i2\pi(0.6)}, \\ 0.1e^{i2\pi(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.71e^{i2\pi(0.61)}, \\ 0.11e^{i2\pi(0.21)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.62)}, \\ 0.12e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.73e^{i2\pi(0.63)}, \\ 0.13e^{i2\pi(0.23)} \end{pmatrix}$	$\begin{pmatrix} 0.74e^{i2\pi(0.64)}, \\ 0.14e^{i2\pi(0.24)} \end{pmatrix}$

Stage 2: We initiated the support $\Re_i$ for the experts, such that

$$\overline{\overline{\Re}}_i^{(1)} = \begin{bmatrix} 0.475 & 0.23 & 0.253 & 0.343 & 0.352 \\ 0.475 & 0.3 & 0.45 & 0.3 & 0.475 \\ 0.2256 & 0.09 & 0.2025 & 0.09 & 0.2256 \\ 0.1072 & 0.027 & 0.0911 & 0.027 & 0.1072 \\ 0.0509 & 0.0081 & 0.041 & 0.0081 & 0.0509 \end{bmatrix}, \overset{=}{\Re}_i^{(2)} = \begin{bmatrix} 0.35 & 0.42 & 0.3452 & 0.453 & 0.453 \\ 0.35 & 0.4 & 0.475 & 0.325 & 0.475 \\ 0.1225 & 0.16 & 0.2256 & 0.1056 & 0.2256 \\ 0.0429 & 0.064 & 0.1072 & 0.0343 & 0.1072 \\ 0.015 & 0.0256 & 0.0509 & 0.0112 & 0.0509 \end{bmatrix},$$

$$\overset{=}{\Re}_i^{(3)} = \begin{bmatrix} 0.243 & 0.3 & 0.524 & 0.24 & 0.27 \\ 0.475 & 0.3 & 0.5 & 0.35 & 0.5 \\ 0.2256 & 0.09 & 0.25 & 0.1225 & 0.25 \\ 0.1072 & 0.027 & 0.125 & 0.0429 & 0.125 \\ 0.0509 & 0.0081 & 0.0625 & 0.015 & 0.0625 \end{bmatrix}, \overset{=}{\Re}_i^{(4)} = \begin{bmatrix} 0.56 & 0.3 & 0.546 & 0.2 & 0.2 \\ 0.45 & 0.3 & 0.45 & 0.3 & 0.45 \\ 0.2025 & 0.09 & 0.2025 & 0.09 & 0.2025 \\ 0.0911 & 0.027 & 0.0911 & 0.027 & 0.0911 \\ 0.041 & 0.0081 & 0.041 & 0.0081 & 0.041 \end{bmatrix}$$

Stage 3: By using the weighted averaging/geometric operators, we determined the support $\overset{=}{\mathcal{T}}_j$ for the experts, such that

$$\overset{=}{\mathcal{T}}_j^{(1)} = \begin{bmatrix} 1.653 \\ 2 \\ 0.8337 \\ 0.3595 \\ 0.159 \end{bmatrix}, \overset{=}{\mathcal{T}}_j^{(2)} = \begin{bmatrix} 2.0212 \\ 2.025 \\ 0.8393 \\ 0.3556 \\ 0.1536 \end{bmatrix}, \overset{=}{\mathcal{T}}_j^{(3)} = \begin{bmatrix} 1.582 \\ 2.125 \\ 0.9381 \\ 0.4271 \\ 0.199 \end{bmatrix}, \overset{=}{\mathcal{T}}_j^{(4)} = \begin{bmatrix} 1.806 \\ 1.95 \\ 0.7875 \\ 0.3273 \\ 0.1392 \end{bmatrix}.$$

Stage 4: By using the CIFSPA and CIFSPG operators, we accumulated the matrices to initiate the exact values under the TN and TCN $\widetilde{\mathfrak{g}}(\bar{\bar{\mathfrak{r}}}) = -\log(\bar{\bar{\mathfrak{r}}})$, which are presented below:

$$u_1 = \left(0.0438e^{i2\pi(0.0218)}, 0.1353e^{i2\pi(0.2932)}\right),$$
$$u_2 = \left(0.2367e^{i2\pi(0.1711)}, 0.4065e^{i2\pi(0.5739)}\right),$$
$$u_3 = \left(0.5372e^{i2\pi(0.4616)}, 0.6839e^{i2\pi(0.7892)}\right),$$
$$u_4 = \left(0.1417e^{i2\pi(0.1383)}, 0.3169e^{i2\pi(0.5099)}\right),$$
$$u_5 = \left(0.1076e^{i2\pi(0.062)}, 0.2447e^{i2\pi(0.4307)}\right)$$
$$u_1 = \left(0.7014e^{i2\pi(0.7795)}, 0.4813e^{i2\pi(0.2777)}\right),$$
$$u_2 = \left(0.8366e^{i2\pi(0.8817)}, 0.6886e^{i2\pi(0.5286)}\right),$$
$$u_3 = \left(0.9184e^{i2\pi(0.9429)}, 0.8359e^{i2\pi(0.7424)}\right),$$
$$u_4 = \left(0.7888e^{i2\pi(0.7098)}, 0.5844e^{i2\pi(0.3874)}\right),$$
$$u_5 = \left(0.6652e^{i2\pi(0.7612)}, 0.4614e^{i2\pi(0.2703)}\right)$$

Stage 5: By using the accumulated values, we explored the SVs, which are presented below.

$$u_1 = 0.3629, u_2 = 0.5727, u_3 = 0.4743, u_4 = 0.5468, u_5 = 0.5058.$$

$$u_1 = 0.7219, u_2 = 0.5012, u_3 = 0.2831, u_4 = 0.6078, u_5 = 0.6948$$

Stage 6: By using the SVs, we initiated the ranking result to find the best optimal, which are presented below.

$$u_2 \geq u_4 \geq u_5 \geq u_3 \geq u_1.$$

$$u_1 \geq u_5 \geq u_4 \geq u_2 \geq u_3.$$

Under the two distinct operators, we obtained two different results: u_2 and u_1. Moreover, to initiate the supremacy and dominancy of the elaborated operators under the CIFSSs, a comparative analysis of the presented works is also discussed.

4.3. Sensitivity Analysis

To contrast the proposed approach and some current methodologies under IFSS climate, a correlation investigation was performed with various methodologies given in [41,43,46,49,51,52]. The acquired outcomes are depicted in Table 6. From this table, it is seen that the qualities assessed by using the predominant techniques matched with the results of the proposed approach. Nonetheless, these administrators did not consider the connection between the boundaries as being just between the specialists by the forced weighting; however, the proposed administrators thought about such connections. In the proposed administrators, the weight of each piece of information's contribution relied on the contributions of other pieces of information. From these realities, we can presume that the proposed strategy was more sensible and substantial for IFSS data. In addition, to settle the DM problem [41,43,46,49,51,52] we needed to expect an ideal outrageous elective that builds the intricacy and thus prompts extra overhead, yet with the proposed administrators we did not need any ideal elective to diminish the expense, calculation time, and intricacy of the specialists. Accordingly, the proposed technique was more reasonable for taking care of the DM issue than the current strategies.

Table 6. Expression of the sensitivity analysis of the initiated works.

Method	Score Values	Ranking Values
Hayat et al. [41]	$u_1 = 0.473, u_2 = 0.683, u_3 = 0.585,$ $u_4 = 0.657, u_5 = 0.616$	$u_2 \geq u_4 \geq u_5 \geq u_3 \geq u_1$
Arora and Garg [43]	$u_1 = 0.251, u_2 = 0.461, u_3 = 0.363,$ $u_4 = 0.435, u_5 = 0.404$	$u_2 \geq u_4 \geq u_5 \geq u_3 \geq u_1$
Garg and Arora [46]	$u_1 = 0.746, u_2 = 0.956, u_3 = 0.858,$ $u_4 = 0.929, u_5 = 0.909$	$u_2 \geq u_4 \geq u_5 \geq u_3 \geq u_1$
Ali et al. [49]	$u_1 = 0.625, u_2 = 0.832, u_3 = 0.747,$ $u_4 = 0.829, u_5 = 0.809$	$u_2 \geq u_4 \geq u_5 \geq u_3 \geq u_1$
Klement et al. [51]	Cannot be Calculated	Cannot be Calculated
Grabisch et al. [52]	Cannot be Calculated	Cannot be Calculated
CIFSPA operator	$u_1 = 0.3629, u_2 = 0.5727, u_3 = 0.4743,$ $u_4 = 0.5468, u_5 = 0.5058$	$u_2 \geq u_4 \geq u_5 \geq u_3 \geq u_1$
CIFSPG operator	$u_1 = 0.7219, u_2 = 0.5012, u_3 = 0.2831,$ $u_4 = 0.6078, u_5 = 0.6948$	$u_1 \geq u_5 \geq u_4 \geq u_2 \geq u_3$

Based on the above theory, we determined the same ranking results; therefore, the initiated operators under the CIFSSs were massive dominant and more flexible than the prevailing operators.

5. Conclusions

The principle of CIFS setting is massive dominant and can be applied more consistently with problematic and inconsistent data in real-world dilemmas. Furthermore, the PAOs based on general t-norm and t-conorm are also more flexible than the averaging, geometric, Einstein, and Hamacher aggregation operators when handling problematic data. Maintaining the benefits of the PAOs and CIFS setting, the results of this analysis are highlighted below.

1. We initiated the theories of CIFSPA, CIFSWPA, CIFSOWPA, CIFSPG, CIFSWPG, and CIFSOWPG, and their flexible laws were elaborated.
2. Certain specific cases (such as averaging, Einstein, and Hamacher operators) of the explored operators were also illustrated with the help of t-norm and t-conorm $\overset{\frown}{\mathfrak{g}}(\bar{\bar{\mathfrak{r}}}) = -\log(\bar{\bar{\mathfrak{r}}}), \overset{\frown}{\mathfrak{g}}(\bar{\bar{\mathfrak{r}}}) = \log\left(\frac{2-\bar{\bar{\mathfrak{r}}}}{\bar{\bar{\mathfrak{r}}}}\right), \bar{\bar{\mathfrak{r}}} \neq 0$, and $\overset{\frown}{\mathfrak{g}}(\bar{\bar{\mathfrak{r}}}) = \log\left(\overset{\frown}{\delta} + \frac{\left(1-\overset{\frown}{\delta}\right)\bar{\bar{\mathfrak{r}}}}{\bar{\bar{\mathfrak{r}}}}\right), \overset{\frown}{\delta} \in (0, \infty), \bar{\bar{\mathfrak{r}}} \neq 0$.
3. MADM processes were presented under the developed operators based on the CIFS environment to investigate the performance of the initiated works.
4. Finally, the advantages, graphically shown, and sensitivity analysis of the initiated operators and numerous prevailing works were also developed to verify the efficiency of the proposed works.

5.1. *Advantages of the Elaborated Works*

The main advantages of the explored works are illustrated below.

1. The invented works based on CIFSSs are more beneficial than the prevailing works elaborated under IFSs, IFSSs, and FSs.
2. The invented works based on CIFSSs are more generalized than the prevailing operators that were initiated under IFSs, IFSSs, and FSs.

5.2. *Advantages of the Elaborated Works*

The main disadvantages of the explored works are illustrated below.

1. In subsequent research, we will address how the presented works were unable to resolve information types that covered the TG, abstinence, and FG; consequently, the

principle of CIFS was neglected because the explored theory had to cope only with the type of data, which only covers TG and FG.

2. For this, we will be elaborate on the principle of power aggregation operators under complex Pythagorean fuzzy soft sets, complex q-rung orthopair fuzzy soft sets, complex picture fuzzy soft sets, complex spherical fuzzy soft sets, and complex T-spherical fuzzy soft sets.

In subsequent research, we will be elaborate on the principle of hesitant fuzzy linguistic aggregation operators [53], complex q-rung orthopair fuzzy sets [54], complex spherical fuzzy sets [55,56], T-spherical fuzzy sets [57], etc. [58–64] to enhance the performance and capacity of the exploration performs.

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Appendix A

Proof. Under the $\widetilde{\mathfrak{f}}$ and $\widetilde{\mathfrak{g}}$, we determine $\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)$ and $\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)$, then

$$\begin{aligned}
&\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{g}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)+\\
&\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)\leq\\
&\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{g}}\left(1-\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)+\\
&\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)\leq\\
&1-\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)+\\
&\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\bar{\bar{\mathfrak{x}}})\right)+\widetilde{\mathfrak{f}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\bar{\bar{\mathfrak{x}}})\right)\right)\leq 1.
\end{aligned}$$

Thus, $\overline{\overline{\mathfrak{C}_{CIF-11}}}\otimes\overline{\overline{\mathfrak{C}_{CIF-12}}}$ is a CIFSN. Similarly, we prove that $\widetilde{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-1j}}}$ and $\overline{\overline{\mathfrak{C}_{CIF-1j}}}^{\widetilde{\delta}}$ are also CIFSNs. □

Appendix B

Proof Part (1), (2), (4), and (6) are trivial. We only prove Equations (3) and (5). Let $\widetilde{\delta}\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overline{\overline{\mathfrak{C}_{CIF-12}}}\right)$, then

$$\overset{\frown}{\delta}\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overline{\overline{\mathfrak{C}_{CIF-12}}}\right)$$

$$=\overset{\frown}{\delta}\begin{pmatrix}\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)},\\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)},\\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)+\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)}\end{pmatrix}$$

$$=\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)},\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)}\right)$$

$$\oplus\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)},\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-12}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-12}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)}\right)$$

$$=\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-12}}}.$$

Hence, $\overset{\frown}{\delta}\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overline{\overline{\mathfrak{C}_{CIF-12}}}\right)=\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}_{CIF-12}}}$. Let, $\overset{\frown}{\delta_1}\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overset{\frown}{\delta_2}\overline{\overline{\mathfrak{C}_{CIF-11}}}$, then

$$l\overset{\frown}{\delta_1}\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overset{\frown}{\delta_2}\overline{\overline{\mathfrak{C}_{CIF-11}}}$$

$$=\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta_1}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta_1}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)},\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta_1}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta_1}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)}\right)$$

$$\oplus\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta_2}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta_2}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)},\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta_2}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta_2}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)}\right)$$

$$=\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta_1}\overset{\frown}{\mathfrak{f}}\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})+\overset{\frown}{\delta_2}\overset{\frown}{\mathfrak{f}}\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta_1}\overset{\frown}{\mathfrak{f}}\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})+\overset{\frown}{\delta_2}\overset{\frown}{\mathfrak{f}}\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)},\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta_1}\overset{\frown}{\mathfrak{g}}\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})+\overset{\frown}{\delta_2}\overset{\frown}{\mathfrak{g}}\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-11}}}}(\overline{\overline{\mathfrak{x}}})\right)e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta_1}\overset{\frown}{\mathfrak{g}}\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})+\overset{\frown}{\delta_2}\overset{\frown}{\mathfrak{g}}\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-11}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)}\right)$$

$$=\left(\overset{\frown}{\delta_1}+\overset{\frown}{\delta_2}\right)\overline{\overline{\mathfrak{C}_{CIF-11}}}.$$

Hence, $\overset{\frown}{\delta_1}\overline{\overline{\mathfrak{C}_{CIF-11}}}\oplus\overset{\frown}{\delta_2}\overline{\overline{\mathfrak{C}_{CIF-11}}}=\left(\overset{\frown}{\delta_1}+\overset{\frown}{\delta_2}\right)\overline{\overline{\mathfrak{C}_{CIF-11}}}$. □

Appendix C

Proof. Let $n = 1$, then under Equation (12), we have

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-1m}}}\right) = \oplus_{j=1}^{m}\left(\breve{\mathbb{M}}_j \overline{\overline{\mathfrak{C}_{PI-1j}}}\right)$$

$$= \begin{pmatrix} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{1} \breve{\mathbb{N}}_1 \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{1} \breve{\mathbb{N}}_1 \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{1} \breve{\mathbb{N}}_1 \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{1} \breve{\mathbb{N}}_1 \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-1j}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}.$$

For $m = 1$, we have

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-n1}}}\right) = \oplus_{i=1}^{n}\left(\breve{\mathbb{N}}_i \overline{\overline{\mathfrak{C}_{PI-i1}}}\right) =$$

$$\begin{pmatrix} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{1} \breve{\mathbb{M}}_1 \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-i1}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{1} \breve{\mathbb{M}}_1 \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-i1}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{1} \breve{\mathbb{M}}_1 \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-i1}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{1} \breve{\mathbb{M}}_1 \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-i1}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}.$$

Equation (12) is corrected for $m = n = 1$. Further, we assume that Equation (12) is corrected for $m = k_1, n = k_2 + 1$ and $m = k_1 + 1, n = k_2$, then for $m = k_1 + 1, n = k_2 + 1$, we have

$$\oplus_{j=1}^{k_1+1}\left(\breve{\mathbb{M}}_j \oplus_{i=1}^{k_2+1}\left(\breve{\mathbb{N}}_i \overline{\overline{\mathfrak{C}_{PI-ij}}}\right)\right) = \oplus_{j=1}^{k_1}\left(\breve{\mathbb{M}}_j \oplus_{i=1}^{k_2+1}\left(\breve{\mathbb{N}}_i \overline{\overline{\mathfrak{C}_{PI-ij}}}\right)\right) \oplus \left(\breve{\mathbb{M}}_{k_1+!} \oplus_{i=1}^{k_2+1}\left(\breve{\mathbb{N}}_i \overline{\overline{\mathfrak{C}_{PI-ij}}}\right)\right)$$

$$= \begin{pmatrix} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{k_1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{k_1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{k_1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{k_1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \overset{\frown}{\mathfrak{f}}^{-1}\left(\breve{\mathbb{M}}_{k_1+1} \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\breve{\mathbb{M}}_{k_1+1} \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\breve{\mathbb{M}}_{k_1+1} \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\breve{\mathbb{M}}_{k_1+1} \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{k_1+1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{k_1+1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{k_1+1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{k_1+1} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{k_2+1} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}.$$

□

Equation (12) is corrected for $m = k_1 + 1, n = k_2 + 1$ by using mathematical induction for any positive integer m, n. Moreover, we discussed the specific properties of Equation (12), which are illustrated below.

Appendix D

Proof of Property 1: If $\overline{\overline{\mathfrak{C}_{CIF-ij}}} = \overline{\overline{\mathfrak{C}_{CIF}}}$, then by using Equation (12), we have

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right)$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_R}}}(\overline{\overline{\mathfrak{x}}})\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_I}}}(\overline{\overline{\mathfrak{x}}})\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_R}}}(\overline{\overline{\mathfrak{x}}})\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_I}}}(\overline{\overline{\mathfrak{x}}})\right)\right)} \end{pmatrix}$$

$$= \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_R}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_I}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}_R}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_I}}}(\overline{\overline{\mathfrak{x}}})\right)}\right) = \overline{\overline{\mathfrak{C}_{CIF}}}.$$

□

Appendix E

Proof of Property 2. If $\overline{\overline{\mathfrak{C}_{CIF-ij}}}$ and $\overline{\overline{\mathfrak{C}_{CIF}}}$ are any two CIFSNs, then

$$\overline{\overline{\mathfrak{C}_{CIF-ij}}} \oplus \overline{\overline{\mathfrak{C}_{CIF}}} = \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_R}}}(\overline{\overline{\mathfrak{x}}})\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_I}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_R}}}(\overline{\overline{\mathfrak{x}}})\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_I}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)} \end{pmatrix}$$

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overline{\overline{\mathfrak{C}_{CIF}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}} \oplus \overline{\overline{\mathfrak{C}_{CIF}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}} \oplus \overline{\overline{\mathfrak{C}_{CIF}}}\right)$$

$$= \left(\begin{array}{c} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) \\ e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) \\ e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right) + \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) + \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) \\ e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\mathfrak{f}}\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) + \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) + \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) \\ e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\mathfrak{g}}\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) + \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \overset{\frown}{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{array} \right)$$

$$\oplus \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I}}(\overline{\overline{\mathfrak{x}}})\right)}\right)$$

$$= CIFSPA\left(\overline{\overline{\mathfrak{C}}}_{CIF-11}, \overline{\overline{\mathfrak{C}}}_{CIF-12}, \ldots, \overline{\overline{\mathfrak{C}}}_{CIF-nm}\right) \oplus \overline{\overline{\mathfrak{C}}}_{CIF}.$$

□

Appendix F

Proof of Property 3. For any $\overset{\frown}{\delta} > 0$, then

$$\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}}}_{CIF-ij} = \left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{f}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)}, \overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{R-ij}}(\overline{\overline{\mathfrak{x}}})\right) e^{i2\pi\left(\overset{\frown}{\mathfrak{g}}^{-1}\left(\overset{\frown}{\delta}\,\overset{\frown}{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}}}_{I-ij}}(\overline{\overline{\mathfrak{x}}})\right)\right)} \right),$$

then

$$CIFSPA\left(\overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}}}_{CIF-11}, \overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}}}_{CIF-12}, \ldots, \overset{\frown}{\delta}\,\overline{\overline{\mathfrak{C}}}_{CIF-nm}\right)$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\widetilde{\mathfrak{f}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\widetilde{\mathfrak{f}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\widetilde{\mathfrak{g}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\widetilde{\mathfrak{g}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overline{\delta}\,\widetilde{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overline{\delta}\,\widetilde{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overline{\delta}\,\widetilde{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\overline{\delta}\,\widetilde{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{f}}\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{f}}\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{f}}\,\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{g}}\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\overline{\delta}\,\widetilde{\mathfrak{g}}\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{g}}\,\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)\right)} \end{pmatrix}$$

$$= \overline{\overline{\delta}}\,CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right).$$

□

Appendix G

Proof of Property 4. If $\overline{\overline{\mathfrak{C}^*_{CIF-ij}}} = \left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}, \mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}}) e^{i2\pi\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)}\right)$, then

$$\overline{\overline{\mathfrak{C}_{CIF-ij}}} \oplus \overline{\overline{\mathfrak{C}^*_{CIF-ij}}} = \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right) e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)} \end{pmatrix},$$

then

$$CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}} \oplus \overline{\overline{\mathfrak{C}^*_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}} \oplus \overline{\overline{\mathfrak{C}^*_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}} \oplus \overline{\overline{\mathfrak{C}^*_{CIF-nm}}}\right)$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) \\ e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{f}}\left(\widetilde{\mathfrak{f}}^{-1}\left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right) \\ e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \widetilde{\mathfrak{g}}\left(\widetilde{\mathfrak{g}}^{-1}\left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right) \\ e^{i2\pi\left(\widetilde{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)}, \\ \widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right) \\ e^{i2\pi\left(\widetilde{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m} \breve{\mathbb{M}}_j \left(\sum_{i=1}^{n} \breve{\mathbb{N}}_i \left(\widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right) + \widetilde{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\overline{\mathfrak{x}}})\right)\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)+\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)+\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right)\right)}, \\ \overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)+\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)+\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right)\right)}, \\ \overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right)\right)} \end{pmatrix} \oplus \begin{pmatrix} \overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overline{\mathfrak{f}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{f}}\left(\mathcal{M}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right)\right)}, \\ \overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{R-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right) \\ e^{i2\pi\left(\overline{\mathfrak{g}}^{-1}\left(\sum_{j=1}^{m}\breve{\mathbb{M}}_j\left(\sum_{i=1}^{n}\breve{\mathbb{N}}_i\left(\overline{\mathfrak{g}}\left(\mathcal{N}_{\overline{\overline{\mathfrak{C}^*_{I-ij}}}}(\overline{\mathfrak{r}})\right)\right)\right)\right)\right)} \end{pmatrix}$$

$$= CIFSPA\left(\overline{\overline{\mathfrak{C}_{CIF-11}}}, \overline{\overline{\mathfrak{C}_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}_{CIF-nm}}}\right) \oplus CIFSPA\left(\overline{\overline{\mathfrak{C}^*_{CIF-11}}}, \overline{\overline{\mathfrak{C}^*_{CIF-12}}}, \ldots, \overline{\overline{\mathfrak{C}^*_{CIF-nm}}}\right).$$

□

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