symmetry

MDPI

Article

On the Fekete–Szegö Problem for Meromorphic Functions Associated with p,q-Wright Type Hypergeometric Function

Adriana Cătaş

Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania; acatas@uoradea.ro

Abstract: Making use of a post-quantum derivative operator, we define two classes of meromorphic analytic functions. For the considered family of functions, we aim to investigate the sharp bounds' values in the case of the Fekete–Szegö problem. The study of the well-known Fekete–Szegö functional in the post-quantum calculus case for meromorphic functions provides new outcomes for research in the field. With the extended p,q-operator, we establish certain inequalities' relations concerning meromorphic functions. In the final part of the paper, a new p,q-analogue of the q-Wright type hypergeometric function is introduced. This function generalizes the classical and symmetrical Gauss hypergeometric function. All the obtained results are sharp.

Keywords: meromorphic functions; Fekete–Szegö problem; p,q-derivative operator; p,q-Wright type hypergeometric function

check for updates

Citation: Cătaş, A. On the Fekete–Szegö Problem for Meromorphic Functions Associated with p,q-Wright Type Hypergeometric Function. *Symmetry* **2021**, *13*, 2143. https://doi.org/10.3390/sym13112143

Academic Editor: Roman Ger

Received: 24 September 2021
Accepted: 5 November 2021
Published: 10 November 2021

1. Introduction

In many branches of mathematics and physics, an important role is played by the theory of q-analysis. We can mention here several areas, such as ordinary fractional calculus, problems of optimal control, q-integral equations, q-difference and q-transform analysis (for example, see [1–4]). The study of q-calculus has experienced accelerated development, especially after the pioneering work of M. E. H. Ismail et al. [5]. This was followed by similar works, such as those of S. Kanas and D. Raducanu [6] and S. Sivasubramanian and M. Govindaraj [7]. In the beginning, we will describe some special concepts of q-calculus. Consider the class of meromorphic functions denoted by Σ on the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k. \tag{1}$$

They are analytic in U^*, the open punctured unit disc:

$$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}.$$

We say that $f \in \Sigma$ is a meromorphic starlike function of order β if the following inequality is satisfied:

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (0 \le \beta < 1).$$

Let us denote by $\Sigma^*(\beta)$ such a class of functions. Pommerenke [8] introduced and studied the class $\Sigma^*(\beta)$ (see also Miller [9]).

Consider the class of meromorphic functions denoted by $\Sigma^*(\varphi)$ having the property

$$-\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

The function φ is an analytic function such that $Re\varphi > 0$, $\varphi(0) = 1$ and $\varphi'(0) > 0$. The domain U is mapped onto a region that is starlike with respect to 1 and is a symmetric region with respect to the real axis. Silverman et al. [10] introduced and studied the class $\Sigma^*(\varphi)$. When the function φ has the special form $\varphi(z) = \frac{1+(1-2\beta)}{1-z}$, $0 \leq \beta < 1$, then the class $\Sigma^*(\beta)$ is the particular case of the class $\Sigma^*(\varphi)$.

The notion of q-derivative $(0 < q < 1)$ is defined by Gasper and Rahman [2] as

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}. \tag{2}$$

From definition (2), we find that $D_q f(z)$ is given by

$$D_q f(z) = -\frac{1}{qz^2} + \sum_{k=0}^{\infty} [k]_q a_k z^{k-1}, \quad z \neq 0,$$

$$[k]_q = \frac{1-q^k}{1-q},$$

for a function f of the form (1). When $q \to 1^-$, then $[k]_q \to k$ and we deduce

$$\lim_{q \to 1^-} D_q f(z) = f'(z).$$

2. Basic Results and Definitions

In the present section, we recall a few well-known statements and related elements in post-quantum calculus. Given the new results in the previous section, we need the required elements of the (p,q)-calculus. Lately, there have been many authors concerned with (p,q)-calculus. We recall here [11,12]. The post-quantum calculus denoted by the (p,q)-calculus is an extension of the q-calculus (see [13,14]).

The theory of post-quantum calculus ([15]) operators is useful in the theory of geometric function and also in many areas of science. The (p,q)-number is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}$$

$$[k]_{p,q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \ldots + pq^{k-2} + q^{k-1}. \tag{3}$$

which is a natural extension of the q-number: we have ([16,17]). Moreover, the

$$\lim_{p \to 1} [k]_{p,q} = [k]_q.$$

Additionally, the (p,q)-factorial is

$$[k]_{p,q}! = [k]_{p,q}[k-1]_{p,q}[k-2]_{p,q} \cdot \ldots \cdot [1]_{p,q}, \quad k \in \mathbb{N}.$$

The (p,q)-Gamma function, for a complex number x, is defined by

$$\Gamma_{p,q}(x) = (p-q)^{1-x} \prod_{k=0}^{\infty} \frac{p^{k+1} - q^{k+1}}{p^{k+x} - p^{k+x}}$$

$$\Gamma_{p,q}(x) = (p-q)^{1-x} \frac{(p;q)_{p,q}^{\infty}}{(p^x;q^x)_{p,q}^{\infty}}, \; 0 < q < p$$

where

$$(a;b)_{p,q}^{\infty} = \prod_{k=0}^{\infty} \left(ap^k - bq^k \right), \; a,b \in (0,1].$$

Remark 1. *The function q-Gamma, denoted by $\Gamma_q(x)$, is a special case of (p,q)-Gamma function if $p=1$.*

Our present paper is based on geometric function theory and implies a given application in the open unit disk. We formulate two analytic function classes depending on the symmetry properties. Using a symmetric operator, we deduce many interesting properties.

Lemma 1 ([18])**.** *Consider the function p such that $Rep(z)>0$ in U and $p(z)=1+c_1z+c_2z^2+\ldots$ Then,*

$$\left|c_2-\mu c_1^2\right|\le 2\max\{1;|2\mu-1|\}$$

where μ is a complex number. This is a sharp result for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2}\quad \text{and}\quad p(z)=\frac{1+z}{1-z}\,.$$

Lemma 2 ([18])**.** *Consider the function p_1 such that $Rep_1(z)>0$ in U and $p_1(z)=1+c_1z+c_2z^2+\ldots$. Then,*

$$\left|c_2-\nu c_1^2\right|\le\begin{cases}-4\nu+2 & if \quad \nu\le 0\\ 2 & if \quad 0\le\nu\le 1\\ 4\nu-2 & if \quad \nu\ge 1\end{cases}.$$

When $\nu<0$ or $\nu>1$, the equality holds if and only if $p_1(z)=\frac{1+z}{1-z}$ or one of its rotations. If $0<\nu<1$, then the equality holds if and only if $p_1(z)=\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu=0$, the equality holds if and only if

$$p_1(z)=\left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{1-z}{1+z},\ 0\le\lambda\le 1,$$

or one of its rotations. If $\nu=1$, the equality holds if and only if

$$\frac{1}{p_1(z)}=\left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{1-z}{1+z},\ 0\le\lambda\le 1,$$

or one of its rotations. Moreover, the above upper bounds are sharp and can be improved as follows when $0<\nu<1:$

$$\left|c_2-\nu c_1^2\right|+\nu\left|c_1^2\right|\le 2,\qquad \left(0<\nu\le\frac{1}{2}\right),$$

and

$$\left|c_2-\nu c_1^2\right|+(1-\nu)\left|c_1^2\right|\le 2,\qquad \left(\frac{1}{2}<\nu<1\right).$$

Next, with the extended idea of the q-difference operator, we have:

Definition 1 ([12])**.** *Let $f\in\Sigma$ and $0<q<p\le 1$. Then, the p,q-derivative operator or p,q-difference operator for the function f of the form (1) is defined by*

$$D_{p,q}f(z)=\frac{f(pz)-f(qz)}{(p-q)z}\quad (z\in U^*). \tag{4}$$

By (2), we obtain

$$D_{p,q}f(z)=-\frac{1}{pqz^2}+\sum_{k=0}^{\infty}[k]_{p,q}a_kz^{k-1},\quad (z\in U^*). \tag{5}$$

We observe that if $p = 1$,

$$[k]_{1,q} = [k]_q = \frac{1 - q^k}{1 - q},\ q \neq 1$$

and we reobtain the operator from (2).

As $q \to p^-$, $[k]_{p,q} \to kp^{k-1}$ and we have

$$\lim_{q \to p^-} D_{p,q} f(z) = -\frac{1}{p^2 z^2} + \sum_{k=0}^{\infty} kp^{k-1} a_k z^{k-1}, \quad (z \in U^*).$$

The findings and properties proven in this work are a natural extension of those analogous in q-calculus theory and also contribute to the theory of inequalities. In this manner, the famous problem of Fekete–Szegö for meromorphic functions starts to develop within the topic of p,q-calculus. Many of the newly established inequalities are natural extensions of some already given inequalities. In the final part of this paper, interesting applications of the new results are considered. A new (p,q)-analogue of the q-Wright type hypergeometric function is introduced and some relations among properties and inequalities are developed.

3. Fekete–Szegö Problem for Meromorphic Functions in the Post-Quantum Calculus Case

Using the p,q-derivative operator, we define the following classes:

Definition 2. *For* $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $0 < q < p \leq 1$, *a function* $f \in \Sigma$ *is said to be in the class* $\Sigma S^*_{p,q}[b;\varphi]$ *if and only if*

$$1 - \frac{1}{b}\left(1 + \frac{pqzD_{p,q}f(z)}{f(z)}\right) \prec \varphi(z). \tag{6}$$

Special cases of this class were studied by different authors as follows:

(i) $\lim\limits_{q \to 1^-} \Sigma S^*_{1,q}\left[b; \frac{1+Az}{1-Bz}\right] := \Sigma S^*[b; A, B]$ (see Bulboacă [19]);

(ii) $\lim\limits_{q \to 1^-} \Sigma S^*_{1,q}\left[1; \frac{1+z}{1-z}\right] := \mathcal{F}^*$ (see Aouf [20]);

(iii) $\lim\limits_{q \to 1^-} \Sigma S^*_{1,q}\left[1; \frac{1+(1-2\alpha)\beta z}{1-\beta z}\right] := \Sigma S[\alpha,\beta]$ (see El-Ashwah and Aouf [21]);

(iv) $\lim\limits_{q \to 1^-} \Sigma S^*_{1,q}\left[(1-\alpha)e^{-i\mu}\cos\mu; \frac{1+z}{1-z}\right] := \Sigma S^{\mu}(\alpha)$, with $\mu \in \mathbb{R}$, $|\mu| \leq \pi/2$, $0 \leq \alpha < 1$, (see [22], for $p = 1$).

Definition 3. *For* $\alpha \in \mathbb{C}\backslash(0,1]$, $0 < q < p \leq 1$, *a function* $f \in \Sigma$ *is said to be in the class* $\mathcal{F}^*_{p,q}[\alpha;\varphi]$ *if and only if*

$$-\frac{\left(1 - \frac{\alpha}{pq}\right)pqzD_{p,q}f(z) + \alpha pqzD_{p,q}[zD_{p,q}f(z)]}{\left(1 - \frac{\alpha}{pq}\right)f(z) + \alpha zD_{p,q}f(z)} \prec \varphi(z),\ z \in U. \tag{7}$$

We recall here some special classes studied earlier by various authors.

(i) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{1,q}[\alpha;\varphi] := \mathcal{F}^*_{\alpha}(\varphi)$ (see Aouf et al. [23]);

(ii) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{p,q}\left[0; \frac{1+(1-2\beta)z}{1-z}\right] := \pm^*(\beta)$, $0 \leq \beta < 1$ (see Pommerenke [8]);

(iii) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{1,q}\left[0; \frac{1+\beta(1-2\gamma\eta)z}{1+\beta(1-2\gamma)z}\right] := \pm(\eta,\beta,\gamma)$, $0 \leq \eta < 1$, $0 < \beta \leq 1$, $1/2 \leq \gamma \leq 1$, (see Kulkarni and Joshi [24]);

(iv) $\lim_{q\to 1^-} \mathcal{F}^*_{1,q}\left[0; \frac{1+Az}{1-Bz}\right] := K_1(A,B), 0 \le B < 1, -B < A < B$, (see Karunakaran [25]).

In 2001, Virchenko et al. (see [26]) studied and investigated a Wright type hypergeometric function. For another special case, see [27]. In order to consider the application in the last section, we propose a new form of this function in terms of (p,q)-calculus.

Definition 4. *For $\Re a > 0, \Re b > 0, \Re c > 0, \tau \in \Re_+ = (0,+\infty)$, the (p,q)-Wright type hypergeometric function is defined by*

$$ {}_2R_1^{p,q}(a,b;c;\tau;z) = \frac{\Gamma_{p,q}(c)}{\Gamma_{p,q}(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma_{p,q}(b+\tau k)}{\Gamma_{p,q}(c+\tau k)} \frac{z^k}{[k]_{p,q}!}. $$

Remark 2. *The above-defined function is an extension of the classical Gauss hypergeometric function denoted by ${}_2F_1(a,b;c;z)$.*

We now deduce an important result of this paper.

Theorem 1. *Consider the function $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If the function f is in the form (1) and belonging to the class $\Sigma S^*_{p,q}[b;\varphi]$, then*

$$ \left|a_1 - \mu a_0^2\right| \le \frac{|B_1|}{1+pq} \max\left\{1, \left|\frac{B_2}{B_1} - [1-\mu(1+pq)]B_1\right|\right\}, \text{ if } B_1 \neq 0 \tag{8} $$

and

$$ |a_1| \le \frac{|B_2|}{1+pq}, \text{ if } B_1 = 0, \tag{9} $$

where μ is a complex number. The result is sharp.

Proof. Let $f \in \Sigma S^*_{p,q}[b;\varphi]$. Then, there exists φ a Schwarz function such that $w(z) = 0$ and $|w(z)| < 1$ in U with

$$ 1 - \frac{1}{b}\left(1 + \frac{pqzD_{p,q}f(z)}{f(z)}\right) = \varphi(w(z)). \tag{10} $$

Let $p_1(z)$ be

$$ p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{11} $$

We note that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$ since $w(z)$ is a Schwarz function. Now, we define

$$ p(z) = 1 - \frac{1}{b}\left(1 + \frac{pqzD_{p,q}f(z)}{f(z)}\right) = 1 + b_1 z + b_2 z^2 + \cdots. \tag{12} $$

Taking into account the relations (10)–(12), we obtain

$$ p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{13} $$

Thus,

$$ \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \cdots\right] $$

one obtains

$$ \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \cdots\right]B_1 $$

$$+\frac{1}{4}\left[c_1 z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\left(c_3+\frac{c_1^3}{4}-c_1c_2\right)z^3+\cdots\right]^2 B_2+\cdots$$
$$=1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right]z^2+\cdots.$$

Using the above relation and (12), we deduce

$$b_1=\frac{1}{2}B_1c_1$$

and

$$b_2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2.$$

Thus, from (1) and (12), we see that

$$b_1=-a_0$$

and

$$b_2=a_0^2-a_1(1+pq)$$

or, equivalently, we obtain

$$a_0=-\frac{1}{2}B_1c_1 \tag{14}$$

and

$$a_1=-\frac{B_1}{2(1+pq)}\left[c_2-\frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}+B_1\right)\right]. \tag{15}$$

Therefore, after computation,

$$a_1-\mu a_0^2=-\frac{B_1}{2(1+pq)}\left[c_2-\nu c_1^2\right], \tag{16}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}+B_1-\mu B_1(1+pq)\right]. \tag{17}$$

Thus, the result (8) is deduced by an application of Lemma 1. If $B_1=0$, then

$$a_0=0 \text{ and } a_1=-\frac{B_2c_1^2}{4(1+pq)}.$$

If $Rep(z)>0$, then $|c_1|\leq 2$ (see Nehari [28]). Hence,

$$|a_1|\leq\frac{|B_2|}{1+pq}$$

proving (9). For the following functions, the result is sharp.

$$1-\frac{1}{b}\left(1+\frac{pqzD_{p,q}f(z)}{f(z)}\right)=\varphi\left(z^2\right)$$

and

$$1-\frac{1}{b}\left(1+\frac{pqzD_{p,q}f(z)}{f(z)}\right)=\varphi(z).$$

Thus, the proof of Theorem 1 is complete. □

Putting $\lim_{q\to 1^-}\Sigma S_{p,q}^*\left[b;\frac{1+z}{1-z}\right]:=\Sigma S_p^*[b]$, one obtains the next corollary.

Corollary 1. *Consider f in the form (1) belonging to the class $\Sigma S_p^*[b]$. Then,*

$$\left|a_1-\mu a_0^2\right| \leq \frac{2}{1+p}\max\{1,|1-2[1-\mu(1+p)]|\} \tag{18}$$

where μ is a complex number. The result is sharp.

The following theorem is obtained by using Lemma 2.

Theorem 2. *Consider the function $\varphi(z)=1+B_1z+B_2z^2+\ldots(B_i>0, i=1,2)$. If the function f in the form (1) belonging to the class $\Sigma S_{p,q}^*[b;\varphi]$, $\mu\in\mathbb{R}$ then*

$$\left|a_1-\mu a_0^2\right| \leq \begin{cases} \frac{-B_2+[1-\mu(1+pq)]B_1^2}{1+pq} & if \quad \mu\leq\sigma_1 \\ \frac{B_1}{1+pq} & if \quad \sigma_1\leq\mu\leq\sigma_2 \\ \frac{B_2-[1-\mu(1+pq)]B_1^2}{1+pq} & if \quad \mu\geq\sigma_2 \end{cases} \tag{19}$$

where

$$\sigma_1=\frac{-B_1-B_2+B_1^2}{(1+pq)B_1^2}, \quad \sigma_2=\frac{B_1-B_2+B_1^2}{(1+pq)B_1^2}.$$

The result is sharp. Further, let $\sigma_3=\frac{-B_2+B_1^2}{(1+pq)B_1^2}$.

(i) If $\sigma_1\leq\mu\leq\sigma_3$, then

$$\left|a_1-\mu a_0^2\right| + \frac{\{(B_1+B_2)+[\mu(1+pq)-1]B_1^2\}|a_0|^2}{(1+pq)B_1^2} \leq \frac{B_1}{1+pq}; \tag{20}$$

(ii) If $\sigma_3\leq\mu\leq\sigma_2$, then

$$\left|a_1-\mu a_0^2\right| + \frac{\{(B_1-B_2)+[1-\mu(1+pq)]B_1^2\}|a_0|^2}{(1+pq)B_1^2} \leq \frac{B_1}{1+pq}. \tag{21}$$

Proof. First, let $\mu\leq\sigma_1$, then

$$\begin{aligned}\left|a_1-\mu a_0^2\right| &\leq \frac{B_1}{1+pq}\left\{-\frac{B_2}{B_1}+[1-\mu(1+pq)]B_1\right\} \\ &\leq \frac{-B_2+[1-\mu(1+pq)]B_1^2}{1+pq}.\end{aligned}$$

Consider $\sigma_1\leq\mu\leq\sigma_2$. Then, we obtain

$$\left|a_1-\mu a_0^2\right| \leq \frac{B_1}{1+pq}.$$

If $\mu\geq\sigma_2$, then

$$\begin{aligned}\left|a_1-\mu a_0^2\right| &\leq \frac{B_1}{1+pq}\left\{\frac{B_2}{B_1}-[1-\mu(1+pq)]B_1\right\} \\ &\leq \frac{B_2-[1-\mu(1+pq)]B_1^2}{1+pq}.\end{aligned}$$

We define the function $K_{\varphi n}$ in order to show that the bounds are sharp $(n \geq 2)$

$$1-\frac{1}{b}\left(1+\frac{pqzD_{p,q}K_{\varphi n}(z)}{K_{\varphi n}}\right)=\varphi\left(z^{n-1}\right),$$

$$z^2K_{\varphi n}(z)|_{z=0}=0=-z^2K'_{\varphi n}(z)|_{z=0}-1.$$

We have also the functions F_γ and G_γ $(0\leq\gamma\leq 1)$ given by

$$1-\frac{1}{b}\left(1+\frac{pqzD_{p,q}F_\gamma(z)}{F_\gamma(z)}\right)=\varphi\left(\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2F_\gamma(z)|_{z=0}=0=-z^2F'_\gamma(z)|_{z=0}-1$$

and

$$1-\frac{1}{b}\left(1+\frac{pqzD_{p,q}G_\gamma(z)}{G_\gamma(z)}\right)=\varphi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2G_\gamma(z)|_{z=0}=0=-z^2G'_\gamma(z)|_{z=0}-1.$$

Obviously, $K_{\varphi n}, F_\gamma, G_\gamma \in \Sigma S^*_{p,q}[b;\varphi]$ and $K_\varphi = K_{\varphi 2.}$. When $\mu<\sigma_1$ or $\mu>\sigma_2$, then we consider the equality if and only if f is K_φ or one of its rotations. If $\sigma_1<\mu<\sigma_2$, then we consider the equality if f is $K_{\varphi 3}$ or one of its rotations. When we consider $\mu=\sigma_1$, then the equality takes place if and only if f is F_γ or one of its rotations. If $\mu=\sigma_2$, then the equality holds if and only if f is G_γ or one of its rotations. Thus, we complete the proof of Theorem 2. □

Remark 3.

(i) *For $p=1, q\to 1^-$ in Theorem 2, we deduce the result obtained by Huo Tang et al. [29];*

(ii) *Putting $p=1, q\to 1^-$ and $\varphi(z)=\frac{1+z}{1-z}$ in Theorem 2, we have a new result for the class $\mathcal{F}^*$.*

Theorem 3. *Consider $\varphi(z)=1+B_1z+B_2z^2+\ldots$. If the function f in the form (1) belonging to the subclass $\mathcal{F}^*_{p,q}[\alpha;\varphi]$ and μ is a complex number, then*

$$\left|a_1-\mu a_0^2\right|\leq\frac{1}{1+pq}\left|\frac{(pq-2\alpha)B_1}{(pq-\alpha+\alpha pq)}\right| \tag{22}$$

$$\times\max\left\{1,\left|\frac{B_2}{B_1}-\left[1-\mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)}{(pq-\alpha)^2}\right]B_1\right|\right\},\ \text{if } B_1\neq 0$$

and

$$|a_1|\leq\frac{1}{1+pq}\left|\frac{(pq-2\alpha)B_2}{(pq-\alpha+\alpha pq)}\right|,\ \text{if }\ B_1=0. \tag{23}$$

The result is sharp.

Proof. If f belongs to the subclass $\mathcal{F}^*_{p,q}[\alpha;\varphi]$, then there exists w a Schwarz function in U such that $w(z)=0$ and $|w(z)|<1$ with

$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z)+\alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z)+\alpha zD_{p,q}f(z)}=\varphi(w(z)). \tag{24}$$

Let $p_1(z)$ be the function

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots. \tag{25}$$

We observe that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$, since w is a Schwarz function. Now, we define the function

$$p(z) = -\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z) + \alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z) + \alpha zD_{p,q}f(z)} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{26}$$

Taking into account the relations (24)–(26), we obtain

$$p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{27}$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \cdots\right]$$

one obtains

$$\begin{aligned}\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) &= 1 + \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \cdots\right]B_1 \\ &\quad + \frac{1}{4}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \cdots\right]^2 B_2 + \cdots \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \cdots.\end{aligned}$$

Using the above relation and (27), we deduce

$$b_1 = \frac{1}{2}B_1 c_1$$

and

$$b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2.$$

Thus, from (1) and (26), we see that

$$b_1 = -\frac{pq-\alpha}{pq-2\alpha}a_0$$

and

$$b_2 = \left(\frac{pq-\alpha}{pq-2\alpha}\right)^2 a_0^2 - \frac{(pq+1)(pq-\alpha+\alpha pq)}{pq-2\alpha}a_1$$

or, equivalently, we obtain

$$a_0 = -\frac{pq-2\alpha}{2(pq-\alpha)}B_1 c_1 \tag{28}$$

and

$$a_1 = -\frac{(pq-2\alpha)B_1}{2(1+pq)(pq-\alpha+\alpha pq)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{B_2}{B_1} + B_1\right)\right]. \tag{29}$$

Therefore, after computation,

$$a_1 - \mu a_0^2 = -\frac{(pq-2\alpha)B_1}{2(1+pq)(pq-\alpha+\alpha pq)}\left[c_2 - \nu c_1^2\right], \tag{30}$$

where
$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + B_1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}B_1\right]. \tag{31}$$

Thus, the result (22) is obtained by applying Lemma 1. If $B_1 = 0$, then
$$a_0 = 0 \text{ and } a_1 = -\frac{(pq-2\alpha)B_2c_1^2}{4(1+pq)(pq-\alpha+\alpha pq)}.$$

Since $Rep(z) > 0$, then $|c_1| \le 2$ (see Nehari [28]). Hence,
$$|a_1| \le \frac{1}{1+pq}\left|\frac{(pq-2\alpha)B_2}{pq-\alpha+\alpha pq}\right|$$
proving (23). The result is sharp for the functions
$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z) + \alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z) + \alpha zD_{p,q}f(z)} = \varphi\left(z^2\right)$$
and
$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z) + \alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z) + \alpha zD_{p,q}f(z)} = \varphi(z).$$

Thus, we complete Theorem 3. □

Letting $\lim_{q\to1^-} \mathcal{F}^*_{p,q}\left[\alpha; \frac{1+z}{1-z}\right] := \mathcal{F}^*_p[\alpha]$, we derive the following corollary.

Corollary 2. *Consider f in the form (1) and belonging to the subclass $\mathcal{F}^*_p[\alpha]$ and μ is a complex number, then*
$$\left|a_1 - \mu a_0^2\right| \le \frac{2}{1+p}\left|\frac{(p-2\alpha)}{(p-\alpha+\alpha p)}\right| \max\left\{1, \left|1 - 2\left[1 - \mu\frac{(p-2\alpha)(p-\alpha+\alpha p)}{(p-\alpha)^2}\right]\right|\right\}. \tag{32}$$

The result is sharp.

Applying Lemma 2, we deduce the following theorem.

Theorem 4. *Consider the function $\varphi(z) = 1 + B_1z + B_2z^2 + \ldots (B_i > 0, i = 1,2)$ with $0 < 2\alpha < \frac{pq}{1+pq}$. If the function f is in the form (1) and belongs to the subclass $\mathcal{F}^*_{p,q}[\alpha;\varphi]$, then*
$$\left|a_1 - \mu a_0^2\right| \le$$
$$\begin{cases} \frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\left\{-B_2 + \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1^2\right\} & if \quad \mu \le \sigma_1 \\ \frac{(pq-2\alpha)B_1}{(1+pq)(pq-\alpha+\alpha pq)} & if \ \sigma_1 \le \mu \le \sigma_2 \\ \frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\left\{B_2 - \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1^2\right\} & if \quad \mu \ge \sigma_2 \end{cases} \tag{33}$$
where
$$\sigma_1 = \frac{(pq-\alpha)^2\left(-B_1 - B_2 + B_1^2\right)}{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)B_1^2},$$
$$\sigma_2 = \frac{(pq-\alpha)^2\left(B_1 - B_2 + B_1^2\right)}{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)B_1^2}.$$

The result is sharp. Further, let $\sigma_3 = \frac{(pq-\alpha)^2\left(-B_2+B_1^2\right)}{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)B_1^2}$.

(i) *If* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\left|a_1 - \mu a_0^2\right| + \frac{(pq-\alpha)^2}{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)B_1^2} \tag{34}$$

$$\times\left\{(B_1+B_2) + \left[\mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2} - 1\right]B_1^2\right\}|a_0|^2$$

$$\leq \frac{(pq-2\alpha)B_1}{(1+pq)(pq-\alpha+\alpha pq)};$$

(ii) *If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_1 - \mu a_0^2\right| + \frac{(pq-\alpha)^2}{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)B_1^2} \tag{35}$$

$$\times\left\{(B_1-B_2) + \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1^2\right\}|a_0|^2$$

$$\leq \frac{(pq-2\alpha)B_1}{(1+pq)(pq-\alpha+\alpha pq)}.$$

Proof. First, let $\mu \leq \sigma_1$, then

$$\left|a_1 - \mu a_0^2\right|$$

$$\leq \frac{(pq-2\alpha)B_1}{(pq-\alpha+\alpha pq)(1+pq)}\left\{-\frac{B_2}{B_1} + \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1\right\}$$

$$\leq \frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\left\{-B_2 + \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1^2\right\}.$$

Consider $\sigma_1 \leq \mu \leq \sigma_2$. Then, we have

$$\left|a_1 - \mu a_0^2\right| \leq \frac{(pq-2\alpha)B_1}{(1+pq)(pq-\alpha+\alpha pq)}.$$

Finally, if $\mu \geq \sigma_2$, then

$$\left|a_1 - \mu a_0^2\right|$$

$$\leq \frac{(pq-2\alpha)B_1}{(pq-\alpha+\alpha pq)(1+pq)}\left\{\frac{B_2}{B_1} + \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1\right\}$$

$$\leq \frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\left\{B_2 - \left[1 - \mu\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\right]B_1^2\right\}.$$

We define the function $K_{\varphi n}$ $(n \geq 2)$ in order to show that the bounds are sharp

$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z) + \alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z) + \alpha zD_{p,q}f(z)} = \varphi\left(z^{n-1}\right),$$

$$z^2K_{\varphi n}(z)|_{z=0} = 0 = -z^2K'_{\varphi n}(z)|_{z=0} - 1$$

and F_γ and G_γ $(0 \le \gamma \le 1)$ by

$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z)+\alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z)+\alpha zD_{p,q}f(z)} = \varphi\left(\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2F_\gamma(z)|_{z=0} = 0 = -z^2F'_\gamma(z)|_{z=0} - 1$$

and

$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}f(z)+\alpha pqzD_{p,q}\left[zD_{p,q}f(z)\right]}{\left(1-\frac{\alpha}{pq}\right)f(z)+\alpha zD_{p,q}f(z)} = \varphi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2G_\gamma(z)|_{z=0} = 0 = -z^2G'_\gamma(z)|_{z=0} - 1.$$

Obviously, $K_{\varphi n}$, F_γ, $G_\gamma \in \mathcal{F}^*_{p,q}[\alpha;\varphi]$ and $K_\varphi = K_{\varphi 2.}$ When $\mu < \sigma_1$ or $\mu > \sigma_2$, consider the equality if and only if f is K_φ or one of its rotations. If $\sigma_1 < \mu < \sigma_2$, then consider the equality if f is $K_{\varphi 3}$ or one of its rotations. When $\mu = \sigma_1$, then the equality holds if and only if f is F_γ or one of its rotations. When $\mu = \sigma_2$, then the equality holds if and only if f is G_γ or one of its rotations. Thus, we complete the proof of Theorem 4. □

Remark 4.

(i) *For $p = 1, q \to 1^-$ in Theorem 4, we reobtain a result from [23];*
(ii) *Putting $p = 1, q \to 1^-$ and $\alpha = 0$ in Theorem 4, we obtain the result deduced by Ali and Ravichandran [30].*

4. Applications to Functions Defined by the p,q-Wright Type Hypergeometric Function

Considering the p,q-Wright type hypergeometric function given in definition 4, let us define

$$\mathcal{L}_c^{a,b,\tau}(z;p,q) = \frac{1}{z}\cdot {}_2R_1^{p,q}(a,b;c;\tau;z(1-pq)) \tag{36}$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty}\frac{\Gamma_{p,q}(c)(a)_{k+1}\Gamma_{p,q}(b+\tau(k+1))(1-pq)^{k+1}}{\Gamma_{p,q}(b)\Gamma_{p,q}(c+\tau(k+1))[k+1]_{p,q}!}z^k.$$

We define the linear operator denoted by $\mathcal{L}$ by using the Hadamard product, $\mathcal{L}:\Sigma\to\Sigma$, as follows:

$$\left(\mathcal{L}_{p,q}f\right)(z) = \mathcal{L}_c^{a,b,\tau}(z;p,q) * f(z) = \tag{37}$$

$$\frac{1}{z} + \sum_{k=0}^{\infty}\frac{\Gamma_{p,q}(c)(a)_{k+1}\Gamma_{p,q}(b+\tau(k+1))(1-pq)^{k+1}}{\Gamma_{p,q}(b)\Gamma_{p,q}(c+\tau(k+1))[k+1]_{p,q}!}a_kz^k.$$

Further, we define two classes by using the operator defined in (37).

With $0 < q < p \le 1$ and $\alpha \in \mathbb{C}\setminus(0,1]$, let us consider $\Sigma S^*_{b,\beta,p,q}[\gamma,\delta;\tau;\varphi]$ and $\mathcal{F}^*_{\beta,p,q}[\alpha,\gamma,\delta;\tau;\varphi]$ two subclasses of Σ consisting of functions f in the form (1) and satisfying the analytic criteria, respectively:

$$1-\frac{1}{b}\left(1+\frac{pqzD_{p,q}\left(\mathcal{L}_{p,q}f\right)(z)}{\left(\mathcal{L}_{p,q}f\right)(z)}\right) \prec \varphi(z)$$

and

$$-\frac{\left(1-\frac{\alpha}{pq}\right)pqzD_{p,q}\left(\mathcal{L}_{p,q}f\right)(z)+\alpha pqzD_{p,q}\left[zD_{p,q}\left(\mathcal{L}_{p,q}f\right)(z)\right]}{\left(1-\frac{\alpha}{pq}\right)\left(\mathcal{L}_{p,q}f\right)(z)+\alpha zD_{p,q}\left(\mathcal{L}_{p,q}f\right)(z)} \prec \varphi(z).$$

Using similar arguments to those in the proof of the above theorems, we obtain the following results related to the classes.

Theorem 5. *Consider $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If the function f in the form (1) belonging to the subclass $\Sigma S^*_{b,\beta,p,q}[\gamma,\delta;\tau;\varphi]$ and μ is a complex number, then*

$$\left|a_1 - \mu a_0^2\right| \le \frac{\left|\Gamma_{p,q}(\delta)\right|\left|\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\Gamma_{p,q}(\beta)\right|\left|\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\frac{|B_1|}{1+pq}\times \tag{38}$$

$$\max\left\{1, \left|\frac{B_2}{B_1} - [1-\mu s]B_1\right|\right\}, \text{ if } B_1 \neq 0$$

where

$$s = \frac{\left|\Gamma_{p,q}(\delta)\right|\left|\Gamma_{p,q}(\beta+2\tau)\right|^2\left|\Gamma_{p,q}(\delta+2\tau)\right|(1+pq)(\gamma+1)}{\gamma\left|\Gamma_{p,q}(\beta)\right|\left|\Gamma_{p,q}(\delta+\tau)\right|^2\left|\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}$$

and

$$|a_1| \le \frac{\left|\Gamma_{p,q}(\delta)\right|\left|\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\Gamma_{p,q}(\beta)\right|\left|\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\frac{|B_2|}{1+pq}, \text{ if } B_1 = 0.$$

The result is sharp.

Theorem 6. *Consider the function $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots (B_i > 0, i = 1,2)$. If the function f given by (1) belonging to the subclass $\Sigma S^*_{b,\beta,p,q}[\gamma,\delta;\tau;\varphi]$, $\mu \in \mathbb{R}$, $\Re\gamma > 0, \Re\delta > 0, \Re\beta > 0, \tau \in \Re_+$, then*

$$\left|a_1 - \mu a_0^2\right| \le \begin{cases} \frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\Gamma_{p,q}(\beta)\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\frac{1}{1+pq}\times \left\{-B_2 + [1-\mu s]B_1^2\right\} & if \quad \mu \le \sigma_1 \\ \frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\Gamma_{p,q}(\beta)\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\frac{B_1}{1+pq} & if \quad \sigma_1 \le \mu \le \sigma_2 \\ \frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\Gamma_{p,q}(\beta)\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\frac{1}{1+pq}\times \left\{B_2 - [1-\mu s]B_1^2\right\} & if \quad \mu \ge \sigma_2 \end{cases} \tag{39}$$

where

$$\begin{aligned} \sigma_1 &= \frac{-B_1 - B_2 + B_1^2}{B_1^2 s}; \quad \sigma_2 = \frac{B_1 - B_2 + B_1^2}{B_1^2 s} \\ s &= \frac{\left|\Gamma_{p,q}(\delta)\right|\left|\Gamma_{p,q}(\beta+2\tau)\right|^2\left|\Gamma_{p,q}(\delta+2\tau)\right|(1+pq)(\gamma+1)}{\gamma\left|\Gamma_{p,q}(\beta)\right|\left(\Gamma_{p,q}(\delta+\tau)\right)^2\left|\Gamma_{p,q}(\beta+2\tau)\right|(p+q)} \end{aligned}$$

The result is sharp. Further, let $\sigma_3 = \frac{-B_2+B_1^2}{B_1^2 s}$.

(i) *If $\sigma_1 \le \mu \le \sigma_3$, then*

$$\left|a_1 - \mu a_0^2\right| + \frac{\gamma\left|\Gamma_{p,q}(\beta)\right|\left|\Gamma_{p,q}(\beta+2\tau)\right|\left|\Gamma_{p,q}(\delta+\tau)\right|^2(p+q)}{(\gamma+1)\left|\Gamma_{p,q}(\delta)\right|\left|\Gamma_{p,q}(\delta+2\tau)\right|\left|\Gamma_{p,q}(\beta+\tau)\right|^2}\frac{1}{(1+pq)B_1^2} \tag{40}$$

$$\times\left\{(B_1+B_2) + [\mu s - 1]B_1^2\right\}|a_0|^2$$

$$\le \frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\Gamma_{p,q}(\beta)\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\frac{B_1}{1+pq};$$

(ii) *If $\sigma_3 \le \mu \le \sigma_2$, then*

$$\left|a_1 - \mu a_0^2\right| + \frac{\gamma\left|\Gamma_{p,q}(\beta)\right|\left|\Gamma_{p,q}(\beta+2\tau)\right|\left|\Gamma_{p,q}(\delta+\tau)\right|^2(p+q)}{(\gamma+1)\left|\Gamma_{p,q}(\delta)\right|\left|\Gamma_{p,q}(\delta+2\tau)\right|\left|\Gamma_{p,q}(\beta+\tau)\right|^2}\frac{1}{(1+pq)B_1^2} \tag{41}$$

$$\times\Big\{(B_1-B_2)+[1-\mu s]B_1^2\Big\}|a_0|^2$$
$$\leq \frac{\big|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\big|(p+q)}{\gamma(\gamma+1)\big|\Gamma_{p,q}(\beta)\Gamma_{p,q}(\delta+2\tau)\big|(1-pq)^2}\frac{B_1}{1+pq}.$$

Theorem 7. *Consider $\varphi(z)=1+B_1z+B_2z^2+\cdots$. If $f(z)$ given by the form (1) belongs to the class $\mathcal{F}^*_{\beta,p,q}[\alpha,\gamma,\delta;\tau;\varphi]$, $\Re\gamma>0$, $\Re\delta>0$, $\Re\beta>0$, $\tau\in\Re_+$, and μ is a complex number, then*

$$\left|a_1-\mu a_0^2\right|\leq\frac{1}{1+pq}\left|\frac{(pq-2\alpha)B_1}{(pq-\alpha+\alpha pq)}\right|\frac{\big|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\big|(p+q)}{\gamma(\gamma+1)\big|\Gamma_{p,q}(\beta)\Gamma_{p,q}(\delta+2\tau)\big|(1-pq)^2} \tag{42}$$
$$\times\max\left\{1,\left|\frac{B_2}{B_1}-[1-\mu w]B_1\right|\right\},\ \textit{if } B_1\neq 0$$

where

$$w=\frac{(pq-2\alpha)(pq-\alpha+\alpha pq)(1+pq)}{(pq-\alpha)^2}\times$$
$$\frac{(\gamma+1)\big|\Gamma_{p,q}(\delta)\big|\big|\Gamma_{p,q}(\beta+\tau)\big|^2\big|\Gamma_{p,q}(\delta+2\tau)\big|}{\gamma\big|\Gamma_{p,q}(\beta)\big|\big|\Gamma_{p,q}(\delta+\tau)\big|^2\big|\Gamma_{p,q}(\beta+2\tau)\big|(p+q)}$$

and if $B_1=0$

$$|a_1|\leq\frac{1}{1+pq}\left|\frac{(pq-2\alpha)B_2}{(pq-\alpha+\alpha pq)}\right|\frac{\big|\Gamma_{p,q}(\delta)\big|\big|\Gamma_{p,q}(\beta+2\tau)\big|(p+q)}{\gamma(\gamma+1)\big|\Gamma_{p,q}(\beta)\big|\big|\Gamma_{p,q}(\delta+2\tau)\big|(1-pq)^2}. \tag{43}$$

The result is sharp.

Theorem 8. *Let the function $\varphi(z)=1+B_1z+B_2z^2+\ldots(B_i>0,i=1,2)$, $\Re\gamma>0$, $\Re\delta>0$, $\Re\beta>0$, $\tau\in\Re_+$, and $0<2\alpha<\frac{pq}{1+pq}$. If $f(z)$ given by (1) belongs to the class $\mathcal{F}^*_{\beta,p,q}[\alpha,\gamma,\delta;\tau;\varphi]$, then*

$$\left|a_1-\mu a_0^2\right|\leq$$
$$\begin{cases}\frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\left|\Gamma_{p,q}(\beta)\right|\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\times & if \quad \mu\leq\sigma_1\\ \left\{-B_2+[1-\mu w]B_1^2\right\} & \\ \frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)B_1}{\gamma(\gamma+1)\left|\left|\Gamma_{p,q}(\beta)\right|\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2} & if \quad \sigma_1\leq\mu\leq\sigma_2\\ \frac{pq-2\alpha}{(1+pq)(pq-\alpha+\alpha pq)}\frac{\left|\Gamma_{p,q}(\delta)\Gamma_{p,q}(\beta+2\tau)\right|(p+q)}{\gamma(\gamma+1)\left|\left|\Gamma_{p,q}(\beta)\right|\Gamma_{p,q}(\delta+2\tau)\right|(1-pq)^2}\times & if \quad \mu\geq\sigma_2\\ \left\{B_2-[1-\mu w]B_1^2\right\} & \end{cases} \tag{44}$$

where

$$\sigma_1=\frac{-B_1-B_2+B_1^2}{B_1^2w};\quad \sigma_2=\frac{B_1-B_2+B_1^2}{B_1^2w}.$$

The result is sharp. Further, let $\sigma_3=\frac{-B_2+B_1^2}{B_1^2w}$.

(i) *If $\sigma_1\leq\mu\leq\sigma_3$, then*

$$\left|a_1-\mu a_0^2\right|+\frac{(pq-\alpha)^2}{(pq-2\alpha)(pq-\alpha+\alpha pq)}\cdot\theta \tag{45}$$
$$\times\Big\{(B_1+B_2)+[\mu w-1]B_1^2\Big\}|a_0|^2$$
$$\leq\frac{(pq-2\alpha)\big|\Gamma_{p,q}(\delta)\big|\big|\Gamma_{p,q}(\beta+2\tau)\big|(p+q)}{(pq-\alpha+\alpha pq)\big|\Gamma_{p,q}(\beta)\big|\big|\Gamma_{p,q}(\delta+2\tau)\big|\gamma(\gamma+1)(1-pq)^2}\frac{B_1}{(1+pq)}$$

where

$$\theta = \frac{\gamma|\Gamma_{p,q}(\beta)||\Gamma_{p,q}(\beta+2\tau)||\Gamma_{p,q}(\delta+\tau)|^2(p+q)}{(\gamma+1)|\Gamma_{p,q}(\delta)||\Gamma_{p,q}(\delta+2\tau)||\Gamma_{p,q}(\beta+\tau)|^2}\frac{1}{(1+pq)B_1^2}.$$

(ii) *If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_1 - \mu a_0^2\right| + \frac{(pq-\alpha)^2}{(pq-2\alpha)(pq-\alpha+\alpha pq)}\theta \tag{46}$$

$$\times\left\{(B_1 - B_2) + [1-\mu w]B_1^2\right\}|a_0|^2$$

$$\leq \frac{(pq-2\alpha)|\Gamma_{p,q}(\delta)||\Gamma_{p,q}(\beta+2\tau)|(p+q)}{(pq-\alpha+\alpha pq)|\Gamma_{p,q}(\beta)||\Gamma_{p,q}(\delta+2\tau)|\gamma(\gamma+1)(1-pq)^2}\frac{B_1}{(1+pq)}.$$

5. Conclusions

In the present paper, with the extended idea of a symmetric p, q-difference operator, we have introduced two subclasses of meromorphic functions. For certain values of the parameters, we reobtain some special classes studied earlier by various authors. The new results estimate the upper bound coefficients expressed in the Fekete–Szegö problem. We also deduce results that generalize and improve several previously known ones. In addition, we also provide certain applications to support our obtained results. Thus, an interesting aspect of the paper consists in defining the p, q-Wright type hypergeometric function, which is used as an application of the new form of this function in terms of (p, q)-calculus. We still intend to continue the work on the present issue. In this direction, for future research:

(i) we intend to investigate new symmetric p, q-differential operators;
(ii) similar results for other special classes of meromorphic functions are anticipated;
(iii) new aspects of the Fekete–Szegö problem for meromorphic functions are targeted;
(iv) the relationship between the results provided here and comparable outcomes in the field can be also considered.

The results of this study can be applied to post-quantum theory and symmetry. We hope that the new ideas and the new techniques given in the present paper will attract interested readers in the field of geometric function theory.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Abu-Risha, M.H.; Annaby, M.H.; Ismail, M.E.H.; Mansour, Z.S. Linear q-difference equations. *Z. Anal. Anwend.* **2007**, *26*, 481–494. [CrossRef]
2. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*; Cambridge University Press: Cambridge, UK, 1990.
3. Jackson, F.H. The application of basic numbers to Bessel's and Legendre's functions (Second paper). *Proc. Lond. Math. Soc.* **1904**, *3*, 1–23. [CrossRef]
4. Kac, V.G.; Cheung, P. *Quantum Calculus. Universitext*; Springer: New York, NY, USA, 2002.
5. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]
6. Kanas, S.; Raducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
7. Sivasubramanian, S.; Govindaraj, M. On a class of analytic functions related to conic domains involving q-calculus. *Anal. Math.* **2017**, *43*, 475–487.
8. Pommerenke, C. On meromorphic starlike functions. *Pac. J. Math.* **1963**, *13*, 221–235. [CrossRef]
9. Miller, J.E. Convex meromorphic mappings and related functions. *Proc. Am. Math. Soc.* **1970**, *25*, 220–228. [CrossRef]

10. Silverman, H.; Suchithra, K.; Stephen, B.A.; Gangadharan, A. Coefficient bounds for certain classes of meromorphic functions. *J. Inequal. Appl.* **2008**, *2008*, 9. [CrossRef]
11. Cotîrlă, L.I. New classes of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 10642–10651. [CrossRef]
12. Li, S.; Ma, L.; Tang, H. Meromorphic harmonic univalent functions related with generalized (p,q)-post quantum calculus operators. *AIMS Math.* **2020**, *6*, 223–234. [CrossRef]
13. Gupta, V: (p,q)-Baskakov-Kantorovich operators. *Appl. Math. Inf. Sci.* **2016**, *10*, 1551–1556. [CrossRef]
14. Milovanović, G.V.; Gupta, V.; Malik, N. (p,q)-beta functions and applications in approximation. *arXiv* **2016**, arXiv:1602.06307v2.
15. Chakrabarti, R.; Jagannathan, R. On the representations of GLp; q(2);GLp; q(1j1) and noncommutative spaces. *J. Phys. A Math. Gen.* **1991**, *24*, 5683–5703. [CrossRef]
16. Corcino, R.B. On p,q-binomial coefficients. *Electron. J. Comb. Number Theory* **2008**, *8*, A29.
17. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *arXiv* **2013**, arXiv:1309.3934.
18. Ma, W.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; International Press Inc.: Cambridge, MA, USA, 1994; pp. 157–169.
19. Bulboacă, T.; Aouf, M.K.; El-Ashwah, R.M. Convolution Properties for Subclasses of Meromorphic Univalent Functions of Complex Order. *Filomat* **2012**, *26*, 153–163. [CrossRef]
20. Aouf, M.K. Coefficient results for some classes of meromorphic functions. *J. Natl. Sci. Math.* **1987**, *27*, 81–97.
21. El-Ashwah, R.M.; Aouf, M.K. Hadamard product of certain meromorphic starlike and convex functions. *Comput. Math. Appl.* **2009**, *57*, 1102–1106. [CrossRef]
22. Ravichandran, V.; Kumar, S.S.; Subramanian, K.G. Convolution conditions for spirallikeness and convex spirallikenesss of certain p-valent meromorphic functions. *J. Ineq. Pure Appl. Math.* **2004**, *5*, 1–7.
23. Aouf, M.K.; El-Ashwah, R.M.; Zayed, H.M. Fekete–Szego inequalities for certain class of meromorphic functions. *J. Egypt. Math. Soc.* **2013**, *21*, 197–200. [CrossRef]
24. Kulkarni, S.R.; Joshi, S.S. On a subclass of meromorphic univalent functions with positive coefficients. *J. Indian Acad. Math.* **2002**, *24*, 197–205.
25. Karunakaran, V. On a class of meromorphic starlike functions in the unit disc. *Math Chron.* **1976**, *4*, 112–121.
26. Virchenko, N.O.; Kalla, S.L.; Al-Zamel, A.M.A.M. Some results on a generalized hypergeometric function. *Integral Transform. Spec. Funct.* **2001**, *12*, 89–100. [CrossRef]
27. Cotîrlă, L.I.; Sălăgean, G.S. Sufficient conditions for starlikeness of some integral operators. *Autom. Comput. Appl. Math.* **2012**, *21*, 3–9.
28. Nehari, Z. *Conformal Mapping*; McGraw-Hill: New York, NY, USA, 1952.
29. Tang, H.; Zayed, H.M.; Mostafa, A.O.; Aouf, M.K. Fekete-Szego Problems for Certain Classes of Meromorphic Functions Using q-Derivative Operator. *J. Math. Res. Appl.* **2018**, *38*, 236–246.
30. Ali, R.M.; Ravichandran, V. Classes of meromorphic α-convex functions. *Taiwan. J. Math.* **2010**, *14*, 1479–1490. [CrossRef]