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An Infinite Family of Compact, Complete, and Locally Affine *k*-Symplectic Manifolds of Dimension Three

Fanich El Mokhtar * and Essabab Said

Laboratory of Analysis, Modeling and Simulation, Department of Mathematics and Computer Science, Ben M'sick's Faculty of Sciences, Hassan II University of Casablanca, Casablanca 20 000, Morocco; Said.ESSABAB@emines.um6p.ma
* Correspondence: elmokhtar.fanich-etu@etu.univh2c.ma

Abstract: We study the complete, compact, locally affine manifolds equipped with a k-symplectic structure, which are the quotients of $\mathbb{R}^{n(k+1)}$ by a subgroup Γ of the affine group $A(n(k+1))$ of $\mathbb{R}^{n(k+1)}$ acting freely and properly discontinuously on $\mathbb{R}^{n(k+1)}$ and leaving invariant the k-symplectic structure, then we construct and give some examples and properties of compact, complete, locally affine two-symplectic manifolds of dimension three.

Keywords: k-symplectic structure; locally affine manifolds; foliations; Lagrangian submanifolds

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1. Introduction

The notion of a k-symplectic structure [1–6] is a natural generalization of the classical notion of a polarized symplectic structure [7]. This last notion plays an important role in the geometric quantization of Kostant–Souriau [8,9] . The study of a k-symplectic structure was motivated by the implementation of a formalism of Nambu's mechanics [10] by analogy with the symplectic geometry. The canonical model of a k-symplectic manifold is the bundle of k^1-covelocities, that is $(T^1_k)^*M$, while the canonical model of a symplectic manifold is the cotangent bundle T^*M. However, the interest in the k-symplectic geometry has increased especially in recent years due to the awareness of its applications in field theories [11,12]. In fact, the k-symplectic formalism is a geometric approach of the mechanics of Y. Nambu, having the same specific features of symplectic geometry as a formalism of the mechanics Hamiltonian [13–15].

The main goal of this work is to obtain some examples of compact manifolds endowed with a k-symplectic structure. This leads to the study of the locally affine manifolds [16–21], which represent the simplest differentiable manifolds because of the changes of coordinates in whose atlas are affine mappings. Similar approaches have been studied by: M. Goze, Y. Haraguchi on the r-systems of contact [22], and T. Sari on the locally affine contact manifolds [20].

We know [23,24] that the complete, compact, locally affine manifolds of dimension n are the quotient $\mathbb{R}^{n(k+1)}/\Gamma$, where Γ is a subgroup of the affine group $A(n)$ of $\mathbb{R}^n$, acting freely and properly discontinuously on $\mathbb{R}^n$ and $\Gamma = \pi_1(M)$ [25].

The affine manifolds have been studied by several authors. See, for example, L. Auslander, D. Fried, W. Goldman, P. Benzecri, Y. Carrière, T. Sari, etc, while our purpose is to give new examples of compact and complete locally affine manifolds equipped with an additional structure, which is the k-symplectic structure.

These manifolds are the quotients $\mathbb{R}^{n(k+1)}/\Gamma$, of $\mathbb{R}^{n(k+1)}$ by a subgroup Γ of the affine group $A(n(k+1))$ of $\mathbb{R}^{n(k+1)}$, acting freely and properly discontinuously on $\mathbb{R}^{n(k+1)}$ and leaving invariant the k-symplectic structure.

By constructing subgroups Γ of the group $A(3)$ acting freely and properly discontinuously on $\mathbb{R}^3$ [26,27] and leaving invariant the canonical 2-symplectic structure of $\mathbb{R}^3$, we obtain an infinite family of 2-symplectic manifolds of dimension 3 not isomorphic to the torus $\mathbb{T}^3$.

2. Preliminaries

2.1. k-Symplectic Manifolds

Let M be a smooth manifold of dimension $n(k+1)$ equipped with a foliation $\mathfrak{F}$ of codimension n, and let $\omega^1, \ldots, \omega^k$ be k differential two-forms on M.

The sub-bundle of TM defined by the tangent vectors of leaves of the foliation $\mathfrak{F}$ is denoted by E, the set of all cross-sections of the M-bundle

$TM \longrightarrow M$ (resp., $E \longrightarrow M$) is denoted by $\mathfrak{X}(M)$ (resp., $\Gamma(E)$), and the set of all differential p-forms on M is denoted by $\Lambda^p(M)$.

We denote by $C_x(\omega^1), \ldots, C_x\left(\omega^k\right)$ the characteristic spaces of the two-forms $\omega^1, \ldots, \omega^k$ at x where $x \in M$ [28]. Recall that:

$$C_x(\omega^p) = \{X_x \in T_x M \mid i(X_x)\omega^p = 0 \text{ and } i(X_x)d\omega^p = 0\}$$

where $i(X_x)\omega^p$ denote the interior product of the vector X_x by the two-form ω^p. Therefore,

$$C_x(\omega^p) = \{X_x \in T_x M \mid i(X_x)\omega^p = 0\}$$

Definition 1. *We say that $\left(\omega^1, \ldots, \omega^k; E\right)$ is a k-symplectic structure on M, if the following conditions are satisfied [28]:*

1. *The two-forms $\omega^1, \ldots, \omega^k$ are closed;*
2. *The system $\left\{\omega^1, \ldots, \omega^k\right\}$ is nondegenerate, that is,*

$$C_x\left(\omega^1\right) \cap \cdots \cap C_x\left(\omega^k\right) = \{0\}$$

for every $x \in M$;

3. *The system $\left\{\omega^1, \ldots, \omega^k\right\}$ is vanishing on the tangent vectors to the foliation $\mathfrak{F}$, that is,*

$$\omega^p(X, Y) = 0 \text{ for all } X,\ Y \in \Gamma(E) \text{ and } p = 1, \ldots, k.$$

Example 1. ***Canonical** $k-$**symplectic structure on** $\mathbb{R}^{n(k+1)}$ [28]:*
Consider the real space $\mathbb{R}^{n(k+1)}$ endowed with its Cartesian coordinates $\left(x^{pi}, y^i\right)_{1\le p\le k, 1\le i\le n}$. Let E be the sub-bundle of $T\mathbb{R}^{n(k+1)}$ defined by the equations:

$$dy^1 = 0, \ldots, dy^n = 0,$$

and let $\omega^p (p = 1, \ldots, k)$ be the differential two-forms on M given by:

$$\omega^p = \sum_{i=1}^{n} dx^{pi} \wedge dy^i.$$

$\left(\omega^1, \ldots, \omega^k; E\right)$ defines a k-symplectic structure on $\mathbb{R}^{n(k+1)}$ called the canonical k-symplectic structure of $\mathbb{R}^{n(k+1)}$. This structure induces a natural k-symplectic structure on the torus $\mathbb{T}^{n(k+1)}$.

2.2. k-Symplectic Affine Manifolds

Let M be a k-symplectic manifold of dimension $n(k+1)$.

Definition 2 ([28])**.** *We say that M is an affine k-symplectic manifold if the Darboux atlas $\mathfrak{A}$ confers upon M a structure of a locally affine manifold.*

Let $Gp(k, n; \mathbb{R})$ be the group of all affine transformations of $\mathbb{R}^{n(k+1)}$ preserving the canonical k-symplectic structure of $\mathbb{R}^{n(k+1)}$. The group $Gp(k, n; \mathbb{R})$ is the set of all affine transformations:

$$X \longmapsto AX + B$$

of $\mathbb{R}^{n(k+1)}$ such that A belongs to the k-symplectic group $Sp(k, n; \mathbb{R})$.

Proposition 1 ([28])**.** *Let M be a complete connected affine k-symplectic manifold of dimension $n(k+1)$. Then, M is just a quotient $\mathbb{R}^{n(k+1)}/\Gamma$ and with a fundamental group Γ:*

$$M = \mathbb{R}^{n(k+1)}/\Gamma, \quad \pi_1(M) = \Gamma,$$

where Γ is a subgroup of $Gp(k, n; \mathbb{R})$ acting freely and properly discontinuously on $\mathbb{R}^{n(k+1)}$.

2.3. Case Where $\mathfrak{F}$ Is of Codimension One

Let $Hp(k, n; \mathbb{R})$ [28] be the group of all matrices:

$$\begin{pmatrix} I_n & & & A_1 & C_1 \\ & \ddots & & \vdots & \vdots \\ 0 & & I_n & A_k & C_k \\ 0 & \cdots & 0 & I_n & B \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

where I_n is the unit matrix of rank, n, $A_1, \cdots, A_k$ are $n \times n$ real symmetric matrices, and $C_1, \ldots, C_k, B$ are column vectors of length n. We denote by (A, B, C) the matrices of the previous form where $A = (A_1, \ldots, A_k)$, $C = (C_1, \ldots, C_k)$.

Proposition 2 ([28])**.** *If M is a complete connected affine k-symplectic manifold of dimension $k+1$, then M is a quotient $\mathbb{R}^{k+1}/\Gamma$ with a fundamental group Γ:*

$$M = \mathbb{R}^{k+1}/\Gamma, \quad \pi_1(M) = \Gamma,$$

where Γ is a subgroup of $Hp(k, 1; \mathbb{R})$ acting freely and properly discontinuously on $\mathbb{R}^{k+1}$.

3. Main Results

Based on the propositions above, we construct an infinite family of k-symplectic manifolds of dimension three by constructing a family of subgroups of $Hp(2, 1; \mathbb{Z})$, which act freely and properly discontinuously on $\mathbb{R}^3$.

3.1. Subgroups of $Hp(2, 1; \mathbb{Z})$ Acting without a Fixed Point

The group $Hp(2, 1; \mathbb{Z})$ is formed by the matrices of the form:

$$\begin{pmatrix} 1 & 0 & a_1 & c_1 \\ 0 & 1 & a_2 & c_2 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where $a_1, a_2, b, c_1, c_2 \in \mathbb{Z}$.

We denote by (A, b, C) the matrices of Type (1) where $A = (a_1;\ a_2), C = (c_1;\ c_2) \in \mathbb{Z}^2$, and $b \in \mathbb{Z}$.

For all $g = (A, b, C), g' = (A', b', C') \in Hp(2, 1; \mathbb{Z})$, we have:

$$gg' = (A + A', b + b', b'A + C + C')$$

$$g^k = (kA, kb, P_k bA + kC)$$

where $k \in \mathbb{Z}$ and $P_k = \frac{k(k-1)}{2}$.

$$g^{-1} = (-A, -b, bA - C)$$

$$\left[g, g'\right] = (0, 0, b'A - bA').$$

Let Γ be a subgroup of $Hp(2,1;\mathbb{Z})$ and Γ_0 a subgroup of Γ defined by: $\Gamma_0 = \{(A, b, C) \in \Gamma \mid b = 0\}$.

Γ_0 is a free Abelian subgroup of the additive group formed by the triplets: $(A,0,C)$, where A,C $\in\mathbb{Z}^2$.

Therefore, the rank of Γ_0 is less than or equal to four.

3.1.1. Case Where $\Gamma_0 = \{0\}$

It results from the definition of Γ_0 that:

For all $g = (A, b, C) \in \Gamma$, we have $b \neq 0$ or $A = C = 0$.

Let $g_1 = (A^1, b_1, C^1)$, $g_2 = (A^2, b_2, C^2) \in \Gamma$ with $b_1, b_2 \in\mathbb{Z}^*$.

The component b of the element $g_1^{b_2} g_2^{-b_1}$ is zero.

Hence, $g_1^{b_2} = g_2^{b_1}$.

Therefore, $g_2^{b_1}$ is in the subgroup $< g_1 >$ of Γ generated by g_1:

$$g_2^{b_1} \in < g_1 >.$$

Proposition 3. Γ *is a monogenous subgroup.*

Proof. Suppose that Γ is not reduced to $\{0\}$; we consider $g = (A,b,C)$ an element of Γ with $b \neq 0$.

Let E be the set:

$$E = \{b \in\mathbb{N}^* \mid \exists\, g \in \Gamma \text{ where } g = (A,b,C)\}.$$

E is a nonempty subset of $\mathbb{N}^*$, so it admits a least element denoted b_1. We consider an element $g_1 = (A^1, b_1, C^1)$ of Γ corresponding to b_1, and we prove that Γ is generated by g_1:

$$\Gamma = \langle g_1 \rangle.$$

Let $g_2 = (A^2, b_2, C^2)$ be an element of Γ such that $b_2 \neq 0$; we prove first that b_1 divides b_2. We can suppose that $b_2 > 0$. We suppose the opposite (b_1 does not divide b_2).

For $b = b_1 \wedge b_2$, we have: $0 < b < b_1$, and by the Bezout identity: there exist $u, v \in\mathbb{Z}$ such that: $ub_1 + vb_2 = b$.

Therefore:

$$g_1^u g_2^v = (A'', ub_1 + vb_2 = b, C'') \in \Gamma$$

and this contradicts that b_1 is the least element in E.

Hence, b_1 divides $b_{2,}$.

Consequently: there exists $m \in\mathbb{Z}$ such that $b_2 = mb_1$.

The component b of the element $g = g_2 g_1^{-m}$ is zero.

Consequently: $g = i_d \Longrightarrow g_2 = g_1^m$, which proves that $\Gamma =< g_1 >$. □

3.1.2. Case Where the Rank $(\Gamma_0) = 1$

We suppose that Γ_0 is generated by an element $g_0 = (A^0, 0, C^0)$ supposed without a fixed point and different from (0;0;0).

Remark 1. *(1) The following properties are equivalent:*
(i) Γ_0 *acts without a fixed point on* $\mathbb{R}^3$*;*
(ii) $C^0 \notin \mathbb{R}.A^0$*;*
(iii) If $A^1 \neq 0 \Rightarrow \det(C^1, A^1) \neq 0$.
(2) For all $(A, 0, C) \in \Gamma_0$*, we have:* $C = 0 \Longrightarrow A = 0$*;*
(3) For all $g = (A, b, C) \in \Gamma$*, we have:* $(b = 0 \Longrightarrow \exists n \in \mathbb{Z}\text{: } g = g_0^n)$.

Proposition 4. *We have that* $\Gamma = \Gamma_0$ *or* Γ *admits two generators* $\Gamma =< g_0, g_1 >$*, where* $g_1 = (A^1, b_1, C^1)$ *with* $b_1 \neq 0$.

Proof. If Γ strictly contains Γ_0, then it contains at least an element $g = (A, b, C)$ with $b \neq 0$.
Let E be the set:

$$E = \{b \in \mathbb{N}^* \mid \exists g \in \Gamma \ : \ g = (A,b,C)\}.$$

E is a nonempty subset of $\mathbb{N}^*$, so it admits a least element denoted b_1. We consider an element $g_1 = (A^1, b_1, C^1)$ of Γ.
We have: b_1 divides b; so there is an $m \in \mathbb{Z}$ such that: $b = mb_1$.
The component b of the element gg_1^{-m} is zero.
Then, there exists $n \in \mathbb{Z}$:$gg_1^{-m} = g_0^n$, hence $g = g_0^n g_1^m$.
Consequently, $\Gamma =< g, g_1 >$. □

Proposition 5. Γ *acts without a fixed point on* $\mathbb{R}^3$.

Proof. Let $g = (A, b, C) \in \Gamma$; the relative integers A, b, C are respectively written in the forms:

$A = (\sum\limits_{i=1}^{n} \lambda_i)A^0 + (\sum\limits_{i=1}^{n} \mu_i)A^1$.

$b = (\sum\limits_{i=1}^{n} \mu_i)b_1$.

$C = (\sum\limits_{k=1}^{n} (\sum\limits_{j=1}^{k} \lambda_j)\mu_k)b_1 A^0 + P_{\sum_{i=1}^{n}\mu_i} b_1 A^1 + (\sum\limits_{i=1}^{n} \lambda_i)C^0 + (\sum\limits_{i=1}^{n} \mu_i)C^1$ where $n \in \mathbb{N}, \lambda_i, \mu_j \in \mathbb{Z}$ and $i = 0, 1, ..., n$.

If g admits a fixed point, then: $\sum\limits_{i=1}^{n} \mu_i = 0$, and there exists $q \in \mathbb{Q}$ such that:

$(\sum\limits_{k=1}^{n} (\sum\limits_{j=1}^{k} \lambda_j)\mu_k)b_1 A^0 + (\sum\limits_{i=1}^{n} \lambda_i)C^0 = q(\sum\limits_{i=1}^{n} \lambda_i)A^0$.

If $A^0 = 0$, then $(\sum\limits_{i=1}^{n} \lambda_i)C^0 = 0$.

$C^0 \neq 0$, so: $\sum\limits_{i=1}^{n} \lambda_i = 0$, and consequently, $g = (0,0,0)$.

In this particular case, we obtain an Abelian group isomorphic to $\mathbb{Z}^2$.

If $A^0 \neq 0$, then $(\sum\limits_{i=1}^{n} \lambda_i)\det(C^0, A^0) = 0$.

$\det(C^0, A^0) \neq 0$, so $\sum\limits_{i=1}^{n} \lambda_i = 0$.

Consequently, $g = (0,0,0)$. □

3.1.3. Case Where Rank $(\Gamma_0) = 2$

We suppose that Γ_0 is generated by two elements: $g_0 = (A^0, 0, C^0)$ and $g_1 = (A^1, 0, C^1)$ both not equal to $(0, 0, 0)$.
If Γ acts without a fixed point on $\mathbb{R}^3$, then:
(i) $A^0 \neq 0 \Longrightarrow \det(A^0, C^0) \neq 0$;
(ii) $A^1 \neq 0 \Longrightarrow \det(A^1, C^1) \neq 0$;
(iii) $\det(C^1, C^0) \neq 0$.
We prove Assertion (iii).

If $\det(C^1, C^0) = 0$, then there exists $n,m \in \mathbb{Z}^*$ such that: $nC^0 + mC^1 = 0$.
We can suppose that n and m are coprime.
The element $g_0^n g_1^{-m} = (nA^0 - mA^1, 0, 0)$ belongs to Γ_0, which has a fixed point.
Hence, $g_0^n g_1^{-m} = id$.
Let $u, v \in \mathbb{Z}$, which satisfy: $un + vm = 1$, and $g_2 = g_0^v g_1^u \in \Gamma_0$.
We have:

$$g_2^m = g_0^{vm} g_1^{um} = g_0^{vm} g_0^{un} = g_0^{un+vm} = g_0$$

and:

$$g_2^n = g_0^{vn} g_1^{un} = g_1^{vm} g_1^{un} = g_1^{un+vm} = g_1$$

which shows that Γ_0 is monogenous and generated by the element g_2.
Then, we come back to the case 3.1.2.

Proposition 6. *We either have $\Gamma = \Gamma_0$ or Γ is a free group with three generators g_0, g_1, and g_2 with $g_2 = (A^2, b_2, C^2)$ and $b_2 \neq 0$.*

Proof. If Γ strictly contains Γ_0, then Γ contains an element $g_2 = (A^2, b_2, C^2)$, where b_2 is the least element of the subset: $E = \{b \in \mathbb{N}^* \mid \exists\, g \in \Gamma \quad \text{where } g = (A, b, C)\}$.
Let $g = (A, b, C) \in \Gamma$ where $b \neq 0$.
As in 3.1.2., we prove that b_2 divides b.
We denote by m an element of $\mathbb{Z}$ such that: $b = mb_1$.
The element gg_2^{-m} of Γ is written in the form $(A'', 0, C'')$.
Consequently, $gg_2^{-m} \in \Gamma_0$, which proves that g belongs to the free group $< g_0, g_1, g_2 >$ generated by g_0, g_1, g_2.
Hence, $\Gamma = < g, g_1, g_2 >$. □

We are now looking for a characterization of the group Γ, which acts without fixed points on $\mathbb{R}^3$.
We denote by $D_1 = \det(A^1, A^0)$, $D_2 = \det(A^0, C^0)$ and, $D_3 = \det(C^1, A^0)$.
(1) **Case where $D_2 = 0$:**
If $A^0 = A^1 = 0$, then Γ contains only the translations: it is isomorphic to $\mathbb{Z}^2$ *or* $\mathbb{Z}^3$, for consequence Γ acts on $\mathbb{R}^3$ without a fixed point.
The case where $A^0 = 0$ and $A^1 \neq 0$ will be discussed in the case where $D_2 \neq 0$;
(2) **Case where $D_2 \neq 0$ and $D_1 = 0$:**
If $D_1 = 0$, then there exists $m \in \mathbb{Q}$ such that $A^1 = mA^0$;
(i) We suppose that $m = 0$. In this case, an element $g = (A, b, C)$ of Γ is written in the forms:

$$A = (\sum_{i=1}^{n} \lambda_i)A^0 + (\sum_{i=1}^{n} \theta_i)A^2.$$

$$b = (\sum_{i=1}^{n} \theta_i)b_2.$$

$$C = (\sum_{k=1}^{n} (\sum_{j=1}^{k} \lambda_j)\theta_k)b_2 A^0 + P_{\sum_{i=1}^{n}\theta_i} b_2 A^2 + (\sum_{i=1}^{n} \lambda_i)C^0 + (\sum_{i=1}^{n} \mu_i)C^1 + (\sum_{i=1}^{n} \theta_i)C^2$$

where $n \in \mathbb{N}, \lambda_i, \mu_i, \theta_i \in \mathbb{Z}$ and $i = 0, 1, ..., n$;
(i.1) $D_3 = 0$:
In this case, we have: $C^1 = \lambda A^0$, $\lambda \in \mathbb{Q}$.
If g has a fixed point, then:
$(*) \quad (\sum_{i=1}^{n} \theta_i) = 0,$

$(**)\ \exists\, q \in \mathbb{Q}$ such that $C = qA$.
The condition $(**)$ is written:

$$\sum_{k=1}^{n} ((\sum_{j=1}^{k} \lambda_j) b_2\theta_k + \lambda\mu_k)A_1^0 + (\sum_{i=1}^{n} \lambda_i)C_1^0 = q(\sum_{i=1}^{n} \lambda_i)A_1^0.$$

$$\sum_{k=1}^{n} ((\sum_{j=1}^{k} \lambda_j) b_2\theta_k + \lambda\mu_k)A_2^0 + (\sum_{i=1}^{n} \lambda_i)C_2^0 = q(\sum_{i=1}^{n} \lambda_i)A_2^0$$

and then, $(\sum_{i=1}^{n} \lambda_i)D_2 = 0$.
Hence, $(\sum_{i=1}^{n} \lambda_i) = 0$
We deduce that: if $D_3 = 0$ and $m = 0$, then Γ acts without a fixed point on $\mathbb{R}^3$;
(i.2) $D_3 \neq 0$:
We will see in (the case where $m \neq 0$) that Γ acts without a fixed point if and only if $D_3 = mD_2$.
We can also see (1) of remak12;
(ii) We suppose that $m \neq 0$. In this case, we have:

Proposition 7. *The group Γ acts without a fixed point on $\mathbb{R}^3$ if and only if $D_3 = mD_2$.*

Proof. If $D_3 = mD_2$, then Γ is generated by:
$g_0 = (A^0, 0, C^0)$, $g_1 = (mA^0, 0, C^1)$ and $g_2 = (A^2, b_2, C^2)$ with $b_2 \neq 0$.
Every element $g = (A, b, C)$ of Γ is written in the form:

$$A = ((\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i))A^0 + (\sum_{i=1}^{n} \theta_i)A^2.$$

$$b = (\sum_{i=1}^{n} \theta_i)b_2.$$

$$C = (\sum_{k=1}^{n} (\sum_{j=1}^{k} (\lambda_j + m\mu_j)\theta_k)b_2A^0 + P_{\sum_{i=1}^{n}\theta_i} b_2A^2 + (\sum_{i=1}^{n} \lambda_i)C^0 + (\sum_{i=1}^{n} \mu_i)C^1 + (\sum_{i=1}^{n} \theta_i)C^2$$

where $n \in \mathbb{N}, \lambda_i, \mu_i, \theta_i \in \mathbb{Z}$ and $i = 0, 1, ..., n$.
We suppose that $g \neq (0, 0, 0)$.
If g admits a fixed point, then:
$(\sum_{i=1}^{n} \theta_i) = 0$ and $\sum_{j=1}^{k} (\lambda_j + m\mu_j) \neq 0$,
and there exists $q \in \mathbb{Q}^*$ such that: $C = qA$.
It follows that: $(\sum_{i=1}^{n} \lambda_i)D_2 + (\sum_{i=1}^{n} \mu_i)D_3 = (\sum_{j=1}^{k} (\lambda_j + m\mu_j))D_2 = 0$.
$D_2 \neq 0$, then: $(\sum_{j=1}^{k} (\lambda_j + m\mu_j)) = 0$.
Hence, $g = (0, 0, 0)$, which is absurd.
As a consequence, Γ acts without being fixed on $\mathbb{R}^3$.
Conversely, we suppose that $D_3 \neq mD_2$.

An element g of Γ not equal to $(0,0,0)$ has a fixed point if and only if there exist: $\lambda_1,\ldots,\lambda_n,\mu_1,\ldots\mu_n \in \mathbb{Z}$, such that the system:

$$\begin{cases} (\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i) \neq 0 \\ (\sum_{i=1}^{n} \lambda_i)D_2 + (\sum_{i=1}^{n} \mu_i)D_3 = 0 \end{cases} \quad (2)$$

is satisfied.

There are many integers $\lambda_1,\ldots,\lambda_n,\mu_1,\ldots\mu_n \in \mathbb{Z}$ such that:
$\delta,(\sum_{i=1}^{n} \mu_i) \in \mathbb{Z}^*$, $(\sum_{i=1}^{n} \mu_i) = \delta D$ and $(\sum_{i=1}^{n} \lambda_i) = -\delta D_3$, and in these conditions, we have:
$(\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i) = \delta(mD_2 - D_3) \neq 0$ and $(\sum_{i=1}^{n} \lambda_i)D_2 + (\sum_{i=1}^{n} \mu_i)D_3 = 0.$
Consequently, Γ has a fixed point on $\mathbb{R}^3$. □

Remark 2. *(1) In the case where* $m = 0$ and $D_3 \neq 0$, *the system* (2) *becomes:*

$$\begin{cases} (\sum_{i=1}^{n} \lambda_i) \neq 0 \\ (\sum_{i=1}^{n} \lambda_i)D_2 + (\sum_{i=1}^{n} \mu_i)D_3 = 0 \end{cases} \quad (3)$$

There exists $\lambda_1\ldots,\lambda_n,\theta_1,\ldots\theta_n \in \mathbb{Z}$ *satisfying (3); hence, (i.2)* $D_3 \neq 0$;
(2) Γ_0 *is a normal subgroup in* Γ, *isomorphic to* $\mathbb{Z}\times\mathbb{Z}$;
(3) For $D_3 = mD_2$, *we have* $A^0 = (C^1 - mC^0)\frac{D_2}{\det(C^0,C^1)}$.

(3) **Case where** $D_2 \neq 0$ **and** $D_1 \neq 0$**:**
In this case, the group Γ admits fixed points.
Let $\lambda_0,\lambda_1,\mu_1,\mu_2,\theta_1,\theta_2,\delta \in \mathbb{Z}$ such that:

$$\theta_1 + \theta_2 = 1,\ \delta D_2 \in \mathbb{Z}^*, \mu_1 = -\mu_0 = -\delta D_2 \text{ and } \lambda_0 + \lambda_1 = -\delta b_2 D_1$$

The element: $g = g_0^{\lambda_0} g_1^{\mu_0} g_2^{\theta_0} g_0^{\lambda_1} g_1^{\mu_1} g_2^{\theta_1}$ is different from $(0,0,0)$ and admits a fixed point.

3.1.4. Case Where the Rank of Γ_0 Is at Least 3

In these conditions, Γ_0 contains three elements $g_0 = (A^0,0,C^0)$,$g_1 = (mA^0,0,C^1)$, and $g_2 = (A^2,0,C^2)$, which are independent of the $\mathbb{Z}$-module Γ_0.
Otherwise, we can find three nonvanishing integers λ,μ,θ, such that: $\lambda C^0 + \mu C^1 + \theta C^2 = 0$ and $\lambda A^0 + \mu A^1 + \theta A^2 \neq 0$.
g_0, g_1, g_2 are free in the $\mathbb{Z}$-module Γ_0.
The element $g_0^{\lambda} g_1^{\mu} g_2^{\theta} = (\lambda A^0 + \mu A^1 + \theta A^2, 0, 0)$ has a fixed point.
The study above is summed up by the following proposition:

Proposition 8. *The subgroups* Γ *of* $Hp(2,1;\mathbb{Z})$ *of the type:*
$\Gamma =< (A^0,0,C^0), (mA^0,0,C^1), (A^2,b_2,C^2) >$ *with* $A^0,C^0,C^1,A^2,C^2 \in \mathbb{Z}^2$ *and* $b_2 \in \mathbb{N}^*$ *satisfying:*
$det(C^0,C^1) \neq 0$, $D_3 = mD_2$
and their subgroups are all subgroups of $Hp(2,1;\mathbb{Z})$ *acting without a fixed point on* $\mathbb{R}^3$.
(D_2 *and* D_3 *denote respectively the determinants* $det(A^0,C^0)$ *and* $\det(A^0,C^1)$.*)*

3.2. An Infinite Family of Compact Complete and Locally Affine k-Symplectic Manifolds of Dimension 3

Proposition 9. *The subgroups* Γ *of* $Hp(2,1;\mathbb{Z})$ *of the type:*

$\Gamma =< (A^0,0,C^0), (mA^0,0,C^1), (A^2,b_2,C^2) >$ *with* $A^0,C^0,C^1,A^2,C^2 \in \mathbb{Z}^2$ *and* $b_2 \in \mathbb{N}^*$, *which satisfy:* $det(C^0,C^1) \neq 0$, $D_3 = mD_2$, *and their subgroups, act freely and properly discontinuously without a fixed point on* $\mathbb{R}^3$.

Proof. We prove that Γ acts properly discontinuously on $\mathbb{R}^3$.

If $A^0 = 0$, the group Γ contains only translations; as the vectors $(0,C^0)$,$(0,C^1)$ and (b_2,C^2) are independent, the quotient space is isomorphic to the torus $\mathbb{T}^3$.

We suppose now that $A^0 \neq 0$:

(a) Let us first prove that any point of $\mathbb{R}^3$ admits an open neighborhood U such that the set $\{g \in \Gamma \,|g(U) \cap U \neq \phi\}$ is finite.

Let $a_0 \in \mathbb{Z}$ and $b_0 \in \mathbb{N}^*$ such that: $m = \frac{a_0}{b_0}$ and M_0 a point of $\mathbb{R}^3$ of coordinates (x_0,y_0,z_0).

We pose $\varepsilon = Min(\frac{1}{4b_2b_0}, \frac{|D_2|}{4b_0(|A_1^0|+|A_2^0|)})$ and U_0 the open ball of center M_0 and radius ε for the norm: $|\,|(x,y,z)|| = sup(|\,x\,|,|\,y\,|,|\,z\,|)$.

Let g be an element of Γ satisfying $g(U) \cap U \neq \phi$; there exists a point $M = (x,y,z) \in \mathbb{R}^3$ such that M and $g(M)$ are in U.

For $(x',y',z') = g(M)$, we must have:

$(\star)$ $|\,x'-x\,| < 2\varepsilon$, $|\,y'-y\,| < 2\varepsilon$ and $|\,z'-z\,| < 2\varepsilon$.

Recall that the element $g = (A,b,C)$ of Γ is written in the form (A,b,C) with:

$$A = ((\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i))A^0 + (\sum_{i=1}^{n} \theta_i)A^2,$$

$$b = (\sum_{i=1}^{n} \theta_i)b_2,$$

$$C = (\sum_{k=1}^{n} (\sum_{j=1}^{k} (\lambda_j + m\mu_j)\theta_k)b_2A^0 + P_{\sum_{i=1}^{n}\theta_i} b_2A^2 + (\sum_{i=1}^{n} \lambda_i)C^0 + (\sum_{i=1}^{n} \mu_i)C^1 + (\sum_{i=1}^{n} \theta_i)C^2$$

where $n \in \mathbb{N}, \lambda_i, \mu_i, \theta_i \in \mathbb{Z}$, and $i = 0,1,...,n$.

By hypothesis, we have:

$|\,z'-z\,| = |(\sum_{i=1}^{n} \theta_i)b_2\,| < 2\varepsilon < 1$ with $(\sum_{i=1}^{n} \theta_i)b_2 \in \mathbb{Z}$ with $b_2 \neq 0$, then $|\,(\sum_{i=1}^{n} \theta_i)\,| = 0$.

The components $x'-x$ and $y'-y$ become:

$x'-x = ((\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i))\,zS_1^0 + (\sum_{k=1}^{n} (\sum_{j=1}^{k} (\lambda_j + m\mu_j)\theta_k)b_2A_1^0 + (\sum_{i=1}^{n} \lambda_i)C_1^0 + (\sum_{i=1}^{n} \mu_i)C_1^1$

$y'-y = ((\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i))\,zS_2^0 + (\sum_{k=1}^{n} (\sum_{j=1}^{k} (\lambda_j + m\mu_j)\theta_k)b_2A_2^0 + (\sum_{i=1}^{n} \lambda_i)C_2^0 + (\sum_{i=1}^{n} \mu_i)C_2^1.$

The inequalities: $|\,x'-x\,| < 2\varepsilon$, $|\,y'-y\,| < 2\varepsilon$ imply that:

$|\,A_2^0(x'-x) - A_1^0(y'-y)\,| \leq 2\varepsilon(|\,A_1^0\,| + |\,A_2^0\,|)$.

Then,

$$|\,(\sum_{i=1}^{n} \lambda_i)det(A^0,C^0) + (\sum_{i=1}^{n} \mu_i)det(A^0,C^1)\,| = |\,(\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i)\,|\,|\,D_2\,| \leq 2\varepsilon(|\,A_1^0\,| + |\,A_2^0\,|)$$

Consequently, $|\,(\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i)\,|\,b_0$ is an integer satisfying:

$|\,(\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i)\,|\,b_0 \leq 2\varepsilon(|\,A_1^0\,| + |\,A_1^0\,|)b_0/\,|\,D_2\,| < \frac{1}{2}$.

Hence, $|\,(\sum_{i=1}^{n} \lambda_i) + m(\sum_{i=1}^{n} \mu_i)\,| = 0$

It follows that $b_0\,|\,x'-x\,|$ and $b_0\,|\,y'-y\,|$ are positive integers less than 1, so they are equal to zero.

Consequently, $g = (0,0,0)$, and this proves that:

$$\{g \in \Gamma \mid g(U) \cap U \neq \phi\} = \{I_d\};$$

(b) First, we prove that Γ acts properly on $\mathbb{R}^3$: for any compact subset K of $\mathbb{R}^3$, the set:

$$\{g \in G \mid g(K) \cap K \neq \phi\}$$

is finite.

Let $M_1 = (x_1, y_1, z_1)$, K be a compact subset of $\mathbb{R}^3$ and a real number $R > 0$ such that K is contained in the open ball $B(M_1, R)$ of center M_1 and radius R.

The following sets are finite:

$$S_1 = \{\mu \in \mathbb{Z} :\mid \mu b_2 \mid < R\}.$$

$$S_2 = \{\lambda + m\theta :\mid \lambda + m\theta \mid < 2R(\mid A_1^0 \mid + \mid A_2^0 \mid) \; where \; \lambda, \theta \in \mathbb{Z}\}.$$

$$S_3^i = \{(\lambda + m\theta)z_1 A_i^0 + \mu C_i^2 + u :\mid (\lambda + m\theta)z_1 A_i^0 + \mu C_i^2 + u \mid < R\},$$

where $(\lambda + m\theta) \in S_2, \mu \in S_1, u \in \mathbb{Z}, and \; i = 1, 2$.

It follows that the set $\Gamma(M_1) \cap K$ is finite; then, Γ acts properly on $\mathbb{R}^3$.

We recall the following theorem:

Theorem [29]: Let G be a discrete group acting continuously on a locally compact topological space E. Each orbit is closed and discrete in E, and the space of orbits E/G is a Hausdorff space.

By this theorem, it follows, in particular, that any points $M_1 = (x_1, y_1, z_1)$ and $M_2 = (x_2, y_2, z_2)$ of $\mathbb{R}^3$ not equivalent by Γ admit two open neighborhoods U_1 and U_2 such that: $\Gamma(U_1) \cap U_2 = \phi$, which proves the proposition. □

4. Conclusions

Proposition 10. *For all* $A^0, C^0, C^1, A^2, C^2 \in \mathbb{Z}^2$ *and* $b \in \mathbb{Z}$ *satisfying:*

$b_2 \neq 0$, $det(C^0, C^1) \neq 0$ *and* $D_3 = mD_2$, *we denote by* $M(A^0, C^0, C^1, b_2)$ *the quotient manifold:*

$$M(A^0, C^0, C^1, b_2) = \mathbb{R}^3 / < (A^0, 0, C^0), (mA^0, 0, C^1), (A^2, b_2, C^2) >.$$

Then:

-The quotient $M(A^0, C^0, C^1, b_2)$ *is a locally affine, compact, and complete* 2*-symplectic manifold whose fundamental domain is the parallelepiped built on the vectors:*

$(C^0, 0)$, $(C^1, 0)$ *and* (C^2, b_2);

-The fundamental group is given by:

$$\pi_1(M(A^0, C^0, C^1, b_2)) = < (A^0, 0, C^0), (mA^0, 0, C^1), (A^2, b_2, C^2) >;$$

-The manifold $M(A^0, C^0, C^1, b_2)$ *is homeomorphic to the torus* $\mathbb{T}^3$ *if and only if* $A^0 = 0$.

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