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Optimal Number of Pursuers in Differential Games on the 1-Skeleton of anOrthoplex

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Abstract: We study a differential game of many pursuers and one evader. All the players move only along the one-skeleton graph of an orthoplex of dimension $d+1$. It is assumed that the maximal speeds of the pursuers are less than the speed of the evader. By definition, the pursuit is completed if the position of a pursuer coincides with the position of the evader. Evasion is said to be possible in the game if the movements of players are started from some initial positions and the position of the evader never coincides with the position of any pursuer. We found the optimal number of pursuers in the game. The symmetry of the orthoplex plays an important role in the construction of the players' strategies.

Keywords: graph of a polyhedron; differential game; pursuit game; evasion game; game in normal form; π-strategy

MSC: Primary: 05C57; Secondary: 91A43

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1. Introduction

There are many papers devoted to pursuit and evasion differential games in $\mathbb{R}^n$ (see, for example, [1–7]).

An essential part of differential games is the differential games of many players. In the case of geometric constraints, interesting results were obtained by [8–19]. The paper [20] was devoted to a survey of such differential games.

In the work [21], a self-triggered pursuit strategy was proposed. It was assumed that the state information was available to the pursuer and evader. In a differential game of two evaders and one faster pursuer considered in [22], the plane was divided into two half-planes, the play and goal regions. The pursuer tries to protect the goal region from the evaders, and the evaders try to reach this region. A strategy for the pursuer was constructed based on the Apollonius circle.

Differential games of many players with integral constraints on the control functions of the players are also of increasing interest. For example, the works [23–28] dealt with the evasion differential games of many pursuers.

The papers [29,30] were devoted to differential games with state constraints.

There are some differential games in $\mathbb{R}^n$ where, for any behavior of the evader, the pursuit can be completed from some initial position. At the same time, the evader, by choosing his/her control, may delay the capture time as long as he/she wishes. However, this does not happen for the pursuit and evasion differential games on finite graphs. For finite graphs, it is possible either that there is a number θ such that, for any initial state of the players, the game ends by time θ or evasion is possible in the game forever.

The following two types of games on graphs should be mentioned. In the first type of games on graphs, players move from one vertex of the graph to an adjacent vertex by jumping [31–37]. In the second type of games, players move along the edges of the graph [38–43]. Both types of games can be formulated in a minimax form, and each of them is a model of the search problem of a moving object [43–45].

In the paper [39], pursuit and evasion differential games of n pursuers and one evader were studied on the one-skeletons of the regular polyhedrons in $\mathbb{R}^3$. All players can move with the speed not exceeding one. It was shown that the optimal number of pursuers for the tetrahedron, cube, and octahedron is two and, for the icosahedron and dodecahedron, is three. A similar differential game was studied in [40] on the one-skeletons of the d-dimensional regular simplex, orthoplex, and cube, and it was proven that the optimal number of pursuers for these polyhedrons are two, two, and $[d/2]+1$, respectively. Later on, it was shown in the work [41] that the optimal number of pursuers for the regular twenty-four-gone and one-hundred-twenty-gone in $\mathbb{R}^4$ is equal to three.

The purpose of the present paper is to study a pursuit and evasion game on the edge graph K_{d+1} of the orthoplex $\Sigma^{2(d+1)}$ in the Euclidean space $\mathbb{R}^{d+1}$. A $(d+1)$-dimensional orthoplex $\Sigma^{2(d+1)}$ is a convex hull of $2(d+1)$ points:

$$e_{2i-1} = (0,\dots,0,1,0,\dots,0), \;\; e_{2i} = (0,\dots,0,-1,0,\dots,0) \in \mathbb{R}^{d+1}, \;\; i = 1,2,\dots,(d+1),$$

where 1 in e_{2i-1} and -1 in e_{2i} is in the i-th place. Its edges of length $\sqrt{2}$ form a finite graph K_{d+1} with $2(d+1)$ vertices.

Let n pursuing points x_k, $k = 1,2,\dots,n$, whose velocities do not exceed in absolute value ρ_k, $\rho_k > 0$, $k = 1,2,\dots,n$, respectively, and one evading point E, whose velocity does not exceed σ, $\sigma > 0$, move along the graph K_{d+1}. These data define the differential game. The main difference of the present work from [40] is that we study differential games of slow pursuers. In [40], the construction of the strategies of pursuers was based on the fact that a pursuer can move symmetrically toward the evader with respect to some hyperplane, but in the case of slow pursuers, this is impossible. The construction of the strategies of pursuers in the present paper is based on the fact that each pursuer with speed $\geq 1/2$ can guard two vertices of an edge of the orthoplex, and each pursuer with speed $< 1/2$ can guard only one vertex of an edge of the orthoplex. In constructing the evader's strategy, this fact plays a key role as well.

Usually, in pursuit differential games, the pursuer has an advantage over the evader. For example, the control set of the pursuer may contain that of the evader, and vice versa, in evasion differential games, it is natural that the evader has an advantage over the pursuer. Note that the fact that the players cannot leave a given finite graph is itself an advantage for the pursuer in differential games on finite graphs. Therefore, we can consider a pursuit differential game with slow pursuers as well. Furthermore, we can obtain a pair of winning strategies of players in pursuit and evasion problems [2,6]. In the present paper, we use the π-strategy (see, for example, [46]) in constructing the pursuit strategies. Note that if a pursuer applies the π-strategy on some time interval $[t_0, t_1]$, then the straight line passing through the states of the pursuer and evader at any time $t \in [t_0, t_1]$ remains parallel with the straight line passing through the states of the pursuer and evader at time t_0.

2. Statement of the Problem

We considered a differential game of n pursuers x_1, x_2, $\dots$, x_n, $n \geq 2$, and one evader y, whose dynamics are given by the following equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \;\; i = 1,\dots,n, \\ &\dot{y} = v, \quad\;\; y(0) = y_0, \end{aligned} \tag{1}$$

where $x_{i0}, y_0 \in K_{d+1}$, $x_{i0} \neq y_0$, $i = 1,\dots,n$; u_i is the control parameter of the i-th pursuer; v is the control parameter of the evader. All the players move along the edges of orthoplex

K_{d+1}. The maximal speeds of the pursuers x_1, x_2,..., x_n are $\rho_1, \rho_2, \ldots, \rho_n$, respectively, and that of the evader y is σ, i.e., $|u_i| \le \rho_i$, $i = 1, \ldots, n$, $|v| \le \sigma$.

A function $u_i(\cdot)$, $u_i : [0, \infty) \to B(\rho_i)$ is called an admissible control of the i-th pursuer, $i \in \{1, \ldots, n\}$, if for the solution $x_i(\cdot)$ of the equation:

$$\dot{x}_i = u_i, \; x_i(0) = x_{i0},$$

we have $x_i(t) \in K_{d+1}$, $t \ge 0$.

A function $v(\cdot)$, $v : [0, \infty) \to B(\sigma)$ is called the admissible control of the evader, if for the solution $y(\cdot)$ of the equation:

$$\dot{y} = v, \; y(0) = y_0,$$

we have $y(t) \in K_{d+1}$, $t \ge 0$.

We considered pursuit and evasion games. In the pursuit differential game, pursuers apply some strategies, and the evader uses an arbitrary admissible control. Let us define the strategies of the pursuers.

Functions $(t, x_1, \ldots, x_n, y, v) \to U_i(t, x_1, ..., x_n, y, v)$, $i = 1, 2, \ldots, n$, are called the strategies of the pursuers x_i, $i = 1, 2, ..., n$, if the initial value problem (1) has a unique solution $x_1(t), \ldots, x_n(t)$, $y(t) \in K_{d+1}$, $t \ge 0$, for $u_i = U_i(t, x_1, \ldots, x_n, y, v)$, $i = 1, 2, \ldots, n$, and for any admissible control $v = v(t)$ of the evader.

If, for some number $T > 0$, there exist strategies of pursuers such that $x_i(\tau) = y(\tau)$ at some $0 < \tau \le T$ and $i \in \{1, \ldots, n\}$, then the pursuit is said to be completed. The pursuers are interested in completing the pursuit as earlier as possible.

A function $(t, x_1, \ldots, x_n, y) \to V(t, x_1, \ldots, x_n, y)$ is called a strategy of the evader y if the initial value problem (1) has a unique solution $x_1(t), \ldots, x_n(t)$, $y(t) \in K_{d+1}$, $t \ge 0$, for $v = V(t, x_1, \ldots, x_n, y)$ and for any admissible controls of the pursuers $u_i = u_i(t)$, $i = 1, 2, \ldots, n$.

If, for some initial states of the players $x_{10}, \ldots, x_{n0}, y_0 \in K_{d+1}$, there exists a strategy of the evader such that $x_i(t) \ne y(t)$ for all $t \ge 0$, and $i = 1, \ldots, n$, then we say that evasion is possible in the game in K_{d+1}. The evader is interested in maintaining the inequality $x_i(t) \ne y(t)$ as long as possible. Since for some initial states, the evader may be trapped by the pursuers and the pursuit can be completed by the pursuers easily, therefore this definition contains the phrase "for some initial states of players $x_{10}, \ldots, x_{n0}, y_0 \in K_{d+1}$".

The number $N = N(K_{d+1})$ is called the optimal number of pursuers for the game on cocube K_{d+1} if, for any initial states of the players, the pursuit can be completed in the game with N pursuers and evasion is possible in the game with $N - 1$ pursuers.

The problem is to find the optimal number of pursuers N in the game, to construct strategies for the pursuers in the pursuit game, and the evasion strategy.

3. Main Result

The one-skeleton K_{d+1} of the orthoplex $\Sigma^{2(d+1)}$ can be obtained as follows. We call the symmetry vertices e_{2i-1} and e_{2i} of $\Sigma^{2(d+1)}$ antipodal. For each $i = 1, 2, \ldots, 2(d+1)$, we connect e_i with all vertices with segments, which are not antipodal to e_i, and we obtain K_{d+1}.

If $\rho_{i_0} > \sigma$ for some $i_0 \in \{1, 2, \ldots, n\}$, then, clearly, only one pursuer x_{i_0} can capture the evader. If $\rho_{i_0} = \sigma$ and $\rho_{j_0} = \sigma$, for some $i_0, j_0 \in \{1, 2, \ldots, n\}$ and $i_0 \ne j_0$, then it is shown that only two pursuers x_{i_0} and x_{j_0} can capture the evader [40]. Furthermore, it can be shown that if $\rho_{i_0} = \sigma$ and $n \ge 2$, then the pursuit can be completed. Therefore, in the present paper, we considered the case where $\rho_1, \rho_2, \ldots, \rho_n$ less than σ, that is $0 < \rho_i < \sigma$.

We denote the vectors corresponding to the points x_1, x_2, ..., x_n, and E by x_1, x_2, ..., x_n, and y, respectively. Let:

$$1/2 \le \rho_i < 1, \;\; i = 1, 2, ..., k; \;\; 0 < \rho_i < 1/2, \;\; i = k+1, ..., n.$$

If the inequality $1/2 \leq \rho_i < 1$ is not satisfied for all $i = 1, 2, \ldots, n$, then we consider $k = 0$. Clearly, $k \leq n$. For the pursuit differential game, the following statement is true.

Theorem 1. *If either (i) $n = k = d + 1$ or (ii) $n \neq k$ and $n + k \geq 2d$, then the pursuit can be completed in the differential game on the orthoplex K_{d+1}.*

Proof. Case 1. Let $n = k = d + 1$, that is the maximum speeds of all pursuers greater than or equal to $1/2$ and the number of pursuers is $d + 1$. We temporarily remove from the orthoplex K_{d+1} the two symmetric with respect to the origin vertices $S = e_{2i-1} = (0, \ldots, 0, 1, 0, \ldots, 0)$, $S_1 = e_{2i} = (0, \ldots, 0, -1, 0, \ldots, 0)$ with $i = d + 1$ and all the edges with the endpoint at S or S_1. As a result, we obtain an orthoplex K_d, which has $2d$ vertices.

Let A_1B_1, A_2B_2, $\ldots$, A_dB_d be the edges of the orthoplex K_d, any two of which have no common vertex (Figure 1), where A_1, B_1, A_2, B_2, $\ldots$, A_d, B_d are all the vertices of the orthoplex K_d. For example, if we denote the points corresponding to the vectors $e_1, e_3, \ldots, e_{2d-1}, e_2, e_4, \ldots, e_{2d}$, respectively, by A_1, B_1, A_2, B_2, $\ldots$, A_d, B_d, then we obtain such edges, where A_i is not antipodal to B_i, $i = 1, 2, \ldots, d$, and so, the vectors e_p and e_q corresponding to A_i and B_i are orthogonal.

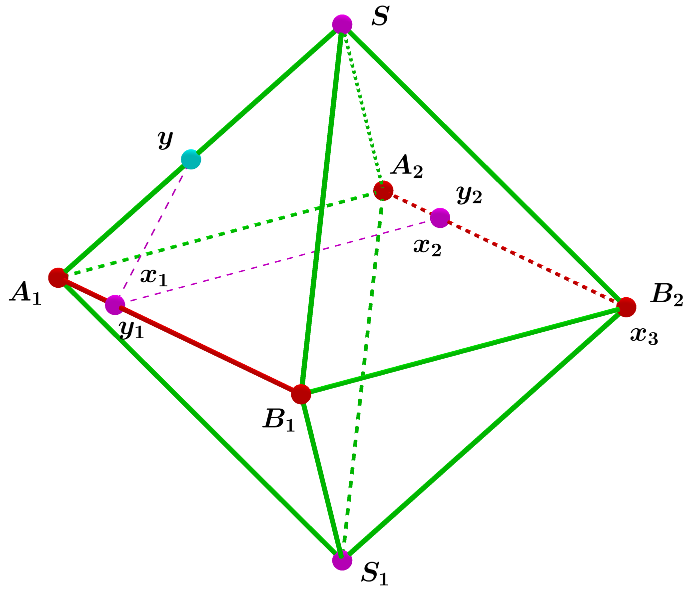

Figure 1. $d = 2$, $k = 2$, and $K_d = K_2 = A_1B_1A_2B_2$.

We show that any two edges of the orthoplex K_d are either parallel, or orthogonal, or form an angle equal to $\pi/3$. To this end, it is sufficient to find the angle between any two vectors $e_i - e_j$ and $e_k - e_l$, where $e_i \neq e_j$, $e_i \neq -e_j$, $e_k \neq e_l$, $e_k \neq -e_l$. Note that any two distinct vectors e_p and e_q are either symmetric with respect to the origin or orthogonal. We have:

$$\cos\alpha = \frac{(e_i - e_j)(e_k - e_l)}{|e_i - e_j||e_k - e_l|} = \frac{e_ie_k - e_ie_l - e_je_k + e_je_l}{2}.$$

A. Let $e_i = -e_k$, $e_j = -e_l$. Then, $e_ie_k = -1$, $e_je_l = -1$, $e_ie_l = 0$, $e_je_k = 0$, and so, $\cos\alpha = -1$. Consequently, $\alpha = \pi$, and so, the vector $(e_i - e_j)$ is parallel with the vector $(e_k - e_l)$.

B. Let $e_i = -e_k$, $e_j \neq -e_l$. Then, $e_ie_k = -1$, $e_je_l = 0$, $e_ie_l = 0$, $e_je_k = 0$, and so, $\cos\alpha = -1/2$. Consequently, $\alpha = 2\pi/3$. This means the angle between the edges with vertices e_i, e_j and e_k, e_l forms an angle equal to $\pi/3$.

C. Let $e_i \neq -e_k$, $e_j = -e_l$. Then, $e_ie_k = 0$, $e_je_l = -1$, $e_ie_l = 0$, $e_je_k = 0$, and so, $\cos\alpha = -1/2$. Consequently, $\alpha = 2\pi/3$. This means the angle between the edges with vertices e_i, e_j and e_k, e_l forms an angle equal to $\pi/3$.

D. Let $e_i \neq -e_k$, $e_j \neq -e_l$. Then, $e_i e_k = 0$, $e_j e_l = 0$, $e_i e_l = 0$, $e_j e_k = 0$, and so, $\cos\alpha = 0$. Consequently, $\alpha = \pi/2$. This means the angle between the edges with vertices e_i, e_j and e_k, e_l forms an angle equal to $\pi/2$.

Thus, the vector $(e_i - e_j)$ is parallel with the vector $(e_k - e_l)$ only in Case 1. Hence, for each edge of K_d, there is at most one edge parallel with this edge.

Next, we construct strategies for the pursuers x_1, x_2,…, x_d as follows. First, we let the pursuers $x_1, x_2, \ldots, x_d$ come to the vertices $A_1, A_2, \ldots, A_d$, respectively. The pursuer reaches his/her vertex and waits until the other pursuers reach their vertices. Let all the pursuers reach the vertices $A_1, A_2, \ldots, A_d$ at some time t_1. Without any loss of generality, we assumed that if the edge $A_{j_0}B_{j_0}$ is parallel with $A_{i_0}B_{i_0}$, $i_0 \neq j_0$, then the vector $\overrightarrow{A_{j_0}B_{j_0}}$ is codirected toward $\overrightarrow{A_{i_0}B_{i_0}}$.

Start from the time t_1 each pursuer x_i walks along the edge A_iB_i with the speed $\rho_i' = \min\{\rho_i, \rho_j\}$, where ρ_j is the maximal speed of the pursuer x_j moving along the edge A_jB_j parallel with A_iB_i (if there is such an edge), until x_i captures the projection y_i of the evader y on the edge A_iB_i, $i = 1, 2, \ldots, d$. Thus, the pursuers x_i and x_j that move in parallel edges A_iB_i and A_jB_j move with the same speed and direction until they capture the projections of the evader on those edges, respectively.

As $x_i(t) = y_i(t)$ at some time $t = t_{2i}$, we suggest the following strategy to the pursuer x_i:

$$u_i(t) = (m_i, v(t))m_i, \;\; t_{2i} < t \leq t_2, \tag{2}$$

where m_i is the unit vector of $\overrightarrow{A_{i_0}B_{i_0}}$, that is $m_i = \frac{1}{\sqrt{2}}\overrightarrow{A_{i_0}B_{i_0}}$; t_2 is the time when $x_i(t) = y_i(t)$ for all $i = 1, \ldots, d$. Then, clearly, (2) ensures that $x_i(t) = y_i(t)$, $t_{2i} < t \leq t_2$.

The strategy of pursuer (2) is admissible since (i) if the evader moves along an edge A_iB_i orthogonal to m_i, then $u_i(t) = 0$ since $(m_i, v(t)) = 0$; (ii) if the edge where the evader moves on forms an angle $\pi/3$ with the edge A_iB_i, then:

$$|u_i(t)| = |(m_i, v(t))m_i| \leq |m_i||v(t)| \cos\frac{\pi}{3} \leq \frac{1}{2} \tag{3}$$

since $|m_i| = 1$ and $|v(t)| \leq 1$.

Note that if the evader is on an edge A_jB_j parallel with A_iB_i on the time interval $t_{2i} \leq t \leq t_2$, then the evader is captured by a pursuer at the time t_{2i} since one of the points $y_i(t_{2i})$ coincides with the real evader and the pursuers x_i and x_j move on the interval $[t_1, t_{2i}]$ on the parallel edges with the same speed and direction. Therefore, the evader cannot be on the edge parallel with A_iB_i on the time interval $t_{2i} \leq t \leq t_2$. Thus, (2) is admissible. Using the strategy (2), the pursuer x_i controls the edge A_iB_i, meaning that the evader is not on A_iB_i, and if he/she reaches one of the points A_i, B_i, he/she is captured by the pursuer x_i.

If the evader is on an edge of K_d at the time t_2, then he/she is trapped by two pursuers and the pursuit can be completed by these pursuers. To this end, these pursuers just move towards the evader to catch him/her. Let now the evader be not in K_d at the time t_2. Then, we let the pursuers x_i, $i = 1, \ldots, d$, further use strategies (2) for $t \geq t_2$. Then, clearly, $x_i(t) = y_i(t)$, $t \geq t_2$.

If we remove K_d from K_{d+1}, then we obtain two trees, one of which contains the vertex S and the other of which contains S_1. Therefore, the pursuer x_{d+1} moving towards the evader will catch him/her or force the evader to reach K_d, in which case the evader will be caught by a pursuer in K_d.

Case 2. Let $n \neq k$ and $n + k \geq 2d$. It follows from $n \neq k$ that $n > k$. It suffices to consider the case $n + k = 2d$, that is if $n + k = d$, then n pursuers can complete the game. Let A_1B_1, A_2B_2, …, A_kB_k be the edges of the orthoplex K_d, any two of which have no common vertex (Figure 2), where A_1, B_1, A_2, B_2, …, A_k, B_k are some distinct vertices of the orthoplex K_d.

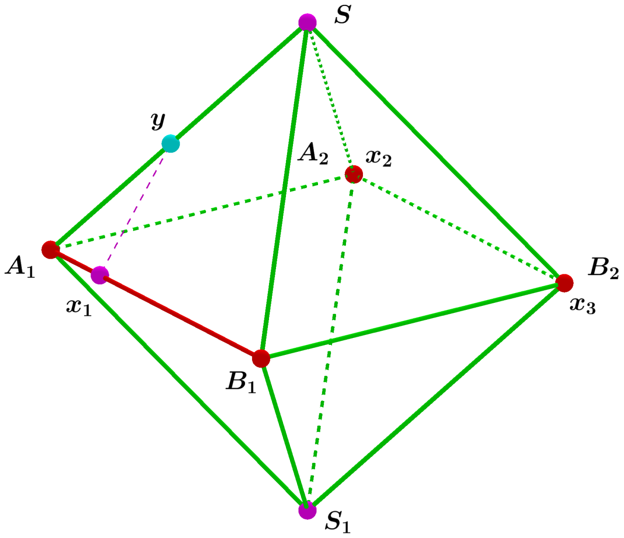

Figure 2. The case where $d = 2$, $k = 1$, $n = 3$, and $K_d = K_2 = A_1B_1A_2B_2$.

First, we bring the pursuers $x_1, x_2, \ldots, x_k$ to the vertices $A_1, A_2, \ldots, A_k$. Then, as in Case 1, these pursuers move towards $B_1, B_2, \ldots, B_k$, respectively, to catch the projections on the edges $A_1B_1, A_2B_2, \ldots, A_kB_k$. As the pursuer x_i catches the projection y_i of the evader y on A_iB_i, this pursuer further moves on the projection, keeping it. In this way, pursuers control the edges $A_1B_1, A_2B_2, \ldots, A_kB_k$ (Figure 2). In particular, these pursuers can control $2k$ vertices $A_1, B_1, A_2, B_2, \ldots, A_k, B_k$ of K_d.

The rest $n - k$ of the pursuers $x_{k+1}, \ldots, x_n$ go to the $2d - 2k$ vertices $A_{k+1}, B_{k+1}, \ldots, A_d, B_d$. This is possible since $n - k = 2d - 2k$. Thus, all the pursuers can now control all the vertices of K_d. $K_{d+1} \setminus K_d$ is a union of two disjoint trees and one of them contains S and another contains S_1. The pursuer x_{k+1} moving towards the evader either catches the evader or forces him/her to reach K_d. In the latter case, the evader will be captured by some of the pursuers $x_1, \ldots, x_k, x_{k+2}, \ldots, x_d$. □

Theorem 2. *If either (i) $n = k < d + 1$ or (ii) $n \neq k$ and $n + k < 2d$, then the evader E can avoid the pursuers $x_1, x_2, \ldots, x_n$ in the game on orthoplex K_{d+1} from some initial states.*

Proof. Case 1. Let $n = k < d + 1$ and the initial state of the evader be at the vertex S of K_{d+1}. Hence, $n = k \leq d$. Let the evader stay at the vertex S until the time t_0 when $\min_{i=1,\ldots,n} |x_i(t_0) - S| \leq 1/3$ for the first time. Furthermore, it is possible that $t_0 = 0$. For the definiteness, assume that $|x_1(t_0) - S| \leq 1/3$.

Pursuer x_1 then cannot control more than one vertex of the orthoplex K_d in $\sqrt{2}$ units of time. Each of the rest $(n - 1)$ of the pursuers $x_2, \ldots, x_n$ can control at most two vertices of K_d, and so, together, they can control at most $2(n - 1) \leq 2(d - 1)$ vertices of K_d in $\sqrt{2}$ units of time.

Thus, all the pursuers $x_1, \ldots, x_n$ can control $2(d - 1) + 1 = 2d - 1$ vertices of orthoplex K_d in $\sqrt{2}$ units of time. Hence, the pursuers cannot control one of the vertices of K_d because K_d has $2d$ vertices, and so, the evader can reach that vertex of K_d in $\sqrt{2}$ units of time not being caught. The evader repeats this procedure over and over and can walk not being caught for an infinite period of time.

Case 2. Let $n \neq k$ and $n + k < 2d$ and the evader be at the vertex S of K_{d+1} at the initial time. Clearly, k pursuers $x_1, x_2, \ldots, x_k$ can control $2k$ vertices of K_d in $\sqrt{2}$ units of time. We now think about the rest $2d - 2k$ of the vertices of K_d.

Each of the rest $n - k$ of the pursuers can control at most one vertex of K_d in $\sqrt{2}$ units of time, and so, all these pursuers together can control at most $(n - k)$ vertices. Hence, all the pursuers can control at most $2k + (n - k) = n + k$ vertices of K_d in $\sqrt{2}$ units of time.

Since $n+k<2d$, therefore, the evader can come to a vertex of K_d not being caught. The evader repeats this procedure over and over and can walk not being caught for an infinite period of time. The proof of the theorem is complete. □

4. Discussion and Conclusions

In the present paper, we studied pursuit and evasion differential games of many slow pursuers and one evader on the edge graph of an orthoplex. If either (i) $n=k=d+1$ or (ii) $n \neq k$ and $n+k \geq 2d$, then we proved that the pursuit can be completed in the differential game on the orthoplex K_{d+1}. If this condition is not satisfied, that is either (i) $n=k<d+1$ or (ii) $n \neq k$ and $n+k<2d$, then we proved that evasion is possible. In the case of pursuit game, we constructed explicit strategies for the pursuers, and in the case of the evasion game, we constructed an evasion strategy for the evader. Thus, we solved the pursuit and evasion game problems on the orthoplex completely.

A differential game on the one-skeletons of the d-dimensional orthoplex studied in [40] considered the case where the dynamical possibilities of the pursuers and evader being equal. It was shown in [40] that the optimal number of pursuers is equal to two. This is not difficult to prove using the same idea of that paper that the two pursuers, one of them having the same maximal speed as the evader and another pursuer having a maximal speed less than that of the evader, can complete the game as well. If the maximal speeds of all pursuers are less than that of evader, then the problem of the optimal number of pursuers is open, and in the present paper, we found a formula for this number.

Based on Theorems 1 and 2, we can conclude that the optimal number of pursuers in the game on the one-skeleton K_{d+1} is $N(K_{d+1}) = d+1$ if $n=k$, and $N(K_{d+1}) = 2d-k$ if $n>k$ (recall that $n \geq k$).

As an open problem, we suggest solving such differential game problems for the n-cube, that is in the n-dimensional cube in $\mathbb{R}^n$.

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