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Existence Results of a Nonlocal Fractional Symmetric Hahn Integrodifference Boundary Value Problem

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Abstract: The existence of solutions of nonlocal fractional symmetric Hahn integrodifference boundary value problem is studied. We propose a problem of five fractional symmetric Hahn difference operators and three fractional symmetric Hahn integrals of different orders. We first convert our nonlinear problem into a fixed point problem by considering a linear variant of the problem. When the fixed point operator is available, Banach and Schauder's fixed point theorems are used to prove the existence results of our problem. Some properties of (q, ω)-integral are also presented in this paper as a tool for our calculations. Finally, an example is also constructed to illustrate the main results.

Keywords: fractional symmetric Hahn; boundary value problems; existence; fixed point theorem

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1. Introduction

Quantum calculus has been proposed in the last three decades. The q-calculus, one type of quantum presented by Jackson [1,2], has been extensively used in the studies of mechanics, calculus of variations, and other problems in physics [3–13]). In 1949, Hahn [14] presented the development of quantum calculus based on two parameters q and ω, which is called Hahn calculus. The Hahn difference operator is formulated from the forward and q-difference operators, as defined by

$$D_{q,\omega} f(t) := \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

The right inverse and properties of Hahn difference operator were further presented (see [15,16]). Hahn operator was widly employed in many studies such as variational calculus [17–19], the initial value problems [20–22], the boundary value problems [23,24], and families of orthogonal polynomials [25–27].

Later, symmetric Hahn difference operator was introduced by Artur et al. [28] as

$$\tilde{D}_{q,\omega} f(t) := \frac{f(qt+\omega) - f(q^{-1}(t-\omega))}{(q-q^{-1})t + (1+q^{-1})\omega}, \quad t \neq \omega_0.$$

For fractional quantum calculus, Agarwal [29] and Al-Salam [30] introduced fractional q-calculus, and Díaz and Osler [31] introduced fractional difference calculus. In 2017, Brikshavana and Sitthiwirattham [32] proposed fractional Hahn difference calculus. In 2020, Soontharanon and Sitthiwirattham [33] introduced fractional (p, q)-calculus and its properties. For some recent developments in fractional calculus, see [34–45] and the references cited therein.

Recently, Patanarapeelert and Sitthiwirattham [46] introduced the fractional symmetric Hahn integral, fractional symmetric Hahn difference of Riemann-Liouville and Caputo types, and their properties. However, after fractional symmetric Hahn operators have been defined, there is one paper on the study of the boundary value problem for fractional symmetric Hahn difference equation [47].

Motivated by the above result, we aim to enrich this research work by presenting the new problem given by

$$\begin{aligned} \tilde{D}^{\alpha}_{q,\omega}u(t) &= F\Big[t, u(t), (\tilde{\Psi}^{\gamma}_{q,\omega}u)(t), \tilde{D}^{\nu}_{q,\omega}u(t), \tilde{D}^{\nu+1}_{q,\omega}u(t)\Big], \\ \lambda_1 u(\eta_1) + \mu_1 \tilde{D}^{\beta_1}_{q,\omega}u(\eta_1) + \kappa_1 \tilde{I}^{\theta_1}_{q,\omega}u(\eta_1) &= \phi_1(u), \\ \lambda_2 u(\eta_2) + \mu_2 \tilde{D}^{\beta_2}_{q,\omega}u(\eta_2) + \kappa_2 \tilde{I}^{\theta_2}_{q,\omega}u(\eta_2) &= \phi_2(u), \quad \eta_1, \eta_2 \in I^T_{q,\omega} - \{\omega_0, T\} \end{aligned} \tag{1}$$

where $t \in I^T_{q,\omega} := \{q^k T + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\alpha \in (1,2]$; $\gamma, \nu, \beta_1, \beta_2, \theta_1, \theta_2 \in (0,1]$, $\nu + 1 \leq \alpha$; $\omega > 0$; $q \in (0,1)$; $\lambda_1, \lambda_2, \mu_1, \mu_2, \kappa_1, \kappa_2 \in \mathbb{R}^+$; $F \in C\big(I^T_{q,\omega} \times \mathbb{R}^4, \mathbb{R}\big)$ is given function; ϕ_1, ϕ_2 are given functionals; and for $\varphi \in C\big(I^T_{q,\omega} \times I^T_{q,\omega}, [0,\infty)\big)$, we define an operator of the product of φu as

$$(\tilde{\Psi}^{\gamma}_{q,\omega}u)(t) := \Big(\tilde{\mathcal{I}}^{\gamma}_{q,\omega}\varphi\, u\Big)(t) = \frac{q^{\binom{\gamma}{2}}}{\tilde{\Gamma}_{q,\omega}(\gamma)} \int_{\omega_0}^{t} \widetilde{\Big(t-s\Big)}^{\gamma-1}_{q,\omega} \varphi(t, \sigma^{\gamma-1}_{q,\omega}(s))\, u(\sigma^{\gamma-1}_{q,\omega}(s))\, \tilde{d}_{q,\omega}s.$$

We note that our problem is in the form of fractional symmetric Hahn difference of Riemann–Liouville type. The function F depends on fractional symmetric Hahn integral and difference operators. Our boundary conditions are nonlocal and are assigned values at two non-local points of both fractional symmetric Hahn integral and difference operators. We investigate a linear variant of (1) in order to obtain fixed point operator and prove the existence of solutions of problem (1) by using a fixed point theorem. Firstly, we convert this nonlinear problem into a fixed point problem related to (1), by considering a linear variant of (1). Once the fixed point operator is available, we use the classical fixed point theorems to establish existence results.

Some definitions and lemmas related to fractional symmetric Hahn calculus are revealed in Section 2. In Section 3, we employ the Banach fixed point theorem to analyze the existence and uniqueness of a solution of the problem (1). The existence of at least one solution of the problem (1) is also studied by using the Schuader's fixed point theorem. Finally, we give an example to show the application of our results.

2. Preliminaries

2.1. Basic Knowledge

Here, we provide the basic knowledge of fractional symmetric Hahn difference calculus as follows [28,46–49].

For $0 < q < 1$, $\omega > 0$, $\omega_0 = \frac{\omega}{1-q}$ and $[k]_q = \frac{1-q^k}{1-q}$, we define some notations as follows.

$$\widetilde{[k]}_q := \begin{cases} \dfrac{1-q^{2k}}{1-q^2} = [k]_{q^2}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}$$

$$\widetilde{[k]}_q! := \begin{cases} \widetilde{[k]}_q \widetilde{[k-1]}_q \cdots \widetilde{[1]}_q = \displaystyle\prod_{i=1}^{k} \frac{1-q^{2i}}{1-q^2}, & k \in \mathbb{N} \\ 1, & k = 0. \end{cases}$$

For $k \in \mathbb{N}$, the q,ω-forward jump and q,ω-backward jump operators are defined by

$$\sigma^k_{q,\omega}(t) := q^k t + \omega[k]_q \quad \text{and} \quad \rho^k_{q,\omega}(t) := \frac{t - \omega[k]_q}{q^k}, \text{ , respectively.}$$

For $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, and $a, b \in \mathbb{R}$,

- The power function of the q-analogue is defined by

$$(a-b)_q^{\underline{0}} := 1, \qquad (a-b)_q^{\underline{n}} := \prod_{i=0}^{n-1}(a - bq^i).$$

- The power function of the q-symmetric analogue is defined by

$$\widetilde{(a-b)}_q^{\underline{0}} := 1, \qquad \widetilde{(a-b)}_q^{\underline{n}} := \prod_{i=0}^{n-1}(a - bq^{2i+1}).$$

- The power function of the q,ω-symmetric analogue is defined by

$$\widetilde{(a-b)}_{q,\omega}^{\underline{0}} := 1, \qquad \widetilde{(a-b)}_{q,\omega}^{\underline{n}} := \prod_{i=0}^{n-1}\Big[a - \sigma_{q,\omega}^{2i+1}(b)\Big].$$

For $\alpha \in \mathbb{R}$, the power functions are defined by

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1 - \left(\frac{b}{a}\right)q^i}{1 - \left(\frac{b}{a}\right)q^{\alpha+i}}, \; a \neq 0,$$

$$\widetilde{(a-b)}_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1 - \left(\frac{b}{a}\right)q^{2i+1}}{1 - \left(\frac{b}{a}\right)q^{2(\alpha+i)+1}}, \; a \neq 0,$$

$$\widetilde{(a-b)}_{q,\omega}^{\underline{\alpha}} = \Big(\widetilde{(a-\omega_0) - (b-\omega_0)}\Big)_q^{\underline{\alpha}} = (a-\omega_0)^\alpha \prod_{i=0}^{\infty} \frac{1 - \left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2i+1}}{1 - \left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2(\alpha+i)+1}}, \; a \neq \omega_0.$$

If $b = 0$, $a_q^{\underline{\alpha}} = \widetilde{a}_q^{\underline{\alpha}} = a^\alpha$ and $\widetilde{(a-\omega_0)}_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$. If $a = b$, $(0)_q^{\underline{\alpha}} = \widetilde{(0)}_q^{\underline{\alpha}} = \widetilde{(\omega_0)}_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$.

The q-symmetric gamma and q-symmetric beta functions are defined by

$$\tilde{\Gamma}_q(x) := \begin{cases} \dfrac{(1-q^2)_q^{\underline{x-1}}}{(1-q^2)^{x-1}} = \dfrac{\widetilde{(1-q)}_q^{\underline{x-1}}}{(1-q^2)^{x-1}}, & x \in \mathbb{R} \setminus \{0, -1, -2, \dots\} \\ \widetilde{[x-1]}_q!, & x \in \mathbb{N} \end{cases}$$

$$\tilde{B}_q(x,y) := \int_0^1 \left(q^{-1}s\right)^{\underline{x-1}} \widetilde{(1-s)}_q^{\underline{y-1}} \tilde{d}_q s = \frac{\tilde{\Gamma}_q(x)\tilde{\Gamma}_q(y)}{\tilde{\Gamma}_q(x+y)},$$

respectively.

Lemma 1 ([46]). *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$,*

(a) $\widetilde{(x - \sigma_{q,\omega}^n(x))}_{q,\omega}^{\underline{\alpha}} = (x-\omega_0)^k \widetilde{(1-q^n)}_q^{\underline{\alpha}}$,

(b) $\widetilde{(\sigma_{q,\omega}^m(x)) - \sigma_{q,\omega}^n(x))}_{q,\omega}^{\underline{\alpha}} = q^{m\alpha}(x-\omega_0)^\alpha \widetilde{(1-q^{n-m})}_q^{\underline{\alpha}}$.

Next, the symmetric Hahn difference and integral are defined as follows.

Definition 1 ([28]). *Let f be a function defined on $I_{q,\omega}^{T} \subseteq \mathbb{R}$ and $q \in (0,1)$, $\omega > 0$. The symmetric Hahn difference of f is defined by*

$$\tilde{D}_{q,\omega} f(t) := \frac{f(qt+\omega) - f(q^{-1}(t-\omega))}{(q-q^{-1})t + (1+q^{-1})\omega} \quad t \in I_{q,\omega}^{T} - \{\omega_0\},$$
$$\tilde{D}_{q,\omega} f(\omega_0) = f'(\omega_0) \text{ where } f \text{ is differentiable at } \omega_0.$$

$\tilde{D}_{q,\omega} f$ is called q,ω-symmetric derivative of f, and f is q,ω-symmetric differentiable on $I_{q,\omega}^{T}$.

For $N \in \mathbb{N}$, we define

$$\tilde{D}_{q,\omega}^{0} f(x) = f(x) \text{ and } \tilde{D}_{q,\omega}^{N} f(x) = \tilde{D}_{q,\omega} \tilde{D}_{q,\omega}^{N-1} f(x).$$

Definition 2 ([28]). *Let $f : I \to \mathbb{R}$ be a given function, where I is any closed interval of $\mathbb{R}$ containing a, b and ω_0. The symmetric Hahn integral of f from a to b is defined by*

$$\int_a^b f(t) \tilde{d}_{q,\omega} t := \int_{\omega_0}^b f(t) \tilde{d}_{q,\omega} t - \int_{\omega_0}^a f(t) \tilde{d}_{q,\omega} t,$$

where

$$\tilde{\mathcal{I}}_{q,\omega} f(t) = \int_{\omega_0}^x f(t) \tilde{d}_{q,\omega} t := (1-q^2)(x-\omega_0) \sum_{k=0}^{\infty} q^{2k} f\Big(\sigma_{q,\omega}^{2k+1}(x)\Big), \quad x \in I.$$

Provided that the above series converges at $x = a$ and $x = b$, we call that function f is symmetric Hahn integrable on $[a,b]$, and it is on I if it is symmetric Hahn integrable on $[a,b]$ for all $a, b \in I$.

For $N \in \mathbb{N}$,

$$\tilde{\mathcal{I}}_{q,\omega}^{0} f(x) = f(x), \ \ \tilde{\mathcal{I}}_{q,\omega}^{N} f(x) = \tilde{\mathcal{I}}_{q,\omega} \tilde{\mathcal{I}}_{q,\omega}^{N-1} f(x),$$
$$\tilde{D}_{q,\omega} \tilde{\mathcal{I}}_{q,\omega} f(x) = f(x), \ \text{ and } \ \tilde{\mathcal{I}}_{q,\omega} \tilde{D}_{q,\omega} f(x) = f(x) - f(\omega_0).$$

We next introduce the fractional symmetric Hahn integral and fractional Riemann–Liouville symmetric Hahn difference as follows.

Definition 3 ([46]). *Let f be a function defined on $I_{q,\omega}^{T}$ and $\alpha, \omega > 0$, $0 < q < 1$. The fractional symmetric Hahn integral is defined by*

$$\begin{aligned}
\widetilde{\mathcal{I}}_{q,\omega}^{\alpha} f(t) &:= \frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}} f\Big(\sigma_{q,\omega}^{\alpha-1}(s)\Big) \tilde{d}_{q,\omega} s \\
&= \frac{(1-q^2) q^{\binom{\alpha}{2}} (t-\omega_0)}{\widetilde{\Gamma}_q(\alpha)} \sum_{k=0}^{\infty} q^{2k} \Big(t - \sigma_{q,\omega}^{2k+1}(t)\Big)_{q,\omega}^{\underline{\alpha-1}} f\Big(\sigma_{q,\omega}^{2k+\alpha}(t)\Big) \\
&= \frac{(1-q^2) q^{\binom{\alpha}{2}} (t-\omega_0)^{\alpha}}{\widetilde{\Gamma}_q(\alpha)} \sum_{k=0}^{\infty} q^{2k} \Big(\widetilde{1-q^{2k+1}}\Big)_{q}^{\underline{\alpha-1}} f\Big(\sigma_{q,\omega}^{2k+\alpha}(t)\Big)
\end{aligned}$$

and $\widetilde{\mathcal{I}}_{q,\omega}^{0} f)(t) = f(t)$.

Definition 4 ([46]). *Let f be a function defined on $I_{q,\omega}^{T}$ and $\alpha, \omega > 0$, $0 < q < 1$. The fractional symmetric Hahn difference operator of Riemann–Liouville type of order α is defined by*

$$\begin{aligned}
\widetilde{D}_{q,\omega}^{\alpha} f(t) &:= \widetilde{D}_{q,\omega}^{N} \widetilde{\mathcal{I}}_{q,\omega}^{N-\alpha} f(t) \\
&= \frac{q^{\binom{-\alpha}{2}}}{\widetilde{\Gamma}_q(-\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{-\alpha-1}} f\Big(\sigma_{q,\omega}^{-\alpha-1}(s)\Big) \tilde{d}_{q,\omega} s
\end{aligned}$$

and $\widetilde{D}^0_{q,\omega} f(t) = f(t)$ *where* $\alpha \in (N-1, N)$, $N \in \mathbb{N}$.

Next, we introduce lemmas that are used in the main results.

Lemma 2 ([46])**.** *Let* $\alpha, \omega > 0$, $0 < q < 1$ *and* $f : I^T_{q,\omega} \to \mathbb{R}$. *Then,*

$$\tilde{\mathcal{I}}^{\alpha}_{q,\omega} \tilde{D}^{\alpha}_{q,\omega} f(t) = f(t) + C_1 (t-\omega_0)^{\alpha-1} + C_2 (t-\omega_0)^{\alpha-2} + \cdots + C_N (t-\omega_0)^{\alpha-N}$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, N$ *and* $\alpha \in (N-1, N)$ *for* $N \in \mathbb{N}$.

Lemma 3 ([50])**.** *(Arzelá–Ascoli theorem) A set of function in* $C[a,b]$ *with the sup norm, is relatively compact if, and only if, it is uniformly bounded and equicontinuous on* $[a,b]$.

Lemma 4 ([50])**.** *If a set is closed and relatively compact, then it is compact.*

Lemma 5 ([51])**.** *(Schauder's fixed point theorem) Let* (D,d) *be a complete metric space, U be a closed convex subset of D, and* $T : D \to D$ *be the map, such that the set* $Tu : u \in U$ *is relatively compact in D. Then, the operator T has at least one fixed point* $u^* \in U$: $Tu^* = u^*$.

2.2. Auxiliary Lemmas

Lemma 6 ([47])**.** *Let* $\alpha > 0, q \in (0,1), \omega > 0$ *and* $n \in \mathbb{N}$. *Then,*

$$(i) \quad \int_{\omega_0}^{t} \tilde{d}_{q,\omega} s = t - \omega_0,$$

$$(ii) \quad \int_{\omega_0}^{t} (s-\omega_0)^n \, \tilde{d}_{q,\omega} s = \frac{q^n}{\widetilde{[n+1]}_q} (t-\omega_0)^{n+1},$$

$$(iii) \quad \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega} \, \tilde{d}_{q,\omega} s = \frac{(t-\omega_0)^{\alpha}}{\widetilde{[\alpha]}_q},$$

$$(iv) \quad \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega} \left(\sigma^{\alpha-1}_{q,\omega}(s) - \omega_0\right)^{\beta} \tilde{d}_{q,\omega} s = q^{\alpha\beta} (t-\omega_0)^{\alpha+\beta} \tilde{B}_q(\beta+1, \alpha).$$

The following lemma describes the properties of the fractional symmetry Hahn integral.

Lemma 7. *Let* $\alpha, \beta > 0$, $q \in (0,1)$, $\omega > 0$ *and* $n \in \mathbb{N}$. *Then,*

$$(i) \quad \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{-\beta-1}}_{q,\omega} \left(\sigma^{-\beta-1}_{q,\omega}(s) - \omega_0\right)^{\alpha-n} \tilde{d}_{q,\omega} s = \frac{(t-\omega_0)^{\alpha-\beta-n}}{q^{\beta(\alpha-n)}} \tilde{B}_q(\alpha-n+1, -\beta),$$

$$(ii) \quad \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{\theta-1}}_{q,\omega} \left(\sigma^{\theta-1}_{q,\omega}(s) - \omega_0\right)^{\alpha-n} \tilde{d}_{q,\omega} s = \frac{(t-\omega_0)^{\alpha+\theta-n}}{q^{-\theta(\alpha-n)}} \tilde{B}_q(\alpha-n+1, \theta),$$

$$(iii) \quad \int_{\omega_0}^{t} \int_{\omega_0}^{\sigma^{-\beta-1}_{q,\omega}(x)} \widetilde{(t-x)}^{\underline{-\beta-1}}_{q,\omega} \left(\widetilde{\sigma^{-\beta-1}_{q,\omega}(x) - s}\right)^{\underline{\alpha-1}}_{q,\omega} \tilde{d}_{q,\omega} s \, \tilde{d}_{p,\omega} x = \frac{(t-\omega_0)^{\alpha-\beta}}{\tilde{[\alpha]}_q q^{\alpha\beta}} \tilde{B}_q(\alpha+1, -\beta),$$

$$(iv) \quad \int_{\omega_0}^{t} \int_{\omega_0}^{\sigma^{\theta-1}_{q,\omega}(x)} \widetilde{(t-x)}^{\underline{\theta-1}}_{q,\omega} \left(\widetilde{\sigma^{\theta-1}_{q,\omega}(x) - s}\right)^{\underline{\alpha-1}}_{q,\omega} \tilde{d}_{q,\omega} s \, \tilde{d}_{p,\omega} x = \frac{(t-\omega_0)^{\alpha+\theta}}{\tilde{[\alpha]}_q q^{-\alpha\theta}} \tilde{B}_q(\alpha+1, \theta).$$

Proof. We use the definition of power function of q,ω-symmetric analogue, Lemma 1, Lemma 6, and Definition 2. Then, we obtain

$$(i) \quad \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{-\beta-1}}_{q,\omega} \left(\sigma^{-\beta-1}_{q,\omega}(s) - \omega_0\right)^{\alpha-n} \tilde{d}_{q,\omega} s$$

$$= (1-q^2)(t-\omega_0) \sum_{k=0}^{\infty} q^{2k} \left(\widetilde{t - \sigma^{2k+1}_{q,\omega}(t)}\right)^{\underline{-\beta-1}}_{q,\omega} \left(q^{-\beta-1}\left(\sigma^{2k+1}_{q,\omega}(t) - \omega_0\right)\right)^{\alpha-n}$$

$$= q^{-\beta(\alpha-n)}(1-q^2)(t-\omega_0)^{\alpha-\beta-n}\sum_{k=0}^{\infty} q^{2k}\widetilde{(1-q^{2k+1})}_q^{\underline{-\beta-1}}(q^{-1}q^{2k+1})^{\alpha-n}$$
$$= q^{-\beta(\alpha-n)}(t-\omega_0)^{\alpha-\beta-n}\int_{\omega_0}^{1}\widetilde{(1-s)}_{q,\omega}^{\underline{-\beta-1}}\left(q^{-1}s\right)^{\alpha-n}\tilde{d}_{q,\omega}s$$
$$= \frac{(t-\omega_0)^{\alpha-\beta-n}}{q^{\beta(\alpha-n)}}\tilde{B}_q(\alpha-n+1,-\beta).$$

Using (i), we have

$$(iii)\quad \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\beta-1}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{-\beta-1}}\left(\widetilde{\sigma_{q,\omega}^{-\beta-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\,\tilde{d}_{p,\omega}x$$
$$= \int_{\omega_0}^{t}\widetilde{(t-x)}_{q,\omega}^{\underline{-\beta-1}}\left[\int_{\omega_0}^{\sigma_{q,\omega}^{-\beta-1}(x)}\left(\widetilde{\sigma_{q,\omega}^{-\beta-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\right]\tilde{d}_{p,\omega}x$$
$$= \frac{1}{[\tilde{\alpha}]_q}\int_{\omega_0}^{t}\widetilde{(t-x)}_{q,\omega}^{\underline{-\beta-1}}\left(\sigma_{q,\omega}^{-\beta-1}(x)-\omega_0\right)^{\alpha}\tilde{d}_{p,\omega}x$$
$$= \frac{(t-\omega_0)^{\alpha-\beta}}{[\tilde{\alpha}]_q q^{\alpha\beta}}\tilde{B}_q(\alpha+1,-\beta).$$

Similarly, we can obain (ii) and (iv). □

2.3. Lemma for Linear Variant Form

In the following lemmas, we establish a linear variant form of the problem (1) and investigate its solution that plays an important role in the upcoming analysis.

Lemma 8. *Let* $\Omega \neq 0$; $\omega > 0$; $q \in (0,1)$; $\alpha \in (1,2]$; $\beta_1,\beta_2,\theta_1,\theta_2 \in (0,1]$; $\lambda_1,\lambda_2,\mu_1,\mu_2,\kappa_1,\kappa_2 \in \mathbb{R}^+$; $h \in C\left(I_{q,\omega}^T,\mathbb{R}\right)$ *be given function; and* $\phi_1,\phi_2 : C\left(I_{q,\omega}^T,\mathbb{R}\right)\to\mathbb{R}$ *be given functionals. Then, the linear variant form*

$$\tilde{D}_{q,\omega}^{\alpha}u(t) = h(t),\quad t\in I_{q,\omega}^T, \tag{2}$$
$$\lambda_i u(\eta_i)+\mu_i\tilde{D}_{q,\omega}^{\beta_i}u(\eta_i)+\kappa_i\tilde{I}_{q,\omega}^{\theta_i}u(\eta_i) = \phi_i(u),\quad \eta_i\in I_{q,\omega}^T-\{\omega_0,T\},\ i=1,2 \tag{3}$$

has the unique solution

$$\begin{aligned} u(t) = {} & \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}h(\sigma_{q,\omega}^{\alpha-1}(s))\,\tilde{d}_{q,\omega}s \\ & +\frac{(t-\omega_0)^{\alpha-1}}{\Omega}\left\{\boldsymbol{B}_1\Phi_2[\phi_2,h]-\boldsymbol{B}_2\Phi_1[\phi_1,h]\right\} \\ & -\frac{(t-\omega_0)^{\alpha-2}}{\Omega}\left\{\boldsymbol{A}_1\Phi_2[\phi_2,h]-\boldsymbol{A}_2\Phi_1[\phi_1,h]\right\} \end{aligned} \tag{4}$$

where the functionals $\Phi_i[\phi_i,h],\ i=1,2$ *are defined by*

$$\begin{aligned} \Phi_i[\phi_i,h] := {} & \phi_i(u)-\frac{\lambda_i q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_i}\widetilde{(\eta_i-s)}_{q,\omega}^{\underline{\alpha-1}}h(\sigma_{q,\omega}^{\alpha-1}(s))\,\tilde{d}_{q,\omega}s-\frac{\mu_i q^{\binom{\alpha}{2}+\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\beta_i)}\times \\ & \int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma_{q,\omega}^{-\beta_i-1}(x)}\widetilde{(\eta_i-x)}_{q,\omega}^{\underline{-\beta_i-1}}\left(\widetilde{\sigma_{q,\omega}^{-\beta_i-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}h(\sigma_{q,\omega}^{\alpha-1}(s))\,\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x \\ & -\frac{\kappa_i q^{\binom{\alpha}{2}+\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_i-1}(x)}\widetilde{(\eta_i-x)}_{q,\omega}^{\underline{\theta_i-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_i-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\times \\ & h(\sigma_{q,\omega}^{\alpha-1}(s))\,\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x, \end{aligned} \tag{5}$$

and the constants $\boldsymbol{A}_i, \boldsymbol{B}_i,\ i = 1,2$ *and* Ω *are defined by*

$$\begin{aligned}
\boldsymbol{A}_i &:= \lambda_i(\eta_i - \omega_0)^{\alpha-1} + \mu_i q^{\binom{-\beta_i}{2} - \beta_i(\alpha-1)}(\eta_i - \omega_0)^{\alpha-\beta_i-1}\frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha-\beta_i)} \\
&\quad + \kappa_i q^{\binom{\theta_i}{2} + \theta_i(\alpha-1)}(\eta_i - \omega_0)^{\alpha+\theta_i-1}\frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha+\theta_i)}, \qquad (6) \\
\boldsymbol{B}_i &:= \lambda_i(\eta_i - \omega_0)^{\alpha-2} + \mu_i q^{\binom{-\beta_i}{2} - \beta_i(\alpha-2)}(\eta_i - \omega_0)^{\alpha-\beta_i-2}\frac{\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\alpha-\beta_i-1)} \\
&\quad + \kappa_i q^{\binom{\theta_i}{2} + \theta_i(\alpha-2)}(\eta_i - \omega_0)^{\alpha+\theta_i-2}\frac{\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\alpha+\theta_i-1)}, \qquad (7) \\
\Omega &:= \boldsymbol{A}_1\boldsymbol{B}_2 - \boldsymbol{A}_2\boldsymbol{B}_1. \qquad (8)
\end{aligned}$$

Proof. We first take fractional symmetric Hahn integral of order α for (2) and use Lemma 2. We find that

$$u(t) = C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\, h\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\tilde{d}_{q,\omega}s. \qquad (9)$$

Then, we take fractional symmetric Hahn difference of order $\beta_i,\ i = 1,2$ for (9); we obtain

$$\begin{aligned}
\tilde{D}_{q,\omega}^{\beta_i}u(t) = {} & \frac{C_1 q^{\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(-\beta_i)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\beta_i-1}}\left(\sigma_{q,\omega}^{-\beta_i-1}(s)-\omega_0\right)^{\alpha-1}\tilde{d}_{q,\omega}s \qquad (10) \\
& + \frac{C_2 q^{\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(-\beta_i)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\beta_i-1}}\left(\sigma_{q,\omega}^{-\beta_i-1}(s)-\omega_0\right)^{\alpha-2}\tilde{d}_{q,\omega}s + \frac{q^{\binom{\alpha}{2}+\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\beta_i)} \times \\
& \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\beta_i-1}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{-\beta_i-1}}\left(\widetilde{\sigma_{q,\omega}^{-\beta_i-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}} h\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}r.
\end{aligned}$$

Next, take fractional symmetric Hahn integral of order $\theta_i,\ i = 1,2$ for (9). Then,

$$\begin{aligned}
\tilde{I}_{q,\omega}^{\theta_i}u(t) = {} & \frac{C_1 q^{\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\theta_i-1}}\left(\sigma_{q,\omega}^{\theta_i-1}(s)-\omega_0\right)^{\alpha-1}\tilde{d}_{q,\omega}s \qquad (11) \\
& + \frac{C_2 q^{\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\theta_i-1}}\left(\sigma_{q,\omega}^{\theta_i-1}(s)-\omega_0\right)^{\alpha-2}\tilde{d}_{q,\omega}s \\
& + \frac{q^{\binom{\alpha}{2}+\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta-1}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{\theta-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_i-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}} h\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}r.
\end{aligned}$$

Substituting $t = \eta_i,\ i = 1,2$ into (9)–(11), and from the conditions of (3), we have

$$\begin{aligned}
\mathbf{A}_1 C_1 + \mathbf{B}_1 C_2 &= \Phi_1[\phi_1, h], \qquad (12) \\
\mathbf{A}_2 C_1 + \mathbf{B}_2 C_2 &= \Phi_2[\phi_2, h]. \qquad (13)
\end{aligned}$$

The solutions of the system of Equations (11)–(13) are given by

$$C_1 = \frac{\mathbf{B}_1\Phi_2[\phi_2,h] - \mathbf{B}_2\Phi_1[\phi_1,h]}{\Omega} \quad \text{and} \quad C_2 = \frac{\mathbf{A}_2\Phi_1[\phi_1,h] - \mathbf{A}_1\Phi_2[\phi_2,h]}{\Omega}.$$

Then, we get the solution (4) after substituting the constants C_1 and C_2 into (9). For the converse, we can prove it by using direct computation. Our proof is complete. □

3. Main Results

In this section, we present the existence results of problem (1).

3.1. Existence and Uniqueness Result

Using Banach fixed point theorem, we first let $\mathcal{C} = C\left(I^T_{q,\omega}, \mathbb{R}\right)$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \|u\| + \|\tilde{D}^{\nu}_{q,\omega}u\| + \|\tilde{D}^{\nu+1}_{q,\omega}u\|,$$

where $\|u\| = \max_{t\in I^T_{q,\omega}} |u(t)|$ and $\|\tilde{D}^{\nu}_{q,\omega}u\| = \max_{t\in I^T_{q,\omega}} \left|(\tilde{D}^{\nu}_{q,\omega}u)(t)\right|$.

By Lemma 8, replacing $h(t)$ by $F\left[t, u(t), (\tilde{\Psi}^{\gamma}_{q,\omega}u)(t), \tilde{D}^{\nu}_{q,\omega}u(t), \tilde{D}^{\nu+1}_{q,\omega}u(t)\right]$, an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is defined as

$$\begin{aligned}(\mathcal{A}u)(t) :=\ & \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}^{\overline{\alpha-1}}_{q,\omega} F\left[\sigma^{\alpha-1}_{q,\omega}(s), u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), (\tilde{\Psi}^{\gamma}_{q,\omega}u)\left(\sigma^{\alpha-1}_{q,\omega}(s)\right),\right.\\ & \left.\tilde{D}^{\nu}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), \tilde{D}^{\nu+1}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\right] \tilde{d}_{q,\omega}s\\ & + \frac{(t-\omega_0)^{\alpha-1}}{\Omega}\left\{\mathbf{B}_1\Phi^*_2[\phi_2, F_u] - \mathbf{B}_2\Phi^*_1[\phi_1, F_u]\right\}\\ & - \frac{(t-\omega_0)^{\alpha-2}}{\Omega}\left\{\mathbf{A}_1\Phi^*_2[\phi_2, F_u] - \mathbf{A}_2\Phi^*_1[\phi_1, F_u]\right\}\end{aligned} \tag{14}$$

where the functionals $\Phi_i{}^*[\phi_i, F_u],\ i = 1, 2$ are given by

$$\begin{aligned}\Phi_i{}^*[\phi_i, F_u] :=\ & \phi_i(u) - \frac{\lambda_i q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{\eta_i} \widetilde{(\eta_i-s)}^{\overline{\alpha-1}}_{q,\omega} F\left[\sigma^{\alpha-1}_{q,\omega}(s), u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), (\tilde{\Psi}^{\gamma}_{q,\omega}u)\left(\sigma^{\alpha-1}_{q,\omega}(s)\right),\right.\\ & \left.\tilde{D}^{\nu}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), \tilde{D}^{\nu+1}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\right] \tilde{d}_{q,\omega}s\\ & - \frac{\mu_i q^{\binom{\alpha}{2}+\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\beta_i)} \int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma^{-\beta_i-1}_{q,\omega}(x)} \widetilde{(\eta_i-x)}^{\overline{-\beta_i-1}}_{q,\omega} \widetilde{\left(\sigma^{-\beta_i-1}_{q,\omega}(x)-s\right)}^{\overline{\alpha-1}}_{q,\omega} \times\\ & F\left[\sigma^{\alpha-1}_{q,\omega}(s), u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), (\tilde{\Psi}^{\gamma}_{q,\omega}u)\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), \tilde{D}^{\nu}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right),\right.\\ & \left.\tilde{D}^{\nu+1}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\right] \tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x,\end{aligned} \tag{15}$$

and the constants $\mathbf{A}_i, \mathbf{B}_i,\ i = 1, 2$ and Ω are defined by (6)–(8), respectively.

In order to show that the solution of problem (1) exist, we need to prove that the operator $\mathcal{A}$ has a fixed point.

Theorem 1. *Let $F : I^T_{q,\omega} \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function, and $\varphi : I^T_{q,\omega} \times I^T_{q,\omega} \to [0, \infty)$ be a continuous function with $\varphi_0 = \max\left\{\varphi(t, s) : (t, s) \in I^T_{q,\omega} \times I^T_{q,\omega}\right\}$. The following hypotheses are assumed.*

(H_1) *There exist positive constants $\ell_i,\ i = 1, 2, 3, 4$, such that for each $t \in I^T_{q,\omega}$ and $u_i, v_i \in \mathbb{R}$,*

$$\left|F[t, u_1, u_2, u_3, u_4] - F[t, v_1, v_2, v_3, v_4]\right| \leq \ell_1|u_1 - v_1| + \ell_2|u_2 - v_2| + \ell_3|u_3 - v_3| + \ell_4|u_4 - v_4|.$$

(H_2) *There exist positive constants $\omega_i,\ i = 1, 2$, such that for each $u, v \in \mathcal{C}$,*

$$|\phi_i(u) - \phi_i(v)| \leq \omega_i\|u - v\|_{\mathcal{C}}.$$

(H_3) $\Xi := \mathcal{L}\chi + \omega_1\mathcal{O}^*_1 + \omega_2\mathcal{O}^*_2 < 1,$

where

$$\mathcal{L} := \max\left\{\ell_1 + \ell_2\varphi_0 \frac{(T-\omega_0)^\gamma}{\tilde{\Gamma}_q(\gamma+1)}, \ell_3, \ell_4\right\} \tag{16}$$

$$\begin{aligned}\chi := &\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha+1)}(T-\omega_0)^\alpha + \frac{q^{\binom{\alpha-\nu}{2}}}{\tilde{\Gamma}_q(\alpha-\nu+1)}(T-\omega_0)^{\alpha-\nu} \\ &+ \frac{q^{\binom{\alpha-\nu-1}{2}}}{\tilde{\Gamma}_q(\alpha-\nu)}(T-\omega_0)^{\alpha-\nu-1} + \Theta_1\mathcal{O}_1^* + \Theta_2\mathcal{O}_2^*\end{aligned} \tag{17}$$

$$\mathcal{O}_i^* := \mathcal{O}_i + \bar{\mathcal{O}}_i + \bar{\bar{\mathcal{O}}}_i,\ i=1,2 \tag{18}$$

$$\mathcal{O}_i := \frac{1}{|\Omega|}\left\{|\boldsymbol{B}_i|(T-\omega_0)^{\alpha-1} + |\boldsymbol{A}_i|(T-\omega_0)^{\alpha-2}\right\},\ i=1,2 \tag{19}$$

$$\begin{aligned}\bar{\mathcal{O}}_i := &\frac{1}{|\Omega|}\left\{|\boldsymbol{B}_i| q^{\binom{-\nu}{2}-\nu(\alpha-1)} \frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha-\nu)}(T-\omega_0)^{\alpha-\nu-1} + |\boldsymbol{A}_i| q^{\binom{-\nu}{2}-\nu(\alpha-2)} \times \right. \\ &\left.\frac{\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\alpha-\nu-1)}(T-\omega_0)^{\alpha-\nu-2}\right\},\ i=1,2\end{aligned} \tag{20}$$

$$\begin{aligned}\bar{\bar{\mathcal{O}}}_i := &\frac{1}{|\Omega|}\left\{|\boldsymbol{B}_i| q^{\binom{-\nu-1}{2}-(\nu+1)(\alpha-1)} \frac{\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\alpha-\nu-1)}(T-\omega_0)^{\alpha-\nu-2} + |\boldsymbol{A}_i| q^{\binom{-\nu-1}{2}-(\nu+1)(\alpha-2)} \times \right. \\ &\left.\frac{\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\alpha-\nu-2)}(T-\omega_0)^{\alpha-\nu-3}\right\},\ i=1,2\end{aligned} \tag{21}$$

$$\Theta_i := \lambda_i q^{\binom{\alpha}{2}} \frac{(\eta_i-\omega_0)^\alpha}{\tilde{\Gamma}_q(\alpha+1)} + \mu_i q^{\binom{\alpha-\beta_i}{2}} \frac{(\eta_i-\omega_0)^{\alpha-\beta_i}}{\tilde{\Gamma}_q(\alpha-\beta_i+1)} + \kappa_i q^{\binom{\alpha+\theta_i}{2}} \frac{(\eta_i-\omega_0)^{\alpha+\theta_i}}{\tilde{\Gamma}_q(\alpha+\theta_i+1)},\ i=1,2. \tag{22}$$

Then, the problem (1) has a unique solution on $I_{q,\omega}^T$*.*

Proof. For each $t \in I_{q,\omega}^T$ and $u, v \in \mathcal{C}$, we have

$$\begin{aligned}\left|\tilde{\Psi}_{q,\omega}^\gamma u - \tilde{\Psi}_{q,\omega}^\gamma v\right| &\le \frac{\varphi_0 q^{\binom{\gamma}{2}}}{\tilde{\Gamma}_q(\gamma)} \int_{\omega_0}^t \widetilde{(t-s)}_{q,\omega}^{\underline{\gamma-1}} \left|u\left(\sigma_{q,\omega}^{\gamma-1}(s)\right) - v\left(\sigma_{q,\omega}^{\gamma-1}(s)\right)\right| \tilde{d}_{q,\omega}s \\ &\le \frac{\varphi_0 q^{\binom{\gamma}{2}}}{\tilde{\Gamma}_q(\gamma)} \|u-v\| \int_{\omega_0}^T \widetilde{(T-s)}_{q,\omega}^{\underline{\gamma-1}}\, \tilde{d}_{q,\omega}s \\ &= \frac{\varphi_0 (T-\omega_0)^\gamma}{\tilde{\Gamma}_q(\gamma+1)}\|u-v\|.\end{aligned}$$

Let

$$\mathcal{F}|u-v|(t) := \left|F\left[t, u(t), (\tilde{\Psi}_{q,\omega}^\gamma u)(t), \tilde{D}_{q,\omega}^\nu u(t), \tilde{D}_{q,\omega}^{\nu+1} u(t)\right] - F\left[t, v(t), (\tilde{\Psi}_{q,\omega}^\gamma v)(t), \tilde{D}_{q,\omega}^\nu v(t), \tilde{D}_{q,\omega}^{\nu+1} v(t)\right]\right|.$$

Then, for $i = 1, 2$, we have

$$\begin{aligned}&\left|\Phi_i^*[\phi_i, F_u] - \Phi_i^*[\phi_i, F_v]\right| \\ &\le |\phi_i(u) - \phi_i(v)| + \frac{\lambda_i q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{\eta_i} \widetilde{(\eta_i - s)}_{q,\omega}^{\underline{\alpha-1}} \mathcal{F}|u-v|\left(\sigma_{q,\omega}^{\alpha-1}(s)\right) \tilde{d}_{q,\omega}s \\ &+ \frac{\mu_i q^{\binom{\alpha}{2}+\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\beta_i)} \int_{\omega_0}^{\eta_i} \int_{\omega_0}^{\sigma_{q,\omega}^{-\beta_i-1}(x)} \widetilde{(\eta_i - x)}_{q,\omega}^{\underline{-\beta_i-1}} \left(\widetilde{\sigma_{q,\omega}^{-\beta_i-1}(x) - s}\right)_{q,\omega}^{\underline{\alpha-1}} \times \\ &\mathcal{F}|u-v|\left(\sigma_{q,\omega}^{\alpha-1}(s)\right) \tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_i q^{\binom{\alpha}{2}+\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_i-1}(x)}\widetilde{(\eta_i-x)}_{q,\omega}^{\underline{\theta_i-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_i-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\times\\
&\mathcal{F}|u-v|\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}x\\
\leq\ &\omega_i\|u-v\|_{\mathcal{C}}+\Big(\ell_1|u-v|+\ell_2\big|\tilde{\Psi}_{q,\omega}^{\gamma}u-\tilde{\Psi}_{q,\omega}^{\gamma}v\big|+\ell_3\Big|\tilde{D}_{q,\omega}^{\nu}u-\tilde{D}_{q,\omega}^{\nu}v\Big|\\
&+\ell_4\Big|\tilde{D}_{q,\omega}^{\nu+1}u-\tilde{D}_{q,\omega}^{\nu+1}v\Big|\Big)\left[\lambda_i q^{\binom{\alpha}{2}}\frac{(\eta_i-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}+\mu_i q^{\binom{\alpha-\beta_i}{2}}\frac{(\eta_i-\omega_0)^{\alpha-\beta_i}}{\tilde{\Gamma}_q(\alpha-\beta_i+1)}\right.\\
&\left.+\kappa_i q^{\binom{\alpha+\theta_i}{2}}\frac{(\eta_i-\omega_0)^{\alpha+\theta_i}}{\tilde{\Gamma}_q(\alpha+\theta_i+1)}\right]\\
\leq\ &(\omega_i+\mathcal{L}\Theta_i)\|u-v\|_{\mathcal{C}}.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\big|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\big|\\
&\leq\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}\mathcal{F}|u-v|\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\tilde{d}_{q,\omega}s\\
&\quad+\frac{(T-\omega_0)^{\alpha-1}}{|\Omega|}\Big\{|\mathbf{B}_1|\big|\Phi_2^*[\phi_2,F_u]-\Phi_2^*[\phi_2,F_v]\big|+|\mathbf{B}_2|\big|\Phi_1^*[\phi_1,F_u]-\Phi_1^*[\phi_1,F_v]\big|\Big\}\\
&\quad+\frac{(T-\omega_0)^{\alpha-2}}{|\Omega|}\Big\{|\mathbf{A}_1|\big|\Phi_2^*[\phi_2,F_u]-\Phi_2^*[\phi_2,F_v]\big|+|\mathbf{A}_2|\big|\Phi_1^*[\phi_1,F_u]-\Phi_1^*[\phi_1,F_v]\big|\Big\}\\
&\leq\left\{\mathcal{L}q^{\binom{\alpha}{2}}\frac{(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}+\frac{\omega_1+\mathcal{L}\Theta_1}{|\Omega|}\Big[|\mathbf{B}_1|(T-\omega_0)^{\alpha-1}+|\mathbf{A}_1|(T-\omega_0)^{\alpha-2}\Big]\right.\\
&\quad\left.+\frac{\omega_2+\mathcal{L}\Theta_2}{|\Omega|}\Big[|\mathbf{B}_2|(T-\omega_0)^{\alpha-1}+|\mathbf{A}_2|(T-\omega_0)^{\alpha-2}\Big]\right\}\|u-v\|_{\mathcal{C}}\\
&\leq\left\{\mathcal{L}\left[q^{\binom{\alpha}{2}}\frac{(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}+\Theta_1\mathcal{O}_1+\Theta_2\mathcal{O}_2\right]+\omega_1\mathcal{O}_1+\omega_2\mathcal{O}_2\right\}\|u-v\|_{\mathcal{C}}.
\end{aligned}\tag{23}
$$

We take the fractional symmetric Hahn difference of order ν and $\nu+1$ for (14). Then, we have

$$
\begin{aligned}
&(\tilde{D}_{q,\omega}^{\nu}\mathcal{A}u)(t)\\
&=\frac{q^{\binom{\alpha}{2}+\binom{-\nu}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\nu-1}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{-\nu-1}}\left(\widetilde{\sigma_{q,\omega}^{-\nu-1}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\times\\
&\quad F\Big[\sigma_{q,\omega}^{\alpha-1}(s),u\left(\sigma_{q,\omega}^{\alpha-1}(s)\right),(\tilde{\Psi}_{q,\omega}^{\gamma}u)\left(\sigma_{q,\omega}^{\alpha-1}(s)\right),(\tilde{Y}_{q,\omega}^{\nu}u)\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\Big]\tilde{d}_{q,\omega}s\\
&\quad+\Big\{\mathbf{B}_1\Phi_2^*[\phi_2,F_u]-\mathbf{B}_2\Phi_1^*[\phi_1,F_u]\Big\}\frac{q^{\binom{-\nu}{2}}}{\Omega\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\nu-1}}\left(\sigma_{q,\omega}^{-\nu-1}(s)-\omega_0\right)^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\\
&\quad-\Big\{\mathbf{A}_1\Phi_2^*[\phi_2,F_u]-\mathbf{A}_2\Phi_1^*[\phi_1,F_u]\Big\}\frac{q^{\binom{-\nu}{2}}}{\Omega\tilde{\Gamma}_q(-\nu)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\nu-1}}\left(\sigma_{q,\omega}^{-\nu-1}(s)-\omega_0\right)^{\underline{\alpha-2}}\tilde{d}_{q,\omega}s,
\end{aligned}\tag{24}
$$

and

$$
\begin{aligned}
&(\tilde{D}_{q,\omega}^{\nu+1}\mathcal{A}u)(t)\\
&=\frac{q^{\binom{\alpha}{2}+\binom{-\nu-1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\nu-1)}\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{-\nu-2}(x)}\widetilde{(t-x)}_{q,\omega}^{\underline{-\nu-2}}\left(\widetilde{\sigma_{q,\omega}^{-\nu-2}(x)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\times
\end{aligned}\tag{25}
$$

$$F\left[\sigma_{q,\omega}^{\alpha-1}(s), u\left(\sigma_{q,\omega}^{\alpha-1}(s)\right), (\tilde{\Psi}_{q,\omega}^{\gamma}u)\left(\sigma_{q,\omega}^{\alpha-1}(s)\right), (\tilde{Y}_{q,\omega}^{\nu}u)\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\right] \tilde{d}_{q,\omega}s$$
$$+\left\{\mathbf{B}_1\Phi_2^*[\phi_2, F_u] - \mathbf{B}_2\Phi_1^*[\phi_1, F_u]\right\}\frac{q^{\binom{-\nu-1}{2}}}{\Omega\tilde{\Gamma}_q(-\nu-1)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\nu-2}}\left(\sigma_{q,\omega}^{-\nu-2}(s)-\omega_0\right)^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s$$
$$-\left\{\mathbf{A}_1\Phi_2^*[\phi_2, F_u] - \mathbf{A}_2\Phi_1^*[\phi_1, F_u]\right\}\frac{q^{\binom{-\nu-1}{2}}}{\Omega\tilde{\Gamma}_q(-\nu-1)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\nu-2}}\left(\sigma_{q,\omega}^{-\nu-2}(s)-\omega_0\right)^{\underline{\alpha-2}}\tilde{d}_{q,\omega}s.$$

Similary, we have

$$\left|(\tilde{D}_{q,\omega}^{\nu}\mathcal{A}u)(t) - (\tilde{D}_{q,\omega}^{\nu}\mathcal{A}v)(t)\right| \le \left\{\mathcal{L}\left[q^{\binom{\alpha-\nu}{2}}\frac{(T-\omega_0)^{\alpha-\nu}}{\tilde{\Gamma}_q(\alpha-\nu+1)} + \Theta_1\bar{\mathcal{O}}_1 + \Theta_2\bar{\mathcal{O}}_2\right] + \omega_1\bar{\mathcal{O}}_1 + \omega_2\bar{\mathcal{O}}_2\right\}\|u-v\|_{\mathcal{C}}, \tag{26}$$

and

$$\left|(\tilde{D}_{q,\omega}^{\nu+1}\mathcal{A}u)(t) - (\tilde{D}_{q,\omega}^{\nu+1}\mathcal{A}v)(t)\right| \le \left\{\mathcal{L}\left[q^{\binom{\alpha-\nu-1}{2}}\frac{(T-\omega_0)^{\alpha-\nu-1}}{\tilde{\Gamma}_q(\alpha-\nu)} + \Theta_1\bar{\bar{\mathcal{O}}}_1 + \Theta_2\bar{\bar{\mathcal{O}}}_2\right] + \omega_1\bar{\bar{\mathcal{O}}}_1 + \omega_2\bar{\bar{\mathcal{O}}}_2\right\}\|u-v\|_{\mathcal{C}}. \tag{27}$$

From (23), (26) and (27), we get

$$\|\mathcal{A}u - \mathcal{A}v\|_{\mathcal{C}} \le \left[\mathcal{L}\chi + \omega_1\mathcal{O}_1^* + \omega_2\mathcal{O}_2^*\right]\|u-v\|_{\mathcal{C}} = \Xi\,\|u-v\|_{\mathcal{C}}.$$

From (H_3), we can conclude that $\mathcal{A}$ is a contraction. Based on Banach fixed point theorem, we can conclude that $\mathcal{A}$ has a fixed point. Thus, problem (1) has a unique solution on $I_{q,\omega}^{T}$. □

3.2. Existence of at Least One Solution

The Schauder's fixed point theorem is employed to prove that problem (1) has at least one solution by using hypothesises (H_1) and (H_3) obtained from Theorem 1 in the previous section.

Theorem 2. *Assume that (H_1) and (H_3) hold. Then, problem (1) has at least one solution on $I_{q,\omega}^{T}$.*

Proof. Step I. Prove that $\mathcal{A}$ map bounded sets into bounded sets in $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le R\}$. Assume that $\max_{t\in I_{q,\omega}^{T}}|F(t,0,0,0)| = F$, $\sup_{u\in\mathcal{C}}|\phi_i(u)| = M_i$ for $i = 1,2$. Then, we choose a constant

$$R \ge \frac{F\chi + M_1\mathcal{O}_1^* + M_2\mathcal{O}_2^*}{1-\mathcal{L}\chi}. \tag{28}$$

Denote that

$$\left|\mathcal{F}(t,u,0)\right| = \left|F\left[t,u(t),\tilde{\Psi}_{q,\omega}^{\gamma}u(t),\tilde{D}_{q,\omega}^{\nu}u(t),\tilde{D}_{q,\omega}^{\nu+1}u(t)\right] - F[t,0,0,0,0]\right| + \left|F[t,0,0,0,0]\right|.$$

For $i = 1,2$, for each $t \in I_{q,\omega}^{T}$ and $u \in B_R$, we have

$$\Phi_i^*[\phi_i, F_u] \le M_i + \frac{\lambda_i q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_i}\widetilde{(\eta_i-s)}_{q,\omega}^{\underline{\alpha-1}}\left|\mathcal{F}\left(\sigma_{q,\omega}^{\alpha-1}(s),u,0\right)\right|\tilde{d}_{q,\omega}s + \frac{\mu_i q^{\binom{\alpha}{2}+\binom{-\beta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(-\beta_i)}\times$$
$$\int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma_{q,\omega}^{-\beta_i-1}(x)}\widetilde{(\eta_i-x)}_{q,\omega}^{\underline{-\beta_i-1}}\left(\sigma_{q,\omega}^{-\beta_i-1}(x)-s\right)_{q,\omega}^{\underline{\alpha-1}}\left|\mathcal{F}(t,u,0)\right|\tilde{d}_{q,\omega}x$$

$$+\frac{\kappa_i q^{\binom{\alpha}{2}+\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{\eta_i}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_i-1}(x)}\widetilde{(\eta_i-x)}_{q,\omega}^{\theta_i-1}\left(\widetilde{\sigma_{q,\omega}^{\theta_i-1}(x)-s}\right)_{q,\omega}^{\alpha-1}|\mathcal{F}(t,u,0)|\,\tilde{d}_{q,\omega}x$$
$$\leq M_i+\left[\left(\ell_1|u|+\ell_2\left|\tilde{\Psi}_{q,\omega}^{\gamma}u\right|+\ell_3\left|\tilde{D}_{q,\omega}^{\nu}u\right|+\ell_4\left|\tilde{D}_{q,\omega}^{\nu+1}u\right|\right)+F\right]\Theta_i$$
$$\leq M_i+[\mathcal{L}\|u\|_{\mathcal{C}}+F]\Theta_i, \tag{29}$$

and

$$\left|(\mathcal{A}u)(t)\right| \leq \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(t-s)}_{q,\omega}^{\alpha-1}\left|\mathcal{F}\left(\sigma_{q,\omega}^{\alpha-1}(s),u,0\right)\right|\tilde{d}_{q,\omega}s \tag{30}$$
$$+\frac{(T-\omega_0)^{\alpha-1}}{|\Omega|}\Big\{|\mathbf{B}_1|\big|\Phi_2^*[\phi_2,F_u]\big|+|\mathbf{B}_2|\big|\Phi_1^*[\phi_1,F_u]\big|\Big\}$$
$$+\frac{(T-\omega_0)^{\alpha-2}}{|\Omega|}\Big\{|\mathbf{A}_1|\big|\Phi_2^*[\phi_2,F_u]\big|+|\mathbf{A}_2|\big|\Phi_1^*[\phi_1,F_u]\big|\Big\}$$
$$\leq (\mathcal{L}\|u\|_{\mathcal{C}}+F)\left[q^{\binom{\alpha}{2}}\frac{(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}+\Theta_1\mathcal{O}_1+\Theta_2\mathcal{O}_2\right]+M_1\mathcal{O}_1+M_2\mathcal{O}_2.$$

Similary,

$$\left|\left(\tilde{D}_{q,\omega}^{\nu}\mathcal{F}u\right)(t)\right| \tag{31}$$
$$\leq (\mathcal{L}\|u\|_{\mathcal{C}}+F)\left[q^{\binom{\alpha-\nu}{2}}\frac{(T-\omega_0)^{\alpha-\nu}}{\tilde{\Gamma}_q(\alpha-\nu+1)}+\Theta_1\bar{\mathcal{O}}_1+\Theta_2\bar{\mathcal{O}}_2\right]+\omega_1\bar{\mathcal{O}}_1+\omega_2\bar{\mathcal{O}}_2,$$

and

$$\left|\left(\tilde{D}_{q,\omega}^{\nu+1}\mathcal{F}u\right)(t)\right| \tag{32}$$
$$\leq (\mathcal{L}\|u\|_{\mathcal{C}}+F)\left[q^{\binom{\alpha-\nu-1}{2}}\frac{(T-\omega_0)^{\alpha-\nu-1}}{\tilde{\Gamma}_q(\alpha-\nu)}+\Theta_1\bar{\bar{\mathcal{O}}}_1+\Theta_2\bar{\bar{\mathcal{O}}}_2\right]+\omega_1\bar{\bar{\mathcal{O}}}_1+\omega_2\bar{\bar{\mathcal{O}}}_2.$$

From (30)–(32), we obtain

$$\|\mathcal{A}u\|_{\mathcal{C}} \leq (\mathcal{L}\|u\|_{\mathcal{C}}+F)\chi+\omega_1\mathcal{O}_1^*+\omega_2\mathcal{O}_2^* \leq R. \tag{33}$$

Hence, $\mathcal{A}$ is uniformly bounded.
Step II. We find that $\mathcal{A}$ is the continuous operator on B_R, since the continuity of F.
Step III. Examine that $\mathcal{A}$ is equicontinuous on B_R.
For any $t_1,t_2\in I_{q,\omega}^{T}$ with $t_1<t_2$, we have

$$\left|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\right|$$
$$\leq \frac{\|F\|q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha+1)}\left|(t_2-\omega_0)^{\alpha}-(t_1-\omega_0)^{\alpha}\right|$$
$$+\frac{\left|(t_2-\omega_0)^{\alpha-1}-(t_1-\omega_0)^{\alpha-1}\right|}{|\Omega|}\Big\{|\mathbf{B}_1|\big|\Phi_2^*[\phi_2,F_u]\big|+|\mathbf{B}_2|\big|\Phi_1^*[\phi_1,F_u]\big|\Big\}$$
$$+\frac{\left|(t_2-\omega_0)^{\alpha-2}-(t_1-\omega_0)^{\alpha-2}\right|}{|\Omega|}\Big\{|\mathbf{A}_1|\big|\Phi_2^*[\phi_2,F_u]\big|+|\mathbf{A}_2|\big|\Phi_1^*[\phi_1,F_u]\big|\Big\}, \tag{34}$$

$$\left|\left(\tilde{D}_{q,\omega}^{\nu}\mathcal{F}u\right)(t_1)-\left(\tilde{D}_{q,\omega}^{\nu}\mathcal{F}u\right)(t_2)\right| \tag{35}$$

$$\leq \frac{\|F\| q^{\binom{\alpha-\nu}{2}}}{\tilde{\Gamma}_q(\alpha-\nu+1)} \left|(t_2-\omega_0)^{\alpha-\nu} - (t_1-\omega_0)^{\alpha-\nu}\right|$$
$$+ \frac{q^{\binom{-\nu}{2}-\nu(\alpha-1)}}{|\Omega|\tilde{\Gamma}_q(\alpha-\nu)} \left|(t_2-\omega_0)^{\alpha-\nu-1} - (t_1-\omega_0)^{\alpha-\nu-1}\right| \left\{ |\mathbf{B}_1||\Phi_2^*[\phi_2, F_u]| + |\mathbf{B}_2||\Phi_1^*[\phi_1, F_u]| \right\}$$
$$+ \frac{q^{\binom{-\nu}{2}-\nu(\alpha-2)}}{|\Omega|\tilde{\Gamma}_q(\alpha-\nu)} \left|(t_2-\omega_0)^{\alpha-\nu-2} - (t_1-\omega_0)^{\alpha-\nu-2}\right| \left\{ |\mathbf{A}_1||\Phi_2^*[\phi_2, F_u]| + |\mathbf{A}_2||\Phi_1^*[\phi_1, F_u]| \right\},$$

and

$$\left|(\tilde{D}_{q,\omega}^{\nu+1}\mathcal{F}u)(t_1) - (\tilde{D}_{q,\omega}^{\nu+1}\mathcal{F}u)(t_2)\right| \tag{36}$$
$$\leq \frac{\|F\| q^{\binom{\alpha-\nu-1}{2}}}{\tilde{\Gamma}_q(\alpha-\nu)} \left|(t_2-\omega_0)^{\alpha-\nu-1} - (t_1-\omega_0)^{\alpha-\nu-1}\right|$$
$$+ \frac{q^{\binom{-(\nu+1)}{2}-(\nu+1)(\alpha-1)}}{|\Omega|\tilde{\Gamma}_q(\alpha-\nu-1)} \left|(t_2-\omega_0)^{\alpha-\nu-2} - (t_1-\omega_0)^{\alpha-\nu-2}\right| \left\{ |\mathbf{B}_1||\Phi_2^*[\phi_2, F_u]| + |\mathbf{B}_2||\Phi_1^*[\phi_1, F_u]| \right\}$$
$$+ \frac{q^{\binom{-(\nu+1)}{2}-(\nu+1)(\alpha-2)}}{|\Omega|\tilde{\Gamma}_q(\alpha-\nu-1)} \left|(t_2-\omega_0)^{\alpha-\nu-3} - (t_1-\omega_0)^{\alpha-\nu-3}\right| \left\{ |\mathbf{A}_1||\Phi_2^*[\phi_2, F_u]| + |\mathbf{A}_2||\Phi_1^*[\phi_1, F_u]| \right\}.$$

The right-hand side of (34)–(36) tends to be zero when $|t_2 - t_1| \to 0$. Thus, $\mathcal{A}$ is relatively compact on B_R. Therefore, the set $\mathcal{A}(B_R)$ is an equicontinuous set. By Steps I to III and the Arzelá-Ascoli theorem, $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is completely continuous. Therefore, problem (1) has at least one solution by Schauder fixed point theorem. □

4. Example

We provide the fractional symmetric Hahn integrodifference equation as

$$\tilde{D}^{\frac{4}{3}}_{\frac{1}{2},\frac{2}{3}} u(t) = \frac{1}{(t^2+100e^2)(1+|u(t)|)} \Bigg[e^{-3t}\Big(u^2+2|u|\Big) + e^{-(2\pi+\cos^2 \pi t)} \left|\tilde{\Psi}^{\frac{1}{2}}_{\frac{1}{2},\frac{2}{3}} u(t)\right|$$
$$+ e^{-(3\pi+\sin^2 t)} \left|\tilde{D}^{\frac{1}{4}}_{\frac{1}{2},\frac{2}{3}} u(t)\right| + e^{-(\pi+cos^2 t)} \left|\tilde{D}^{\frac{5}{4}}_{\frac{1}{2},\frac{2}{3}} u(t)\right| \Bigg], \quad t \in I^{10}_{\frac{1}{2},\frac{2}{3}} \tag{37}$$

with nonlocal fractional (p, q)-integrodifferene boundary condition

$$\frac{1}{20\pi} u\left(\frac{15}{8}\right) + 200\pi \tilde{D}^{\frac{1}{3}}_{\frac{1}{2},\frac{2}{3}} u\left(\frac{15}{8}\right) + 20\pi \tilde{I}^{\frac{3}{4}}_{\frac{1}{2},\frac{2}{3}} u\left(\frac{15}{8}\right) = \sum_{i=1}^{\infty} \frac{C_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i = \sigma^i_{\frac{1}{2},\frac{2}{3}}(10)$$
$$100e\, u\left(\frac{77}{48}\right) + \frac{1}{10e} \tilde{D}^{\frac{2}{3}}_{\frac{1}{2},\frac{2}{3}} u\left(\frac{77}{48}\right) + \frac{1}{100e} \tilde{I}^{\frac{1}{4}}_{\frac{1}{2},\frac{2}{3}} u\left(\frac{77}{48}\right) = \sum_{i=1}^{\infty} \frac{D_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i = \sigma^i_{\frac{1}{2},\frac{2}{3}}(10)$$

where $\varphi(t,s) = \frac{e^{-|s-t|}}{(t+\pi)^2}$, $\frac{1}{200t^2} \leq \sum_{i=1}^{\infty} C_i \leq \frac{\pi}{200t^2}$, and $\frac{1}{100t^3} \leq \sum_{i=1}^{\infty} D_i \leq \frac{e}{100t^3}$.

We let $\alpha = \frac{4}{3}$, $\gamma = \frac{1}{2}$, $\nu = \frac{1}{4}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{2}{3}$, $\theta_1 = \frac{3}{4}$, $\theta_2 = \frac{1}{4}$, $q = \frac{1}{2}$, $\omega = \frac{2}{3}$, $\omega_0 = \frac{\omega}{1-q} = \frac{4}{3}$, $T = 10$, $\eta_1 = \sigma^4_{\frac{1}{2},\frac{2}{3}}(10) = \frac{15}{8}$, $\eta_2 = \sigma^5_{\frac{1}{2},\frac{2}{3}}(10) = \frac{77}{48}$, $\lambda_1 = \frac{1}{20\pi}$, $\lambda_2 = 100e$, $\mu_1 = 200\pi$, $\mu_2 = \frac{1}{10e}$, $\phi_1(u) = \sum_{i=1}^{\infty} \frac{C_i|u(t_i)|}{1+|u(t_i)|}$, $\phi_2(u) = \sum_{i=1}^{\infty} \frac{D_i|u(t_i)|}{1+|u(t_i)|}$ and

$$F\Big[t, u(t), (\tilde{\Psi}^{\gamma}_{q,\omega}u)(t), (\tilde{\Upsilon}^{\nu}_{q,\omega}u)(t)\Big] = \frac{1}{(t^2+100e^2)(1+|u(t)|)} \Big[e^{-3t}(u^2+2|u|) + e^{-(2\pi+\cos^2 \pi t)} \times$$
$$\left|\tilde{\Psi}^{\frac{1}{2}}_{\frac{1}{2},\frac{2}{3}} u(t)\right| + e^{-(3\pi+\sin^2 t)} \left|\tilde{D}^{\frac{1}{4}}_{\frac{1}{2},\frac{2}{3}} u(t)\right| + e^{-(\pi+cos^2 t)} \left|\tilde{D}^{\frac{5}{4}}_{\frac{1}{2},\frac{2}{3}} u(t)\right| \Big].$$

For all $t \in I^{10}_{\frac{1}{2},\frac{2}{3}}$ and $u, v \in \mathbb{R}$, we have

$$\begin{aligned}&\left|F\left[t,u,\tilde{\Psi}^{\gamma}_{q,\omega}u,D^{\nu}_{q,\omega}u,D^{\nu+1}_{q,\omega}u\right]-F\left[t,v,\tilde{\Psi}^{\gamma}_{q,\omega}v,,D^{\nu}_{q,\omega}v,D^{\nu+1}_{q,\omega}v\right]\right|\\ \le\ &\frac{1}{e^{4}\left(\frac{16}{9}+100e^{2}\right)}|u-v|+\frac{1}{e^{2\pi}\left(\frac{16}{9}+100e^{2}\right)}\left|\tilde{\Psi}^{\gamma}_{q,\omega}u-\tilde{\Psi}^{\gamma}_{q,\omega}v\right|\\ &+\frac{1}{e^{3\pi}\left(\frac{16}{9}+100e^{2}\right)}\left|\tilde{D}^{\nu}_{q,\omega}u-\tilde{D}^{\nu}_{q,\omega}v\right|+\frac{1}{e^{\pi}\left(\frac{16}{9}+100e^{2}\right)}\left|\tilde{D}^{\nu}_{q,\omega}u-\tilde{D}^{\nu}_{q,\omega}v\right|.\end{aligned}$$

Thus, (H_1) holds with $\ell_1 = 0.00002473$, $\ell_2 = 2.5212 \times 10^{-6}$, $\ell_3 = 1.0895 \times 10^{-7}$ and $\ell_4 = 0.00005834$.

For all $u, v \in \mathcal{C}$, we have

$$|\phi_1(u)-\phi_1(v)| \le \frac{9\pi}{3200}\|u-v\|_{\mathcal{C}} \quad \text{and} \quad |\phi_2(u)-\phi_2(v)| \le \frac{27\pi}{6400}\|u-v\|_{\mathcal{C}}.$$

Thus, (H_2) holds with $\omega_1 = 0.00884$, $\omega_2 = 0.01147$.

After calculating, we obtain

$$\begin{gathered}\varphi_0 = 0.04994, \quad |\mathbf{A}_1| = 473.16723, \quad |\mathbf{A}_2| = 175.89586,\\ |\mathbf{B}_1| = 0.02395, \quad |\mathbf{B}_2| = 649.28092 \quad \text{and} \quad |\Omega| = 307214.2417.\end{gathered}$$

In addition, we find that

$$\begin{gathered}\mathcal{L} = 0.00005834, \quad \Theta_1 = 340.34486, \quad \Theta_2 = 38.32822,\\ \mathcal{O}_1 = 0.000365, \quad \mathcal{O}_2 = 0.004477, \quad \bar{\mathcal{O}}_1 = 0.00004869, \quad \bar{\mathcal{O}}_2 = 0.00235,\\ \bar{\bar{\mathcal{O}}}_1 = 0.0003764, \quad \bar{\bar{\mathcal{O}}}_2 = 0.0028,\\ \mathcal{O}^*_1 = 0.0007901, \quad \mathcal{O}^*_2 = 0.009716 \quad \text{and} \quad \chi = 24.63249.\end{gathered}$$

Therefore, (H_3) holds with

$$\Xi = 0.01567 < 1.$$

Hence, by Theorem 1, problem (37) and (38) have a unique solution, and by Theorem 2, it has at least one solution.

5. Conclusions

The new knowledge of fractional symmetric Hanh calculus was used in the studying of existence results of the nonlocal fractional symmetric Hahn integrodifference boundary value problems. After proving an auxiliary result concerning a linear variant of the our problem, this problem is transformed into a fixed point problem related to (1). In Theorem 1, we found the conditions for the existence and uniqueness of solution for problem (1) by using the Banach fixed point theorem. We also found the conditions of at least one solution by using Schauder's fixed point theorem in Theorem 2. Some properties of fractional symmetric Hahn integral needed in our study are also discussed. The results of the paper are new and enrich the subject of boundary value problems for symmetric Hahn integrodifference equations. In future work, we may extend this work by considering solutions of new boundary value problems or study inequalities of symmetric Hahn calculus.

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