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On Conditions for L^2-Dissipativity of an Explicit Finite-Difference Scheme for Linearized 2D and 3D Barotropic Gas Dynamics System of Equations with Regularizations

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Abstract: We deal with 2D and 3D barotropic gas dynamics system of equations with two viscous regularizations: so-called quasi-gas dynamics (QGD) and quasi-hydrodynamics (QHD) ones. The system is linearized on a constant solution with any velocity, and an explicit two-level in time and symmetric three-point in each spatial direction finite-difference scheme on the uniform rectangular mesh is considered for the linearized system. We study L^2-dissipativity of solutions to the Cauchy problem for this scheme by the spectral method and present a criterion in the form of a matrix inequality containing symbols of symmetric matrices of convective and regularizing terms. Analyzing these inequality and matrices, we also derive explicit sufficient conditions and necessary conditions in the Courant-type form which are rather close to each other. For the QHD regularization, such conditions are derived for the first time in 2D and 3D cases, whereas, for the QGD regularization, they improve those that have recently been obtained. Explicit formulas for a scheme parameter that guarantee taking the maximal time step are given for these conditions. An important moment is a new choice of an "average" spatial mesh step ensuring the independence of the conditions from the ratios of the spatial mesh steps and, for the QGD regularization, from the Mach number as well.

Keywords: barotropic gas dynamics equations; regularization; explicit two-level scheme; dissipativity

MSC: 65M12; 76M20; 35K45

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1. Introduction

Numerical methods for gas dynamics problems play an important role in computational mathematics, and a vast literature is devoted to them, see, in particular, the monographs [1–5] and references therein. Stability conditions for such methods, including those explicit in time, are of great practical and theoretical importance. For the explicit schemes, computational costs are inversely proportional to the time step Δt but the maximal value of Δt is restricted by a Courant-type stability condition involving steps of the spatial mesh and the sound speed.

Among the mentioned methods there exist a family of mesh methods based on preliminary viscous regularization of the equations involving so-called quasi-gas dynamics (QGD) and quasi-hydrodynamics (QHD) regularizations [6–9]. These methods are widely used for numerical solution of various applied problems. In the barotropic case, gas dynamics systems of equations with such regularizations were introduced and studied in [10–13], and their numerous applications to computer simulation were given for various 1D and 2D shallow water models [14–19], some 2D astrophysical problems [20] and the 2D and 3D compressible Navier–Stokes–Cahn–Hilliard models [21–23], etc. Despite of a lot of applications, almost nothing was known until recent years on rigorous theoretical conditions for stability of schemes with the QGD and QHD regularizations thus leading to additional time-consuming preliminary numerical experiments in order to choose the adequate parameters of schemes allowing to use larger values of Δt. Notice that there exist also other regularizations for gas dynamics equations, see in [24–26], etc., however, they have not yet undergone such detailed practical testing.

The present paper is a contribution to the theoretical basis of schemes with the QGD and QHD regularizations. We study an explicit two-level in time and symmetric three-point in each spatial direction finite-difference scheme for the 2D and 3D barotropic gas dynamics system of equations with such regularizations linearized on a constant solution (with any velocity, though rather often previously much simpler case of the zero velocity was considered only). The possibility of applying symmetric approximation in space is related namely to application of the regularized equations. For this scheme, we give a matrix criterion as well as simpler and rather close to each other sufficient conditions and necessary conditions for L^2-dissipativity of solutions to the Cauchy problem, for any Mach number and the uniform rectangular mesh. To this end, we apply the spectral method [4,27,28] and analyze matrix inequalities containing symbols of symmetric matrices of convective and regularizing terms. Notice that we apply the version of the spectral method based on the Fourier series [4,28] rather than the integral Fourier transform [27]. Moreover, the analysis of namely the L^2-dissipativity is natural since the corresponding results are known for the linearized QGD and QHD systems of equations [10,11], though rather often only the von Neumann-type necessary stability conditions are studied that is simpler but does not ensure stability in any norm. Furthermore, the advantage of the spectral method versus an alternative energy approach is the possibility to derive not only sufficient conditions for stability but necessary conditions as well. In this paper, for the first time this analysis is generally performed in the unified manner for the both regularizations, but for the simpler QHD regularization it turns out to be more complicated since its regularizing terms are weaker with respect to the convective ones than in the QGD regularization.

Explicit formulas for the scheme regularizing parameter that guarantee taking the maximal time step Δt are given for the derived conditions. We also present the choice of the "average" spatial step in the Courant-type conditions on the time step and in the regularizing parameter which ensures that the sufficient conditions and necessary conditions are uniform with respect to the rectangular spatial mesh and, for the QGD regularization, with respect to the Mach number as well (that can be valuable for simulation of super- and hypersonic flows). This choice depends not only on the steps of the rectangular spatial mesh, but also the Mach numbers in the respective directions. These results are valuable for practical applications helping one to choose adequately the scheme parameters allowing to use the maximal time step.

For the QHD regularization, these 2D and 3D results are derived for the first time and, for the QGD regularization, they improve those that have recently been obtained in [29]. Previously, the L^2-dissipativity analysis of similar schemes in the much simpler 1D barotropic case was accomplished for zero and any Mach number in [30–32], respectively. In this paper, we base on the papers in [29,32] and aim to develop further their technique. We also include the complete proof of the formula for the norm of the level-to-level transition operator in terms of the eigenvalues of its symmetrized symbol that is essential for the spectral technique; the main items of its proof have recently been presented in [33]. Importantly, the technique is general enough and applicable for various schemes, other regularizations (for example, see in [34]) and in more general or different statements of the equations, and some such studies are planned for the near future.

2. The Barotropic Gas Dynamics System of Equations with Two Regularizations, Its Linearization and the Corresponding Difference Scheme

We write the barotropic gas dynamics system of equations with the QGD and QHD regularizations in the form from [12] in a unified manner, setting for them $\ell = 1$ and $\ell = 0$, respectively. They consist of the mass and momentum balance equations which in the absence of external forces have the form

$$\partial_t\rho + \operatorname{div}\mathbf{j}_\ell = 0,\ \ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\mathbf{j}_\ell\otimes\mathbf{u}) + \nabla p(\rho) = \operatorname{div}\Pi_\ell \tag{1}$$

in $\mathbb{R}^n$, $n = 2,3$, for $t \geqslant 0$. The sought functions are the density $\rho > 0$ and velocity $\mathbf{u} = (u_1,\ldots,u_n)$ of a gas which depend on $x = (x_1,\ldots,x_n)$ and t, also $p(\rho) \in C^2(0,+\infty)$ is the pressure with $p'(\rho) > 0$. The operators div and $\nabla = (\partial_1,\ldots,\partial_n)$ are taken in x, also $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$. The symbol $\otimes$ denote the tensor product of vectors. The divergence of a tensor is taken with respect to its first index.

The regularized mass flux $\mathbf{j}_\ell$ and the viscous stress tensor $\Pi_\ell = \Pi^{NS} + \Pi_\ell^\tau$ are as follows:

$$\mathbf{j}_\ell = \rho\mathbf{u} - \mathbf{m}_\ell,\ \ \mathbf{m}_\ell = \tau\big[\ell\operatorname{div}(\rho\mathbf{u})\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p(\rho)\big], \tag{2}$$

$$\Pi^{NS} = \mu\big[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\big] + \big(\lambda - \tfrac{2}{3}\mu\big)(\operatorname{div}\mathbf{u})\mathbb{I},\ \ \Pi_\ell^\tau = \mathbf{u}\otimes\mathbf{m}_0 + \ell\tau p'(\rho)\operatorname{div}(\rho\mathbf{u})\mathbb{I}, \tag{3}$$

where $\mathbf{u}\cdot\nabla = u_i\partial_i$, and hereafter the summation from 1 to n over the repeated indices i,j (and only over them) is assumed. Furthermore, Π^{NS} is the Navier–Stokes tensor with $\nabla\mathbf{u} = \{\partial_i u_j\}_{i,j=1}^n$ and the artificial viscosity coefficients $\mu = \alpha_s\tau\rho p'(\rho)$ and $\lambda = \alpha_{1s}\tau\rho p'(\rho)$ with the parameters $\alpha_s \geqslant 0$ for $\ell = 1$ or $\alpha_s > 0$ for $\ell = 0$ (the Schmidt number) and $\alpha_{1s} \geqslant 0$, $\mathbb{I}$ is the unit tensor, $\mathbf{m}_\ell$ and Π_ℓ^τ are the regularizing momentum and viscosity tensor, and $\tau = \tau(\rho,\mathbf{u}) > 0$ is the regularizing parameter.

For $\tau = 0$ and given $\mu > 0$ and $\lambda \geqslant 0$, this system becomes the barotropic compressible Navier–Stokes system of equations. For $\tau = \mu = \lambda = 0$, it is simplified as the barotropic Euler system of equations.

We recall the linearization of system (1)–(3) on a constant solution $\rho(x,t) \equiv \rho_* > 0$ and $\mathbf{u}(x,t) \equiv \mathbf{u}_* = (u_{*1},\ldots,u_{*n})$ [10,11]. Let

$$c_* = \sqrt{p'(\rho_*)},\ \ \tau_* = \tau(\rho_*,\mathbf{u}_*),\ \mu_* = \alpha_s\tau_*\rho_* c_*^2,\ \lambda_* = \alpha_{1s}\tau_*\rho_* c_*^2$$

be the background sound speed and values of the parameters τ, μ and λ. We substitute the solution in the form $\rho = \rho_* + \rho_*\tilde\rho$ and $\mathbf{u} = \mathbf{u}_* + c_*\tilde{\mathbf{u}}$ into Equations (1)–(3), where $\tilde\rho$ and $\tilde{\mathbf{u}}$ are the dimensionless small perturbations. We omit the terms of higher than the first order of smallness with respect to them and get the linearized system of equations for $\mathbf{z} := (\tilde\rho,\tilde{\mathbf{u}})^T$. Its vector symmetrized form is as follows:

$$\partial_t\mathbf{z} + c_* B^{(i)}\partial_i\mathbf{z} - \tau_* c_*^2 A_\ell^{(ij)}\partial_i\partial_j\mathbf{z} = 0 \tag{4}$$

in $\mathbb{R}^n$ for $t \geqslant 0$, where $B^{(i)}$ and $A^{(ij)}$ are matrices of the order $n+1$ (we omit the derivation details).

In order to write down these matrices, we introduce the column vectors $\mathbf{e}_0,\ldots,\mathbf{e}_n$ of the canonical basis in $\mathbb{R}^{n+1}$, then

$$B^{(k)} = M_k I_{n+1} + \mathbf{e}_0\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_0^T,$$

$$A_\ell^{(kk)} = \operatorname{diag}\{1,\alpha_s,\ldots,\alpha_s\} + M_k^2 I^{(\ell)} + (\ell+1)M_k(\mathbf{e}_0\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_0^T) + a_\ell\mathbf{e}_k\mathbf{e}_k^T, \tag{5}$$

$$A_\ell^{(ij)} = M_iM_jI^{(\ell)} + \tfrac{\ell+1}{2}M_i(\mathbf{e}_0\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_0^T) + \tfrac{\ell+1}{2}M_j(\mathbf{e}_0\mathbf{e}_i^T + \mathbf{e}_i\mathbf{e}_0^T) + \tfrac{a_\ell}{2}(\mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T) \tag{6}$$

for any k, i and j from 1 to n and $i \neq j$. Hereafter

$$M_k = \frac{u_{*k}}{c_*},\ \ \mathbf{M} = (M_1,\ldots,M_n)^T,\ \ a_\ell = \frac{1}{3}\alpha_s + \alpha_{1s} + \ell,$$

also $M = |\mathbf{M}|$ is the Mach number, I_k is the unit matrix of the order k, the matrix $I^{(\ell)} = \operatorname{diag}\{\ell,1,\ldots,1\}$ has the order $n+1$, where $\operatorname{diag}\{p_1,\ldots,p_k\}$ is the diagonal matrix of the order k with diagonal elements listed sequentially. Clearly $B^{(k)}$, $A_\ell^{(kk)}$ and $A_\ell^{(ij)}$ are symmetric matrices and $A_\ell^{(ij)} = A_\ell^{(ji)}$ that is essential below.

For the solution to systems of equations like (4) supplemented with the initial condition $\mathbf{z}|_{t=0} = \mathbf{z}_0$, the following uniform in $t \geqslant 0$ bound is known [10,11]:

$$\sup_{t\geqslant 0} \|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leqslant \|\mathbf{z}_0\|_{L^2(\mathbb{R}^n)} \ \ \forall\, \mathbf{z}_0 \in L^2(\mathbb{R}^n). \tag{7}$$

We define the uniform mesh ω_{kh} in x_k with the nodes lh_k, $l \in \mathbb{Z}$, and the step $h_k > 0$, $1 \leqslant k \leqslant n$, and the mesh $\bar{\omega}^{\Delta t}$ in t with the nodes $t_m = m\Delta t$, $m \geqslant 0$, and the step $\Delta t > 0$. We introduce the symmetric difference operators in x_k and the forward difference operator in t

$$\mathring{\delta}_k v_l = \frac{v_{l+1} - v_{l-1}}{2h_k}, \ \ (\delta_k^* \delta_k v)_l = \frac{v_{l+1} - 2v_l + v_{l-1}}{h_k^2}, \ \ \delta_t y = \frac{y^+ - y}{\Delta t}, \ \ y^{+,m} = y^{m+1},$$

where $v_l = v(lh_k)$ and $y^m = y(t_m)$.

We define the rectangular mesh $\omega_{\mathbf{h}} := \omega_{1h} \times \ldots \times \omega_{nh}$ in $\mathbb{R}^n$ with $\mathbf{h} = (h_1, \ldots, h_n)$. Let H be the Hilbert space of vector-functions $\mathbf{v}$: $\omega_{\mathbf{h}} \to \mathbb{C}^{n+1}$ defined and square summable on $\omega_{\mathbf{h}}$, equipped with the inner product

$$(\mathbf{v}, \mathbf{y})_H = h_1 \ldots h_n \textstyle\sum_{\mathbf{k}\in\mathbb{Z}^n} (\mathbf{v_k}, \mathbf{y_k})_{\mathbb{C}^{n+1}}, \ \ \mathbf{k} = (k_1, \ldots, k_n),$$

(where, for example, $\mathbf{v_k} = \mathbf{v}(k_1 h_1, \ldots, k_n h_n)$) and the corresponding norm $\|\cdot\|_H$.

We approximate the system of Equation (4) using the defined difference operators and get the explicit in t and symmetric three-point in each direction $x_1, \ldots, x_n$ difference scheme

$$\delta_t \mathbf{y} + c_* B^{(i)} \mathring{\delta}_i \mathbf{y} - \tau_* c_*^2 \big[A^{(ii)} \delta_i^* \delta_i + (1 - \delta^{(ij)}) A^{(ij)} \mathring{\delta}_i \mathring{\delta}_j\big] \mathbf{y} = 0 \tag{8}$$

on $\omega_{\mathbf{h}} \times \omega^{\Delta t}$, where $\delta^{(ij)}$ is the Kronecker symbol. A similar difference scheme arises after the linearization of schemes for the original Equations (1)–(3) given, in particular, in [35].

We pose the question on conditions for the validity of the mesh counterpart of bound (7), namely,

$$\sup_{m\geqslant 0} \|\mathbf{y}^m\|_H \leqslant \|\mathbf{y}^0\|_H \ \ \forall\, \mathbf{y}^0 \in H. \tag{9}$$

We define the level-to-level transition operator acting in H:

$$\mathcal{A} := I - \Delta t \big\{ c_* B^{(i)} \mathring{\delta}_i - \tau_* c_*^2 [A_\ell^{(ii)} \delta_i^* \delta_i + (1 - \delta^{(ij)}) A_\ell^{(ij)} \mathring{\delta}_i \mathring{\delta}_j] \big\},$$

where I is the unit operator. Bound (9) is equivalent to the properties $\|\mathcal{A}\|_{\mathcal{L}(H)} := \sup_{\|\mathbf{y}\|_H = 1} \|\mathcal{A}\mathbf{y}\|_H \leqslant 1$ and the H–dissipativity (i.e., L^2–dissipativity) of the scheme

$$\|\mathbf{y}^m\|_H \leqslant \|\mathbf{y}^{m-1}\|_H \leqslant \ldots \leqslant \|\mathbf{y}^0\|_H \ \ \forall \mathbf{y}^0 \in H, \ m \geqslant 1.$$

Notice that in the case of more general than (8) non-homogeneous scheme

$$\delta_t \mathbf{y} + c_* B^{(i)} \mathring{\delta}_i \mathbf{y} - \tau_* c_*^2 \big[A^{(ii)} \delta_i^* \delta_i + (1 - \delta^{(ij)}) A^{(ij)} \mathring{\delta}_i \mathring{\delta}_j\big] \mathbf{y} = \mathbf{f}$$

on $\omega_{\mathbf{h}} \times \omega^{\Delta t}$, with any given $\mathbf{y}^0$ and $\mathbf{f}$, under the above mentioned property $\|\mathcal{A}\|_{\mathcal{L}(H)} \leqslant 1$, it is easy to derive the following more general stability bound:

$$\max_{0\leqslant m\leqslant \overline{m}} \|\mathbf{y}^m\|_H \leqslant \|\mathbf{y}^0\|_H + \Delta t \sum_{m=0}^{\overline{m}-1} \|\mathbf{f}^m\|_H \ \ \forall\, \overline{m} \geqslant 1.$$

3. An Analysis of the L^2–Dissipativity of the Scheme

Let Δt and τ_* be given by the formulas of the typical form [6–8]

$$\Delta t = \frac{\beta \hat{h}}{c_*}, \ \ \tau_* = \frac{\alpha \hat{h}}{c_*} \tag{10}$$

with the parameters $\beta > 0$ (like the Courant number) and $\alpha > 0$. Here, the "average" spatial step $\hat{h} = \hat{h}(\mathbf{h}) > 0$ is arbitrary during the whole analysis and can depend also on $\mathbf{M}$ and other parameters. Its adequate choice arises as a result of the analysis, and it will be given at the end of the paper. Below we derive conditions on β in dependence on α related to the validity of bound (9).

According to the spectral method [4,28], we consider particular solutions of scheme (8) in the form

$$\mathbf{y}_{\mathbf{k}}^m(\boldsymbol{\xi}) = e^{\mathbf{i}\mathbf{k}\boldsymbol{\xi}}\mathbf{w}^m(\boldsymbol{\xi}),\ \ \mathbf{k} \in \mathbb{Z}^n,\ \ \boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T \in D := [-\pi, \pi]^n,\ \ m \geqslant 0,$$

where $\mathbf{i}$ is the imaginary unit and $\boldsymbol{\xi}$ is the vector parameter. We substitute them into (8) taking into account the formulas

$$\mathring{\delta}_k e^{\mathbf{i}k\xi_k} = \mathbf{i}\frac{1}{h_k}(\sin\xi_k)e^{\mathbf{i}k\xi_k} = \mathbf{i}\frac{2}{h_k}\Big(\sin\frac{\xi_k}{2}\Big)\Big(\cos\frac{\xi_k}{2}\Big)e^{\mathbf{i}k\xi_k},\ \ -\delta_k^*\delta_k e^{\mathbf{i}k\xi_k} = \frac{4}{h_k^2}\Big(\sin^2\frac{\xi_k}{2}\Big)e^{\mathbf{i}k\xi_k}$$

together with Formula (10) and derive the explicit recurrent formula

$$\mathbf{w}^+ = G_{\mathbf{s}}\mathbf{w}\ \text{ on }\ \bar{\omega}^{\Delta t}.$$

In it, $G_{\mathbf{s}}$ is the matrix-symbol of the operator $\mathcal{A}$ having the form

$$G_{\mathbf{s}} = I_{n+1} - \beta F_{\mathbf{s}},\ \ F_{\mathbf{s}} = 4\alpha A_{\mathbf{s}\ell} + 2\mathbf{i}B_{\mathbf{s}},$$
$$B_{\mathbf{s}} = d_i s_i B^{(i)},\ \ A_{\mathbf{s}\ell} = d_i^2 A_\ell^{(ii)} + (1-\delta^{(ij)})d_i d_j s_i s_j A_\ell^{(ij)},$$

where the matrices $B_{\mathbf{s}}$ and $A_{\mathbf{s}\ell}$ are proportional to the symbols of convective and viscous (regularizing) terms as well as $\mathbf{s} = (s_1, \ldots, s_n)$ and

$$d_k = r_k\sqrt{\sigma_k},\ \ r_k = \frac{\hat{h}}{h_k},\ \ \sigma_k = \sin^2\frac{\xi_k}{2} \in [0,1],\ \ s_k = \operatorname{sgn}\xi_k\sqrt{1-\sigma_k},\ \ 1 \leqslant k \leqslant n.$$

Hereafter, mainly it seems more convenient to take $\mathbf{s} \in S := [-1,1]^n$ as a parameter instead of $\boldsymbol{\xi}$; obviously, $\sigma_k = 1 - s_k^2$.

We define the row vector and number

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{s}) \equiv (\zeta_1, \ldots, \zeta_n)\ \text{ with }\ \zeta_k = d_k s_k = r_k\sqrt{\sigma_k}s_k,\ \ d = \big(d_1^2 + \ldots + d_n^2\big)^{1/2}.$$

Lemma 1. *The matrices $B_{\mathbf{s}}$ and $A_{\mathbf{s}\ell}$ can be written in the 2×2-block symmetric form*

$$B_{\mathbf{s}} = \begin{pmatrix} \boldsymbol{\zeta}\mathbf{M} & \boldsymbol{\zeta} \\ \boldsymbol{\zeta}^T & (\boldsymbol{\zeta}\mathbf{M})I_n \end{pmatrix},$$
$$A_{\mathbf{s}\ell} = \begin{pmatrix} \ell a_{\mathbf{M}} + d^2 & (\ell+1)\big[(\boldsymbol{\zeta}\mathbf{M})\boldsymbol{\zeta} + \mathbf{M}^T Q\big] \\ (\ell+1)\big[(\boldsymbol{\zeta}\mathbf{M})\boldsymbol{\zeta}^T + Q\mathbf{M}\big] & (a_{\mathbf{M}} + \alpha_s d^2)I_n + a_\ell(\boldsymbol{\zeta}^T\boldsymbol{\zeta} + Q) \end{pmatrix}, \tag{11}$$

where

$$a_{\mathbf{M}} = (\boldsymbol{\zeta}\mathbf{M})^2 + \mathbf{M}^T Q\mathbf{M},\ \ Q = \operatorname{diag}\{q_1, \ldots, q_n\}\ \textit{ with }\ q_k = d_k^2\sigma_k = r_k^2\sigma_k^2,\ \ 1 \leqslant k \leqslant n.$$

Proof. For $\ell = 1$, the result was proven in ([29], Lemma 1).

For $\ell = 0$, by virtue of Formulas (5) and (6) for the matrices $A_\ell^{(ij)}$ together with $d_k^2 = d_k^2\sigma_k + \zeta_k^2$, we can write down

$$A_{\mathbf{s}0} = d_i^2\sigma_i A_0^{(ii)} + |\boldsymbol{\zeta}|^2\operatorname{diag}\{1, \alpha_s, \ldots, \alpha_s\} + d_i s_i d_j s_j A_0^{(ij)}.$$

Further, we obtain

$$\begin{aligned} & d_i^2\sigma_i A_0^{(ii)} + |\zeta|^2 \operatorname{diag}\{1,\alpha_s,\dots,\alpha_s\} \\ &= q_i M_i^2 I^{(\ell)} + d^2 \operatorname{diag}\{1,\alpha_s,\dots,\alpha_s\} + q_i M_i(\mathbf{e}_0\mathbf{e}_i^T + \mathbf{e}_i\mathbf{e}_0^T) + a_0 q_i \mathbf{e}_i\mathbf{e}_i^T \\ &= \mathbf{M}^T Q \mathbf{M} I^{(\ell)} + d^2\begin{pmatrix} 1 & 0 \\ 0 & \alpha_s I_n \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{M}^T Q \\ Q\mathbf{M} & 0 \end{pmatrix} + a_0\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} & d_i s_i d_j s_j A_0^{(ij)} \\ &= \zeta_i\zeta_j\big[M_iM_j I^{(\ell)} + \tfrac{1}{2}M_i(\mathbf{e}_0\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_0^T) + \tfrac{1}{2}M_j(\mathbf{e}_0\mathbf{e}_i^T + \mathbf{e}_i\mathbf{e}_0^T) + \tfrac{a_0}{2}(\mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T)\big] \\ &= (\boldsymbol{\zeta}\mathbf{M})^2 I^{(\ell)} + \boldsymbol{\zeta}\mathbf{M}\begin{pmatrix} 0 & \boldsymbol{\zeta} \\ \boldsymbol{\zeta}^T & 0 \end{pmatrix} + a_0\begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\zeta}^T\boldsymbol{\zeta} \end{pmatrix}. \end{aligned}$$

These formulas imply form (11) with $\ell = 0$ of the matrix $A_{\mathbf{s}0}$. □

In ([29], Lemma 1), the important matrix inequality was proven:

$$B_{\mathbf{s}}^2 \leqslant A_{\mathbf{s}1} \quad \forall \mathbf{s} \in S. \tag{12}$$

Recall that it follows from the formula

$$A_{\mathbf{s}1} - B_{\mathbf{s}}^2 = \begin{pmatrix} \mathbf{M}^T Q\mathbf{M} + \operatorname{tr} Q & 2\mathbf{M}^T Q \\ 2Q\mathbf{M} & (\mathbf{M}^T Q \mathbf{M} + \alpha_s d^2) I_n + (a_1 - 1)\boldsymbol{\zeta}^T\boldsymbol{\zeta} + a_1 Q \end{pmatrix}, \tag{13}$$

where $\operatorname{tr} Q = q_1 + \ldots + q_n$ is the trace of Q, which can be straightforwardly verified.

Now, we derive the corresponding more complicated inequality between the matrices $B_{\mathbf{s}}^2$ and $A_{\mathbf{s}0}$.

Lemma 2. *1. The following inequality holds:*

$$b_k(B^{(k)})^2 \leqslant A_0^{(kk)} \ \ \textit{with} \ \ b_k = \frac{\widehat{a}_0}{q(|M_k|)}, \ \ \widehat{a}_0 := \alpha_s + a_0 = \frac{4}{3}\alpha_s + \alpha_{1s}, \ \ 1 \leqslant k \leqslant n,$$

where the given constant $0 < b_k \leqslant 1$ is maximal in this inequality, with

$$q(m) = \varkappa(m^2 - 1), \ \ \varkappa(\theta) := \varkappa_0(\theta) + \sqrt{\varkappa_0^2(\theta) - \widehat{a}_0\theta^2}, \ \ \varkappa_0(\theta) := \tfrac{1}{2}[\theta^2 + \widehat{a}_0(\theta + 2)] \tag{14}$$

for $m \geqslant 0$. Herewith there holds the inequality

$$\varkappa_0^2(\theta) - \widehat{a}_0\theta^2 = \tfrac{1}{4}(\theta^2 + \widehat{a}_0\theta)^2 + \widehat{a}_0(1 + \theta) \geqslant 0 \ \ \textit{for} \ \ \theta \geqslant -1. \tag{15}$$

2. The following inequality holds:

$$b^{(0)} B_{\mathbf{s}}^2 \leqslant A_{\mathbf{s}0} \ \ \forall \mathbf{s} \in S, \ \ \textit{with} \ \ b^{(0)} := \frac{\widehat{a}_0}{\max\limits_{0 \leqslant m \leqslant M} q(m)}. \tag{16}$$

Proof. 1. The following formula holds:

$$(B^{(k)})^2 = M_k^2 I_{n+1} + 2M_k(\mathbf{e}_0\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_0^T) + \mathbf{e}_0\mathbf{e}_0^T + \mathbf{e}_k\mathbf{e}_k^T.$$

It is not difficult to check that the inequality $b(B^{(k)})^2 \leqslant A_0^{(kk)}$ is equivalent to the corresponding inequality for the 2×2-blocks of the matrices $(B^{(k)})^2$ and $A_0^{(kk)}$:

$$b\begin{pmatrix} M_k^2+1 & 2M_k \\ 2M_k & M_k^2+1 \end{pmatrix} \leqslant \begin{pmatrix} 1 & M_k \\ M_k & M_k^2+\widehat{a}_0 \end{pmatrix}. \tag{17}$$

In the proof of ([32], Theorem 3), it was shown that the maximal constant in this inequality equals $b = b_k$, where b_k is given in Item 1 of this Lemma. Herewith inequality (17) implies that $b(M_k^2+1) \leqslant 1$ and therefore knowingly $b_k \leqslant 1$.

2. Let $\boldsymbol{\zeta} \neq 0$. Due to the formula $d^2 = \operatorname{tr} Q + |\boldsymbol{\zeta}|^2$, we can write down

$$B_{\mathbf{s}} = |\boldsymbol{\zeta}|\tilde{B}_{\mathbf{s}}, \;\; \tilde{B}_{\mathbf{s}}^2 = \begin{pmatrix} m^2+\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T & 2m\tilde{\boldsymbol{\zeta}} \\ 2m\tilde{\boldsymbol{\zeta}}^T & \tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{\zeta}}+m^2 I_n \end{pmatrix}, \tag{18}$$

$$A_{\mathbf{s}0} = A_{\mathbf{s}0}^{(0)} + |\boldsymbol{\zeta}|^2\tilde{A}_{\mathbf{s}0}^{(1)},$$

$$A_{\mathbf{s}0}^{(0)} = \begin{pmatrix} \operatorname{tr} Q & \mathbf{M}^T Q \\ Q\mathbf{M} & (\mathbf{M}^T Q\mathbf{M} + \alpha_s \operatorname{tr} Q)I_n + a_0 Q \end{pmatrix}, \;\; \tilde{A}_{\mathbf{s}0}^{(1)} = \begin{pmatrix} 1 & m\tilde{\boldsymbol{\zeta}} \\ m\tilde{\boldsymbol{\zeta}}^T & (m^2+\alpha_s)I_n + a_0\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{\zeta}} \end{pmatrix}, \tag{19}$$

where $\tilde{\boldsymbol{\zeta}} = \boldsymbol{\zeta}/|\boldsymbol{\zeta}|$ and $m = \tilde{\boldsymbol{\zeta}}\mathbf{M}$.

The inequalities $A_{\mathbf{s}0}^{(0)} \geqslant 0$ and $\tilde{A}_{\mathbf{s}0}^{(1)} - b\tilde{B}_{\mathbf{s}}^2 \geqslant 0$ mean that

$$(\operatorname{tr} Q)v_0^2 + 2v_0\mathbf{M}^T Q\mathbf{v} + (\mathbf{M}^T Q\mathbf{M} + \alpha_s \operatorname{tr} Q)|\mathbf{v}|^2 + a_0\mathbf{v}^T Q\mathbf{v} \geqslant 0,$$
$$\left[1 - b(m^2+1)\right]v_0^2 + 2(1-2b)mv_0\tilde{\boldsymbol{\zeta}}\mathbf{v} + \left[m^2+\alpha_s - bm^2)\right]|\mathbf{v}|^2 + (a_0 - b)|\tilde{\boldsymbol{\zeta}}\mathbf{v}|^2 \geqslant 0 \tag{20}$$

for any $v_0 \in \mathbb{R}$ and $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$. The first of these inequalities is a consequence of the inequality

$$(\operatorname{tr} Q)v_0^2 + 2v_0\mathbf{M}^T Q\mathbf{v} + \mathbf{M}^T Q\mathbf{M}|\mathbf{v}|^2 \geqslant 0$$

that follows from the estimates (already used in [29])

$$2|v_0\mathbf{M}^T Q\mathbf{v}| \leqslant 2|v_0|(\operatorname{tr} Q)^{1/2}\left(q_1 M_1^2 v_1^2 + \ldots + q_n M_n^2 v_n^2\right)^{1/2} \leqslant (\operatorname{tr} Q)v_0^2 + \mathbf{M}^T Q\mathbf{M}|\mathbf{v}|^2.$$

The second inequality (20) for $m^2 + \alpha_s - bm^2 \geqslant 0$ and due to the lower estimate $|\mathbf{v}|^2 \geqslant (\tilde{\boldsymbol{\zeta}}\mathbf{v})^2$ follows from the matrix inequality

$$\begin{pmatrix} 1 & m \\ m & m^2+\widehat{a}_0 \end{pmatrix} - b\begin{pmatrix} m^2+1 & 2m \\ 2m & m^2+1 \end{pmatrix} \geqslant 0. \tag{21}$$

It coincides with (17) for $m = M_k^2$ and, therefore, is valid for $b = \widehat{a}_0/q(|m|) \leqslant 1$. Consequently, we have $m^2 + \alpha_s - bm^2 \geqslant 0$. On the other hand, for $\mathbf{v} = w\tilde{\boldsymbol{\zeta}}$, inequality (20) implies the inequality with any v_0 and w that is equivalent to inequality (21), thus the found constant b is maximal in (20).

Finally, we get

$$A_{\mathbf{s}} - b^{(0)}B_{\mathbf{s}}^2 = A_{\mathbf{s}0}^{(0)} + |\boldsymbol{\zeta}|^2(\tilde{A}_{\mathbf{s}0}^{(1)} - b^{(0)}\tilde{B}_{\mathbf{s}}^2) \geqslant 0 \;\; \forall \mathbf{s} \in S,$$

with $b^{(0)} = \widehat{a}_0/\max\limits_{|m|\leqslant M} q(|m|)$, since $|m| = |\tilde{\boldsymbol{\zeta}}\mathbf{M}|$ runs over $[0, M]$ when $\mathbf{s}$ runs over S.

Herewith, in the inequality $\tilde{A}_{\mathbf{s}0}^{(1)} - b\tilde{B}_{\mathbf{s}}^2 \geqslant 0$ for any $\mathbf{s} \in S$, the specified constant $b = b^{(0)}$ is maximal. This additional result is essential below. □

To apply Lemma 2, it is required to study behavior of the function $q(m)$.

Lemma 3. *1. For $\widehat{a}_0 \geqslant 2/3$, the function $q(m)$ increases in $m \geqslant 0$.*

2. Let $\widehat{a}_0 < 2/3$. Then $q(m)$ decreases in $[0, m_*]$ and increases in $m \geqslant m_*$ and, moreover, $q(m_0) = q(0) = 1$, where

$$0 < m_* = \left\{ -\frac{\widehat{a}_0}{4} + \left[\left(\frac{\widehat{a}_0}{4} + 1 \right)^2 - 2\widehat{a}_0 \right]^{1/2} \right\}^{1/2} < m_0 = \left(\frac{2 - 3\widehat{a}_0}{1 - \widehat{a}_0} \right)^{1/2} < \sqrt{2}.$$

3. As a consequence of Items 1–2, the following formula holds:

$$\max_{0 \leqslant m \leqslant M} q(m) = \begin{cases} q(M) & \text{for } \widehat{a}_0 \geqslant \frac{2}{3} \text{ or } M \geqslant m_0,\ \widehat{a}_0 < \frac{2}{3} \\ 1 & \text{for } M \leqslant m_0,\ \widehat{a}_0 < \frac{2}{3}. \end{cases}$$

Herewith, $q(0) = \max\{\widehat{a}_0, 1\}$ *and* $q(1) = 2\widehat{a}_0$.

Proof. Let $\theta \geqslant -1$. The function $\varkappa(\theta)$ is the larger root of the quadratic equation

$$\varkappa^2(\theta) - 2\varkappa_0(\theta)\varkappa(\theta) + \widehat{a}_0\theta^2 = 0. \tag{22}$$

Notice that due to Formula (15), we have $\varkappa_0^2(\theta) - \widehat{a}_0\theta^2 > 0$ excluding the particular case $\theta = -1 = -\widehat{a}_0$. From (14) and (15), it also follows that $\varkappa(\theta)$ increases in $\theta \geqslant 0$.

Except for the specified case, there exists $\varkappa'(\theta)$, and the differentiation of (22) gives

$$2\varkappa'(\theta)\left[\varkappa(\theta) - \varkappa_0(\theta)\right] + 2\widehat{a}_0\theta = 0.$$

As $\varkappa(\theta) > \varkappa_0(\theta)$, the property $\varkappa'(\theta) = 0$ is equivalent to $\varkappa(\theta) = 2\widehat{a}_0\theta/(2\theta + \widehat{a}_0)$; moreover, $2\theta + \widehat{a}_0 \neq 0$ (otherwise $\theta = 0$ and $\widehat{a}_0 = 0$). As $\varkappa(\theta) > 0$, here $\theta \neq 0$. Inserting this expression for $\varkappa(\theta)$ into (22), after a series of simplifications we come to the quadratic equation

$$2\theta^2 + (\widehat{a}_0 + 4)\theta + 4\widehat{a}_0 = 0.$$

For $\widehat{a}_0 \geqslant 12 + 8\sqrt{2}$ and $\widehat{a}_0 \leqslant 12 - 8\sqrt{2} \approx 0.686$ it has the real roots

$$\theta_\pm = -\left(\tfrac{1}{4}\widehat{a}_0 + 1\right) \pm \left[\left(\tfrac{1}{4}\widehat{a}_0 + 1\right)^2 - 2\widehat{a}_0\right)\right]^{1/2}.$$

Clearly $\theta_- < 1$ and $\theta_- \leqslant \theta_+ < 0$. It is not difficult to check that the property $\theta_+ > -1$ is equivalent to $\widehat{a}_0 < 2/3$.

Therefore, for $\widehat{a}_0 < 2/3$, there exists a unique root θ_+ of the equation $\varkappa'(\theta) = 0$ for $\theta > -1$, and as $\varkappa(\theta)$ increases in $\theta \geqslant 0$, also $\varkappa(\theta)$ decreases in $[-1, \theta_+]$ and increases in $\theta \geqslant \theta_+$. Moreover, in this case, solving of the equation $\varkappa(\theta_0) = \varkappa(-1) = 1$ for $\theta_0 > -1$ after simple algebraic transformations leads to the quadratic equation

$$(1 - \widehat{a}_0)\theta_0^2 + \widehat{a}_0\theta + 2\widehat{a}_0 - 1 = 0.$$

One of its roots is -1, therefore $\theta_0 = (1 - 2\widehat{a}_0)/(1 - \widehat{a}_0)$. In this case, we set $m_* = \sqrt{\theta_+ + 1}$ and $m_0 = \sqrt{\theta_0 + 1}$, and get formulas given in Item 2 of this Lemma.

In the opposite case $\widehat{a}_0 \geqslant 2/3$, the derivative $\varkappa'(\theta)$ does not vanish for $\theta > -1$, thus $\varkappa(\theta)$ increases for $\theta \geqslant -1$. Notice also that here $q(0) = \frac{1}{2}(\widehat{a}_0 + 1) + \frac{1}{2}|1 - \widehat{a}_0| = \max\{\widehat{a}_0, 1\}$. Thus, the results of Lemma are valid. □

Notice that in the case $\widehat{a}_0 \geqslant \frac{2}{3}$ (in particular, for $\alpha_s \geqslant \frac{1}{2}$), we get

$$q(0) = \max\{\widehat{a}_0, 1\} \leqslant \max_{0 \leqslant m \leqslant M} q(m) \leqslant q(1) = 2\widehat{a}_0, \quad \frac{1}{2} \leqslant b^{(0)} \leqslant \frac{\widehat{a}_0}{\max\{\widehat{a}_0, 1\}}$$

in the entire subsonic region $M \leqslant 1$. On the other hand, in the case $\widehat{a}_0 = \frac{1}{2}$ (for example, for $\alpha_s = \frac{3}{8}$ and $\alpha_{1s} = 0$), the quantity $\max_{0 \leqslant m \leqslant M} q(m) = 1 = q(0)$ is minimal in the entire subsonic region $M \leqslant m_0 = 1$ and $b^{(0)} = \widehat{a}_0 = \frac{1}{2}$.

Moreover, the following properties hold:

$$\frac{1}{2} \leqslant \frac{q(M)}{(M^2 - 1)^2 + \widehat{a}_0(M^2 + 1)} \leqslant 1, \quad b^{(0)} \sim \frac{\widehat{a}_0}{M^4} \quad \text{for } M \to +\infty,$$

thus, $b^{(0)}$ decreases rapidly as M grows.

We denote by $\lambda_{\max}(A)$ the maximal eigenvalue of the Hermitian matrix A. In the following theorem, we give the formula for the norm of the level-to-level transition operator $\mathcal{A}$ in terms of the eigenvalues of its symmetrized symbol $G^*(\mathbf{s})G(\mathbf{s})$ that is essential for the applied spectral technique. It will be more convenient to consider that $G_{\mathbf{s}} = G(\boldsymbol{\xi})$. Let $\mathcal{H} = [L^2(D)]^{n+1}$ be the Hilbert space of vector functions $\mathbf{w}(\boldsymbol{\xi})$: $D \to \mathbb{C}^{n+1}$, equipped with the norm $\|\mathbf{w}\|_{\mathcal{H}} = \||\mathbf{w}(\boldsymbol{\xi})|\|_{L^2(D)}$, where $|\cdot| = \|\cdot\|_{\mathbb{C}^{n+1}}$.

Theorem 1. *The following chain of equalities hold:*

$$\|\mathcal{A}\|_{\mathcal{L}(H)} = \|G(\cdot)\|_{\mathcal{L}(\mathcal{H})} = \max_{\boldsymbol{\xi}\in D} \|G(\boldsymbol{\xi})\| = \max_{\boldsymbol{\xi}\in D} \lambda_{\max}^{1/2}\big(G^*(\boldsymbol{\xi})G(\boldsymbol{\xi})\big), \tag{23}$$

where

$$\|G(\cdot)\|_{\mathcal{L}(\mathcal{H})} := \sup_{\|\mathbf{w}\|_{\mathcal{H}}=1} \|G(\boldsymbol{\xi})\mathbf{w}(\boldsymbol{\xi})\|_{\mathcal{H}}, \ \ \|G\| := \sup_{|\mathbf{v}|=1} |G\mathbf{v}|.$$

Proof. The complex Hilbert spaces H and $\mathcal{H}$ are isomorphic by means of the multiple complex Fourier series: to any mesh function $\mathbf{y} \in H$ corresponds the function $\mathbf{w} \in \mathcal{H}$ such that $\mathbf{w}(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{y}_{\mathbf{k}} e^{-\mathbf{i}\mathbf{k}\boldsymbol{\xi}}$, and vice versa, for the Fourier coefficients of this function, the following formula

$$\mathbf{y}_{\mathbf{k}} = \frac{1}{(2\pi)^n} \int_D \mathbf{w}(\boldsymbol{\xi}) e^{\mathbf{i}\mathbf{k}\boldsymbol{\xi}}\, d\boldsymbol{\xi}, \ \ \mathbf{k} \in \mathbb{Z}^n, \tag{24}$$

is valid. Herewith, the Parseval equality holds:

$$\|\mathbf{w}\|_{\mathcal{H}} = (2\pi)^n \|\mathbf{y}\|_{(\ell^2)^{n+1}} \equiv (2\pi)^n \Big(\sum_{\mathbf{k}\in\mathbb{Z}^n} |\mathbf{y}_{\mathbf{k}}|^2\Big)^{1/2}.$$

Due to Formula (24) and definition of $G(\boldsymbol{\xi})$, we obtain

$$(\mathcal{A}\mathbf{y})_{\mathbf{k}} = \frac{1}{(2\pi)^n} \int_D \mathcal{A}\Big(\mathbf{w}(\boldsymbol{\xi}) e^{\mathbf{i}\mathbf{k}\boldsymbol{\xi}}\Big)\, d\boldsymbol{\xi} = \frac{1}{(2\pi)^n} \int_D G(\boldsymbol{\xi})\mathbf{w}(\boldsymbol{\xi}) e^{\mathbf{i}\mathbf{k}\boldsymbol{\xi}}\, d\boldsymbol{\xi}, \ \ \mathbf{k} \in \mathbb{Z}^n,$$

i.e., $\{(\mathcal{A}\mathbf{y})_{\mathbf{k}}\}$ are the Fourier coefficients of the function $G(\boldsymbol{\xi})\mathbf{w}(\boldsymbol{\xi})$. Thus, due to the Parseval equality and the equality $\|\mathbf{y}\|_H = (h_1 \ldots h_n)^{1/2} \|\mathbf{y}\|_{(\ell^2)^{n+1}}$ we derive

$$\|\mathcal{A}\|_{\mathcal{L}(H)} = \sup_{\mathbf{y}\in(\ell^2)^{n+1}, \mathbf{y}\neq 0} \frac{\|\mathcal{A}\mathbf{y}\|_{(\ell^2)^{n+1}}}{\|\mathbf{y}\|_{(\ell^2)^{n+1}}} = \sup_{\mathbf{w}\in\mathcal{H}, \mathbf{w}\neq 0} \frac{\|G\mathbf{w}\|_{\mathcal{H}}}{\|\mathbf{w}\|_{\mathcal{H}}} = \|G(\cdot)\|_{\mathcal{L}(\mathcal{H})}.$$

Further, clearly $|G(\boldsymbol{\xi})\mathbf{w}(\boldsymbol{\xi})| \leqslant \big(\max_{\boldsymbol{\xi}\in D} \|G(\boldsymbol{\xi})\|\big)|\mathbf{w}(\boldsymbol{\xi})|$, and thus

$$\|G(\cdot)\|_{\mathcal{L}(\mathcal{H})} \leqslant \max_{\boldsymbol{\xi}\in D} \|G(\boldsymbol{\xi})\|;$$

here we take into account the continuity of $G(\boldsymbol{\xi})$ on D.

It remains to prove the inequality of the opposite type

$$\max_{\boldsymbol{\xi}\in D} \|G(\boldsymbol{\xi})\| \leqslant \|G(\cdot)\|_{\mathcal{L}(\mathcal{H})}, \tag{25}$$

as the formula $\|G(\boldsymbol{\xi})\|^2 = \lambda_{\max}\big(\hat{G}(\boldsymbol{\xi})\big)$ with $\hat{G}(\boldsymbol{\xi}) := G^*(\boldsymbol{\xi})G(\boldsymbol{\xi})$ is well known. To this end, we first write down the formula

$$\max_{\boldsymbol{\xi}\in D} \|G(\boldsymbol{\xi})\| = \|G(\boldsymbol{\xi}_{\mathbf{0}})\| = \lambda_{\max}^{1/2}\big(\hat{G}(\boldsymbol{\xi}_{\mathbf{0}})\big) \tag{26}$$

for some $\boldsymbol{\xi_0} \in D$. Let $\mathbf{w}_0 \in \mathbb{C}^{n+1}$ be an eigenvector corresponding to the eigenvalue $\lambda_{\max}(\hat{G}(\boldsymbol{\xi_0}))$:

$$\hat{G}(\boldsymbol{\xi}_0)\mathbf{w}_0 = \lambda_{\max}(\hat{G}(\boldsymbol{\xi_0}))\mathbf{w}_0, \ \ \mathbf{w}_0 \neq 0.$$

We construct the function $\mathbf{w}(\boldsymbol{\xi}) = \chi_\varepsilon(\boldsymbol{\xi})\mathbf{w}_0$, where $\chi_\varepsilon(\boldsymbol{\xi})$ is the characteristic function of the ball $B_\varepsilon = \{|\boldsymbol{\xi} - \boldsymbol{\xi_0}|\} \leqslant \varepsilon$ with $\varepsilon > 0$. By definition of $\|G(\cdot)\|_{\mathcal{L}(\mathcal{H})}$, we can write

$$\|G(\cdot)\|^2_{\mathcal{L}(\mathcal{H})} \geqslant \frac{\|G(\chi_\varepsilon \mathbf{w}_0)\|^2_{\mathcal{H}}}{\|\chi_\varepsilon \mathbf{w}_0\|^2_{\mathcal{H}}} = \frac{1}{|\mathbf{w}_0|^2|D_\varepsilon|}\int_{D_\varepsilon} |G(\boldsymbol{\xi})\mathbf{w}_0|^2\,d\boldsymbol{\xi},$$

where $D_\varepsilon = D \cap B_\varepsilon$ and $|D_\varepsilon|$ is the measure of D_ε. Next, we transform and bound from below the arisen integral as follows:

$$\frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} |G(\boldsymbol{\xi})\mathbf{w}_0|^2\,d\boldsymbol{\xi} = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} (\hat{G}(\boldsymbol{\xi})\mathbf{w}_0, \mathbf{w}_0)_{\mathbb{C}^{n+1}}\,d\boldsymbol{\xi}$$
$$= \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} (\hat{G}(\boldsymbol{\xi}_0)\mathbf{w}_0, \mathbf{w}_0)_{\mathbb{C}^{n+1}}\,d\boldsymbol{\xi} + \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} \big((\hat{G}(\boldsymbol{\xi}) - \hat{G}(\boldsymbol{\xi}_0))\mathbf{w}_0, \mathbf{w}_0\big)_{\mathbb{C}^{n+1}}\,d\boldsymbol{\xi}$$
$$\geqslant \lambda_{\max}(\hat{G}(\boldsymbol{\xi_0}))|\mathbf{w}_0|^2 - \max_{\boldsymbol{\xi}\in D_\varepsilon} \|\hat{G}(\boldsymbol{\xi}) - \hat{G}(\boldsymbol{\xi}_0)\||\mathbf{w}_0|^2.$$

Therefore, using the right Formula (26), we derive the lower bound

$$\|G(\cdot)\|^2_{\mathcal{L}(\mathcal{H})} \geqslant \lambda_{\max}(\hat{G}(\boldsymbol{\xi_0})) - \max_{\boldsymbol{\xi}\in D_\varepsilon} \|\hat{G}(\boldsymbol{\xi}) - \hat{G}(\boldsymbol{\xi}_0)\|$$
$$= \|G(\boldsymbol{\xi_0})\|^2 - \max_{\boldsymbol{\xi}\in D_\varepsilon} \|\hat{G}(\boldsymbol{\xi}) - \hat{G}(\boldsymbol{\xi}_0)\|.$$

Due to continuity of $G(\boldsymbol{\xi})$, the matrix $\hat{G}(\boldsymbol{\xi})$ is continuous on D as well, thus we get

$$\lim_{\varepsilon\to 0}\max_{\boldsymbol{\xi}\in D_\varepsilon} \|\hat{G}(\boldsymbol{\xi}) - \hat{G}(\boldsymbol{\xi}_0)\| = 0.$$

Therefore, $\|G(\cdot)\|^2_{\mathcal{L}(\mathcal{H})} \geqslant \|G(\boldsymbol{\xi_0})\|^2$, and due to the left Formula (26) we get inequality (25). $\square$

The main items of the presented full proof have recently been described briefly in [33] (Theorem 1). Clearly, neither the specific form of $\mathcal{A}$ nor the specific dimension $n+1$ of the involved vectors are essential in it (though the continuity of $G(\boldsymbol{\xi})$ on D has been exploited).

Now, we present a criterion, sufficient conditions and necessary conditions for the validity of bound (9). Recall that the quantity $b^{(0)}$ has been introduced in (12), and we set $b^{(1)} := 1$ to unify the form of inequalities (12) and (16).

Theorem 2. *Let $\ell = 0, 1$.*

1. For the validity of bound (9), the matrix inequality

$$\beta\big(2\alpha A^2_{\mathbf{s}\ell} + \tfrac{1}{2\alpha}B^2_{\mathbf{s}} + \mathbf{i}[A_{\mathbf{s}\ell}, B_{\mathbf{s}}]\big) \leqslant A_{\mathbf{s}\ell} \ \ \forall \mathbf{s} \in S \tag{27}$$

serves as a criterion. Here, $[A_{\mathbf{s}\ell}, B_{\mathbf{s}}] = A_{\mathbf{s}\ell}B_{\mathbf{s}} - B_{\mathbf{s}}A_{\mathbf{s}\ell}$ is the commutator of the matrices $A_{\mathbf{s}\ell}$ and $B_{\mathbf{s}}$, and the matrix $\mathbf{i}[A_{\mathbf{s}\ell}, B_{\mathbf{s}}]$ is Hermitian.

2. For the validity of bound (9), the matrix inequality

$$\beta\big[2\alpha(1+\varepsilon)A^2_{\mathbf{s}\ell} + \tfrac{1}{2\alpha}(1+\varepsilon^{-1})B^2_{\mathbf{s}}\big] \leqslant A_{\mathbf{s}\ell} \ \ \forall \mathbf{s} \in S, \tag{28}$$

with any $\varepsilon > 0$, is a sufficient condition.

Consequently, for $\max_{\mathbf{s}\in S}\lambda_{\max}(A_{\mathbf{s}\ell}) \leqslant \bar{\lambda}_\ell$, the same is valid concerning the inequality

$$\beta \leqslant \beta_{suf}(\alpha) := \frac{1}{\big[(2\alpha\bar{\lambda}_\ell)^{1/2} + (2\alpha b^{(\ell)})^{-1/2}\big]^2} = \frac{1}{\big[2\alpha\bar{\lambda}_\ell + 2(\bar{\lambda}_\ell/b^{(\ell)})^{1/2} + (2\alpha b^{(\ell)})^{-1}\big]}. \tag{29}$$

3. *For the validity of bound* (9), *the inequalities*

$$2\beta\alpha\underline{\lambda}_{\,\ell} \leqslant 1 \;\; with \;\; \underline{\lambda}_{\,\ell} = \max\left\{r_i^2(\ell M_i^2+1), r_i^2(M_i^2+\alpha_s)+a_\ell r_{\max}^2\right\}, \tag{30}$$

$$\beta \leqslant 2b^{(\ell)}\alpha \tag{31}$$

with $r_{\max} := \max_{1\leqslant k\leqslant n} r_k$ *are necessary conditions.*

Proof. 1. Justifications of criterion (27) and the sufficient condition (28) (as well as (29) for $\ell = 1$) were given in ([29], Theorem 2). Herewith, the derivation of the criterion is based on Theorem 1 and the equivalence of the properties $\lambda_{\max}\big(G^*(\boldsymbol{\xi})G(\boldsymbol{\xi})\big) \leqslant 1$ and $G^*(\boldsymbol{\xi})G(\boldsymbol{\xi}) \leqslant I_{n+1}$.

By virtue of (12) and (16) inequality (28) follows from the inequality

$$\beta\big[2\alpha(1+\varepsilon)A_{\mathbf{s}\ell}^2 + \tfrac{1}{2\alpha b^{(\ell)}}(1+\varepsilon^{-1})A_{\mathbf{s}\ell}\big] \leqslant A_{\mathbf{s}\ell} \;\; \forall \mathbf{s} \in S.$$

The last inequality means the following inequality for the eigenvalues $\lambda_k(A_{\mathbf{s}\ell})$ of the matrix $A_{\mathbf{s}\ell}$:

$$\beta\big[2\alpha(1+\varepsilon)\lambda_k^2(A_{\mathbf{s}\ell}) + \tfrac{1}{2\alpha b^{(\ell)}}(1+\varepsilon^{-1})\lambda_k(A_{\mathbf{s}\ell})\big] \leqslant \lambda_k(A_{\mathbf{s}\ell}) \;\; \forall \mathbf{s} \in S.$$

As $\lambda_k(A_{\mathbf{s}\ell}) \geqslant 0$, it is equivalent to

$$\beta\big[2\alpha(1+\varepsilon)\max_{\mathbf{s}\in S}\lambda_{\max}(A_{\mathbf{s}\ell}) + \tfrac{1}{2\alpha b^{(\ell)}}(1+\varepsilon^{-1})\big] \leqslant 1.$$

For $\max_{\mathbf{s}\in S}\lambda_{\max}(A_{\mathbf{s}\ell}) \leqslant \bar{\lambda}_\ell$, choosing $\varepsilon = 1/\big[2\alpha(b^{(\ell)}\bar{\lambda}_\ell)^{1/2}\big]$ we get the sufficient condition (29).

2. For $\mathbf{s} = 0$, we find $\sigma_1 = \ldots = \sigma_n = 1$ and

$$d^2 = r^2 := r_1^2 + \ldots + r_n^2, \;\; \boldsymbol{\zeta} = 0, \;\; Q = \operatorname{diag}\{r_1^2, \ldots, r_n^2\}, \;\; a_{\mathbf{M}} = \mathbf{M}^T Q\mathbf{M} = r_i^2 M_i^2,$$

thus $B_{\mathbf{s}} = 0$ and

$$A_{0\ell} := A_{\mathbf{s}\ell}|_{\mathbf{s}=0} = \begin{pmatrix} \ell r_i^2 M_i^2 + r^2 & (\ell+1)\mathbf{M}^T Q \\ (\ell+1)Q\mathbf{M} & (r_i^2 M_i^2 + \alpha_s r^2)I_n + a_\ell Q \end{pmatrix}. \tag{32}$$

Now criterion (27) leads to the necessary condition $2\beta\alpha A_{0\ell}^2 \leqslant A_{0\ell}$, i.e., $2\beta\alpha\lambda_{\max}(A_{0\ell}) \leqslant 1$. We write down the lower bound by the maximal diagonal element:

$$\max\left\{r_i^2(\ell M_i^2+1), r_i^2(M_i^2+\alpha_s)+a_\ell r_{\max}^2\right\} \leqslant \lambda_{\max}(A_{0\ell}), \tag{33}$$

by means of which the latter necessary condition implies (30). Note that, for $M = 0$, the matrix $A_{0\ell}$ is diagonal, and bound (33) turns into the equality $\max\{r^2, r^2\alpha_s + a_\ell r_{\max}^2\} = \lambda_{\max}(A_{0\ell})$.

To derive condition (31), we set $\boldsymbol{\xi} = 2\varepsilon\tilde{\boldsymbol{\xi}}$ with $\varepsilon \to 0$ and $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n) \neq 0$ similarly to the proof of ([36], Theorem 4). Then, we obtain

$$\sigma_k \sim \varepsilon^2\tilde{\xi}_k^2, \;\; d_k \sim \varepsilon|\eta_k|, \;\; \boldsymbol{\zeta} \sim \varepsilon\boldsymbol{\eta}, \;\; q_k = O(\varepsilon^4), \;\; 1 \leqslant k \leqslant n,$$

with $\eta_k := r_k\tilde{\xi}_k$, $\boldsymbol{\eta} := (\eta_1, \ldots, \eta_n)$ and therefore $B_{\mathbf{s}} \sim \varepsilon B^{(0)}$ and $A_{\mathbf{s}\ell} \sim \varepsilon^2 A_\ell^0$ with

$$B^{(0)} = \begin{pmatrix} \boldsymbol{\eta}\mathbf{M} & \boldsymbol{\eta} \\ \boldsymbol{\eta}^T & (\boldsymbol{\eta}\mathbf{M})I_n \end{pmatrix}, \; A_\ell^0 = \begin{pmatrix} |\boldsymbol{\eta}|^2 & (\ell+1)(\boldsymbol{\eta}\mathbf{M})\boldsymbol{\eta} \\ (\ell+1)(\boldsymbol{\eta}\mathbf{M})\boldsymbol{\eta}^T & \big[(\boldsymbol{\eta}\mathbf{M})^2 + \alpha_s|\boldsymbol{\eta}|^2\big]I_n + a_\ell\boldsymbol{\eta}^T\boldsymbol{\eta} \end{pmatrix}, \tag{34}$$

where the asymptotic relations for vectors and matrices are understood componentwise.

Herewith, $A^2_{\mathbf{s}\ell} = O(\varepsilon^4)$ and $[A_{\mathbf{s}\ell}, B_{\mathbf{s}}] = O(\varepsilon^3)$, therefore here criterion (27), after division by ε^2 and passing to the limit as $\varepsilon \to 0$, leads to the necessary condition

$$\frac{\beta}{2\alpha}(B^{(0)})^2 \leqslant A^0_\ell \ \ \forall \boldsymbol{\eta} \in \mathbb{R}^n,\ \boldsymbol{\eta} \neq 0.$$

After division of the both sides by $|\boldsymbol{\eta}|^2$ and the formal replacement of $\boldsymbol{\eta}/|\boldsymbol{\eta}|$ by $\tilde{\boldsymbol{\zeta}}$, it takes the form

$$\frac{\beta}{2\alpha}\tilde{B}^2_{\mathbf{s}} \leqslant \tilde{A}^{(1)}_{\mathbf{s}\ell} \ \ \forall \tilde{\boldsymbol{\zeta}} \in \mathbb{R}^n \ \text{ with } \ |\tilde{\boldsymbol{\zeta}}| = 1 \tag{35}$$

with the symmetric matrix

$$\tilde{A}^{(1)}_{\mathbf{s}\ell} = \begin{pmatrix} 1 & (\ell+1)(\tilde{\boldsymbol{\zeta}}\mathbf{M})\tilde{\boldsymbol{\zeta}} \\ (\ell+1)(\tilde{\boldsymbol{\zeta}}\mathbf{M})\tilde{\boldsymbol{\zeta}}^T & [(\tilde{\boldsymbol{\zeta}}\mathbf{M}^2 + \alpha_s] I_n + a_\ell \tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{\zeta}} \end{pmatrix}.$$

Recall that the matrices $\tilde{B}_{\mathbf{s}}$ and $\tilde{A}^{(1)}_{\mathbf{s}\ell}$ for $\ell = 0$ have been introduced in (18) and (19). In the inequality

$$\tilde{A}^{(1)}_{\mathbf{s}\ell} - b\tilde{B}^2_{\mathbf{s}} \geqslant 0 \ \ \forall \tilde{\boldsymbol{\zeta}} \in \mathbb{R}^n \ \text{ with } \ |\tilde{\boldsymbol{\zeta}}| = 1,$$

the constant $b = b^{(\ell)}$ is maximal. For $\ell = 0$, this has been established in the proof of Lemma 2, whereas for $\ell = 1$ this is a consequence of the relations

$$\tilde{A}^{(1)}_{\mathbf{s}1} - \tilde{B}^2_{\mathbf{s}} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_s I_n + a_0 \tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{\zeta}} \end{pmatrix} \geqslant 0$$

which follow from (13). Therefore, condition (35) implies the necessary condition (31). □

Corollary 1. *For the validity of bound* (9)*, the following inequality*

$$\beta \leqslant \beta_{nec}(\alpha) := \min\left\{2b^{(\ell)}\alpha, \frac{1}{2\underline{\lambda}_\ell \alpha}\right\} \tag{36}$$

is a necessary condition.

Notice that $\beta_{suf}(\alpha) \to 0$ and $\beta_{nec}(\alpha) \to 0$ as $\alpha \to +0$ or $\alpha \to +\infty$. The maximum of the right-hand side of the sufficient condition (29) is attained at $\alpha = \frac{1}{2}(b^{(\ell)}\bar{\lambda}_\ell)^{-1/2}$ and equals

$$\max_{\alpha>0} \beta_{suf}(\alpha) = \frac{1}{4}\left(\frac{b^{(\ell)}}{\bar{\lambda}_\ell}\right)^{1/2}.$$

The maximum of the right-hand side of the necessary condition (36) is attained at $\alpha = \frac{1}{2}(b^{(\ell)}\underline{\lambda}_\ell)^{-1/2}$ and equals

$$\max_{\alpha>0} \beta_{nec}(\alpha) = \left(\frac{b^{(\ell)}}{\underline{\lambda}_\ell}\right)^{1/2}.$$

For comparison, notice that for the validity of bound (9), the matrix inequalities

$$2\beta\alpha r_k^2 A^{(kk)}_\ell \leqslant I,\ \ \frac{\beta}{2\alpha}(B^{(k)})^2 \leqslant A^{(kk)}_\ell,\ \ 1 \leqslant k \leqslant n,$$

serve as necessary conditions as well. For $\ell = 1$, this was proved in ([29], Theorem 2) by reducing to the 1D case and, for $\ell = 0$, this is proved in the same way (recall that the 1D case was previously studied in [32]). The form of matrices $A_\ell^{(ij)}$, $i \neq j$, is inessential in this proof.

Due to Lemma 2, Item 1, for $\ell = 0$, and due to ([32], Theorem 1 and Remark 2), for $\ell = 1$, these inequalities are equivalent to the number inequalities

$$2\beta\alpha \max_{1\leqslant k\leqslant n} r_k^2 \lambda_{\max}(A_\ell^{(kk)}) \leqslant 1, \;\; \beta \leqslant 2\underline{b}^{(\ell)}\alpha, \tag{37}$$

where $\underline{b}^{(0)} = \min_{1\leqslant k\leqslant n} b_k$ and $\underline{b}^{(1)} = 1$. As $\max_{1\leqslant k\leqslant n} q(|M_k|) \leqslant \max_{0\leqslant m\leqslant M} q(M)$, for $\ell = 0$, condition (31) is more sharp than the second condition (37), whereas for $\ell = 1$ they coincide.

It is not difficult to calculate eigenvalues of the matrix $A_\ell^{(kk)}$, in particular, the maximal one, which is the maximal eigenvalue of its 2×2 block:

$$A_{\ell k}^{(kk)} := \begin{pmatrix} \ell M_k^2 + 1 & (\ell+1)M_k \\ (\ell+1)M_k & M_k^2 + \widehat{a}_\ell \end{pmatrix} \text{ with } \widehat{a}_\ell = a_\ell + \alpha_s = \frac{4}{3}\alpha_s + \alpha_{1s} + \ell$$

(recall that $\widehat{a}_0$ first appeared in Lemma 2), namely,

$$\lambda_{\max}(A_1^{(kk)}) = M_k^2 + \tfrac{1}{2}(\widehat{a}_1 + 1) + \big\{4M_k^2 + \big[\tfrac{1}{2}(\widehat{a}_1 - 1)\big]^2\big\}^{1/2},$$
$$\lambda_{\max}(A_0^{(kk)}) = \tfrac{1}{2}(M_k^2 + \widehat{a}_0 + 1) + \big\{\big[\tfrac{1}{2}(M_k^2 + \widehat{a}_0 + 1)\big]^2 - \widehat{a}_0\big\}^{1/2}, \;\; 1 \leqslant k \leqslant n,$$

see in [29] and ([32], Theorem 3). For them, the following two-sided bounds hold:

$$\max\big\{(|M_k|+1)^2 + \tfrac{1}{2}(\widehat{a}_1+1), M_k^2 + \widehat{a}_1\big\} \leqslant \lambda_{\max}(A_1^{(kk)}) \leqslant (|M_k|+1)^2 + \widehat{a}_1,$$
$$\max\{M_k^2 + \widehat{a}_0, 1\} \leqslant \lambda_{\max}(A_0^{(kk)}) \leqslant M_k^2 + \widehat{a}_0 + 1 \leqslant 2\max\{M_k^2 + \widehat{a}_0, 1\}.$$

Furthermore, the following lower bounds hold:

$$\max_{1\leqslant k\leqslant n} r_k^2(M_k^2 + \widehat{a}_1) \leqslant \underline{\lambda}_\ell = \max\big\{r_i^2(M_i^2+1), r_i^2(M_i^2 + \alpha_s) + a_1 r_{\max}^2\big\} \text{ for } \ell = 1,$$
$$\max_{1\leqslant k\leqslant n} r_k^2 \max\{M_k^2 + \widehat{a}_0, 1\} \leqslant \underline{\lambda}_\ell = \max\big\{r^2, r_i^2(M_i^2 + \alpha_s) + a_0 r_{\max}^2\big\} \text{ for } \ell = 0,$$

therefore, the necessary condition (30) is qualitatively stronger than the first condition (37). It is not difficult to ensure this property on the quantitative level as well. To this end, it is required to strengthen bound (33) by means of using the 2×2 blocks of matrix (32):

$$\lambda_{\max}(A_{0\ell k}) \leqslant \lambda_{\max}(A_{0\ell}) \text{ with } A_{0\ell k} := \begin{pmatrix} \ell r_i^2 M_i^2 + r^2 & (\ell+1)M_k r_k^2 \\ (\ell+1)M_k r_k^2 & r_i^2 M_i^2 + \alpha_s r^2 + a_\ell r_k^2 \end{pmatrix},$$

for $1 \leqslant k \leqslant n$, since then

$$r_k^2 \lambda_{\max}(A_{\ell k}^{(kk)}) = \lambda_{\max}(r_k^2 A_{\ell k}^{(kk)}) \leqslant \lambda_{\max}(A_{0\ell k}), \;\; 1 \leqslant k \leqslant n.$$

This has not been implemented above in order not to complicate bound (30) essentially.

To apply the sufficient condition (29), we present the uniform in $\mathbf{s}$ upper bound for $\lambda_{\max}(A_{\mathbf{s}\ell})$.

Theorem 3. *For $\ell = 0, 1$ and $n = 2, 3$, for the eigenvalues of the matrix $A_{\mathbf{s}\ell}$, see (11), the following bound holds:*

$$\max_{\mathbf{s}\in S} \lambda_{\max}(A_{\mathbf{s}\ell}) \leqslant \bar{\lambda}_\ell := r_i^2\big[(1+\ell\varepsilon)c_n M_i^2 + \max\{1, \alpha_s\}\big] + c_n(a_\ell + \varepsilon^{-1})r_{\max}^2, \tag{38}$$

where $c_2 = 1$, $c_3 = \frac{9}{8}$ and also $\varepsilon = 1$ for $\ell = 0$ or $\varepsilon > 0$ is arbitrary for $\ell = 1$.

In the particular case $M = 0$, it holds that $\bar{\lambda}_\ell = r^2 \max\{1, \alpha_s\} + c_n a_1 r^2_{\max}$.

Proof. The following quadratic form corresponds to the matrix $A_{\mathbf{s}\ell}$:

$$\mathcal{A}_{\mathbf{s}\ell}(v_0, \mathbf{v}) =$$
$$= (\ell a_{\mathbf{M}} + d^2)v_0^2 + 2(\ell+1)v_0\big[(\boldsymbol{\zeta}\mathbf{M})\boldsymbol{\zeta}\mathbf{v} + \mathbf{M}^T Q\mathbf{v}\big] + \big(a_{\mathbf{M}} + \alpha_s d^2\big)|\mathbf{v}|^2 + a_\ell\big[(\boldsymbol{\zeta}\mathbf{v})^2 + \mathbf{v}^T Q\mathbf{v}\big]$$

with any $v_0 \in \mathbb{R}$ and $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$. As $Q = Q^T \geqslant 0$, we have

$$|\mathbf{M}^T Q\mathbf{v}| \leqslant (\mathbf{M}^T Q\mathbf{M})^{1/2}(\mathbf{v}^T Q\mathbf{v})^{1/2},$$

and therefore the following bound holds:

$$2v_0\big[(\boldsymbol{\zeta}\mathbf{M})\boldsymbol{\zeta}\mathbf{v} + \mathbf{M}^T Q\mathbf{v}\big] \leqslant 2|v_0|\big[(\boldsymbol{\zeta}\mathbf{M})^2 + \mathbf{M}^T Q\mathbf{M}\big]^{1/2}\big[(\boldsymbol{\zeta}\mathbf{v})^2 + \mathbf{v}^T Q\mathbf{v}\big]^{1/2}. \tag{39}$$

As $\sigma_i + s_i^2 = 1$ and $0 \leqslant \sigma_i \leqslant 1$, due to the Cauchy inequality we obtain

$$(\boldsymbol{\zeta}\mathbf{v})^2 + \mathbf{v}^T Q\mathbf{v} = d_i s_i v_i d_j s_j v_j + d_i^2\sigma_i v_i^2 = d_i^2 v_i^2 + (1 - \delta^{(ij)})d_i s_i v_i d_j s_j v_j$$
$$\leqslant d_i^2 v_i^2 + (n-1)d_i^2 s_i^2 v_i^2 = r_i^2\sigma_i\big[n - (n-1)\sigma_i\big]v_i^2 \leqslant c_n r_i^2 v_i^2 \leqslant c_n r^2_{\max}|\mathbf{v}|^2 \tag{40}$$

using the formula $\max_{0\leqslant\sigma\leqslant 1}\sigma[n - (n-1)\sigma] = c_n$ with c_n introduced in the Lemma, similarly to the proof of ([29], Theorem 4). Replacing $\mathbf{v}$ with $\mathbf{M}$ here, we also obtain

$$a_{\mathbf{M}} = (\boldsymbol{\zeta}\mathbf{M})^2 + \mathbf{M}^T Q\mathbf{M} \leqslant c_n r_i^2 M_i^2 =: \bar{a}_{\mathbf{M}}.$$

Therefore, using bound (39) we get

$$\mathcal{A}_{\mathbf{s}\ell}(v_0, \mathbf{v}) \leqslant (\ell\bar{a}_{\mathbf{M}} + r^2)v_0^2 + 2(\ell+1)\big(\bar{a}_{\mathbf{M}} c_n r^2_{\max}\big)^{1/2}|v_0||\mathbf{v}| + \varkappa_\ell|\mathbf{v}|^2 \tag{41}$$

with $\varkappa_\ell := \bar{a}_{\mathbf{M}} + \alpha_s r^2 + c_n a_\ell r^2_{\max}$, and due to the classical Rayleigh formula for $\lambda_{\max}(A)$, we derive

$$\max_{\mathbf{s}\in S}\lambda_{\max}(A_{\mathbf{s}\ell}) \leqslant \lambda_{\max}(C_\ell),\ \ C_\ell := \begin{pmatrix} \ell\bar{a}_{\mathbf{M}} + r^2 & (\ell+1)\big(\bar{a}_{\mathbf{M}} c_n r^2_{\max}\big)^{1/2} \\ (\ell+1)\big(\bar{a}_{\mathbf{M}} c_n r^2_{\max}\big)^{1/2} & \varkappa_\ell \end{pmatrix}.$$

It is not difficult to calculate that

$$\lambda_{\max}(C_\ell) = \tfrac{1}{2}\Big\{\varkappa_\ell + \ell\bar{a}_{\mathbf{M}} + r^2 + \big[\big(\varkappa_\ell - (\ell\bar{a}_{\mathbf{M}} + r^2)\big)^2 + 4(\ell+1)^2\bar{a}_{\mathbf{M}} c_n r^2_{\max}\big]^{1/2}\Big\}$$
$$\leqslant \tfrac{1}{2}\big(\varkappa_\ell + \ell\bar{a}_{\mathbf{M}} + r^2 + |\varkappa_\ell - (\ell\bar{a}_{\mathbf{M}} + r^2)|\big) + (\ell+1)(\bar{a}_{\mathbf{M}} c_n)^{1/2} r_{\max}$$
$$= \max\{\varkappa_\ell, \ell\bar{a}_{\mathbf{M}} + r^2\} + (\ell+1)(\bar{a}_{\mathbf{M}} c_n)^{1/2} r_{\max}.$$

Further, for $\ell = 1$, we write down the estimates

$$\max\{\varkappa_1, \bar{a}_{\mathbf{M}} + r^2\} + 2(\bar{a}_{\mathbf{M}} c_n)^{1/2} r_{\max}$$
$$\leqslant (1+\varepsilon)\bar{a}_{\mathbf{M}} + \max\{\alpha_s r^2 + c_n a_\ell r^2_{\max}, r^2\} + \varepsilon^{-1} c_n r^2_{\max}$$
$$\leqslant \bar{\lambda}_1 := (1+\varepsilon)\bar{a}_{\mathbf{M}} + \max\{\alpha_s, 1\}r^2 + c_n(a_\ell + \varepsilon^{-1})r^2_{\max}\ \ \forall\varepsilon > 0.$$

On the other hand, for $\ell = 0$, the right-hand side of inequality (41) can be estimated as follows:

$$r^2 v_0^2 + 2\big(\bar{a}_{\mathbf{M}} c_n r^2_{\max}\big)^{1/2}|v_0||\mathbf{v}| + \varkappa_0|\mathbf{v}|^2 \leqslant (\bar{a}_{\mathbf{M}} + r^2)v_0^2 + (\varkappa_0 + c_n r^2_{\max})|\mathbf{v}|^2$$
$$\leqslant \big(\bar{a}_{\mathbf{M}} + \max\big\{r^2, \alpha_s r^2 + c_n(a_0+1)r^2_{\max}\big\}\big)(v_0^2 + |\mathbf{v}|^2)$$
$$\leqslant \bar{\lambda}_0(v_0^2 + |\mathbf{v}|^2)\ \text{ with }\ \bar{\lambda}_0 := \bar{a}_{\mathbf{M}} + \max\{\alpha_s, 1\}r^2 + c_n(a_0+1)r^2_{\max}.$$

Consequently, due to the classical Rayleigh formula for $\lambda_{\max}(C_0)$, we get $\lambda_{\max}(C_0) \leqslant \bar{\lambda}_0$. $\square$

Of course, on the right-hand side of bound (38), we could strengthen $\bar{\lambda}_\ell$ to the value $\lambda_{\max}(C_\ell)$ given in the proof; above this has not been done in order to avoid too cumbersome result.

In accordance with the derived bounds, the natural choice of $\hat{h}$, depending on $\mathbf{h}$ and $\mathbf{M}$ only, is as follows:

$$\frac{1}{\hat{h}^2} = \frac{M_1^2+1}{h_1^2} + \ldots + \frac{M_n^2+1}{h_n^2}.$$

Such a formula is suggested for the first time for schemes based on the QGD or QHD regularizations. For it, the following two-sided bound holds:

$$\frac{1}{\sqrt{n}} \leqslant \frac{\hat{h}}{\min\limits_{1\leqslant k\leqslant n} \frac{h_k}{\sqrt{M_k^2+1}}} \leqslant 1$$

together with the equality

$$r_i^2(M_i^2+1) \equiv \frac{\hat{h}^2}{h_i^2}(M_i^2+1) = 1.$$

Due to this equality, we can estimate the constant on the right-hand side of bound (38) as follows:

$$\bar{\lambda}_\ell \leqslant \max\{(1+\ell\varepsilon)c_n, \max\{1,\alpha_s\} + c_n(a_\ell + \varepsilon^{-1})\}$$

and, furthermore,

$$\bar{\lambda}_\ell \leqslant \max\{1,\alpha_s\} + c_n(a_\ell + 1).$$

For $\ell = 0$ and $\varepsilon = 1$ the latter estimate is obvious, whereas for $\ell = 1$ it arises after taking the minimum in $\varepsilon > 0$ and simplifying the result a little.

Furthermore, for $\underline{\lambda}_\ell$ in the necessary condition (30), the following lower bounds hold:

$$\underline{\lambda}_0 \geqslant \min\left\{1, \alpha_s + \tfrac{1}{n}a_0\right\}, \ \ \underline{\lambda}_1 \geqslant 1.$$

Importantly, the given bounds lead to the sufficient condition and necessary condition independent of $\mathbf{h}$. Moreover, in the case $\ell = 1$, i.e., for the QGD regularization, they are also uniform in the Mach number $M \geqslant 0$ that can be valuable for computing super- and hypersonic gas flows.

In addition, Formula (10) can be rewritten in the form

$$\frac{c_*}{\hat{h}}\Delta t = c_*\left(\frac{M_1^2+1}{h_1^2} + \ldots + \frac{M_n^2+1}{h_n^2}\right)^{1/2}\Delta t = \beta, \ \ \tau = \frac{\alpha}{c_*\left(\frac{M_1^2+1}{h_1^2} + \ldots + \frac{M_n^2+1}{h_n^2}\right)^{1/2}}.$$

For some other schemes, a similar formula for β but without powers 2 and $1/2$ is contained in ([1], Chapter 2).

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