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Some Hermite–Hadamard-Type Fractional Integral Inequalities Involving Twice-Differentiable Mappings

Soubhagya Kumar Sahoo [1], Muhammad Tariq [2,*], Hijaz Ahmad [3,4,*], Ayman A. Aly [5], Bassem F. Felemban [5] and Phatiphat Thounthong [6]

[1] Department of Mathematics, Institute of Technical Education and Research, Siksha 'O' Anusandhan University, Bhubaneswar 751030, India; soubhagyakumarsahoo@soa.ac.in
[2] Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan
[3] Department of Computer Engineering, Biruni University, Istanbul 34025, Turkey
[4] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
[5] Department of Mechanical Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; aymanaly@tu.edu.sa (A.A.A.); B.felemban@tu.edu.sa (B.F.F.)
[6] Renewable Energy Research Centre (RERC), Department of Teacher Training in Electrical Engineering, Faculty of Technical Education, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; phatiphat.t@fte.kmutnb.ac.th
* Correspondence: captaintariq2187@gmail.com (M.T.); hijaz555@gmail.com (H.A.)

Abstract: The theory of fractional analysis has been a focal point of fascination for scientists in mathematical science, given its essential definitions, properties, and applications in handling real-life problems. In the last few decades, many mathematicians have shown their considerable interest in the theory of fractional calculus and convexity due to their wide range of applications in almost all branches of applied sciences, especially in numerical analysis, physics, and engineering. The objective of this article is to establish Hermite-Hadamard type integral inequalities by employing the *k*-Riemann-Liouville fractional operator and its refinements, whose absolute values are twice-differentiable h-convex functions. Moreover, we also present some special cases of our presented results for different types of convexities. Moreover, we also study how **q**-digamma functions can be applied to address the newly investigated results. Mathematical integral inequalities of this class and the arrangements associated have applications in diverse domains in which symmetry presents a salient role.

Keywords: convex function; Hermite-Hadamard inequality; h-convex function; Riemann-Liouville *k*-fractional integrals

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1. Introduction

The idea of convex analysis has a strong background and has been the inspiration for excellent research for more than a century in the field of mathematics. Various augmentations, variations, and speculations of the theory of convexity have been taken into consideration by numerous researchers. This theory develops and provides numerical procedures to handle and study complex problems in the field of mathematics. This theory has been very inspirational and popular among mathematicians as it possesses a wide range of potential applications in pure and applied sciences.

The idea of inequalities is perhaps one of the most important elements of science having various applications in different branches of mathematics, engineering, and physics. Currently, the theory of inequalities is still intensively developed. In this regard, the Hermite-Hadamard type inequality is broadly notable and has been read and generalized for various sorts of convex functions under different parameters and conditions. In recent times, the correlation between convexity and inequalities has acquired a great deal of consideration among mathematicians because of their basic definitions and properties. Numerous mathematicians and researchers are working in the direction of this inequality for estimating the fractional version of the Hermite-Hadamard inequality utilizing various types of convexity (see, for example, refs. [1–10]).

The theory of fractional differential equations was initiated in the 19th century by Riemann and Liouville, who introduced the preliminary concepts of the theory. Since then, many new versions of these definitions, such as Gruenwald-Letnikov derivatives, Caputo derivatives, and their multidimensional analogs, have appeared in the literature. This hypothesis moreover has been accepted as a critical part in the progression of the idea of inequalities. In research activities, the theory of inequalities has an extraordinary arrangement of employment in financial issues, numerical analysis, probability theory, and many more. Fractional differentiable inequalities have applications in fractional differential equations, the most important ones being to establish the uniqueness of the solution of initial-value problems and give upper bounds to their solution. These applications have motivated many researchers in the field of integral inequalities to investigate a few extensions and generalizations using different fractional differential and integral operators. For some related articles, the readers can see [11–16].

One of the main objectives of this article is to present a new fractional version of the Hermite-Hadamard inequality, where Minkowski and Hölder's inequality is used to prove the right-hand side of the inequality. We derive general Hermite-Hadamard-type inequalities for functions whose second derivatives are h-convex by using the k-fractional operator. Next, some new integral identities are studied and employing these and with the help of some well-known fundamental inequalities, such as the Hölder, Hölder-İscan, power-mean inequality, we establish some refinements of the Hermite-Hadamard-type inequality for twice-differentiable mappings. Moreover, some interesting applications related to **q**-digamma functions are discussed.

2. Preliminaries and Basic Concept

The main objective of this section is to recall some known definitions and concepts.

Definition 1 ([17])**.** *A real-valued function* $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ *is known as convex on the interval* I*, if:*

$$\wp(\sigma\ell_1 + (1-\sigma)\ell_2) \leq \sigma\wp(\ell_1) + (1-\sigma)\wp(\ell_2),$$

holds for all $\ell_1, \ell_2 \in \mathrm{I}$ *and* $\sigma \in [0,1]$.

Theorem 1 ([18])**.** *Let* $\wp : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *be a convex function with* $\ell_1 < \ell_2$ *and* $\ell_1, \ell_2 \in \mathrm{I}$*. Then, the Hermite-Hadamard inequality is expressed as follows:*

$$\wp\left(\frac{\ell_1+\ell_2}{2}\right) \leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2} \wp(x)dx \leq \frac{\wp(\ell_1)+\wp(\ell_2)}{2}. \quad (1)$$

In the recent past, the classical Hermite-Hadamard inequality (1) was generalized and extended extensively by numerous mathematicians under the assumption of some interesting new definitions as a generalization of the convex function.

In the year 2007, Varošanec [19] introduced and investigated the term h-convexity.

Definition 2 ([19])**.** *Let* $\mathrm{h} : \mathrm{I} \longrightarrow \mathbb{R} \subseteq \mathbb{R}$ *be a positive function, then a non-negative function* $\wp : \mathrm{I} \longrightarrow \mathbb{R} \subseteq \mathbb{R}$ *is an "*h*-convex function" if* $\forall\ \ell_1, \ell_2 \in \mathrm{I}, \sigma \in (0,1)$*, we have:*

$$\wp(\sigma\ell_1 + (1-\sigma)\ell_2) \leq \mathrm{h}(\sigma)\wp(\ell_1) + \mathrm{h}(1-\sigma)\wp(\ell_2). \quad (2)$$

Definition 3 ([20])**.** *Let* $\forall\ \ell_1, \ell_2 \in \mathrm{I}$*, then an inequality of the form:*

$$\mathrm{h}(\ell_1+\ell_2) \geq \mathrm{h}(\ell_1) + \mathrm{h}(\ell_2),$$

is said to be a super-additive function.

In the field of fractional calculus, several mathematicians worked on the concept of h-convexity and presented different types of Hermite-Hadamard type inequalities. For the readers, see [21–36] and the references cited therein.

Definition 4 ([11])**.** *Let* $\wp \in L_1([\ell_1, \ell_2])$*. The fractional integrals* $\mathfrak{J}^{\xi}_{\ell_1^+}$ *and* $\mathfrak{J}^{\xi}_{\ell_2^-}$ *of order* $\xi > 0$ *are defined by:*

$$\mathfrak{J}^{\xi}_{\ell_1^+}\wp(x) := \frac{1}{\Gamma(\xi)}\int_{\ell_1}^{x}(x-\sigma)^{\xi-1}\wp(\sigma)d\sigma, 0 \le \ell_1 < x < \ell_2$$

$$\mathfrak{J}^{\xi}_{\ell_2^-}\wp(x) := \frac{1}{\Gamma(\xi)}\int_{x}^{\ell_2}(\sigma-x)^{\xi-1}\wp(\sigma)d\sigma, 0 \le \ell_1 < x < \ell_2,$$

respectively.

In [11], Sarikaya et al. proved the following Hadamard-type inequalities for Riemann-Liouville fractional integrals as follows:

Theorem 2 ([11])**.** *Let* $\wp : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ *be a positive mapping with* $0 \le \ell_1 \le \ell_2$*,* $\wp \in \mathcal{L}[\ell_1, \ell_2]$ *and* $\mathrm{I}^{\xi}_{(\ell_1)^+}\wp$ *and* $\mathrm{I}^{\xi}_{(\ell_2)^-}\wp$ *be fractional operator. Then, the following inequality for fractional integral holds if* $\wp$ *is a convex function.*

$$\wp\left(\frac{\ell_1+\ell_2}{2}\right) \le \frac{\Gamma(\xi+1)}{2(\ell_2-\ell_1)^{\xi}}\left[\mathrm{I}^{\xi}_{\ell_1^+}\wp(\ell_2) + \mathrm{I}^{\xi}_{\ell_2^-}\wp(\ell_1)\right] \le \frac{\wp(\ell_1)+\wp(\ell_2)}{2}.$$

Definition 5 ([25])**.** *Let* $\wp \in L_1([\ell_1, \ell_2])$ *and* $k > 0$*, then k-fractional integrals* ${}_kI^{\xi}_{\ell_1^+}$ *and* ${}_kI^{\xi}_{\ell_2^-}$ *of order* $\xi > 0$ *are defined by:*

$${}_k\mathfrak{J}^{\xi}_{\ell_1^+}\wp(x) := \frac{1}{k\Gamma_k(\xi)}\int_{\ell_1}^{x}(x-\sigma)^{\frac{\xi}{k}-1}\wp(\sigma)d\sigma, 0 \le \ell_1 < x < \ell_2,$$

$${}_k\mathfrak{J}^{\xi}_{\ell_2^-}\wp(x) := \frac{1}{k\Gamma_k(\xi)}\int_{x}^{\ell_2}(\sigma-x)^{\frac{\xi}{k}-1}\wp(\sigma)d\sigma, 0 \le \ell_1 < x < \ell_2,$$

respectively, where

$$\Gamma_k(x) := \int_0^{\infty}\sigma^{x-1}e^{\frac{\sigma^k}{k}}d\sigma,\ \mathrm{Re}(\sigma) > 0.$$

Theorem 3 ([25])**.** *Let* $k > 0$*,* $\wp : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ *be a positive mapping with* $0 \le \ell_1 \le \ell_2$*,* $\wp \in \mathcal{L}[\ell_1, \ell_2]$ *and* $\mathrm{I}^{\xi}_{(\ell_1)^+}\wp$ *and* $\mathrm{I}^{\xi}_{(\ell_2)^-}\wp$ *be fractional operator. Then, the following inequality for fractional integral holds if* $\wp$ *is a convex function.*

$$\wp\left(\frac{\ell_1+\ell_2}{2}\right) \le \frac{\Gamma_k(\xi+k)}{2(\ell_2-\ell_1)^{\frac{\xi}{k}}}\left[\mathrm{I}^{\xi,k}_{\ell_1^+}\wp(\ell_2) + \mathrm{I}^{\xi,k}_{\ell_2^-}\wp(\ell_1)\right] \le \frac{\wp(\ell_1)+\wp(\ell_2)}{2}.$$

Lemma 1 ([25])**.** *Let* $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ *be a differentiable mapping on* I°*, where* $\ell_1, \ell_2 \in \mathrm{I}^{\circ}$ *with* $0 \le \ell_1 \le \ell_2$*. If* $\wp' \in \mathcal{L}[\ell_1, \ell_2]$*, then the following equality for the fractional integral holds:*

$$\frac{\wp(\ell_1)+\wp(\ell_2)}{2} - \frac{\Gamma_k(\xi+k)}{2(\ell_2-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{(\ell_1)^+}\wp(\ell_2) + \mathfrak{J}^{\xi,k}_{(\ell_2)^-}\wp(\ell_1)\right]$$
$$= \frac{(\ell_2-\ell_1)}{2}\left\{\int_0^1\left[(1-\sigma)^{\frac{\xi}{k}} - \sigma^{\frac{\xi}{k}}\right]\wp'(\sigma\ell_1 + (1-\sigma)\ell_2)d\sigma\right\}.$$

For some recent generalization of Hermite-Hadamard type inequalities via fractional operators, readers can refer to [26–30] and the references cited therein. Recently, in [27,28], the authors introduced a new class of convex functions and presented the Hermite-Hadamard inequality using a generalized Riemann-Liouville fractional integral operator concerning a monotonic function. Cortez et al. in [29] presented some trapezium-type inequalities for generalized coordinated convex function via a new form of the Riemann-Liouville fractional operator using Raina's special function. Kashuri et al. [30,31] worked on the k-Riemann-Liouville fractional operator to study the inequalities of Hermite-Hadamard type. Farid et al. in [32,33] extended their work on k-fractional inequalities for quasi-convexity and exponential convexity. Interested readers can also refer to the references cited in the above-mentioned articles to obtain detailed knowledge about generalized fractional operators and inequalities.

Motivated by the above results, the article is structured as follows: In Section 3, a new version of the Hermite-Hadamard inequality by using the concept of the h-convex function is presented. In Section 4, refinements of the Hermite-Hadamard type inequality for twice-differentiable functions are discussed. Section 5 deals with the application of **q**-digamma functions and related results. In the end, a conclusion is given in Section 6.

3. Hermite-Hadamard Inequality via Fractional Operator

The main focus of this section is to investigate and prove the Hermite-Hadamard inequality via the fractional operator, namely the k-fractional operator.

Theorem 4. *Let a function* $\wp : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ *be* h*-convex with* $0 \leq \ell_1 \leq \ell_2$*. If* $\wp \in \mathcal{L}[\ell_1, \ell_2]$*, then the following inequality for the fractional integral holds:*

$$\frac{\Gamma_k(\xi+k)}{(\ell_2-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\ell_2)+\Im^{\xi,k}_{\ell_2^-}\wp(\ell_1)\right] \leq \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k}\int_0^1 \sigma^{\frac{\xi}{k}-1}[\mathrm{h}(\sigma)+\mathrm{h}(1-\sigma)]d\sigma$$
$$\leq \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k^{\frac{p-1}{p}}}\left(\frac{1}{p\xi-pk+k}\right)^{\frac{1}{p}}\left[\left(\int_0^1(\mathrm{h}(1-\sigma))^r d\sigma\right)^{\frac{1}{r}}+\left(\int_0^1 \mathrm{h}(\sigma)^r d\sigma\right)^{\frac{1}{r}}\right], \quad (3)$$

where $\frac{1}{p}+\frac{1}{r}=1$.

Proof. Employing the definition of h-convexity, we have:

$$\wp(\sigma x+(1-\sigma)y) \leq \mathrm{h}(\sigma)\wp(x)+\mathrm{h}(1-\sigma)\wp(y),$$

and:

$$\wp((1-\sigma)x+\sigma y) \leq \mathrm{h}(1-\sigma)\wp(x)+\mathrm{h}(\sigma)\wp(y).$$

Adding these inequalities, one has:

$$\wp(\sigma x+(1-\sigma)y)+\wp((1-\sigma)x+\sigma y) \leq [\wp(x)+\wp(y)][\mathrm{h}(\sigma)+\mathrm{h}(1-\sigma)]. \quad (4)$$

Substituting $x=\ell_1$ and $y=\ell_2$ and multiplying (4) by $\sigma^{\frac{\xi}{k}-1}$, $\xi>0$, then integrating the result over $[0,1]$, we then obtain:

$$\int_0^1 \sigma^{\frac{\xi}{k}-1}\wp(\sigma\ell_1+(1-\sigma)\ell_2)d\sigma+\int_0^1 \sigma^{\frac{\xi}{k}-1}\wp((1-\sigma)\ell_1+\sigma\ell_2)d\sigma$$
$$\leq [\wp(\ell_1)+\wp(\ell_2)]\int_0^1 \sigma^{\frac{\xi}{k}-1}[\mathrm{h}(\sigma)+\mathrm{h}(1-\sigma)]d\sigma,$$

which simplifies to:

$$\frac{\Gamma_k(\xi+k)}{(\ell_2-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi}_{\ell_1^+}\wp(\ell_2)+\mathfrak{J}^{\xi}_{\ell_2^-}\wp(\ell_1)\right]$$
$$\le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k}\int_0^1 \sigma^{\frac{\xi}{k}-1}[\mathrm{h}(\sigma)+\mathrm{h}(1-\sigma)]d\sigma. \tag{5}$$

Hence, the proof of the first part of the theorem is complete.

For the second part of Theorem 4, we use Hölder's inequality and then Minkowski's inequality for the RHS of (5):

$$\frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k}\int_0^1 \sigma^{\frac{\xi}{k}-1}[\mathrm{h}(1-\sigma)+\mathrm{h}(\sigma)]d\sigma$$
$$\le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k}\left(\int_0^1 (\sigma^{\frac{\xi}{k}-1})^p d\sigma\right)^{\frac{1}{p}}\left(\int_0^1 [\mathrm{h}(1-\sigma)+\mathrm{h}(\sigma)]^r d\sigma\right)^{\frac{1}{r}}$$
$$= \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k}\left(\frac{k}{p\xi-pk+k}\right)^{\frac{1}{p}}\left(\int_0^1 [\mathrm{h}(1-\sigma)+\mathrm{h}(\sigma)]^r d\sigma\right)^{\frac{1}{r}}$$
$$\le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k^{\frac{p-1}{p}}}\left(\frac{1}{p\xi-pk+k}\right)^{\frac{1}{p}}\left[\left(\int_0^1 (\mathrm{h}(1-\sigma))^r d\sigma\right)^{\frac{1}{r}}+\left(\int_0^1 \mathrm{h}(\sigma)^r d\sigma\right)^{\frac{1}{r}}\right].$$

This completes the proof. □

If we put $\mathrm{h}(\sigma)=\sigma$ in Theorem 4, we obtain a new inequality for convex functions.

Corollary 1. *Let $\wp:[\ell_1,\ell_2]\longrightarrow\mathbb{R}$ be a convex function with $0\le\ell_1\le\ell_2$. If $\wp\in\mathcal{L}[\ell_1,\ell_2]$, then the following inequality for the fractional integral holds:*

$$\frac{\Gamma_k(\xi+k)}{2(\ell_2-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{\ell_1^+}\wp(\ell_2)+\mathfrak{J}^{\xi,k}_{\ell_2^-}\wp(\ell_1)\right]\le\left[\frac{\wp(\ell_1)+\wp(\ell_2)}{2}\right]$$
$$\le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k^{\frac{p-1}{p}}}\left(\frac{1}{p\xi-pk+k}\right)^{\frac{1}{p}}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}, \tag{6}$$

where $\frac{1}{p}+\frac{1}{r}=1$.

Remark 1. *If we use $k=1$, in Corollary 1, then the following inequality is obtained:*

$$\frac{\Gamma(\xi+1)}{2(\ell_2-\ell_1)^{\xi}}\left[\mathfrak{J}^{\xi}_{\ell_1^+}\wp(\ell_2)+\mathfrak{J}^{\xi}_{\ell_2^-}\wp(\ell_1)\right]\le\left[\frac{\wp(\ell_1)+\wp(\ell_2)}{2}\right]$$
$$\le \xi[\wp(\ell_1)+\wp(\ell_2)]\left(\frac{1}{p\xi-p+1}\right)^{\frac{1}{p}}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}, \tag{7}$$

where $\frac{1}{p}+\frac{1}{r}=1$.

Remark 2. *If we use $k=2$, in Corollary 1, then the following inequality is obtained:*

$$\frac{\xi 2^{\frac{\xi}{2}}\Gamma(\frac{\xi}{2})}{2(\ell_2-\ell_1)^{\frac{\xi}{2}}}\left[\mathfrak{J}^{\xi,2}_{\ell_1^+}\wp(\ell_2)+\mathfrak{J}^{\xi,2}_{\ell_2^-}\wp(\ell_1)\right]\le\left[\frac{\wp(\ell_1)+\wp(\ell_2)}{2}\right]$$
$$\le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{2^{\frac{p-1}{p}}}\left(\frac{1}{p\xi-2p+2}\right)^{\frac{1}{p}}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}, \tag{8}$$

where $\frac{1}{p}+\frac{1}{r}=1$.

If we put $h(\sigma) = \sigma^s$ in Theorem 4, then we obtain the following inequality for s-convex functions.

Corollary 2. *Let $\wp : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ be a s-convex function with $0 \le \ell_1 \le \ell_2$. If $\wp \in \mathcal{L}[\ell_1, \ell_2]$, then the following inequality for the fractional integral holds:*

$$\frac{\Gamma_k(\xi+k)}{(\ell_2-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{\ell_1^+}\wp(\ell_2)+\mathfrak{J}^{\xi,k}_{\ell_2^-}\wp(\ell_1)\right] \le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{\xi+sk}$$
$$\le \frac{\xi[\wp(\ell_1)+\wp(\ell_2)]}{k^{\frac{p-1}{p}}}\left(\frac{1}{p\xi-pk+k}\right)^{\frac{1}{p}}\left(\frac{1}{sr+1}\right)^{\frac{1}{r}}, \tag{9}$$

where $\frac{1}{p}+\frac{1}{r}=1$.

Corollary 3. *If we put $k=\xi=1$ in Theorem 4, then the following inequality holds:*

$$\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\wp(x)dx \le [\wp(\ell_1)+\wp(\ell_2)]\int_0^1[h(\sigma)+h(1-\sigma)]d\sigma$$
$$\le [\wp(\ell_1)+\wp(\ell_2)]\left[\left(\int_0^1(h(1-\sigma))^r d\sigma\right)^{\frac{1}{r}}+\left(\int_0^1 h(\sigma)^r d\sigma\right)^{\frac{1}{r}}\right],$$

where $\frac{1}{p}+\frac{1}{r}=1$.

4. Main Results

Lemma 2. *Let $\wp : \left[\ell_1, \frac{\ell_2}{b}\right] \longrightarrow \mathbb{R}$ be a differentiable mapping with $0 \le \ell_1 \le \frac{\ell_2}{b}$, $0 < b < 1$, and $\mathfrak{J}^{\xi,k}_{(\ell_1)^+}\wp(\frac{\ell_2}{b})$, $\mathfrak{J}^{\xi,k}_{(\frac{\ell_2}{b})^-}\wp(\ell_1)$ be right and left fractional operators. If $\wp' \in \mathcal{L}\left[\ell_1, \frac{\ell_2}{b}\right]$, then the following equality holds:*

$$\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2}-\frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{(\ell_1)^+}\wp(\frac{\ell_2}{b})+\mathfrak{J}^{\xi,k}_{(\frac{\ell_2}{b})^-}\wp(\ell_1)\right]$$
$$=\frac{(\frac{\ell_2}{b}-\ell_1)}{2}\left\{\int_0^1\left[(1-\sigma)^{\frac{\xi}{k}}-\sigma^{\frac{\xi}{k}}\right]\wp'(\sigma\ell_1+(1-\sigma)\frac{\ell_2}{b})d\sigma\right\}.$$

Proof. We can set that:

$$\frac{(\frac{\ell_2}{b}-\ell_1)}{2}\left\{\int_0^1\left[(1-\sigma)^{\frac{\xi}{k}}-\sigma^{\frac{\xi}{k}}\right]\wp'(\sigma\ell_1+(1-\sigma)\frac{\ell_2}{b})d\sigma\right\}$$
$$=\frac{(\frac{\ell_2}{b}-\ell_1)}{2}\left\{\int_0^1(1-\sigma)^{\frac{\xi}{k}}\wp'(\sigma\ell_1+(1-\sigma)\frac{\ell_2}{b})d\sigma\right.$$
$$\left.-\int_0^1\sigma^{\frac{\xi}{k}}\wp'(\sigma\ell_1+(1-\sigma)\frac{\ell_2}{b})d\sigma\right\}$$
$$=\frac{(\frac{\ell_2}{b}-\ell_1)}{2}[L_1+L_2]. \tag{10}$$

Considering,

$$
\begin{aligned}
L_1 &= \int_0^1 (1-\sigma)^{\frac{\xi}{k}} \wp'(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b}) d\sigma \\
&= (1-\sigma)^{\frac{\xi}{k}} \frac{\wp(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})}{\ell_1 - \frac{\ell_2}{b}} \Bigg|_0^1 \\
&\quad + \int_0^1 \frac{\xi}{k}(1-\sigma)^{\frac{\xi}{k}-1} \frac{\wp(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})}{\ell_1 - \frac{\ell_2}{b}} d\sigma \\
&= \frac{\wp(\frac{\ell_2}{b})}{\frac{\ell_2}{b} - \ell_1} - \frac{\Gamma_k(\xi+k)}{(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}+1}} \mathfrak{J}^{\xi,k}_{(\frac{\ell_2}{b})^-} \wp(\ell_1). \qquad (11)
\end{aligned}
$$

Similarly,

$$
L_2 = \frac{\wp(\ell_1)}{\frac{\ell_2}{b} - \ell_1} - \frac{\Gamma_k(\xi+k)}{(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}+1}} \mathfrak{J}^{\xi,k}_{(\ell_1)^+} \wp(\frac{\ell_2}{b}). \qquad (12)
$$

Using (11) and (12) in (10), the proof of the desired lemma is complete. □

Before establishing our main results, we need the following lemmas.

Lemma 3. *Let $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ be a differentiable mapping on I°, where $\ell_1, \ell_2 \in I^\circ$ with $0 \le \ell_1 \le \ell_2$. If $\wp' \in \mathcal{L}[\ell_1, \ell_2]$, then the following equality for the fractional integral holds:*

$$
\begin{aligned}
&\frac{\wp(\ell_1) + \wp(\ell_2)}{2} - \frac{\Gamma_k(\xi+k)}{2(\ell_2 - \ell_1)^{\frac{\xi}{k}}} \left[\mathfrak{J}^{\xi,k}_{(\ell_1)^+} \wp(\ell_2) + \mathfrak{J}^{\xi,k}_{(\ell_2)^-} \wp(\ell_1) \right] \\
&= \frac{k(\ell_2 - \ell_1)^2}{2(\xi+k)} \left\{ \int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1} \right] \wp''(\sigma\ell_1 + (1-\sigma)\ell_2) d\sigma \right\}.
\end{aligned}
$$

Proof. To investigate the require equality, we apply the result by Wu et al. [25]:

$$
\begin{aligned}
&\frac{\wp(\ell_1) + \wp(\ell_2)}{2} - \frac{\Gamma_k(\xi+k)}{2(\ell_2 - \ell_1)^{\frac{\xi}{k}}} \left[\mathfrak{J}^{\xi,k}_{(\ell_1)^+} \wp(\ell_2) + \mathfrak{J}^{\xi,k}_{(\ell_2)^-} \wp(\ell_1) \right] \\
&= \frac{(\ell_2 - \ell_1)}{2} \left\{ \int_0^1 \left[(1-\sigma)^{\frac{\xi}{k}} - \sigma^{\frac{\xi}{k}} \right] \wp'(\sigma\ell_1 + (1-\sigma)\ell_2) d\sigma \right\}.
\end{aligned}
$$

It is enough to verify that:

$$
\begin{aligned}
&\frac{(\ell_2 - \ell_1)}{2} \left\{ \int_0^1 \left[(1-\sigma)^{\frac{\xi}{k}} - \sigma^{\frac{\xi}{k}} \right] \wp'(\sigma\ell_1 + (1-\sigma)\ell_2) d\sigma \right\} \\
&= \frac{k(\ell_2 - \ell_1)^2}{2(\xi+k)} \left\{ \int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1} \right] \wp''(\sigma\ell_1 + (1-\sigma)\ell_2) d\sigma \right\}.
\end{aligned}
$$

Consequently, integration by parts gives,

$$
\begin{aligned}
&\frac{(\ell_2 - \ell_1)}{2} \left\{ \int_0^1 \left[(1-\sigma)^{\frac{\xi}{k}} - \sigma^{\frac{\xi}{k}} \right] \wp'(\sigma\ell_1 + (1-\sigma)\ell_2) d\sigma \right\} \\
&\quad = \frac{(\ell_2 - \ell_1)}{2} \left\{ \frac{k(\wp'(\ell_2) - \wp'(\ell_1))}{\xi+k} - k(\ell_1 - \ell_1) \right. \\
&\left. \times \int_0^1 \frac{\left[(1-\sigma)^{\frac{\xi}{k}+1} + \sigma^{\frac{\xi}{k}+1} \right]}{\xi+k} \wp''(\sigma\ell_1 + (1-\sigma)\ell_2) d\sigma \right\}.
\end{aligned}
$$

Now, using the fact:

$$\wp'(\ell_2) - \wp'(\ell_1) = (\ell_2 - \ell_1)\int_0^1 \wp''(\sigma\ell_1 + (1-\sigma)\ell_2)d\sigma,$$

we attain the required equality, and the proof is complete. □

Remark 3. *If we put $k = 1$ in Lemma 3, then [[36], Lemma 2.1, page-2243], is recovered.*

Remark 4. *If we put $\xi = k = 1$ in Lemma 3, then [[37], Lemma 1, page 1066], is recovered.*

Lemma 4. *Let $\wp : \left[\ell_1, \frac{\ell_2}{b}\right] \longrightarrow \mathbb{R}$ be a differentiable mapping with $0 \leq \ell_1 \leq \frac{\ell_2}{b}$, $0 < b \leq 1$, and $I^{\xi,k}_{(\ell_1)^+}\wp(\frac{\ell_2}{b})$, $I^{\xi,k}_{(\frac{\ell_2}{b})^-}\wp(\ell_1)$ be right and left fractional operators. If $\wp'' \in \mathcal{L}\left[\ell_1, \frac{\ell_2}{b}\right]$, then the following equality holds:*

$$\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{(\ell_1)^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{(\frac{\ell_2}{b})^-}\wp(\ell_1)\right]$$
$$= \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left\{\int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})d\sigma\right\}.$$

Proof. The proof can made in a similar manner as that of Lemma 3 and using the result of Lemma 2. □

Theorem 5. *Let $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ be a differentiable function on I°, where $\ell_1, \frac{\ell_2}{b} \in I^\circ$ with $0 \leq \ell_1 \leq \frac{\ell_2}{b}$ and $\wp'' \in \mathcal{L}\left[\ell_1, \frac{\ell_2}{b}\right]$. If $|\wp''|$ is an* h*-convex function, then the following fractional integral inequality holds:*

$$\left|\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left\{|\wp''(\ell_1)|\int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]\mathrm{h}(\sigma)d\sigma\right.$$
$$\left. + |\wp''\left(\frac{\ell_2}{b}\right)|\int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]\mathrm{h}(1-\sigma)d\sigma\right\}.$$

Proof. Using Lemma 4 and employing the h-convexity of $|\wp''|$, we attain:

$$\left|\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left\{\int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]|\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})d\sigma|\right\}$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left\{|\wp''(\ell_1)|\int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]\mathrm{h}(\sigma)d\sigma\right.$$
$$\left. + |\wp''(\frac{\ell_2}{b})|\int_0^1 \left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]\mathrm{h}(1-\sigma)d\sigma\right\}.$$

□

Corollary 4. *Choosing* $\mathrm{h}(\sigma) = \sigma$ *in Theorem 5, then the following inequality for the convex function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp\left(\frac{\ell_2}{b}\right)}{2}-\frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b}-\ell_1\right)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left\{\left(\frac{1}{2}-\beta\left(2,\frac{\xi}{k}+2\right)-\frac{k}{\xi+3k}\right)\left[|\wp''(\ell_1)|+\left|\wp''\left(\frac{\ell_2}{b}\right)\right|\right]\right\}.$$

Remark 5. *For an m-convex function, we obtain a new result from Theorem 5.*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2}-\frac{\Gamma_k(\xi+k)}{2(b-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left\{\left(\frac{1}{2}-\beta\left(2,\frac{\xi}{k}+2\right)-\frac{k}{\xi+3k}\right)\left[|\wp''(\ell_1)|+m\left|\wp''\left(\frac{\ell_2}{mb}\right)\right|\right]\right\}.$$

Corollary 5. *Choosing* $\mathrm{h}(\sigma) = \sigma^s$ *in Theorem 5, then the following inequality for the s-convex function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp\left(\frac{\ell_2}{b}\right)}{2}-\frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b}-\ell_1\right)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left\{\left(\frac{1}{s+1}-\beta\left(s+1,\frac{\xi}{k}+2\right)-\frac{k}{\xi+ks+2k}\right)\left[|\wp''(\ell_1)|+\left|\wp''\left(\frac{\ell_2}{b}\right)\right|\right]\right\}.$$

Corollary 6. *Choosing* $\mathrm{h}(\sigma) = \sigma(1-\sigma)$ *in Theorem 5, then the following inequality for the tgs-convex function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp\left(\frac{\ell_2}{b}\right)}{2}-\frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b}-\ell_1\right)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left\{\left(\frac{1}{6}-\beta\left(2,\frac{\xi}{k}+3\right)-\beta\left(\frac{\xi}{k}+3,2\right)\right)\left[|\wp''(\ell_1)|+\left|\wp''\left(\frac{\ell_2}{b}\right)\right|\right]\right\}.$$

Corollary 7. *Choosing* $\mathrm{h}(\sigma) = 1$ *in Theorem 5, then the following inequality for the P-function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp\left(\frac{\ell_2}{b}\right)}{2}-\frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b}-\ell_1\right)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left\{\left(1-\frac{2k}{\xi+2k}\right)\left[|\wp''(\ell_1)|+\left|\wp''\left(\frac{\ell_2}{b}\right)\right|\right]\right\}.$$

Theorem 6. *Let* $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ *be a twice-differentiable function on* I°*, where* $\ell_1, \frac{\ell_2}{b} \in I^\circ$ *with* $0 \leq \ell_1 \leq \frac{\ell_2}{b}$ *and* $\wp'' \in \mathcal{L}[\ell_1, \frac{\ell_2}{b}]$*. If* $|\wp''|^r$*,* $p, r \geq 1$ $\frac{1}{p} + \frac{1}{r} = 1$ *is an* h*-convex function, then:*

$$\left|\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left(1 - \frac{2k}{p(\xi + k) + k}\right)^{\frac{1}{p}}\left(|\wp''(\ell_1)|^r\int_0^1 \mathrm{h}(\sigma)d\sigma + |\wp''(\frac{\ell_2}{b})|^r\int_0^1 \mathrm{h}(1-\sigma)d\sigma\right)^{\frac{1}{r}}.$$

Proof. Employing Lemma 4, Hölder's inequality, and the h-convexity of $|\wp''|^r$, we have:

$$\left|\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left\{\int_0^1\left[1 - (1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]|\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})d\sigma|\right\}$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left\{\left(\int_0^1\left[1 - (1-\sigma)^{p\left(\frac{\xi}{k}+1\right)} - \sigma^{p\left(\frac{\xi}{k}+1\right)}\right]d\sigma\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\int_0^1 |\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})|^r d\sigma\right)^{\frac{1}{r}}\right\}$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left(1 - \frac{2k}{p(\xi + k) + k}\right)^{\frac{1}{p}}$$
$$\times\left(|\wp''(\ell_1)|^r\int_0^1 \mathrm{h}(\sigma)d\sigma + |\wp''(\frac{\ell_2}{b})|^r\int_0^1 \mathrm{h}(1-\sigma)d\sigma\right)^{\frac{1}{r}}.$$

□

Corollary 8. *Choosing* $\mathrm{h}(\sigma) = \sigma$ *in Theorem 6, then the following inequality for the convex function is obtained:*

$$\left|\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left(1 - \frac{2k}{p(\xi + k) + k}\right)^{\frac{1}{p}}\left(\frac{|\wp''(\ell_1)|^r + |\wp''(\frac{\ell_2}{b})|^r}{2}\right)^{\frac{1}{r}}.$$

Corollary 9. *Choosing* $\mathrm{h}(\sigma) = \sigma^s$ *in Theorem 6, then the following inequality for the s-convex function is obtained:*

$$\left|\frac{\wp(\ell_1) + \wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)}\left(1 - \frac{2k}{p(\xi + k) + k}\right)^{\frac{1}{p}}\left(\frac{|\wp''(\ell_1)|^r + |\wp''(\frac{\ell_2}{b})|^r}{s+1}\right)^{\frac{1}{r}}.$$

Corollary 10. *Choosing* $\mathrm{h}(\sigma) = e^{\sigma} - 1$ *in Theorem 6, then the following inequality for the exponential-type convex function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{J}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right| \leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left(1-\frac{2k}{p(\xi+k)+k}\right)^{\frac{1}{p}}\left([e-2]\left(|\wp''(\ell_1)|^q + |\wp''(\frac{\ell_2}{b})|^r\right)\right)^{\frac{1}{r}}.$$

Corollary 11. *Choosing* $\mathrm{h}(\sigma) = \sigma(1-\sigma)$ *in Theorem 6, then the following inequality for the tgs-convex function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{J}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right| \leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left(1-\frac{2k}{p(\xi+k)+k}\right)^{\frac{1}{p}}\left(\frac{|\wp''(\ell_1)|^r + |\wp''(\frac{\ell_2}{b})|^r}{6}\right)^{\frac{1}{r}}.$$

Theorem 7. *Let* $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ *be a differentiable function on* I°*, where* $\ell_1, \frac{\ell_2}{b} \in \mathrm{I}^\circ$ *with* $0 \leq \ell_1 \leq \frac{\ell_2}{b}$ *and* $\wp'' \in \mathcal{L}\left[\ell_1, \frac{\ell_2}{b}\right]$*. If* $|\wp''|^r$*,* $p, r \geq 1$*,* $\frac{1}{p} + \frac{1}{r} = 1$ *is an* h*-convex function, then we have:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{J}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right| \leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left(\int_0^1\left[1-(1-\sigma)^{p(\frac{\xi}{k}+1)} - \sigma^{p(\frac{\xi}{k}+1)}\right]^r\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r + \mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]d\sigma\right)^{\frac{1}{r}}.$$

Proof. Employing Lemma 4, Hölder's inequality, and the h-convexity of $|\wp''|^r$, we have:

$$\begin{aligned}
&\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{J}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{J}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right| \\
&\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\int_0^1\left[1-(1-\sigma)^{\frac{\xi}{k}+1} - \sigma^{\frac{\xi}{k}+1}\right]|\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})d\sigma|\right\} \\
&\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\int_0^1 1d\sigma\right)^{\frac{1}{p}}\left(\begin{array}{c}\int_0^1\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)} - \sigma^{\left(\frac{\xi}{k}+1\right)}\right]^r \\ \times|\wp''(\sigma\ell_1+(1-\sigma)\frac{\ell_2}{b})|^r d\sigma\end{array}\right)^{\frac{1}{r}}\right\} \\
&\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left(\begin{array}{c}\int_0^1\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)} - \sigma^{\left(\frac{\xi}{k}+1\right)}\right]^r \\ \times\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r + \mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]d\sigma\end{array}\right)^{\frac{1}{r}}.
\end{aligned}$$

□

Corollary 12. *Choosing* $\mathrm{h}(\sigma) = \sigma$ *in Theorem 7, then the following inequality for the convex function is obtained:*

$$\begin{aligned}&\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2}-\frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b})+\Im^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|\\&\leq\frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{S^1(\alpha,\sigma,r)|\wp''(\ell_1)|^r+S^2(\alpha,\sigma,r)|\wp''(\frac{\ell_2}{b})|^r\right\}^{\frac{1}{r}}\\&\textit{where, } S^1(\alpha,\sigma,r)=\int_0^1\sigma\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]^r d\sigma\\&S^2(\alpha,\sigma,r)=\int_0^1(1-\sigma)\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]^r d\sigma.\end{aligned}$$

Corollary 13. *Choosing* $\mathrm{h}(\sigma) = \sigma^s$ *in Theorem 7, then the following inequality for the s-convexity is obtained:*

$$\begin{aligned}&\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2}-\frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b})+\Im^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|\\&\leq\frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{S^1(\alpha,\sigma,r,s)|\wp''(\ell_1)|^r+S^2(\alpha,\sigma,r,s)|\wp''(\frac{\ell_2}{b})|^r\right\}^{\frac{1}{r}}\\&\textit{where, } S^1(\alpha,\sigma,r,s)=\int_0^1\sigma^s\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]^r d\sigma\\&S^2(\alpha,\sigma,r,s)=\int_0^1(1-\sigma)^s\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]^r d\sigma.\end{aligned}$$

Theorem 8. *Let* $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ *be a differentiable function on* I°*, where* $\ell_1, \ell_2/b \in \mathrm{I}^\circ$ *with* $0 \leq \ell_1 \leq \frac{\ell_2}{b}$ *and* $\wp'' \in \mathcal{L}[\ell_1, \frac{\ell_2}{b}]$*. If* $|\wp''|^r$*, for* $r > 1$ *is an* h*-convex function, then:*

$$\begin{aligned}&\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2}-\frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b})+\Im^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|\\&\leq\frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left(\frac{\xi}{\xi+2k}\right)^{1-\frac{1}{r}}\left(\int_0^1\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]\right.\\&\qquad\left.\times\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r+\mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]d\sigma\right)^{\frac{1}{r}}.\end{aligned}$$

Proof. Employing Lemma 4, the power-mean inequality, and the h-convexity of $|\wp''|^r$,

$$\begin{aligned}&\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2}-\frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b})+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|\\&\leq\frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\int_0^1\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]d\sigma\right)^{1-\frac{1}{r}}\right.\\&\left.\times\left(\int_0^1\left(1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right)|\wp''(\sigma\ell_1+(1-\sigma)\frac{\ell_2}{b})|^r d\sigma\right)^{\frac{1}{r}}\right\}\\&\leq\frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left(\frac{\xi}{\xi+2k}\right)^{1-\frac{1}{r}}\left(\int_0^1\left[1-(1-\sigma)^{\left(\frac{\xi}{k}+1\right)}-\sigma^{\left(\frac{\xi}{k}+1\right)}\right]\right.\\&\left.\times\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r+\mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]d\sigma\right)^{\frac{1}{r}}.\end{aligned}$$

□

Corollary 14. *Choosing* $\mathrm{h}(\sigma)=\sigma$ *in Theorem 8, then the following inequality for the convex function is obtained:*

$$\begin{aligned}&\left|\frac{\wp(\ell_1)+\wp\left(\frac{\ell_2}{b}\right)}{2}-\frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b}-\ell_1\right)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|\\&\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left(\frac{\xi}{\xi+2k}\right)^{1-\frac{1}{r}}\left\{\left(\frac{1}{2}-\beta\left(2,\frac{\xi}{k}+2\right)-\frac{k}{\xi+3k}\right)\left[|\wp''(\ell_1)|^r+\left|\wp''\left(\frac{\ell_2}{b}\right)\right|^r\right]\right\}^{\frac{1}{r}}.\end{aligned}$$

Corollary 15. *Choosing* $\mathrm{h}(\sigma)=\sigma^s$ *in Theorem 8, then the following new inequality for the s-convexity is obtained:*

$$\begin{aligned}&\left|\frac{\wp(\ell_1)+\wp\left(\frac{\ell_2}{b}\right)}{2}-\frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b}-\ell_1\right)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp\left(\frac{\ell_2}{b}\right)+\mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|\\&\leq\frac{k\left(\frac{\ell_2}{b}-\ell_1\right)^2}{2(\xi+k)}\left(\frac{\xi}{\xi+2k}\right)^{1-\frac{1}{r}}\\&\times\left\{\left(\frac{1}{s+1}-\beta\left(s+1,\frac{\xi}{k}+2\right)-\frac{k}{\xi+k(s+2)}\right)\left[|\wp''(\ell_1)|^r+\left|\wp''\left(\frac{\ell_2}{b}\right)\right|^r\right]\right\}^{\frac{1}{r}}.\end{aligned}$$

Theorem 9. *Let $\wp : \mathrm{I} \longrightarrow \mathbb{R}$ be a differentiable function on I°, where $\ell_1, \frac{\ell_2}{b} \in I^\circ$ with $0 \leq \ell_1 \leq \frac{\ell_2}{b}$ and $\wp'' \in \mathcal{L}[\ell_1, \frac{\ell_2}{b}]$. If $|\wp''|^r,\ r \geq 0$ is an* h*-convex function, then:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\frac{1}{2} - \beta(2, p(\frac{\xi}{k}+1)+1) - \frac{k}{p(\xi+k)+2k}\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\int_0^1 (1-\sigma)\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r + \mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]\right)^{\frac{1}{r}}\right.$$
$$\left.\left. + \left(\int_0^1 \sigma\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r + \mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]d\sigma\right)^{\frac{1}{r}}\right]\right\}.$$

Proof. Employing Lemma 4, the Hölder-Íscan integral inequality, and the h-convexity of $|\wp''|^r$, we have:

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\mathfrak{I}^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \mathfrak{I}^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\int_0^1 (1-\sigma)\left[1-(1-\sigma)^{p\left(\frac{\xi}{k}+1\right)} - \sigma^{p\left(\frac{\xi}{k}+1\right)}\right]d\sigma\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1 (1-\sigma)|\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})|^r\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1 \sigma\left[1-(1-\sigma)^{p\left(\frac{\xi}{k}+1\right)} - \sigma^{p\left(\frac{\xi}{k}+1\right)}\right]d\sigma\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_0^1 \sigma|\wp''(\sigma\ell_1 + (1-\sigma)\frac{\ell_2}{b})|^r d\sigma\right)^{\frac{1}{r}}\right\}$$
$$\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\frac{1}{2} - \beta(2, p(\frac{\xi}{k}+1)+1) - \frac{k}{p(\xi+k)+2k}\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\int_0^1 (1-\sigma)\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r + \mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]\right)^{\frac{1}{r}}\right.$$
$$\left.\left. + \left(\int_0^1 \sigma\left[\mathrm{h}(\sigma)|\wp''(\ell_1)|^r + \mathrm{h}(1-\sigma)|\wp''(\frac{\ell_2}{b})|^r\right]d\sigma\right)^{\frac{1}{r}}\right]\right\}.$$

□

Corollary 16. *Choosing* $\mathrm{h}(\sigma) = \sigma$ *in Theorem 9, then the following inequality for the convex function is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \Im^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\frac{1}{2} - \beta(2, p(\frac{\xi}{k}+1)+1) - \frac{k}{p(\xi+k)+2k}\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left[\left(\frac{|\wp''(\ell_1)|^r}{6} + \frac{|\wp''(\frac{\ell_2}{b})|^r}{3}\right)^{\frac{1}{r}} + \left(\frac{|\wp''(\ell_1)|^r}{3} + \frac{|\wp''(\frac{\ell_2}{b})|^r}{6}\right)^{\frac{1}{r}}\right]\right\}.$$

Corollary 17. *Choosing* $\mathrm{h}(\sigma) = e^\sigma - 1$ *in Theorem 9, then the following inequality for the exponential-type convexity is obtained:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \Im^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\frac{1}{2} - \beta(2, p(\frac{\xi}{k}+1)+1) - \frac{k}{p(\xi+k)+2k}\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\left(\frac{2e-5}{2}\right)|\wp''(\ell_1)|^r + \left(\frac{1}{2}\right)|\wp''(\frac{\ell_2}{b})|^r\right)^{\frac{1}{r}}\right.$$
$$\left.\left.+\left(\left(\frac{1}{2}\right)|\wp''(\ell_1)|^r + \left(\frac{2e-5}{2}\right)|\wp''(\frac{\ell_2}{b})|^r\right)^{\frac{1}{r}}\right]\right\}.$$

Corollary 18. *Choosing* $\mathrm{h}(\sigma) = \sigma^s$ *in Theorem 9, then the following inequality for the s-convexity isobtained:*

$$\left|\frac{\wp(\ell_1)+\wp(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi+k)}{2(\frac{\ell_2}{b}-\ell_1)^{\frac{\xi}{k}}}\left[\Im^{\xi,k}_{\ell_1^+}\wp(\frac{\ell_2}{b}) + \Im^{\xi,k}_{\frac{\ell_2}{b}^-}\wp(\ell_1)\right]\right|$$
$$\leq \frac{k(\frac{\ell_2}{b}-\ell_1)^2}{2(\xi+k)}\left\{\left(\frac{1}{2} - \beta(2, p(\frac{\xi}{k}+1)+1) - \frac{k}{p(\xi+k)+2k}\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left[\left(\beta(s+1,2)|\wp''(\ell_1)|^r + \frac{|\wp''(\frac{\ell_2}{b})|^r}{s+2}\right)^{\frac{1}{r}} + \left(\frac{|\wp''(\ell_1)|^r}{s+2} + \beta(s+1,2)|\wp''(\frac{\ell_2}{b})|^r\right)^{\frac{1}{r}}\right]\right\}.$$

5. Applications to Special Functions

Jolevska-Tuneska et al. [38] summed up the digamma function for non-negative integers. Further, the polygamma function was generalized for negative integers by Salem and Kilicman [39]. Salem in his articles [40,41] elaborated the idea of the neutrix and neutrix limit and also defined the $\mathbf{q}$-gamma, the incomplete gamma functions, and their derivatives for negative values of x. Later, Krattenthaler and Srivastava [42] investigated the concept of the $\mathbf{q}$-digamma function $\psi_{\mathbf{q}}(x)$. They expressed that $\psi_{\mathbf{q}}(x)$ tends to the digamma function $\psi(x)$, if $\mathbf{q} \to 1$. Salem [43] again studied some fundamental properties and generalizations of $\mathbf{q}$-digamma functions.

For any complex number a, we define $[a]_{\mathbf{q}} = \frac{1-\mathbf{q}^a}{1-\mathbf{q}}, \quad \mathbf{q} \neq 1; \quad [n]_{\mathbf{q}}! = [n]_{\mathbf{q}}[n-1]_{\mathbf{q}} \cdots [1]_{\mathbf{q}}, \quad n = 1, 2, \ldots; \quad [0]_{\mathbf{q}} = 1$

The $\mathbf{q}$-analogue of the gamma function is:

$$\Gamma_{\mathbf{q}}(z) = \frac{(\mathbf{q};\mathbf{q})_\infty}{(\mathbf{q}^z;\mathbf{q})_\infty}(1-\mathbf{q})^{1-z},\ z \neq 0, -1, -2, \ldots |\mathbf{q}| < 1.$$

The **q**-integral representation is given as: $\Gamma_{\mathbf{q}}(z) = \int_0^{\frac{1}{1-\mathbf{q}}} t^{z-1} E_{\mathbf{q}}(-\mathbf{q}t) d_{\mathbf{q}}t, \quad R(z) > 0.$

q-digamma function: Suppose $0 < \mathbf{q} < 1$; the **q**-digamma(psi) function $\psi_{\mathbf{q}}$ is the **q**-analogue of the Psi or digamma function ψ defined by:

$$\begin{aligned}\psi_{\mathbf{q}} &= -\ln(1-\mathbf{q}) + \ln \mathbf{q} \sum_{s=0}^{\infty} \frac{\mathbf{q}^{s+x}}{1-\mathbf{q}^{s+x}} \\ &= -\ln(1-\mathbf{q}) + \ln \mathbf{q} \sum_{s=0}^{\infty} \frac{\mathbf{q}^{sx}}{1-\mathbf{q}^{sx}}.\end{aligned}$$

For $\mathbf{q} > 1$ and $x > 0$, the **q**-digamma function $\psi_{\mathbf{q}}$ is defined by:

$$\begin{aligned}\psi_{\mathbf{q}} &= -\ln(\mathbf{q}-1) + \ln \mathbf{q} \left[x - \frac{1}{2} - \sum_{s=0}^{\infty} \frac{\mathbf{q}^{-(s+x)}}{1-\mathbf{q}^{-(s+x)}} \right] \\ &= -\ln(\mathbf{q}-1) + \ln \mathbf{q} \left[x - \frac{1}{2} - \sum_{s=0}^{\infty} \frac{\mathbf{q}^{-sx}}{1-\mathbf{q}^{-sx}} \right].\end{aligned}$$

In [42], it was proven that $\lim_{\mathbf{q}\to 1^+} \psi_{\mathbf{q}}(x) = \lim_{\mathbf{q}\to 1^-} \psi_{\mathbf{q}}(x) = \psi(x)$.
We also use the fact that,

$$\psi_{\mathbf{q}}^{n}(x) = \frac{d^n}{dx^n} \psi_{\mathbf{q}}(x); \;\; x > 0, \;\; 0 < \mathbf{q} < 1.$$

Proposition 1. *For* $\mathbf{q} \in (0,1)$ *and* $0 < \ell_1 < \ell_2$*, then as a result, we have:*

$$\begin{aligned}&\left| \frac{\wp_{\mathbf{q}}'(\ell_1) + \wp_{\mathbf{q}}'\left(\frac{\ell_2}{b}\right)}{2} - \frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b} - \ell_1\right)^{\frac{\xi}{k}}} \left[\mathfrak{I}_{\ell_1^+}^{\xi,k} \wp_{\mathbf{q}}'\left(\frac{\ell_2}{b}\right) + \mathfrak{I}_{(\frac{\ell_2}{b})^-}^{\xi,k} \wp_{\mathbf{q}}'(\ell_1) \right] \right| \\ &\le \frac{k\left(\frac{\ell_2}{b} - \ell_1\right)^2}{2(\xi+k)} \left\{ \left(\frac{1}{2} - \beta\left(2, \frac{\xi}{k} + 2\right) - \frac{k}{\xi+3k} \right) \left[|\wp_{\mathbf{q}}^3(\ell_1)| + \left| \wp_{\mathbf{q}}^3\left(\frac{\ell_2}{b}\right) \right| \right] \right\}.\end{aligned} \tag{13}$$

Proof. Setting $\wp = \wp_{\mathbf{q}}'$, we have that $\wp'' = \wp_{\mathbf{q}}^3$ is a completely monotone function on $(0,\infty)$ for each $\mathbf{q} \in (0,1)$. Now, using Corollary 4, we acquire Inequality (13). □

Proposition 2. *For* $\mathbf{q} \in (0,1)$ *and* $0 < \ell_1 < \ell_2$*, then as a result, we have:*

$$\begin{aligned}&\left| \frac{\wp_{\mathbf{q}}'(\ell_1) + \wp_{\mathbf{q}}'\left(\frac{\ell_2}{b}\right)}{2} - \frac{\Gamma_k(\xi+k)}{2\left(\frac{\ell_2}{b} - \ell_1\right)^{\frac{\xi}{k}}} \left[\mathfrak{I}_{\ell_1^+}^{\xi,k} \wp_{\mathbf{q}}'\left(\frac{\ell_2}{b}\right) + \mathfrak{I}_{(\frac{\ell_2}{b})^-}^{\xi,k} \wp_{\mathbf{q}}'(\ell_1) \right] \right| \\ &\le \frac{k\left(\frac{\ell_2}{b} - \ell_1\right)^2}{2(\xi+k)} \left\{ \left(1 - \frac{2k}{\xi+2k} \right) \left[\left|\wp_{\mathbf{q}}^3(\ell_1)\right| + \left| \wp_{\mathbf{q}}^3\left(\frac{\ell_2}{b}\right) \right| \right] \right\}.\end{aligned}$$

Proof. The proof is completed in a similar fashion as that of Proposition 1 and applying Corollary 7. □

Proposition 3. *For* $\mathbf{q} \in (0,1)$ *and* $0 < \ell_1 < \ell_2$*, then as a result, we have:*

$$\left| \frac{\wp_{\mathbf{q}}'(\ell_1) + \wp_{\mathbf{q}}'(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}} \left[\Im_{\ell_1^+}^{\xi,k} \wp_{\mathbf{q}}'(\frac{\ell_2}{b}) + \Im_{(\frac{\ell_2}{b})^-}^{\xi,k} \wp_{\mathbf{q}}'(\ell_1) \right] \right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)} \left(1 - \frac{2k}{p(\xi + k) + k} \right)^{\frac{1}{p}} \left(\frac{|\wp_{\mathbf{q}}^3(\ell_1)|^r + |\wp_{\mathbf{q}}^3(\frac{\ell_2}{b})|^r}{2} \right)^{\frac{1}{r}}.$$

Proof. The proof is completed in a similar fashion as that of Proposition 1 and applying Corollary 8. □

Proposition 4. *For* $\mathbf{q} \in (0,1)$ *and* $0 < \ell_1 < \ell_2$*, then as a result, we have:*

$$\left| \frac{\wp_{\mathbf{q}}'(\ell_1) + \wp_{\mathbf{q}}'(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}} \left[\Im_{\ell_1^+}^{\xi,k} \wp_{\mathbf{q}}'(\frac{\ell_2}{b}) + \Im_{(\frac{\ell_2}{b})^-}^{\xi,k} \wp_{\mathbf{q}}'(\ell_1) \right] \right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)} \left(1 - \frac{2k}{p(\xi + k) + k} \right)^{\frac{1}{p}} \left(\frac{|\wp_{\mathbf{q}}^3(\ell_1)|^r + |\wp_{\mathbf{q}}^3(\frac{\ell_2}{b})|^r}{6} \right)^{\frac{1}{r}}.$$

Proof. The proof is completed in a similar fashion as that of Proposition 1 and applying Corollary 11. □

Proposition 5. *For* $\mathbf{q} \in (0,1)$ *and* $0 < \ell_1 < \ell_2$*, then as a result, we have:*

$$\left| \frac{\wp_{\mathbf{q}}'(\ell_1) + \wp_{\mathbf{q}}'\left(\frac{\ell_2}{b}\right)}{2} - \frac{\Gamma_k(\xi + k)}{2\left(\frac{\ell_2}{b} - \ell_1\right)^{\frac{\xi}{k}}} \left[\Im_{\ell_1^+}^{\xi,k} \wp_{\mathbf{q}}'\left(\frac{\ell_2}{b}\right) + \Im_{(\frac{\ell_2}{b})^-}^{\xi,k} \wp_{\mathbf{q}}'(\ell_1) \right] \right|$$
$$\leq \frac{k\left(\frac{\ell_2}{b} - \ell_1\right)^2}{2(\xi + k)} \left(\frac{\xi}{\xi + 2k} \right)^{1-\frac{1}{r}} \left\{ \left(\frac{1}{2} - \beta\left(2, \frac{\xi}{k} + 2\right) - \frac{k}{\xi + 3k} \right) \left[|\wp_{\mathbf{q}}^3(\ell_1)|^r + \left| \wp_{\mathbf{q}}^3\left(\frac{\ell_2}{b}\right) \right|^r \right] \right\}^{\frac{1}{r}}.$$

Proof. The proof is completed in a similar fashion as that of Proposition 1 and applying Corollary 14. □

Proposition 6. *For* $\mathbf{q} \in (0,1)$ *and* $0 < \ell_1 < \ell_2$*, then as a result, we have:*

$$\left| \frac{\wp_{\mathbf{q}}'(\ell_1) + \wp_{\mathbf{q}}'(\frac{\ell_2}{b})}{2} - \frac{\Gamma_k(\xi + k)}{2(\frac{\ell_2}{b} - \ell_1)^{\frac{\xi}{k}}} \left[\Im_{\ell_1^+}^{\xi,k} \wp_{\mathbf{q}}'(\frac{\ell_2}{b}) + \Im_{(\frac{\ell_2}{b})^-}^{\xi,k} \wp_{\mathbf{q}}'(\ell_1) \right] \right|$$
$$\leq \frac{k(\frac{\ell_2}{b} - \ell_1)^2}{2(\xi + k)} \left\{ \left(\frac{1}{2} - \beta(2, p(\frac{\xi}{k} + 1) + 1) - \frac{k}{p(\xi + k) + 2k} \right)^{\frac{1}{p}} \right.$$
$$\left. \times \left[\left(\frac{|\wp_{\mathbf{q}}^3(\ell_1)|^r}{6} + \frac{|\wp_{\mathbf{q}}^3(\frac{\ell_2}{b})|^r}{3} \right)^{\frac{1}{r}} + \left(\frac{|\wp_{\mathbf{q}}^3(\ell_1)|^r}{3} + \frac{|\wp_{\mathbf{q}}^3(\frac{\ell_2}{b})|^r}{6} \right)^{\frac{1}{r}} \right] \right\}.$$

Proof. The proof is completed in a similar fashion as that of Proposition 1 and applying Corollary 16. □

6. Conclusions

In this article, we presented a new fractional version of the Hermite-Hadamard type inequality using the Hölder and the Minkowski inequality. Next, we established new integral identities for differentiable mappings, and employing these identities, we proved our main results. Some special cases for different types of convexities were derived as well. Additionally, some applications of our presented results were investigated through **q**-digamma functions. The techniques and ideas employed in this article can be generalized on the coordinates, quantum calculus, interval analysis, and preinvexity.

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