symmetry

MDPI

Article

Some New Simpson's-Formula-Type Inequalities for Twice-Differentiable Convex Functions via Generalized Fractional Operators

Muhammad Aamir Ali [1,*], Hasan Kara [2], Jessada Tariboon [3], Suphawat Asawasamrit [3], Hüseyin Budak [2] and Fatih Hezenci [2]

1 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
2 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; hasan96929@ogr.duzce.edu.tr (H.K.); huseyinbudak@duzce.edu.tr (H.B.); fatihhezenci@duzce.edu.tr (F.H.)
3 Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jessada.t@sci.kmutnb.ac.th (J.T.); suphawat.a@sci.kmutnb.ac.th (S.A.)
* Correspondence: mahr.muhammad.aamir@gmail.com

Abstract: From the past to the present, various works have been dedicated to Simpson's inequality for differentiable convex functions. Simpson-type inequalities for twice-differentiable functions have been the subject of some research. In this paper, we establish a new generalized fractional integral identity involving twice-differentiable functions, then we use this result to prove some new Simpson's-formula-type inequalities for twice-differentiable convex functions. Furthermore, we examine a few special cases of newly established inequalities and obtain several new and old Simpson's-formula-type inequalities. These types of analytic inequalities, as well as the methodologies for solving them, have applications in a wide range of fields where symmetry is crucial.

Keywords: Simpson-type inequalities; convex function; fractional integrals

check for updates

Citation: Ali, M.A.; Kara, H.; Tariboon, J.; Asawasamrit, S.; Budak, H.; Hezenci, H. Some New Simpson's-Formula-Type Inequalities for Twice-Differentiable Convex Functions via Generalized Fractional Operators. *Symmetry* **2021**, *13*, 2249. https://doi.org/10.3390/sym13122249

Academic Editor: Nicusor Minculete

Received: 29 October 2021
Accepted: 11 November 2021
Published: 25 November 2021

1. Introduction

Simpson's inequality is widely used in many areas of mathematics. For four times continuously differentiable functions, the classical Simpson's inequality is expressed as follows:

Theorem 1. *Suppose that $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a,b), and suppose also that $\left\| f^{(4)} \right\|_\infty = \sup_{x\in(a,b)} \left| f^{(4)}(x) \right| < \infty$. Then, one has the inequality*

$$\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{1}{2880}\left\| f^{(4)} \right\|_\infty (b-a)^4.$$

Many researchers have studied various Simpson's inequalities. More precisely, some studies have focused on Simpson's type for the convex function, because this focus has been an effective and powerful way to solve many problems in inequality theory and other areas of mathematics. For example, Alomari et al. established some inequalities of Simpson's type for s-convex functions by using differentiable functions [1]. Subsequently, Sarikaya et al. established new variants of Simpson's-type inequalities based on differentiable convex functions in [2,3]. Additionally, some papers have listed Simpson's-type inequalities in various convex classes [4–8]. Moreover, in the papers [9,10], researchers extended the Simpson inequalities for differentiable functions to Riemann–Liouville fractional integrals. Thereupon, several mathematicians studied fractional Simpson inequalities for these kinds of fractional integral operators [11–19]. For more studies related to different integral operator inequalities, one can see [20–31]. In addition, Sarikaya et al. obtained several Simpson-type inequalities for mappings whose second derivatives are convex [32]. In this article, after giving the definition of the generalized fractional integral operators, we construct a new identity for twice-differentiable functions. Using this equality, we prove several Simpson-type inequalities for functions whose second derivatives are convex. Then, with the help of special choices, the main results in this paper are shown to generalize many studies. In addition to all these, new results for k-Riemann–Liouville fractional integrals are also obtained.

First of all, general definitions and theorems that are used throughout the article are presented.

Definition 1. *Let us consider $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

For further information and several properties of Riemann–Liouville fractional integrals, please refer to [33–35].

In [36], Budak et al. prove the following identity for twice-differentiable functions and they also prove corresponding Simpson-type inequalities.

Lemma 1 ([36]). *Let $f : [a,b] \to \mathbb{R}$ be a twice-differentiable mapping (a,b) such that $f'' \in L_1([a,b])$. Then, the following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + J_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a)\right]$$
$$= \frac{(b-a)^2}{6} \int_0^1 w(t) f''(tb + (1-t)a)dt,$$

where

$$w(t) = \begin{cases} t\left(1 - \frac{3 \cdot 2^{\alpha}}{\alpha+1} t^{\alpha}\right), & t \in \left[0, \frac{1}{2}\right], \\ (1-t)\left(1 - \frac{3 \cdot 2^{\alpha}}{\alpha+1}(1-t)^{\alpha}\right), & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

In [37], Hezenci et al. prove another version of the results given in [36].

However, the generalized fractional integrals were introduced by Sarikaya and Ertuğral as follows:

Definition 2 ([38]). *Let us note that a function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the following condition:*

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We consider the following left-sided and right-sided generalized fractional integral operators

$$_{a+}I_{\varphi} f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \ \ x > a, \tag{1}$$

and

$$_{b-}I_{\varphi}f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b, \tag{2}$$

respectively.

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Hadamard fractional integrals, Katugampola fractional integrals, conformable fractional integrals, etc. These significant special cases of the integral operators (1) and (2) are used as follows:

1. For $\varphi(t) = t$, the operators (1) and (2) reduce to the Riemann integral.
2. If we assign $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1) and (2) reduce to the Riemann–Liouville fractional integrals $J^{\alpha}_{a+}f(x)$ and $J^{\alpha}_{b-}f(x)$, respectively. Here, Γ is the gamma function.
3. Let us consider $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$. Then, the operators (1) and (2) reduce to the k-Riemann–Liouville fractional integrals $J^{\alpha}_{a+,k}f(x)$ and $J^{\alpha}_{b-,k}f(x)$, respectively. Here, Γ_k is k-gamma function.

In recent years, several papers have been devoted to obtaining inequalities for generalized fractional integrals; for some of them please refer to [39–45].

Inspired by the ongoing studies, we give the generalized fractional version of the inequalities proved by Budak et al. in [36] for twice-differentiable convex functions. The fundamental benefit of these inequalities is that they can be turned into classical integral inequalities of Simpson's type [32], Riemann–Liouville fractional integral inequalities of Simpson's type [36], and k-Riemann–Liouville fractional integral inequalities of Simpson's type without having to prove each one separately.

2. Simpson's-Type Inequalities for Twice-Differentiable Functions

In this section, we prove some new inequalities of Simpson's type for twice-differentiable convex functions via the generalized fractional integrals. For brevity in the rest of the paper, we define

$$A(t) = \int_0^t T(s)ds,$$

where

$$T(s) = \int_0^s \frac{\varphi((b-a)u)}{u} du.$$

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be a twice-differentiable mapping (a,b) such that $f'' \in L_1([a,b])$. Then, the following equality for generalized fractional integrals holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{2T\left(\frac{1}{2}\right)}\left[\,_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b) + \,_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]$$

$$= \frac{(b-a)^2}{6} \int_0^1 \varpi(t) f''(tb + (1-t)a)dt,$$

where

$$\varpi(t) = \begin{cases} t - \frac{3A(t)}{T(1/2)}, & t \in \left[0, \frac{1}{2}\right], \\ 1 - t - \frac{3A(1-t)}{T(1/2)}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. Using integration by parts, we obtain

$$\begin{aligned}
I_1 &= \int_0^{1/2}\left(t-\frac{3A(t)}{T(1/2)}\right)f''(tb+(1-t)a)dt \\
&= \left(t-\frac{3A(t)}{T(1/2)}\right)\frac{f'(tb+(1-t)a)}{b-a}\Bigg|_0^{\frac{1}{2}} \\
&\quad -\frac{1}{b-a}\int_0^{1/2}\left(1-\frac{3T(t)}{T(1/2)}\right)f'(tb+(1-t)a)dt \\
&= \frac{1}{b-a}\left(\frac{1}{2}-\frac{3A(1/2)}{T(1/2)}\right)f'\left(\frac{a+b}{2}\right) \\
&\quad -\frac{1}{b-a}\left[\left(1-\frac{3T(t)}{T(1/2)}\right)\frac{f(tb+(1-t)a)}{(b-a)}\Bigg|_0^{1/2}\right. \\
&\quad \left.+\frac{1}{b-a}\int_0^{1/2}\frac{3}{T(1/2)}\frac{\varphi((b-a)t)}{t}f(tb+(1-t))dt\right] \\
&= \frac{1}{b-a}\left(\frac{1}{2}-\frac{3A(1/2)}{T(1/2)}\right)f'\left(\frac{a+b}{2}\right)+\frac{2}{(b-a)^2}f\left(\frac{a+b}{2}\right) \\
&\quad +\frac{f(a)}{(b-a)^2}+\frac{3}{T(1/2)(b-a)^2}\,{}_{\left(\frac{a+b}{2}\right)-}I_\varphi f(a).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \int_{1/2}^{1}\left(1-t-\frac{3A(1-t)}{T(1/2)}\right)f''(tb+(1-t)a)dt \\
&= -\frac{1}{b-a}\left(\frac{1}{2}-\frac{3A(1/2)}{T(1/2)}\right)f'\left(\frac{a+b}{2}\right)+\frac{2}{(b-a)^2}f\left(\frac{a+b}{2}\right) \\
&\quad +\frac{f(b)}{(b-a)^2}+\frac{3}{T(1/2)(b-a)^2}\,{}_{\left(\frac{a+b}{2}\right)+}I_\varphi f(b).
\end{aligned}$$

If I_1 and I_2 are added and then multiplied by $\frac{(b-a)^2}{6}$, the desired result is obtained. □

Remark 1. *If we take $\varphi(t)=t$ in Lemma 2, then Lemma 2 reduces to [32] (Lemma 2.1).*

Remark 2. *Let us note that $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha>0$ in Lemma 2, then Lemma 2 reduces to Lemma 1.*

Corollary 1. *If we choose $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha,k>0$ in Lemma 2, then the following equality for k-Riemann–Liouville fractional integrals holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+,k}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)-,k}f(a)\right] \\
&= \frac{(b-a)^2}{6}\int_0^1 m(t)f''(tb+(1-t)a)dt,
\end{aligned}$$

where

$$m(t)=\begin{cases} t\left(1-\frac{3k\cdot 2^{\frac{\alpha}{k}}}{\alpha+k}t^{\frac{\alpha}{k}}\right), & t\in\left[0,\frac{1}{2}\right], \\ (1-t)\left(1-\frac{3k\cdot 2^{\frac{\alpha}{k}}}{\alpha+k}(1-t)^{\frac{\alpha}{k}}\right), & t\in\left(\frac{1}{2},1\right]. \end{cases}$$

Proof. For $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have

$$\Lambda(s)=\frac{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)}s^{\frac{\alpha}{k}}=\frac{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}s^{\frac{\alpha}{k}}, \tag{3}$$

$$\Lambda(1/2)=\frac{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \tag{4}$$

and

$$\Delta(\tau)=\frac{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\int_0^{\tau}s^{\frac{\alpha}{k}}ds=\frac{k(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}{(\alpha+k)\Gamma_k(\alpha+k)}\tau^{\frac{\alpha}{k}+1}. \tag{5}$$

Then it follows that

$$\varpi(\tau)=m(\tau)$$

which completes the proof. □

Theorem 2. *Assume that the assumptions of Lemma 2 hold. Assume also that the mapping $|f''|$ is convex on $[a,b]$. Then, we have the following Simpson-type inequality for generalized fractional integrals*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right| \\ \leq\;& \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)\left[|f''(a)|+|f''(b)|\right].\end{aligned}$$

Proof. By taking the modulus in Lemma 2, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right| \\ \leq\;& \frac{(b-a)^2}{6}\int_0^1|\varpi(t)||f''(tb+(1-t)a)|dt \\ =\;& \frac{(b-a)^2}{6}\left[\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right||f''(tb+(1-t)a)|dt\right. \\ &\left.+\int_{\frac{1}{2}}^1\left|1-t-\frac{3A(1-t)}{T(1/2)}\right||f''(tb+(1-t)a)|dt\right].\end{aligned} \tag{6}$$

With the help of the convexity of $|f''|$, we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right|$$

$$
\begin{aligned}
&\leq \frac{(b-a)^2}{6}\left[\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|\left[t|f''(b)|+(1-t)|f''(a)|\right]dt\right.\\
&\left.+\int_{\frac{1}{2}}^{1}\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|\left[t|f''(b)|+(1-t)|f''(a)|\right]dt\right]\\
&= \frac{(b-a)^2}{6}\left\{\left[\int_0^{\frac{1}{2}}t\left|t-\frac{3A(t)}{T(1/2)}\right|dt+\int_{\frac{1}{2}}^{1}t\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|dt\right]|f''(b)|\right.\\
&\left.+\left[\int_0^{\frac{1}{2}}(1-t)\left|t-\frac{3A(t)}{T(1/2)}\right|dt+\int_{\frac{1}{2}}^{1}(1-t)\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|dt\right]|f''(a)|\right\}\\
&= \frac{(b-a)^2}{6}\left[\int_0^{\frac{1}{2}}t\left|t-\frac{3A(t)}{T(1/2)}\right|dt+\int_{\frac{1}{2}}^{1}t\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|dt\right]\left[|f''(a)|+|f''(b)|\right]\\
&= \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)\left[|f''(a)|+|f''(b)|\right].
\end{aligned}
$$

This completes the proof of Theorem 2. □

Remark 3. *Consider $\varphi(t)=t$ in Theorem 2, then Theorem 2 reduces to [32] (Theorem 2.2).*

Remark 4. *If we assign $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha>0$ in Theorem 2, then we obtain the following Simpson-type inequality for Riemann–Liouville fractional integrals*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)\right]\right|\\
&\leq \frac{(b-a)^2}{6}\Theta(\alpha)\left[|f''(a)|+|f''(b)|\right].
\end{aligned}
$$

Here,

$$
\Theta(\alpha)=\frac{1}{4(\alpha+2)}\left(\alpha\left(\frac{\alpha+1}{3}\right)^{\frac{2}{\alpha}}+\frac{3}{\alpha+1}\right)-\frac{1}{8}, \tag{7}
$$

which is given by Budak et al. in [36].

Corollary 2. *For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $k,\alpha>0$ in Theorem 2, we have the following Simpson-type inequality for k-Riemann–Liouville fractional integrals*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+,k}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)-,k}f(a)\right]\right|\\
&\leq \frac{(b-a)^2}{6}\Theta(\alpha,k)\left[|f''(a)|+|f''(b)|\right],
\end{aligned}
$$

where

$$
\Theta(\alpha,k)=\frac{k}{4(\alpha+2k)}\left(\frac{\alpha}{k}\left(\frac{\alpha+k}{3k}\right)^{\frac{2k}{\alpha}}+\frac{3k}{\alpha+k}\right)-\frac{1}{8}. \tag{8}
$$

Proof. Let $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$. By the equalities (3)–(5), we have

$$\begin{aligned}\int_0^{\frac{1}{2}} \left|\tau - \frac{3\Delta(\tau)}{\Lambda(1/2)}\right| d\tau &= \int_0^{\frac{1}{2}} \left|\tau - \frac{3k \cdot 2^{\frac{\alpha}{k}}}{\alpha+k}\tau^{\frac{\alpha}{k}}\right| d\tau \\ &= \frac{k}{4(\alpha+2k)}\left(\frac{\alpha}{k}\left(\frac{\alpha+k}{3k}\right)^{\frac{2k}{\alpha}} + \frac{3k}{\alpha+k}\right) - \frac{1}{8}.\end{aligned}$$

This completes the proof. □

Theorem 3. *Suppose that the assumptions of Lemma 2 hold. Suppose also that the mapping $|f''|^q$, $q > 1$, is convex on $[a,b]$. Then, the following Simpson-type inequality for generalized fractional integrals*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b) + {}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}\left|t - \frac{3A(t)}{T(1/2)}\right|^p dt\right)^{\frac{1}{p}} \\ &\times\left[\left(\frac{|f''(b)|^q + 3|f''(a)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|f''(b)|^q + |f''(a)|^q}{8}\right)^{\frac{1}{q}}\right]\end{aligned}$$

is valid. Here, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By applying the Hölder inequality in inequality (6), we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b) + {}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{6}\left[\left(\int_0^{\frac{1}{2}}\left|t - \frac{3A(t)}{T(1/2)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f''(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1\left|1 - t - \frac{3A(1-t)}{T(1/2)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f''(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

By using the convexity of $|f''|^q$, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b) + {}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}\left|t - \frac{3A(t)}{T(1/2)}\right|^p dt\right)^{\frac{1}{p}}\end{aligned}$$

$$\times\left[\left(\int_0^{\frac{1}{2}}\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{8}\right)^{\frac{1}{q}}\right].$$

This finishes the proof of Theorem 3. □

Remark 5. *If we choose $\varphi(t)=t$ in Theorem 3, then we obtain*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx\right|$$

$$\leq\frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}t^p|1-3t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{8}\right)^{\frac{1}{q}}\right],$$

which is given by Budak et al. in [36].

Remark 6. *Let us consider $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha>0$ in Theorem 3, then the Simpson-type inequality for Riemann–Liouville fractional integrals*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\right|$$
$$\leq\frac{(b-a)^2}{6}\Upsilon(\alpha,p)\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{8}\right)^{\frac{1}{q}}\right]$$

is valid. Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\Upsilon(\alpha,p)=\left(\int_0^{\frac{1}{2}}t^p\left|1-\frac{3\cdot 2^\alpha}{\alpha+1}t^\alpha\right|^p dt\right)^{\frac{1}{p}}.$$

which is given by Budak et al. in [36].

Corollary 3. *If we choose $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha,k>0$ in Theorem 3, then we have the following Simpson-type inequality for k-Riemann–Liouville fractional integrals*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+,k}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-,k}f(a)\right]\right|$$
$$\leq\frac{(b-a)^2}{6}\Upsilon(\alpha,p,k)\left[\left(\frac{|f''(b)|^q+3|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{8}\right)^{\frac{1}{q}}\right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\Upsilon(\alpha,p,k)=\left(\int_0^{\frac{1}{2}} t^p\left|1-\frac{3k\cdot 2^{\frac{\alpha}{k}}}{\alpha+k}t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}.$$

Proof. For $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, the proof can be seen easily by the equalities (3)–(5). □

Theorem 4. *Assume that the assumptions of Lemma 2 hold. If the mapping* $|f''|^q$, $q\geq 1$ *is convex on* $[a,b]$, *then we have the following Simpson-type inequality for generalized fractional integrals*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{(b-a)^2}{6}\left(\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)^{1-\frac{1}{q}}\\&\times\left\{\left[\left(\int_0^{\frac{1}{2}}t\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)|f''(b)|^q+\left(\int_0^{\frac{1}{2}}(1-t)\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)|f''(a)|^q\right]^{\frac{1}{q}}\right.\\&\left.+\left[\left(\int_0^{\frac{1}{2}}(1-t)\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)|f''(b)|^q+\left(\int_0^{\frac{1}{2}}t\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)|f''(a)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. By applying the power-mean inequality in (6), we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2T\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right| \quad (9)\\&\leq\frac{(b-a)^2}{6}\left[\left(\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|dt\right)^{1-\frac{1}{q}}\right.\\&\times\left(\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right||f''(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^{1}\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|dt\right)^{1-\frac{1}{q}}\\&\left.\times\left(\int_{\frac{1}{2}}^{1}\left|1-t-\frac{3A(1-t)}{T(1/2)}\right||f''(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $|f''|^q$ is convex, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|\left|f''(tb+(1-t)a)\right|^q dt \qquad (10)\\
\leq\ &\int_0^{\frac{1}{2}}\left|t-\frac{3A(t)}{T(1/2)}\right|\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt\\
=\ &\left|f''(b)\right|^q\int_0^{\frac{1}{2}}t\left|t-\frac{3A(t)}{T(1/2)}\right|dt+\left|f''(a)\right|^q\int_0^{\frac{1}{2}}(1-t)\left|t-\frac{3A(t)}{T(1/2)}\right|dt
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|\left|f''(tb+(1-t)a)\right|^q dt \qquad (11)\\
\leq\ &\left|f''(b)\right|^q\int_{\frac{1}{2}}^{1}t\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|dt+\left|f''(a)\right|^q\int_{\frac{1}{2}}^{1}(1-t)\left|1-t-\frac{3A(1-t)}{T(1/2)}\right|dt\\
=\ &\left|f''(b)\right|^q\int_0^{\frac{1}{2}}(1-t)\left|t-\frac{3A(t)}{T(1/2)}\right|dt+\left|f''(a)\right|^q\int_0^{\frac{1}{2}}t\left|t-\frac{3A(t)}{T(1/2)}\right|dt.
\end{aligned}$$

If we substitute the inequalities (10) and (11) in (9), then we obtain the desired result. □

Remark 7. *Consider $\varphi(t)=t$ in Theorem 4, then Theorem 4 reduces to [32] (Theorem 2.5).*

Remark 8. *If we take $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha>0$ in Theorem 4, then we obtain the following Simpson-type inequality for Riemann–Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)\right]\right|\\
\leq\ &\frac{(b-a)^2}{6}(\Theta(\alpha))^{1-\frac{1}{q}}\left[\left(\Xi(\alpha)\left|f''(b)\right|^q+\Omega(\alpha)\left|f''(a)\right|^q\right)^{\frac{1}{q}}+\left(\Omega(\alpha)\left|f''(b)\right|^q+\Xi(\alpha)\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Here, $\Theta(\alpha)$ is defined as in (7) and

$$\begin{aligned}
\Xi(\alpha) &= \frac{1}{4(\alpha+3)}\left[\frac{\alpha}{3}\left(\frac{\alpha+1}{3}\right)^{\frac{3}{\alpha}}+\frac{3}{2(\alpha+1)}\right]-\frac{1}{24},\\
\Omega(\alpha) &= \Theta(\alpha)-\Xi(\alpha)\\
&= \frac{1}{4(\alpha+2)}\left(\alpha\left(\frac{\alpha+1}{3}\right)^{\frac{2}{\alpha}}+\frac{3}{\alpha+1}\right)\\
&\quad -\frac{1}{4(\alpha+3)}\left[\frac{\alpha}{3}\left(\frac{\alpha+1}{3}\right)^{\frac{3}{\alpha}}+\frac{3}{2(\alpha+1)}\right]-\frac{1}{12},
\end{aligned}$$

which is given by Budak et al. in [36].

Corollary 4. *Let us consider* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha, k > 0$ *in Theorem 4, then the following Simpson-type inequality for k-Riemann–Liouville fractional integrals holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+,k}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-,k}f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{6}(\Theta(\alpha,k))^{1-\frac{1}{q}}\left[\left(\Xi(\alpha,k)|f''(b)|^q+\Omega(\alpha,k)|f''(a)|^q\right)^{\frac{1}{q}}+\left(\Omega(\alpha,k)|f''(b)|^q+\Xi(\alpha,k)|f''(a)|^q\right)^{\frac{1}{q}}\right],$$

where $\Theta(\alpha,k)$ *is defined as in (8) and*

$$\begin{aligned}\Xi(\alpha,k) &= \frac{k}{4(\alpha+3k)}\left[\frac{\alpha}{3k}\left(\frac{\alpha+k}{3k}\right)^{\frac{3k}{\alpha}}+\frac{3k}{2(\alpha+k)}\right]-\frac{1}{24},\\ \Omega(\alpha,k) &= \Theta(\alpha,k)-\Xi(\alpha,k)\\ &= \frac{k}{4(\alpha+2k)}\left(\frac{\alpha}{k}\left(\frac{\alpha+k}{3k}\right)^{\frac{2k}{\alpha}}+\frac{3k}{\alpha+k}\right)\\ &\quad -\frac{k}{4(\alpha+3k)}\left[\frac{\alpha}{3k}\left(\frac{\alpha+k}{3k}\right)^{\frac{3k}{\alpha}}+\frac{3k}{2(\alpha+k)}\right]-\frac{1}{12}.\end{aligned}$$

Proof. Let $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$. By the equalities (3)–(5), we have

$$\begin{aligned}&\int_0^{\frac{1}{2}}\tau\left|\tau-\frac{3\Delta(\tau)}{\Lambda(1/2)}\right|d\tau\\ &= \int_0^{\frac{1}{2}}\tau\left|\tau-\frac{3k\cdot 2^{\frac{\alpha}{k}}}{\alpha+k}\tau^{\frac{\alpha}{k}}\right|d\tau\\ &= \frac{k}{4(\alpha+3k)}\left[\frac{\alpha}{3k}\left(\frac{\alpha+k}{3k}\right)^{\frac{3k}{\alpha}}+\frac{3k}{2(\alpha+k)}\right]-\frac{1}{24}\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\frac{1}{2}}(1-\tau)\left|\tau-\frac{3\Delta(\tau)}{\Lambda(1/2)}\right|d\tau\\ &= \int_0^{\frac{1}{2}}(1-\tau)\left|\tau-\frac{3k\cdot 2^{\frac{\alpha}{k}}}{\alpha+k}\tau^{\frac{\alpha}{k}}\right|d\tau\\ &= \frac{k}{4(\alpha+2k)}\left(\frac{\alpha}{k}\left(\frac{\alpha+k}{3k}\right)^{\frac{2k}{\alpha}}+\frac{3k}{\alpha+k}\right)\\ &\quad -\frac{k}{4(\alpha+3k)}\left[\frac{\alpha}{3k}\left(\frac{\alpha+k}{3k}\right)^{\frac{3k}{\alpha}}+\frac{3k}{2(\alpha+k)}\right]-\frac{1}{12}.\end{aligned}$$

□

3. Conclusions

For twice-differentiable functions, we have developed a generalized fractional version of the Simpson-type inequality in this paper. After that, we explained how our findings generalize a number of inequalities found in previous research. For *k*-Riemann–Liouville fractional integrals, we additionally provided novel Simpson-type inequalities. The findings of this study can be utilized in symmetry. The results for the case of symmetric convex functions can be obtained in future studies. In future studies, researchers can obtain generalized versions of our results by utilizing other kinds of convex function classes or different types of generalized fractional integral operators.

Author Contributions: All authors contributed equally in the preparation of the present work. Theorems and corollaries: M.A.A., H.K., J.T., S.A., H.B. and F.H.; review of the articles and books cited: M.A.A., H.K., J.T., S.A., H.B. and F.H.; formal analysis: M.A.A., H.K., J.T., S.A., H.B. and F.H.; writing—original draft preparation and writing—review and editing: M.A.A., H.K., J.T., S.A., H.B. and F.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology, North Bangkok. Contract no. KMUTNB-62-KNOW-29.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the referees for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of Simpson's type for s-convex functions with applications. *RGMIA Res. Rep. Coll.* **2009**, *12*, 1–18.
2. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson's type for convex functions. *RGMIA Res. Rep. Coll.* **2010**, *13*, 2.
3. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson's type for s-convex functions. *Comput. Math. Appl.* **2020**, *60*, 2191–2199. [CrossRef]
4. Du, T.; Li, Y.; Yang, Z. A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions. *Appl. Math. Comput.* **2017**, *293*, 358–369. [CrossRef]
5. İşcan, İ. Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions. *J. Math.* **2014**, *2014*, 346305. [CrossRef]
6. Matloka, M. Some inequalities of Simpson type for h-convex functions via fractional integrals. *Abstr. Appl. Anal.* **2015**, *2015*, 956850. [CrossRef]
7. Ozdemir, M.E.; Akdemir, A.O.; Kavurmacı, H. On the Simpson's inequality for convex functions on the coordinates. *Turk. J. Anal. Number Theory* **2014**, *2*, 165–169. [CrossRef]
8. Park, J. On Simpson-like type integral inequalities for differentiable preinvex functions. *Appl. Math. Sci.* **2013**, *7*, 6009–6021. [CrossRef]
9. Chen, J.; Huang, X. Some new inequalities of Simpson's type for s-convex functions via fractional integrals. *Filomat* **2017**, *31*, 4989–4997. [CrossRef]
10. Iqbal, M.; Qaisar, S.; Hussain, S. On Simpson's type inequalities utilizing fractional integrals. *J. Comput. Anal. Appl.* **2017**, *23*, 1137–1145.
11. Abdeljawad, T.; Rashid, S.; Hammouch, Z.; Chu, İ.Y.M. Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications. *Adv. Differ. Equ.* **2020**, *2020*, 1–26. [CrossRef]
12. Ertuğral, F.; Sarikaya, M.Z. Simpson type integral inequalities for generalized fractional integral. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Matemáticas* **2019**, *113*, 3115–3124. [CrossRef]
13. Hussain, S.; Khalid, J.; Chu, Y.M. Some generalized fractional integral Simpson's type inequalities with applications. *AIMS Math.* **2020**, *5*, 5859–5883. [CrossRef]
14. Kermausuor, S. Simpson's type inequalities via the Katugampola fractional integrals for s-convex functions. *Kragujev. J. Math.* **2021**, *45*, 709–720. [CrossRef]
15. Lei, H.; Hu, G.; Nie, J.; Du, T. Generalized Simpson-type inequalities considering first derivatives through the k-Fractional Integrals. *IAENG Int. J. Appl. Math.* **2021**, *50*, 1–8.
16. Luo, C.; Du, T. Generalized Simpson type inequalities involving Riemann-Liouville fractional integrals and their applications. *Filomat* **2020**, *34*, 751–760. [CrossRef]
17. Rashid, S.; Akdemir, A.O.; Jarad, F.; Noor, M.A.; Noor, K.I. Simpson's type integral inequalities for κ-fractional integrals and their applications. *AIMS Math.* **2019**, *4*, 1087–1100. [CrossRef]

18. Sarıkaya, M.Z.; Budak, H.; Erden, S. On new inequalities of Simpson's type for generalized convex functions. *Korean J. Math.* **2019**, *27*, 279–295.
19. Set, E.; Akdemir, A.O.; Özdemir, M.E. Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. *Filomat* **2017**, *31*, 4415–4420. [CrossRef]
20. Asawasamrit, S.; Ali, M.A.; Ntouyas, S.K.; Tariboon, J. Some Parameterized Quantum Midpoint and Quantum Trapezoid Type Inequalities for Convex Functions with Applications. *Entropy* **2021**, *23*, 996. [CrossRef]
21. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Methods Appl. Sci.* **2021**, *44*, 378–390. [CrossRef]
22. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On Simpson's inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [CrossRef]
23. Hua, J.; Xi, B.Y.; Qi, F. Some new inequalities of Simpson type for strongly s-convex functions. *Afr. Mat.* **2015**, *26*, 741–752. [CrossRef]
24. Hussain, S.; Qaisar, S. More results on Simpson's type inequality through convexity for twice differentiable continuous mappings. *SpringerPlus* **2016**, *5*, 1–9. [CrossRef]
25. Wu, H.K.J.-D.; Hussain, S.; Latif, M.A. Simpson's Type Inequalities for Co-Ordinated Convex Functions on Quantum Calculus. *Symmetry* **2019**, *11*, 768.
26. Li, Y.; Du, T. Some Simpson type integral inequalities for functions whose third derivatives are (α, m)-GA-convex functions. *J. Egypt. Math. Soc.* **2016**, *24*, 175–180. [CrossRef]
27. Liu, B.Z. An inequality of Simpson type. *Proc. R. Soc. A* **2005**, *461*, 2155–2158. [CrossRef]
28. Liu, W. Some Simpson type inequalities for h-convex and (α, m)-convex functions. *J. Comput. Anal. Appl.* **2014**, *16*, 1005–1012.
29. Simic, S.; Bin-Mohsin, B. Simpson's Rule and Hermite-Hadamard Inequality for Non-Convex Functions. *Mathematics* **2020**, *8*, 1248. [CrossRef]
30. Siricharuanun, P.; Erden, S.; Ali, M.A.; Budak, H.; Chasreechai, S.; Sitthiwirattham, T. Some New Simpson's and Newton's Formulas Type Inequalities for Convex Functions in Quantum Calculus. *Mathematics* **2021**, *9*, 1992. [CrossRef]
31. Vivas-Cortez, M.; Abdeljawad, T.; Mohammed, P.O.; Rangel-Oliveros, Y. Simpson's integral inequalities for twice differentiable convex functions. *Math. Probl. Eng.* **2020**, *2020*, 1936461. [CrossRef]
32. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex. *J. Appl. Math. Inform.* **2013**, *9*, 37–45. [CrossRef]
33. Gorenflo, R.; Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*; Springer: Wien, Austria, 1997; pp. 223–276.
34. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Nertherland, 2006.
35. Miller, S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
36. Budak, H.; Kara, H.; Hezenci, F. Fractional Simpson type inequalities for twice differentiable functions. *Turk. J. Math.* **2021**, submitted.
37. Hezenci, F.; Budak, H.; Kara, H. New version of Fractional Simpson type inequalities for twice differentiable functions. *Adv. Differ. Equ.* **2021**, *2021*, 460. [CrossRef]
38. Sarikaya, M.Z.; Ertugral, F. On the generalized Hermite-Hadamard inequalities. *Ann. Univ. Craiova Math. Comput. Sci. Ser.* **2020**, *47*, 193–213.
39. Budak, H.; Yildirim, S.K.; Kara, H.; Yildirim, H. On new generalized inequalities with some parameters for coordinated convex functions via generalized fractional integrals. *Math. Methods Appl. Sci.* **2021**, 13069–13098. [CrossRef]
40. Mohammed, P.O.; Sarikaya, M.Z. On generalized fractional integral inequalities for twice differentiable convex functions. *J. Comput. Appl. Math.* **2020**, *372*, 112740. [CrossRef]
41. Budak, H.; Ertuğral, F.; Pehlivan, E. Hermite-Hadamard type inequalities for twice differantiable functions via generalized fractional integrals. *Filomat* **2019**, *33*, 4967–4979. [CrossRef]
42. Budak, H.; Pehlivan, E.; Kösem, P. On New Extensions of Hermite-Hadamard inequalities for generalized fractional integrals. *Sahand Commun. Math. Anal.* **2021**, *18*, 73–88.
43. Kashuri, A.; Set, E.; Liko, R. Some new fractional trapezium-type inequalities for preinvex functions. *Fractal Fract.* **2019**, *3*, 12. [CrossRef]
44. Zhao, D.; Ali, M.A.; Kashuri, A.; Budak, H.; Sarikaya, M.Z. Hermite–Hadamard-type inequalities for the interval-valued approximately h-convex functions via generalized fractional integrals. *J. Inequal. Appl.* **2020**, *2020*, 222. [CrossRef]
45. You, X.X.; Ali, M.A.; Budak, H.; Agarwal, P.; Chu, Y.M. Extensions of Hermite–Hadamard inequalities for harmonically convex functions via generalized fractional integrals. *J. Inequal. Appl.* **2021**, *2021*, 102. [CrossRef]