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Spectrum of Fractional and Fractional Prabhakar Sturm–Liouville Problems with Homogeneous Dirichlet Boundary Conditions

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Abstract: In this study, we consider regular eigenvalue problems formulated by using the left and right standard fractional derivatives and extend the notion of a fractional Sturm–Liouville problem to the regular Prabhakar eigenvalue problem, which includes the left and right Prabhakar derivatives. In both cases, we study the spectral properties of Sturm–Liouville operators on function space restricted by homogeneous Dirichlet boundary conditions. Fractional and fractional Prabhakar Sturm–Liouville problems are converted into the equivalent integral ones. Afterwards, the integral Sturm–Liouville operators are rewritten as Hilbert–Schmidt operators determined by kernels, which are continuous under the corresponding assumptions. In particular, the range of fractional order is here restricted to interval $(1/2, 1]$. Applying the spectral Hilbert–Schmidt theorem, we prove that the spectrum of integral Sturm–Liouville operators is discrete and the system of eigenfunctions forms a basis in the corresponding Hilbert space. Then, equivalence results for integral and differential versions of respective eigenvalue problems lead to the main theorems on the discrete spectrum of differential fractional and fractional Prabhakar Sturm–Liouville operators.

Keywords: fractional derivatives; fractional Prabhakar derivatives; fractional differential equations; fractional Sturm–Liouville problems; eigenfunctions and eigenvalues

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1. Introduction

The aim of this paper is to study the fundamental properties of fractional eigenvalue problems developed by the construction of the Sturm–Liouville operator (SLO) with left and right fractional derivatives. In classical differential equations theory, this is a linear differential operator of the second order and yields an eigenvalue problem of the form (here, $x \in [0, b]$ in the case when we consider the problem on a finite interval):

$$\mathcal{L}_q y(x) = -\frac{d}{dx} p(x) \frac{dy(x)}{dx} + q(x)y(x) = \lambda w(x)y(x)$$

with boundary conditions appearing as follows:

$$c_1 y(0) + c_2 \frac{dy(0)}{dx} = 0, \quad d_1 y(b) + d_2 \frac{dy(b)}{dx} = 0. \tag{1}$$

Let us point out that, depending on the choice of coefficient functions and boundary conditions, such problems provide various systems of orthogonal eigenfunctions, orthogonal polynomials and families of special functions. Orthogonal systems of the solutions of classical Sturm–Liouville problems are widely applied in the analysis and solving of fundamental differential equations of mathematics, physics, mechanics , and economics.

In most of the FSLPs presented at the beginning of fractional Sturm–Liouville theory, first-order derivatives in a standard Sturm–Liouville problem were replaced with fractional order derivatives. The resulting equations were solved using some numerical schemes [1–4]. However, in these works, the essential properties, such as the orthogonality of the eigenfunctions of the fractional operator, were not investigated. In addition, the question of whether the associated eigenvalues are real or not is not addressed. Some results concerning these properties have been obtained in papers [5,6], where the discussed equations contain a classical SLO extended by including a sum of the left and the right derivatives. Then, in paper [7], we proposed the construction of a fractional Sturm–Liouville operator which preserves the orthogonality of the eigenfunctions corresponding to distinct eigenvalues and provides real eigenvalues. The FSLO contains both the left and right derivatives and is a symmetric operator on function space restricted by fractional boundary conditions which generalize conditions (1).

A fractional version of Bessel SLO has been developed and applied to anomalous diffusion in [8], where the space-fractional differential operator has a form analogous to the FSLO proposed in a general form in [7]. Some special cases of singular fractional Sturm–Liouville problems were also studied in [9,10], where exact solutions and eigenvalues were calculated.

In our earlier works [7,11–14], we focused on the construction of a fractional version of operator $\mathcal{L}_q$, which includes standard fractional derivatives. The characteristic feature of the proposed approach is the mixture of the left and right fractional derivatives in the fractional Sturm–Liouville operator (FSLO). This construction provides eigenvalue problems with orthogonal eigenfunctions and discrete spectra under the appropriate homogeneous boundary conditions.

In recent years, fractional eigenvalue problems have also been discussed within the framework of tempered and conformable fractional calculus. In the papers [15,16], a fractional Sturm–Liouville operator is built by using the left and right tempered derivatives. Next, in [17,18], an FSLO is constructed as a composition of conformable fractional derivatives. In addition, in paper [19], the authors show how to build an FSLO with composite fractional derivatives.

Here, we add the generalization of fractional eigenvalue problems to problems with operators, including Prabhakar derivatives. The regular fractional and fractional Prabhakar Sturm–Liouville operators considered here include the left and the right derivatives, and the derived equations are in fact of a variational nature; i.e., they are Euler–Lagrange equations for respective actions (compare [11,20] and the references therein for FSLE). The properties of the spectra and eigenfunctions' systems of FSLP can be studied by applying the variational method [12,21]. Here, we shall develop the transformation method for FSLP and PSLP with Dirichlet boundary conditions, which means that we rewrite the FSLP/PSLP as the equivalent integral eigenvalue problem.

The paper is organized as follows. In the next section, we present the necessary definitions and properties of fractional and fractional Prabhakar operators, as well as the formulation of a regular fractional Sturm–Liouville problem with its generalization to the Prabhakar Sturm–Liouville problem. In Section 3, we define the problems with homogeneous Dirichlet boundary conditions and derive equivalence results for both types of fractional eigenvalue problems. It appears that by applying composition rules for derivatives and integrals, they can be converted into the equivalent integral ones. Spectral properties of integral versions of fractional and fractional Prabhakar Sturm–Liouville operators are discussed in Section 4. We shall prove that these operators are Hilbert–Schmidt integral operators, which are compact and self-adjoint on the $L^2_w(0,b)$ space. Applying the spectral Hilbert–Schmidt theorem, we derive results on discrete spectra both for fractional and fractional Prabhakar Sturm–Liouville operators. The equivalence of differential and integral versions of eigenvalue problems leads to the corresponding spectral results for differential operators.

The paper closes with a brief discussion of results and future investigations. The Appendix A contains two parts. First, we present results on Hölder continuity of kernels defining integral Sturm–Liouville operators. Then, we prove a useful theorem on the convergence of convolutions' series in a general case, which is applied in the construction of integral Sturm–Liouville operators.

2. Preliminaries

We start with a summary of definitions and properties of fractional integrals and derivatives which shall be applied in the construction of fractional and fractional Prabhakar eigenvalue problems. First, we recall the left and right Riemann–Liouville fractional derivatives of order $\alpha \in (0,1)$ [22,23]:

$$D^{\alpha}_{0+}\, y(x) := \frac{d}{dx} I^{1-\alpha}_{0+} y(x), \quad D^{\alpha}_{b-}\, y(x) := -\frac{d}{dx} I^{1-\alpha}_{b^-} y(x), \tag{2}$$

where the operators I^{α}_{0+} and I^{α}_{b-} are respectively the left and the right fractional Riemann–Liouville integrals of order $\alpha > 0$ defined by the following formulas

$$I^{\alpha}_{0+} y(x) := \int_0^x \frac{(x-t)^{\alpha-1} y(t)}{\Gamma(\alpha)} dt, \qquad x > 0, \tag{3}$$

$$I^{\alpha}_{b-} y(x) := \int_x^b \frac{(t-x)^{\alpha-1} y(t)}{\Gamma(\alpha)} dt, \qquad x < b. \tag{4}$$

Next, we have Caputo fractional derivatives:

$$^{c}D^{\alpha}_{0+} y(x) = D^{\alpha}_{0+}(y(x) - y(0)), \quad ^{c}D^{\alpha}_{b-} y(x) = D^{\alpha}_{b-}(y(x) - y(b)) \tag{5}$$

and we note that when $y(0) = y(b) = 0$, both types of derivatives coincide, i.e.,

$$^{c}D^{\alpha}_{0+} y(x) = D^{\alpha}_{0+} y(x), \qquad ^{c}D^{\alpha}_{b-} y(x) = D^{\alpha}_{b-} y(x).$$

We also recall some of the composition rules of fractional operators for the case of order $\alpha \in (0,1]$; namely, for the left-sided Caputo derivative and left-sided fractional integral, we have

$$I^{\alpha}_{0+}\, {}^{c}D^{\alpha}_{0+} y(x) = y(x) - y(0), \tag{6}$$

$$^{c}D^{\alpha}_{0+} I^{\alpha}_{0+} y(x) = y(x), \tag{7}$$

while for the right-sided Riemann–Liouville derivatives, the following relations are valid

$$I^{\alpha}_{b-} D^{\alpha}_{b-} y(x) = y(x) - I^{1-\alpha}_{b-} y(b) \cdot \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}, \tag{8}$$

$$D^{\alpha}_{b-} I^{\alpha}_{b-} y(x) = y(x). \tag{9}$$

All of the above rules are fulfilled for all points $x \in [0,b]$ when function y is a continuous one. Let us note that for the continuous function fulfilling condition $y(0) = 0$, rules (6) and (8) look as follows:

$$I^{\alpha}_{0+}\, {}^{c}D^{\alpha}_{0+} y(x) = y(x), \quad I^{\alpha}_{b-} D^{\alpha}_{b-} y(x) = y(x). \tag{10}$$

The fractional operators, described above, are generalized to Prabhakar integrals and derivatives. They are defined using a three-parameter Mittag–Leffler function [22,24]:

$$E^{\gamma}_{\rho,\mu}(z) := \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\rho k + \mu)} \cdot \frac{z^k}{k!} \tag{11}$$

and Prabhakar function [24,25]:

$$e^{\gamma}_{\rho,\mu}(\omega z^{\rho}) := z^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega z^{\rho}), \tag{12}$$

both defined on the complex space when $Re(\rho) > 0$ and $Re(\mu) > 0$.

These functions lead to the left and right Prabhakar derivatives [24]:

$$D^{\alpha}_{\rho,\gamma,\omega,0+}\, y(x) := \frac{d}{dx} E^{1-\alpha}_{\rho,-\gamma,\omega,0+} y(x), \quad D^{\alpha}_{\rho,\gamma,\omega,b-}\, y(x) := -\frac{d}{dx} E^{1-\alpha}_{\rho,-\gamma,\omega,b-} y(x), \tag{13}$$

where operators $E^{\alpha}_{\rho,-\gamma,\omega,0+}$ and $E^{\alpha}_{\rho,-\gamma,\omega,b-}$ are respectively the left and the right fractional Prabhakar integrals:

$$E^{\alpha}_{\rho,-\gamma,\omega,0+} y(x) := \int_0^x e^{-\gamma}_{\rho,\alpha}(\omega(x-t)^{\rho}) y(t) dt, \quad x > 0, \tag{14}$$

$$E^{\alpha}_{\rho,-\gamma,\omega,b-} y(x) := \int_x^b e^{-\gamma}_{\rho,\alpha}(\omega(x-t)^{\rho}) y(t) dt, \quad x < b. \tag{15}$$

Similar to Caputo derivatives, given in (5), we have Caputo-type Prabhakar derivatives defined as follows

$${}^{c}D^{\alpha}_{\rho,\gamma,\omega,0+}\, y(x) = D^{\alpha}_{\rho,\gamma,\omega,0+}(y(x) - y(0)), \tag{16}$$

$${}^{c}D^{\alpha}_{\rho,\gamma,\omega,b-}\, y(x) = D^{\alpha}_{\rho,\gamma,\omega,b-}(y(x) - y(b)) \tag{17}$$

coinciding with Prabhakar derivatives (13) when $y(0) = 0$ or $y(b) = 0$, respectively. Restricting function space to continuous functions fulfilling condition $y(0) = 0$, we arrive at composition rules of Prabhakar operators analogous to (7), (9), and (10):

$${}^{c}D^{\alpha}_{\rho,\gamma,\omega,0+}\, E^{\alpha}_{\rho,\gamma,\omega,0+} y(x) = y(x), \tag{18}$$

$$E^{\alpha}_{\rho,\gamma,\omega,0+}\, {}^{c}D^{\alpha}_{\rho,\gamma,\omega,0+} y(x) = y(x), \tag{19}$$

$$D^{\alpha}_{\rho,\gamma,\omega,b-}\, E^{\alpha}_{\rho,\gamma,\omega,b-} y(x) = y(x), \tag{20}$$

$$E^{\alpha}_{\rho,\gamma,\omega,b-} D^{\alpha}_{\rho,\gamma,\omega,b-} y(x) = y(x). \tag{21}$$

Now, we shall quote the general formulation of the fractional eigenvalue problem, introduced and investigated in papers [7,11–14,21].

Definition 1 (compare Definition 5 in [7])**.** *Let $\alpha \in (0,1]$. With the notation*

$$\mathcal{L}_q := D^{\alpha}_{b-} p(x)\ {}^{c}D^{\alpha}_{0+} + q(x), \tag{22}$$

consider the fractional Sturm–Liouville equation (FSLE)

$$\mathcal{L}_q y_{\lambda}(x) = \lambda w(x) y_{\lambda}(x), \tag{23}$$

where $p(x) \neq 0, w(x) > 0 \quad \forall x \in [0,b]$, functions p, q, w are real-valued continuous functions in $[0,b]$ and boundary conditions are:

$$c_1 y_{\lambda}(0) + c_2 I^{1-\alpha}_{b-} p(x) D^{\alpha}_{0+} y_{\lambda}(x) \mid_{x=0} = 0, \tag{24}$$

$$d_1 y_{\lambda}(b) + d_2 I^{1-\alpha}_{b-} p(x) D^{\alpha}_{0+} y_{\lambda}(x) \mid_{x=b} = 0 \tag{25}$$

with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. The problem of finding number λ (eigenvalue) such that the BVP has a non-trivial solution, y_{λ} (eigenfunction) will be called the regular fractional Sturm–Liouville eigenvalue problem (FSLP).

We include Prabhakar derivatives into the construction of FSLO and formulate below the Prabhakar Sturm–Liouville problem.

Definition 2. *Let $\alpha \in (0,1]$. With the notation*

$$\mathcal{L}'_q := D^{\alpha}_{\rho,\gamma,\omega,b-} p(x) \ {}^{c}D^{\alpha}_{\rho,\gamma,\omega,0+} + q(x), \tag{26}$$

consider the fractional Prabhakar Sturm–Liouville equation (PSLE)

$$\mathcal{L}'_q y_\lambda(x) = \lambda w(x) y_\lambda(x), \tag{27}$$

where $p(x) \neq 0, w(x) > 0 \quad \forall x \in [0,b]$, functions p, q, w are real-valued continuous functions in $[0,b]$ and boundary conditions are:

$$c_1 y_\lambda(0) + c_2 E^{1-\alpha}_{\rho,-\gamma,\omega,b-} p(x) D^{\alpha}_{\rho,\gamma,\omega,0+} y_\lambda(x) \mid_{x=0} = 0, \tag{28}$$

$$d_1 y_\lambda(b) + d_2 E^{1-\alpha}_{\rho,-\gamma,\omega,b-} p(x) D^{\alpha}_{\rho,\gamma,\omega,0+} y_\lambda(x) \mid_{x=b} = 0 \tag{29}$$

with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. The problem of finding number λ (eigenvalue) such that the BVP has a non-trivial solution, y_λ (eigenfunction) will be called the regular fractional Prabhakar Sturm–Liouville eigenvalue problem (PSLP).

3. Formulation of the Problem and Methods

In this section, we shall focus on fractional eigenvalue problems subjected to the homogeneous Dirichlet boundary conditions. We choose values $c_2 = d_2 = 0$ in Definitions 1 and 2 and formulate the corresponding definitions of FSLP and PSLP. First, we have the fractional Sturm–Liouville problem with Dirichlet boundary conditions.

Definition 3. *Let $\alpha \in (0,1]$. With the notation*

$$\mathcal{L}_q := D^{\alpha}_{b-} p(x) \ {}^{c}D^{\alpha}_{0+} + q(x), \tag{30}$$

consider the fractional Sturm–Liouville Equation (23), where $p(x) \neq 0, w(x) > 0 \quad \forall x \in [0,b]$, functions p, q, w are real-valued continuous functions in $[0,b]$ and the boundary conditions are:

$$y_\lambda(0) = y_\lambda(b) = 0.$$

The problem of finding number λ (eigenvalue) such that the BVP has a non-trivial solution, y_λ (eigenfunction) will be called the regular fractional Sturm–Liouville eigenvalue problem (FSLP) with homogeneous Dirichlet boundary conditions.

Next, we formulate the definition of the Prabhakar Sturm–Liouville problem with Dirichlet boundary conditions.

Definition 4. *Let $\alpha \in (0,1]$. With the notation*

$$\mathcal{L}'_q := D^{\alpha}_{\rho,\gamma,\omega,b-} p(x) \ {}^{c}D^{\alpha}_{\rho,\gamma,\omega,0+} + q(x), \tag{31}$$

consider the fractional Prabhakar Sturm–Liouville Equation (27), where $p(x) \neq 0$, $w(x) > 0 \quad \forall x \in [0,b]$, functions p, q, w are real-valued continuous functions in $[0,b]$ and the boundary conditions are:

$$y_\lambda(0) = y_\lambda(b) = 0.$$

The problem of finding number λ (eigenvalue) such that the BVP has a non-trivial solution, y_λ (eigenfunction) is the regular fractional Prabhakar Sturm–Liouville eigenvalue problem (PSLP) with homogeneous Dirichlet boundary conditions.

We shall study the spectral properties of the eigenvalue problems described in the above definitions. Let us point out that an FSLP with a Dirichlet boundary condition spectrum was investigated in papers [12,21] using variational methods. Here, we extend the study to the Prabhakar Sturm–Liouville problem and develop the results by transforming both differential fractional problems into the respective equivalent integral ones. Then, we analyse properties of the integral versions of fractional Sturm–Liouville operators (22) and (26) and apply the Hilbert–Schmidt spectral theorem to prove that their spectrum is purely discrete. Equivalence of the respective differential and integral fractional eigenvalue problems yields the theorems on spectra of the differential fractional and fractional Prabhakar eigenvalue problems given by Definitions 3 and 4. We begin our considerations with the case when $q = 0$.

3.1. Equivalence Results for Differential and Integral FSLP, PSLP: Case $q = 0$

Here, we shall prove equivalence results for the FSLP/PSLP with an equation containing the fractional differential operators (22) and (26) and investigate the properties of the integral eigenvalue problem connected to the FSLE/PSLE in the case of order α fulfilling condition $1 \geq \alpha > 1/2$ and solutions' space restricted by the homogeneous Dirichlet boundary conditions.

In the first part, we transformed the differential fractional Sturm–Liouville problem (Definition 3) into the integral one on the subspace of the continuous functions defined below:

$$C_D[0,b] := \{y \in C[0,b]; \;\; y(0) = y(b) = 0\}. \tag{32}$$

Let us note that the composition rules of fractional operators (7) and (9) allow us a to write a fractional Sturm–Liouville Equation (23) on the $C_D[0,b]$ space in the case of $q = 0$ as follows:

$$\mathcal{L}_0\left(1 - \lambda I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)\right)y(x) = 0$$

which leads to the integral equation

$$\left(1 - \lambda I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)\right)y(x) = C^w_1 + C^w_2 I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}.$$

Constants C^w_1 and C^w_2 are determined by the homogeneous Dirichlet boundary conditions

$$C^w_1 = 0, \quad C^w_2 = -\lambda\frac{I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)y(x)|_{x=b}}{I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}|_{x=b}}. \tag{33}$$

The above calculations lead to the integral form of FSLE (23) with $q = 0$

$$\frac{1}{\lambda}y(x) = T_w y(x), \tag{34}$$

where linear integral operator T_w is built using the left and right Riemann–Liouville integrals and acts as follows:

$$T_w y(x) = I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)y(x) - \frac{I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)y(x)|_{x=b}}{I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}|_{x=b}} \cdot I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}. \tag{35}$$

Similar considerations yield the integral form of PSLE (27) when $q = 0$

$$\frac{1}{\lambda}y(x) = T_w y(x), \tag{36}$$

where linear integral operator T_w is constructed using the left and right Prabhakar integrals and acts as follows

$$T_w y(x) = E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{1}{p}E^{\alpha}_{\rho,\gamma,\omega,b-}w(x)y(x) \tag{37}$$

$$-\frac{E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{1}{p}E^{\alpha}_{\rho,\gamma,\omega,b-}w(x)y(x)|_{x=b}}{E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{e^{-\gamma}_{\rho,\alpha}(\omega(b-x)^{\rho})}{p(x)}|_{x=b}}\cdot E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{e^{-\gamma}_{\rho,\alpha}(\omega(b-x)^{\rho})}{p(x)}.$$

We note that the above integral operators (35) and (37) can be rewritten as operators indexed by the arbitrary continuous function r (here, $r = w$) and determined by the corresponding kernels—G^1 for FSLP and G^2 for PSLP:

$$T_r y(x) := \int_0^b G^j(x,s)r(s)y(s)dx, \quad j = 1,2, \tag{38}$$

where kernels are of the form:

$$G^1(x,s) := K_1(x,s) - \tfrac{K_1(b,x)K_1(b,s)}{K_1(b,b)}, \tag{39}$$

$$G^2(x,s) := K_1^P(x,s) - \tfrac{K_1^P(b,x)K_1^P(b,s)}{K_1^P(b,b)}, \tag{40}$$

$$K_1(x,s) = \textstyle\int_0^{\min\{x,s\}} \tfrac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\cdot\tfrac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}\tfrac{1}{p(t)}dt, \tag{41}$$

$$K_1^P(x,s) = \textstyle\int_0^{\min\{x,s\}} \tfrac{e^{-\gamma}_{\rho,\alpha}(\omega(x-t)^{\rho})e^{-\gamma}_{\rho,\alpha}(\omega(s-t)^{\rho})}{p(t)}dt. \tag{42}$$

It is easy to check the following properties of kernels. First, they are symmetric functions on square $\Delta = [0,b]\times[0,b]$

$$K_1(x,s) = K_1(s,x), \quad K_1^P(x,s) = K_1^P(s,x), \quad G^j(x,s) = G^j(s,x) \tag{43}$$

and, in addition, we have

$$K_1(0,s) = K_1(b,0) = 0, \quad K_1^P(0,s) = K_1^P(b,0) = 0, \quad G^j(0,s) = G^j(b,s) = 0. \tag{44}$$

In our results developed in this paper, we apply two types of assumptions.

Hypothesis 1 (H1). $1 \geq \alpha > 1/2$, $\frac{1}{p} \in C[0,b]$ *and function* $\frac{1}{p}$ *be positive on* $[0,b]$ *or negative.*

Hypothesis 2 (H2). $1 \geq \alpha > 1/2$, $\frac{1}{p} \in C[0,b]$ *and function* $\frac{1}{p}$ *be positive on* $[0,b]$ *or negative. In addition, let the real parameters* $\alpha,\rho,\gamma,\omega$ *fulfil the conditions:*

$$\min\{\rho,\gamma\} > 0, \quad \omega < 0, \quad \alpha \geq \rho\gamma, \quad \rho < 1.$$

Proposition 1. *If (H1) is fulfilled and function* $y \in L^2(0,b)$*, then its image* $T_r y \in C_D[0,b]$ *for any function* $r \in C[0,b]$ *and operator defined by kernel (39).*

If (H2) is fulfilled and function $y \in L^2(0,b)$*, then its image* $T_r y \in C_D[0,b]$ *for any function* $r \in C[0,b]$ *and operator defined by kernel (40).*

Proof. We sketch here the proof of the first part of the discussed proposition and omit the proof of the second one as it is analogous. By Corollary A1, kernel G^1 fulfills the Hölder condition; therefore, we find

$$|T_r y(x') - T_r y(x)| \leq \int_0^b |G^1(x',s') - G^1(x,s)|\cdot|r(s)y(s)|ds$$

$$\leq M_1|x'-x|^{\beta}\int_0^b |r(s)y(s)|ds \leq M_1\sqrt{b}\cdot||r||\cdot||y||_{L^2}\cdot|x'-x|^{\beta}$$

and we infer that image $T_r y$ is a continuous function and is even uniformly continuous on interval $[0,b]$.

We check that it obeys the homogeneous Dirichlet boundary conditions as well, because kernel G^1 fulfils the conditions (44):

$$T_r y(0) = \int_0^b G^1(0,s) y(s) r(s) ds = 0,$$

$$T_r y(b) = \int_0^b G^1(b,s) y(s) r(s) ds = 0.$$

□

For functions belonging to the $C_D[0,\ b]$ space, we can prove the equivalence of the differential and integral form of the FSLP and PSLP, respectively. That is, the following two propositions are valid when $q = 0$. The first one concerns differential and integral fractional Sturm–Liouville problems.

Proposition 2. *If (H1) is fulfilled and $w \in C[0,b]$, then the following equivalence is valid on the $C_D[0,b]$ space*

$$\mathcal{L}_0 y(x) = \lambda w(x) y(x) \Longleftrightarrow T_w y(x) = \frac{1}{\lambda} y(x), \tag{45}$$

where operator $\mathcal{L}_0$ is defined in (22) and operator T_w contains kernel (39).

Proof. Assuming that $y \in C_D[0,b]$ is an eigenfunction corresponding to eigenvalue λ:

$$\frac{1}{w(x)} \mathcal{L}_0 y(x) = \lambda y(x)$$

we act with the T_w operator on both sides of this equation:

$$T_w \frac{1}{w(x)} \mathcal{L}_0 y(x) = \lambda T_w y(x) \tag{46}$$

and by applying composition rules (10), we obtain the integral eigenvalue equation

$$\frac{1}{\lambda} y(x) = T_w y(x). \tag{47}$$

Next, we assume that function $y \in L^2(0,b)$ is an eigenfunction of the integral FSLP, i.e., Equation (47) is fulfilled. According to Proposition 1, eigenfunction y is a continuous one and belongs to the $C_D[0,b]$ space. Then, we calculate composition $\mathcal{L}_0 T_w$ using the composition rules (7) and (9)

$$\mathcal{L}_0 T_w y(x) = w(x) y(x) \tag{48}$$

and by applying Equation (47), we arrive at the implication

$$\mathcal{L}_0 T_w y(x) = w(x) y(x) = \frac{1}{\lambda} \mathcal{L}_0 y(x) \Longrightarrow \mathcal{L}_0 y(x) = \lambda w(x) y(x).$$

Therefore, we conclude that on the $C_D[0,b]$ space, the equivalence of the differential and integral FSLP is valid. □

Below, we formulate the extended version of Proposition 2, where we describe the appropriate equivalence for Prabhakar Sturm–Liouville operators. Its proof is analogous to that presented above.

Proposition 3. *If (H2) is fulfilled and $w \in C[0,b]$, then the following equivalence*

$$\mathcal{L}'_0 f(x) = \lambda w(x) f(x) \Longleftrightarrow T_w f(x) = \frac{1}{\lambda} f(x), \tag{49}$$

is valid on the $C_D[0,b]$ space, where the $\mathcal{L}_0'$ operator is defined in (26) and the T_w operator contains kernel (40).

Equivalence of the integral and differential fractional and fractional Prabhakar eigenvalue problems is an important step in deriving results on the spectrum for the problems described in Definitions 3 and 4. In the next section, we shall extend the equivalence results to the case where $q \neq 0$.

3.2. Equivalence Results for Differential and Integral FSLP, PSLP: General Case $q \neq 0$

We begin our discussion with the fractional Sturm–Liouville problem. We write Equation (23) in the following form

$$\left(\frac{1}{w}\mathcal{L}_q - \lambda\right)y(x) = 0$$

and apply composition rules for fractional operators (7) and (9)

$$\frac{1}{w}\mathcal{L}_0\left(1 + I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}q(x) - \lambda I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)\right)y(x) = 0.$$

The fractional differential Sturm–Liouville Equation (23) now takes the form of integral equation

$$y(x) + I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}q(x) + C^q_1 + C^q_2 I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}$$
$$= \lambda I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}w(x)y(x) + C^w_1 + C^w_2 I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)},$$

where constants are determined by the homogeneous Dirichlet boundary conditions; namely, C^w_1, C^w_2 are given by (33) and for C^q_1, C^q_2, we have

$$C^q_1 = 0, \quad C^q_2 = -\frac{I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}q(x)y(x)|_{x=b}}{I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}|_{x=b}}.$$

To conclude, Equation (23) is now an integral equation

$$(1 + T_q)y(x) = \lambda T_w y(x), \tag{50}$$

where the T_w operator is given in (35) and the T_q operator is given by the formula below

$$T_q y(x) = I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}q(x)y(x) - \frac{I^{\alpha}_{0+}\frac{1}{p}I^{\alpha}_{b-}q(x)y(x)|_{x=b}}{I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}|_{x=b}} \cdot I^{\alpha}_{0+}\frac{(b-x)^{\alpha-1}}{p(x)}. \tag{51}$$

Let us point out that, similar to the calculations presented in the previous part, both of the above integral operators can also be rewritten as integral operators (38) with kernel (39) for $r = w$ and $r = q$, respectively.

Our aim is to reformulate the intermediate integral Equation (50) to the form of an eigenvalue equation. We apply Theorem A1 to invert the operator on the left-hand side. First, we check the assumption of Theorem A1, particularly when condition (H1) is fulfilled and $w \in C[0,b]$. We then apply Corollary A1, denoting $K(x,s) = G^1(x,s)$, and obtain:

$$||G_w(\cdot,s))|| = \sup_{v\in[0,b]}|G_w(v,s)|$$

$$= \sup_{v\in[0,b]}|G^1(v,s)w(s)| \leq ||w|| \sup_{v\in[0,b]}|G^1(v,s)|$$

$$\leq ||w|| \sup_{v\in[0,b]} \Big(|G^1(v,s) - G^1(0,s)| + |G^1(0,s)|\Big)$$

$$\leq ||w|| \cdot M_1 \sup_{v\in[0,b]} v^{\alpha-1/2} = ||w|| \cdot M_1 \cdot b^{\alpha-1/2} < \infty.$$

Next, we write condition (A8) in the explicit form:

$$\xi = \sup_{x\in[0,b]} \int_0^b |q(s)G^1(x,s)|ds \tag{52}$$

$$= \sup_{x\in[0,b]} \int_0^b |q(s)| \cdot \left| K_1(x,s) - \frac{K_1(b,x)K_1(b,s)}{K_1(b,b)} \right| ds < 1.$$

All the above considerations lead to the proposition on convergence of the series associated with the intermediate fractional integral eigenvalue problem given in (50) and (A5). Analogous convolutions' series were also studied on the $C[a,b]$ and $L^2(a,b)$ function spaces for FSLPs with homogeneous mixed and Robin boundary conditions, respectively [13,14].

Proposition 4. *Let (H1) be fulfilled, $w, q \in C[0,b]$ and function w be positive. If condition (52) is fulfilled, then for any function $y \in L^2(0,b)$ series on the right-hand side of the formula below is uniformly convergent on interval $[0,b]$:*

$$Ty(x) := (1+T_q)^{-1} T_w y(x) = T_w y(x) + \sum_{n=1}^{\infty} (-T_q)^n T_w y(x), \tag{53}$$

where operators T_q, T_w are defined in (A6) and (A7) with $K(x,s) = G^1(x,s)$. In addition, series (A9) determining the kernel of integral operator T in (53) is uniformly convergent on square Δ and kernel G is continuous on Δ.

Proof. Let us observe that the composition of operators $T_q T_w$ is an integral operator

$$T_q T_w y(x) = \int_0^b ds \left(G_q(x,s) \int_0^b G_w(s,u) y(u) du \right)$$

$$= \int_0^b du \;\; y(u) \left(\int_0^b G_q(x,s) G_w(s,u) ds \right) = \int_0^b G_q * G_w(x,u) y(u) du,$$

where the kernel is defined by the following convolution:

$$A * B(x,u) := \int_0^b A(x,s)B(s,u)ds.$$

We shall prove that the compositions $(T_q)^n T_w$ are also defined by convolutions of kernels G_q and G_w. We start with the induction hypothesis:

$$(T_q)^n T_w y(x) = \int_0^b (G_q^{*n}) * G_w(x,u) y(u) du \tag{54}$$

and we prove that this formula is valid for the next step $n+1$ as well:

$$(T_q)^{n+1} T_w y(x) = \int_0^b (G_q^{*(n+1)}) * G_w(x,u) y(u) du.$$

We begin with the left-hand side, applying the induction hypothesis and associativity property of the convolutions of continuous functions:

$$(T_q)^{n+1} T_w y(x) = \int_0^b ds G_q(x,s)(T_q)^n T_w y(s)$$

$$= \int_0^b ds G_q(x,s) \left(\int_0^b G_q^{*n} * G_w(s,u) y(u) du \right)$$

$$= \int_0^b du \;\; y(u) \left(G_q * G_q^{*n} * G_w(x,u) \right)$$

$$= \int_0^b G_q^{*(n+1)} * G_w(x,u) y(u) du.$$

As inductive hypothesis (54) leads to the validity of the next step $n+1$; we infer that formula (54) holds for any natural number $n \geq 1$.

Now, we apply Theorem A1 and calculate kernel G for integral operator $T := (1 + T_q)^{-1} T_w$:

$$Ty(x) = T_w y(x) + \sum_{n=1}^{\infty} (-T_q)^n T_w y(x)$$

$$= \int_0^b G_w(x,s) y(s) ds + \sum_{n=1}^{\infty} (-1)^n \int_0^b G_q^{*n} * G_w(x,s) y(s) ds$$

$$= \int_0^b \left(G_w(x,s) + \sum_{n=1}^{\infty} (-1)^n G_q^{*n} * G_w(x,s) \right) y(s) ds = \int_0^b G(x,s) y(s) ds.$$

The above calculations lead to the thesis of Proposition 4; namely, operator T, defined by series (53), is correctly defined on space $L^2_w(0,b) = L^2(0,b)$ as an integral operator with a continuous kernel G:

$$Ty(x) = \int_0^b G(x,s) y(s) ds.$$

□

Having constructed operator T, we now prove the equivalence result, connecting the differential and integral fractional Sturm–Liouville problems in the general case.

Proposition 5. *If (H1) and condition (52) are fulfilled, $w, q \in C[0,b]$ and function w is positive, then the following equivalence is valid on the $C_D[0,b]$ space*

$$\mathcal{L}_q y(x) = \lambda w(x) y(x) \Longleftrightarrow Ty(x) = \frac{1}{\lambda} y(x), \tag{55}$$

where the $\mathcal{L}_q$ operator is defined in (22) and the T operator is given in (53) with a kernel determined by series (A9) with $K(x,s) = G^1(x,s)$.

Proof. We recall that for any function $y \in C_D[0,b]$, we have (proof of Proposition 2)

$$T_w \frac{1}{w(x)} \mathcal{L}_0 y(x) = y(x),$$

and we extend this equality to the analogous formula for operators T and $\mathcal{L}_q$

$$T \frac{1}{w(x)} \mathcal{L}_q y(x) = T \frac{1}{w(x)} \mathcal{L}_0 y(x) + T \frac{q(x)}{w(x)} y(x)$$

$$= \left(T_w + \sum_{n=1}^{\infty} (-T_q)^n T_w \right) \frac{1}{w(x)} \mathcal{L}_0 y(x) + T_q y(x) + \sum_{n=1}^{\infty} (-T_q)^n T_q y(x)$$

$$= y(x) + \sum_{n=1}^{\infty} (-T_q)^n y(x) + T_q y(x) + \sum_{n=1}^{\infty} (-T_q)^n T_q y(x)$$
$$= y(x),$$

where we calculate the corresponding formulas for series by using the fact that operator T is a uniformly convergent series (Proposition 4) when acting on the $C_D[0, b]$ space. For differential FSLE,

$$\frac{1}{w(x)} \mathcal{L}_q y(x) = \lambda y(x)$$

after calculating the image of the T operator of functions on both sides of FSLE

$$T \frac{1}{w(x)} \mathcal{L}_q y(x) = y(x) = \lambda T y(x),$$

we obtain the integral fractional Sturm–Liouville equation in the form of

$$T y(x) = \frac{1}{\lambda} y(x).$$

In the next step, we assume that the above integral FSLE is fulfilled. Then, function $y \in C_D[0, b]$. We apply the differential operator $\mathcal{L}_q$ to both sides of the integral FSLE

$$\mathcal{L}_q T y(x) = \frac{1}{\lambda} \mathcal{L}_q y(x).$$

For the composition of operators on the left-hand side, we get for continuous functions $f, y \in C_D[0, b]$

$$\mathcal{L}_0 T_w f(x) = w(x) f(x), \quad \mathcal{L}_0 T_q f(x) = q(x) f(x),$$
$$\mathcal{L}_0 (-T_q)^n T_w y(x) = -q(x) (-T_q)^{n-1} T_w y(x).$$

Applying Proposition 4 again, we obtain the following result for the composition of the $\mathcal{L}_q$ and T operators

$$\mathcal{L}_q T y(x) = (q(x) + \mathcal{L}_0) \left(T_w y(x) + \sum_{n=1}^{\infty} (-T_q)^n T_w y(x) \right)$$

$$= q(x) T_w y(x) + q(x) \sum_{n=1}^{\infty} (-T_q)^n T_w y(x) + w(x) y(x) - q(x) \sum_{n=1}^{\infty} (-T_q)^{n-1} T_w y(x)$$
$$= w(x) y(x).$$

From this relation, we derive the differential fractional eigenvalue equation

$$w(x) y(x) = \frac{1}{\lambda} \mathcal{L}_q y(x)$$

which leads to the differential fractional Sturm–Liouville equation:

$$\mathcal{L}_q y(x) = \lambda w(x) y(x)$$

and this ends the proof of equivalence (55). □

Now, we generalize the Sturm–Liouville operator $\mathcal{L}_q$ by introducing Prabhakar derivatives and we move on to the Prabhakar Sturm–Liouville problem (PSLP) determined in

Definitions 2 and 4 and discussed in [26] in the case when the solutions' space is restricted by the mixed homogeneous boundary conditions.

$$\left(\frac{1}{w}\mathcal{L}'_q - \lambda\right)y(x) = 0$$

We obtain the intermediate form of the integral fractional Prabhakar eigenvalue equation applying composition rules (18)–(21)

$$(1 + T_q)y(x) = T_w y(x), \tag{56}$$

where integral operator T_w is given in Formula (37) and operator T_q looks as follows:

$$T_q y(x) = E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{1}{p}E^{\alpha}_{\rho,\gamma,\omega,b-}q(x)y(x) \tag{57}$$

$$-\frac{E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{1}{p}E^{\alpha}_{\rho,\gamma,\omega,b-}q(x)y(x)|_{x=b}}{E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{e^{-\gamma}_{\rho,\alpha}(\omega(b-x)^{\rho})}{p(x)}|_{x=b}}\cdot E^{\alpha}_{\rho,\gamma,\omega,0+}\frac{e^{-\gamma}_{\rho,\alpha}(\omega(b-x)^{\rho})}{p(x)}.$$

Similar to the previous calculations for FSLP, operators (37) and (57) can be rewritten as integral operators (38), with kernel G^2 given in (40) for $r = w$ and $r = q$, respectively. Again, we apply Theorem A1 to invert operator $1 + T_q$. First, we check the assumption of Theorem A1, assuming that (H2) is fulfilled and applying Corollary A1 with $K(x,s) = G^2(x,s)$:

$$||G_w(\cdot,s))|| = \sup_{v\in[0,b]}|G_w(v,s)|$$

$$= \sup_{v\in[0,b]}|G^2(v,s)w(s)| \le ||w|| \sup_{v\in[0,b]}|G^2(v,s)|$$

$$\le ||w|| \sup_{v\in[0,b]}\left(|G^2(v,s) - G^2(0,s)| + |G^2(0,s)|\right)$$

$$\le ||w||\cdot M_2 \sup_{v\in[0,b]}|v|^{\beta} = ||w||\cdot M_2\cdot b^{\beta} < \infty.$$

Next, we write condition (A8) in the explicit form:

$$\xi = \sup_{x\in[0,b]}\int_0^b |q(s)G^2(x,s)|ds \tag{58}$$

$$= \sup_{x\in[0,b]}\int_0^b |q(s)|\cdot\left|K_1^P(x,s) - \frac{K_1^P(b,x)K_1^P(b,s)}{K_1^P(b,b)}\right|ds < 1.$$

In the proposition below, we describe the inverse operator $(1 + T_q)^{-1}$ connected to the intermediate Equation (56). We omit the proof as it is a straightforward corollary of Theorem A1, and the full proof is analogous to that of Proposition 4.

Proposition 6. *Let (H2) be fulfilled, $w, q \in C[0,b]$ and function w be positive. If condition (58) is fulfilled, then for any function $y \in L^2(0,b)$ series on the right-hand side of the formula below is uniformly convergent on interval $[0,b]$:*

$$Ty(x) := (1 + T_q)^{-1}T_w y(x) = T_w y(x) + \sum_{n=1}^{\infty}(-T_q)^n T_w y(x), \tag{59}$$

where operators T_q, T_w are defined in (A6) and (A7) with $K(x,s) = G^2(x,s)$. In addition, series (A9) determining kernel of integral operator T in (59) is uniformly convergent on square Δ and kernel G is continuous on Δ.

Similar to Proposition 5, we formulate the equivalence result for integral and differential version of eigenvalue equations corresponding to PSLP. The proof is based on the composition rules (18) and (19) and on Proposition 6, which describes inverse integral operator $(1+T_q)^{-1}$. We omit the proof as it is analogous to the proof of Proposition 5.

Proposition 7. *If (H2) and condition (58) are fulfilled, $w, q \in C[0,b]$ and function w is positive, then the following equivalence is valid on the $C_D[0,b]$ space*

$$\mathcal{L}'_q y(x) = \lambda w(x) y(x) \Longleftrightarrow Ty(x) = \frac{1}{\lambda} y(x), \tag{60}$$

where the $\mathcal{L}'_q$ operator is defined in (26), T operator is given in (59) with the kernel determined by the series (A9) and $K(x,s) = G^2(x,s)$.

4. Results on the Spectrum of Integral and Differential Fractional and Fractional Prabhakar Sturm–Liouville Problems

In the previous section, we discussed and proved the results on the equivalence of differential and integral forms of fractional eigenvalue problems. First, Propositions 2 and 3 describe the equivalence for fractional and fractional Prabhakar Sturm–Liouville problems when fractional differential operators are respectively $\mathcal{L}_0$ and $\mathcal{L}'_0$, i.e., $q = 0$. In this case, the corresponding integral operators are T_w with kernels G^1 and G^2. We prove the spectral results for these operators by applying the Hilbert–Schmidt theorem.

4.1. Case: $q = 0$

Theorem 1. *If (H1) is fulfilled and $w \in C[0,b]$ is a positive function, then the spectrum of operator T_w defined by (38) and (39) is a discrete one, enclosed in the interval $(-1,1)$, with 0 being its only limit point. Eigenfunctions belong to the $C_D[0,b]$ space and form an orthogonal basis in the $L^2_w(0,b)$ space.*

If (H2) is fulfilled and $w \in C[0,b]$ is a positive function, then the spectrum of operator T_w defined by (38) and (40) is a discrete one, enclosed in the interval $(-1,1)$, with 0 being its only limit point. Eigenfunctions belong to the $C_D[0,b]$ space and form an orthogonal basis in the $L^2_w(0,b)$ space.

Proof. Let us observe that when weight function fulfils the assumptions of the theorem, we have for functions spaces

$$L^2(0,b) = L^2_w(0,b), \quad L^2(\Delta) = L^2_{w\otimes w}(\Delta).$$

The integral Hilbert–Schmidt operator T_w, defined by kernel G^1, is a compact one, as this kernel is a function continuous on square Δ and $G^1 \in L^2_{w\otimes w}(\Delta)$.

It is also a self-adjoint operator on $L^2_w(0,b)$, because kernel G^1 is a symmetric function on square Δ, and for an arbitrary pair of functions $f, g \in L^2_w(0,b)$, we obtain:

$$\begin{aligned} \langle g, T_w f\rangle_w &= \int_0^b dx \left(w(x)\overline{g(x)} \int_0^b G^1(x,s) f(s) w(s) ds \right) \\ &= \int_0^b ds \left(w(s) f(s) \int_0^b \overline{G^1(s,x) g(x)} w(x) dx \right) \\ &= \overline{\langle f, T_w g\rangle_w} = \langle T_w g, f\rangle_w. \end{aligned}$$

The thesis is a straightforward result of the Hilbert–Schmidt spectral theorem. We omit the proof of the second part as it is analogous to the one presented above. □

The spectral theorem for integral fractional and Prabhakar Sturm–Liouville operators together with the equivalence results, included in Propositions 2 and 3, lead to the theorem on the spectrum of differential fractional eigenvalue problems subjected to the homogeneous Dirichlet boundary conditions in the case when $q = 0$.

Theorem 2. *If (H1) is fulfilled and $w \in C[0, b]$ is a positive function, then the spectrum of operator $\mathcal{L}_0$ defined by (22) and considered on the $C_D[0, b]$ space is a discrete one, and $|\lambda_n| \to \infty$. Eigenfunctions belonging to the $C_D[0, b]$ space form an orthogonal basis in the $L^2_w(0, b)$ space.*

If (H2) is fulfilled and $w \in C[0, b]$ is a positive function, then the spectrum of operator $\mathcal{L}'_0$, defined by (26) and considered on the $C_D[0, b]$ space is a discrete one and $|\lambda_n| \to \infty$. Eigenfunctions belonging to the $C_D[0, b]$ space form an orthogonal basis in the $L^2_w(0, b)$ space.

4.2. General Case $q \neq 0$

We observe that the analogous equivalence of differential and integral FSLP holds in the general case $q \neq 0$ as well. This result is given by Proposition 5. Analogously, Proposition 7 gives the equivalence relation of both versions of the fractional Prabhakar Sturm–Liouville problem. The results, included in the mentioned propositions, allow us to rewrite eigenvalue equations, replacing the differential FSLO and PSLO with the corresponding integral operators T. These operators, first determined as operator series with convergence described in Propositions 4 and 6, are in fact integral Hilbert–Schmidt operators. Their kernels—sums of a uniformly convergent series of convolutions—are continuous functions on square Δ. The theorem below describes the spectrum of fractional integral operators T with kernel G, determined by kernels G^1 and G^2, respectively.

Theorem 3. *If (H1) and condition (52) are fulfilled, $w, q \in C[0, b]$ and w is a positive function; then the spectrum of operator T defined by (53) with kernel G given in (A9) with $K(x, s) = G^1(x, s)$ is a discrete one, enclosed in interval $(-1, 1)$, with 0 being its only limit point. Eigenfunctions belong to the $C_D[0, b]$ space and form an orthogonal basis in the $L^2_w(0, b)$ space.*

If (H2) and condition (58) are fulfilled, $w, q \in C[0, b]$ and w is a positive function, then the spectrum of operator T is defined by (59), with kernel G given in (A9) and with $K(x, s) = G^2(x, s)$ is a discrete one, enclosed in interval $(-1, 1)$, with 0 being its only limit point. Eigenfunctions belong to the $C_D[0, b]$ space and form an orthogonal basis in the $L^2_w(0, b)$ space.

Proof. Let us again observe that when the weight function fulfils assumptions of the theorem; we have for spaces considered as sets of functions

$$L^2(0, b) = L^2_w(0, b), \quad L^2(\Delta) = L^2_{w\otimes w}(\Delta).$$

Integral Hilbert–Schmidt operator T, defined by kernel G, is a compact one as this kernel is a continuous function on square Δ and $G \in L^2_{w\otimes w}(\Delta)$.

We recall (proof of Theorem 1) that on the $L^2_w(0, b)$ space, the following equality holds for the arbitrary pair of functions $f, g \in L^2_w(0, b)$:

$$\langle g, T_w f\rangle_w = \langle T_w g, f\rangle_w$$

because kernel G^1 is a symmetric function on square Δ. Next, for the composition of operators $T_q T_w$, we obtain the relation

$$\langle g, T_q T_w f\rangle_w = \langle g, T_w \frac{q}{w} T_w f\rangle_w = \langle \frac{q}{w} T_w g, T_w f\rangle_w = \langle T_q T_w g, f\rangle_w.$$

Now, we apply the mathematical induction principle to prove that such relations hold for arbitrary $n > 1$ natural. We formulate an induction hypothesis in the form of

$$\langle g, (T_q)^n T_w f\rangle_w = \langle (T_q)^n T_w g, f\rangle_w \tag{61}$$

and for step $n + 1$, we achieve

$$\langle g, (T_q)^{n+1} T_w f \rangle_w = \langle g, T_q (T_q)^n T_w f \rangle_w = \langle \frac{q}{w} T_w g, (T_q)^n T_w f \rangle_w$$

$$= \langle (T_q)^n T_w \frac{q}{w} T_w g, f \rangle_w = \langle (T_q)^{n+1} T_w g, f \rangle_w.$$

Applying the mathematical induction principle, we infer that Formula (61) is valid for all natural numbers $n \geq 1$. We use this formula in the proof of the fact that integral operator T is a self-adjoint one. Remembering that it is represented by a series, uniformly convergent on the Hilbert space (Proposition 4), we calculate the scalar product term by term

$$\langle g, Tf \rangle_w = \left\langle g, \left(T_w + \sum_{n=1}^{\infty} (-T_q)^n T_w \right) f \right\rangle_w$$

$$= \langle T_w g, f \rangle_w + \sum_{n=1}^{\infty} (-1)^n \langle (T_q)^n T_w g, f \rangle_w$$

$$= \left\langle \left(T_w + \sum_{n=1}^{\infty} (-T_q)^n T_w \right) g, f \right\rangle_w$$

$$= \langle Tg, f \rangle_w.$$

To conclude, the integral operator T with a kernel G given in (A9) with $K(x,s) = G^1(x,s)$ is a compact and self-adjoint operator on Hilbert space $L^2_w(0,b)$. Therefore, the thesis of the first part of the theorem holds by the Hilbert–Schmidt spectral theorem.

Proof of the second part for operator T, associated with the integral PSLP with homogeneous Dirichlet boundary conditions, is analogous. □

Now, we apply the above spectral theorem for integral fractional eigenvalue problems, with equivalence results enclosed in Propositions 5 and 7 to formulate a theorem on discrete spectra for differential fractional and fractional Prabhakar Sturm–Liouville problems.

Theorem 4. *If (H1) and condition (52) are fulfilled, $w, q \in C[0,b]$ and w is a positive function, then the spectrum of operator $\mathcal{L}_q$ defined by (22) and considered on the $C_D[0,b]$ space is a discrete one, and $|\lambda_n| \to \infty$. Eigenfunctions, belonging to the $C_D[0,b]$ space, form an orthogonal basis in the $L^2_w(0,b)$ space.*

If (H2) and condition (58) are fulfilled, $w, q \in C[0,b]$ and w is a positive function; then the spectrum of operator $\mathcal{L}'_q$ defined by (26) and considered on the $C_D[0,b]$ space, is a discrete one and $|\lambda_n| \to \infty$. Eigenfunctions, belonging to the $C_D[0,b]$ space, form an orthogonal basis in the $L^2_w(0,b)$ space.

5. Discussion

In this paper, we presented results on the discrete spectrum of fractional and fractional Prabhakar Sturm–Liouville problems in a case when eigenfunctions' space is subjected to the homogenous Dirichlet boundary conditions. First, we extended the idea of the fractional to the fractional Prabhakar eigenvalue problem, where the Sturm–Liouville operator was constructed by using the left and right Prabhakar derivatives.

Prabhakar derivatives, with respect to time, were recently applied in anomalous diffusion models [27,28]. The derived spectral results for regular PSLP with Dirichlet boundary conditions will be used in developing equations with fractional partial derivatives with respect to the space–variable.

It appears that the method of converting the differential eigenvalue problem into the equivalent integral one can be applied to both types of Sturm–Liouville operator. This approach, developed in [13,14] for fractional eigenvalue problems subject to the homogeneous mixed and Robin boundary conditions, is extended to the case of FSLP with Dirichlet boundary conditions and generalized to PSLP with the same type of conditions.

Let us point out that the spectrum and eigenfunctions of fractional eigenvalue problems with Dirichlet boundary conditions were also studied in [11,16] by applying variational methods. The first of these papers describes the spectrum of FSLP for a fractional order in the range $(1/2, 1]$, and the spectral result was extended to range $(0, 1/2]$ in [16]. Comparing both of the methods—the variational one and the transformation into integral FSLP/PSLP—we observe that in the case of Dirichlet boundary conditions, the range of order is wider in the variational method. Nevertheless, the approach proposed here has an advantage of providing the spectral results for regular PSLP as well. Simultaneously, we obtain eigenfunctions' systems for both types of eigenvalue problems, which provide orthogonal bases in the corresponding Hilbert spaces. Such bases are a meaningful tool in applications in constructing and solving partial differential fractional equations, for example, space-fractional diffusion equations in the finite domain, as well as fractional equations governing control systems (compare references and examples in [29]).

6. Conclusions

The results developed in this paper describe the spectrum and eigenfunctions properties for FSLP and PSLP subjected to homogeneous Dirichlet boundary conditions. It seems that the conversion method can also be easily applied to other Prabhakar Sturm–Liouville problems; in particular, we shall construct the corresponding mixed, Robin, and Neumann boundary conditions and develop the equivalence results. Then, we will construct the integral PSLO with kernels analogous to those from the papers [13,14] and study the spectral properties, both for the integral and differential PSLPs.

Regarding the extension of the range of fractional order for the conversion method, we observe that so far we proved equivalence results on the space of continuous solutions. This restriction is connected to the version of Hölder condition for kernels, as discussed in Lemma A1 and Corollary A1. Thus, the aim of our future work will be to weaken this condition and to extend the range of fractional order.

Further, our investigations will include numerical simulations in order to derive approximate values of eigenvalues and eigenfunctions. As was shown in the papers [13,14], the integral form of the fractional Sturm–Liouville eigenvalue equation is particularly useful as a first step of the numerical method of solving FSLP. Thus, our aim will be to discretize integral eigenvalue problems and apply the equivalence results, enclosed in Propositions 2 and 3 for the case $q = 0$, and in Propositions 5 and 7, when $q \neq 0$. In this way, we shall arrive at numerical solutions of differential FSLP and PSLP with Dirichlet boundary conditions.

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Appendix A

Appendix A.1

Let us point out that the three-parameter Mittag–Leffler function (11), which appears in the definition of the Prabhakar function (12), is a completely monotone function [30], and this property leads to the following two inequalities. First, when parameters $\alpha, \rho, \gamma, \omega$ are real and obey conditions

$$\alpha \in (0, 1], \quad \min\{\rho, \gamma\} > 0, \quad \omega < 0, \quad \alpha \geq \rho\gamma, \quad \rho \leq 1,$$

the three-parameter Mittag–Leffler function is bounded on any interval $[0, b]$ (M_e is a constant)

$$|E^{\gamma}_{\rho,\alpha}(\omega x^{\rho})| \leq M_e$$

and it fulfills the Lipschitz condition on this interval (M_L is a constant)

$$|E^{\gamma}_{\rho,\alpha}(\omega (x')^{\rho}) - E^{\gamma}_{\rho,\alpha}(\omega x^{\rho})| \leq M_L |(x')^{\rho} - x^{\rho}|.$$

In addition, we remember that the power function obeys the Hölder condition on interval $[0, 1]$ when $\rho \leq 1$ (M_ρ is a constant)

$$|(y')^{\rho} - y^{\rho}| \leq M_\rho |y' - y|^{\rho}, \quad y', y \in [0, 1].$$

All the above inequalities will be applied to derive properties of a fractional integral operator associated to the differential Prabhakar Sturm–Liouville operator (PSLO). In particular, they are important in the study of Hölder continuity and the continuity of kernels of integral versions of Prabhakar Sturm–Liouville operators. The lemma below summarizes the Hölder continuity properties of kernels K_1, K_1^P and was proven in [26] (compare Properties 3.2 and 3.3).

Lemma A1. *If (H1) is fulfilled, then kernel K_1, given by (41), obeys the Hölder-type condition, i.e., there exists coefficient $\beta \in (0, 1]$ and function $m \in L^2(0, b)$ such that*

$$|K_1(x', s) - K_1(x, s)| \leq m(s)|x' - x|^{\beta}, \tag{A1}$$

where $\beta = \alpha - 1/2$ and

$$m(s) = \frac{2b^{\alpha-1/2}||1/p||}{(\Gamma(\alpha))^2(\alpha - 1/2)}$$

is a constant function.

If (H2) is fulfilled, then kernel K_1^P, given by (42), obeys the Hölder-type condition, i.e., there exists coefficient $\beta \in (0, 1]$ and function $m \in L^2(0, b)$ such that

$$|K_1^P(x', s) - K_1^P(x, s)| \leq m(s)|x' - x|^{\beta}, \tag{A2}$$

where $\beta = \min\{\alpha - 1/2, \rho\}$ and

$$m(s) = \max\{b^{\alpha-1/2}, b^{2\alpha-1-\rho}\} \cdot \frac{||1/p|| \cdot M_e}{\alpha - 1/2} \cdot (2M_e + M_L M_\rho b^{\rho})$$

is a constant function.

Analyzing the construction of kernels G^1 and G^2, we obtain the following corollary.

Corollary A1. *If (H1) is fulfilled, then kernel G^1, defined by Formulas (39) and (41), obeys the Hölder-type condition, i.e., there exists coefficient $\beta \in (0, 1]$ and constant M_1 such that*

$$|G^1(x', s) - G^1(x, s)| \leq M_1 |x' - x|^{\beta}, \tag{A3}$$

where $\beta = \alpha - 1/2$ and

$$M_1 = \frac{2b^{\alpha-1/2}||1/p||(1 + ||1/p|| \cdot ||p||)}{(\Gamma(\alpha))^2(\alpha - 1/2)}.$$

If (H2) is fulfilled, then kernel G^2, defined by Formulas (40) and (42), obeys the Hölder-type condition, i.e., there exists coefficient $\beta \in (0, 1]$ and constant M_2 such that

$$|G^2(x', s) - G^2(x, s)| \leq M_2 |x' - x|^{\beta}, \tag{A4}$$

where $\beta = \min\{\alpha - 1/2, \rho\}$ *and*

$$M_2 = \max\{b^{\alpha-1/2}, b^{2\alpha-1-\rho}\} \cdot \frac{||1/p||(1+||1/p||\cdot||p||)\cdot M_e}{\alpha - 1/2} \cdot (2M_e + M_L M_\rho b^\rho).$$

Proof. We prove the Hölder-type condition for kernel G^1 by applying Lemma A1 and the symmetry property of kernel K_1 given in (43). We begin by estimating values $K_1(b,b)$ and $K_1(b,s)$:

$$|K_1(b,b)| = \left|\int_a^b \frac{(b-t)^{2\alpha-2}}{(\Gamma(\alpha))^2 p(t)}dt\right| = \int_a^b \frac{(b-t)^{2\alpha-2}}{(\Gamma(\alpha))^2|p(t)|}dt$$
$$\geq \frac{(b-a)^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)||p||},$$
$$|K_1(b,s)| = \left|\int_a^s \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{p(t)}dt\right|$$
$$\leq \left|\left|\frac{1}{p}\right|\right| \cdot \frac{(b-a)^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}.$$

Now, we apply the derived inequalities and condition (A1)

$$|G^1(x',s) - G^1(x,s)| =$$
$$= \left|K_1(x',s) - \frac{K_1(b,x')K_1(b,s)}{K_1(b,b)} - K_1(x,s) + \frac{K_1(b,x)K_1(b,s)}{K_1(b,b)}\right|$$
$$\leq |K_1(x',s) - K_1(x,s)| + |K_1(x',b) - K_1(x,b)| \cdot \left|\frac{K_1(b,s)}{K_1(b,b)}\right|$$
$$\leq |x'-x|^\beta m(s)\left(1 + \left|\frac{K_1(b,s)}{K_1(b,b)}\right|\right)$$
$$\leq m(s)(1+||1/p||\cdot||p||)|x'-x|^\beta$$
$$= M_1|x'-x|^\beta,$$

where

$$M_1 = \frac{2b^{\alpha-1/2}||1/p||(1+||1/p||\cdot||p||)}{(\Gamma(\alpha))^2(\alpha-1/2)}.$$

The proof of the Hölder condition for kernel G^2 is analogous. □

The next corollary results from the Hölder conditions (A3) and (A4) and symmetry properties of kernels G^1, G^2 given in (43) and yields continuity of both kernels on square $\Delta = [0,b] \times [0,b]$.

Corollary A2. *If (H1) is fulfilled, then kernel* G^1*, defined by Formulas (39) and (41), is continuous on square* $\Delta = [0,b] \times [0,b]$.

If (H2) is fulfilled, then kernel G^2*, defined by Formulas (40) and (42) is continuous on square* $\Delta = [0,b] \times [0,b]$.

Proof. Let us note that the symmetry of kernel G^1 allows us to write condition (A3) in the following form

$$|G^1(x',s') - G^1(x,s)| \leq M_1\left(|x'-x|^\beta + |s'-s|^\beta\right).$$

To prove continuity of the kernel, we apply the Cauchy definition of continuous function, i.e., we take arbitrary $\epsilon > 0$ and assume that the distance between points (x',s'), (x,s) is smaller than $\delta(\epsilon) = \left(\frac{\epsilon}{2M_1}\right)^{1/\beta}$, which means

$$|(x', s') - (x, s)| = \sqrt{(x' - x)^2 + (s' - s)^2} < \delta(\epsilon).$$

We observe that the following inequalities are then valid

$$|x' - x| < \delta(\epsilon), \quad |s' - s| < \delta(\epsilon).$$

Now, we check the distance between the values of function G^1:

$$|G^1(x', s') - G^1(x, s)| \le M_1 \left(|x' - x|^\beta + |s' - s|^\beta \right)$$
$$\le M_1 \cdot 2(\delta(\epsilon))^\beta = \epsilon.$$

We see that for arbitrary $\epsilon > 0$, bound $\delta(\epsilon)$ for the distance of points exists, such that the implication below is valid

$$|(x', s') - (x, s)| < \delta(\epsilon) \Longrightarrow |G^1(x', s') - G^1(x, s)| < \epsilon.$$

Thus, kernel G^1 is a continuous function on square Δ by the Cauchy definition of continuity.

Proof for kernel G^2 is analogous. □

Appendix A.2

We shall study properties of integral equations of the form:

$$(1 + T_q)y(x) = \lambda T_w y(x) \tag{A5}$$

determined on the $L^2_w(a, b)$ function space. Such an equation is the intermediate stage of transformation of the fractional differential eigenvalue problems into the equivalent integral ones (see examples in papers [13,14]). In cases where the integral operator on the left-hand side of (A5) is invertible, we can convert fractional differential Sturm–Liouville operator into an integral one. Then, we can study spectral properties of the integral operator and derive results for the spectrum and eigenfunctions of the fractional differential Sturm–Liouville problems connected to various homogeneous boundary conditions.

Operators T_q and T_w are integral ones, with kernels given in the form of

$$T_q y(x) := \int_a^b G_q(x, s) y(s) ds, \quad G_q(x, s) = K(x, s) q(s), \tag{A6}$$
$$T_w y(x) := \int_a^b G_w(x, s) y(s) ds, \quad G_w(x, s) = K(x, s) w(s). \tag{A7}$$

We formulate below a theorem which we shall apply to analyse integral eigenvalue problems associated with the fractional differential ones.

Theorem A1. *Let function $q \in C[a, b]$ and function $||G_w(\cdot, s)|| := \sup_{v \in [a,b]} |G_w(v, s)|$ be bounded on interval $[a, b]$. If condition*

$$\xi := \sup_{x \in [a,b]} \int_a^b |G_q(x, v)| dv < 1 \tag{A8}$$

is fulfilled, then the series

$$G(x, s) := G_w(x, s) + \sum_{n=1}^{\infty} (-1)^n G_q^{*n} * G_w(x, s) \tag{A9}$$

is uniformly convergent on square Δ; i.e., the sum of this series G is determined for all points $(x, s) \in \Delta$.

If, in addition, kernels $G_q, G_w \in C(\Delta)$, then sum $G \in C(\Delta)$.

Proof. We shall apply the mathematical induction principle to estimate all terms of series (A9). First, we estimate the absolute value of the first convolution term:

$$|G_q * G_w(x,s)| = \left| \int_a^b G_q(x,v) G_w(v,s) dv \right| \tag{A10}$$

$$\leq \int_a^b |G_q(x,v) G_w(v,s)| dv \leq ||G_w(\cdot,s)|| \cdot \sup_{x\in[a,b]} \int_a^b |G_q(x,v)| dv = \xi \cdot ||G_w(\cdot,s)||$$

and for the second term, we obtain

$$|G_q * G_q * G_w(x,s)| \leq \xi \sup_{v\in[a,b]} |G_q * G_w(v,s)| \tag{A11}$$

$$\leq \xi^2 \cdot ||G_w(\cdot,s)||.$$

Now, we formulate the induction hypothesis (here, $n > 2$ is a natural number):

$$|G_q^{*n} * G_w(x,s)| \leq \xi^n \cdot ||G_w(\cdot,s)|| \tag{A12}$$

and we shall prove that it holds for the next step $n+1$

$$|G_q^{*(n+1)} * G_w(x,s)| \leq \xi^{n+1} \cdot ||G_w(\cdot,s)||.$$

We begin from the left-hand side of the above inequality and we find

$$|G_q^{*(n+1)} * G_w(x,s)| = |G_q * \left(G_q^{*n} * G_w\right)(x,s)|$$

$$\leq \xi \sup_{v\in[a,b]} |\left(G_q^{*n} * G_w\right)(v,s)|$$

$$\leq \xi^{n+1} \sup_{v\in[a,b]} |G_w(v,s)| \leq \xi^{n+1} \cdot ||G_w(\cdot,s)||.$$

The induction hypothesis (A12) implies the validity of the next step for $n+1$; therefore, we infer that estimation (A12) is valid for all terms indexed by $n \geq 1$. Now, we are ready to consider the convergence of the function series (A9) by using the Weierstrass convergence test and inequality (A12). We observe that the majorant number series (a geometric one) is absolutely convergent under the assumption (A8). Thence, the function series (A9) is absolutely and uniformly convergent, as we achieve for any point $(x,s) \in \Delta$

$$\left| G_w(x,s) + \sum_{n=1}^{\infty} (-1)^n G_q^{*n} * G_w(x,s) \right|$$

$$\leq |G_w(x,s)| + \sum_{n=1}^{\infty} \xi^n ||G_w(\cdot,s)|| = |G_w(x,s)| + \frac{||G_w(\cdot,s)|| \cdot \xi}{1-\xi}.$$

In the second part of Theorem 2, we note that continuity of kernels G_q, G_w implies that all terms of the series (A9) are continuous as convolutions of continuous functions. The absolutely and uniformly convergent series (A9) leads to sum G, which is also continuous on Δ. □

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