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Some Remarks about Exponential and Semi-Exponential Operators

Monika Herzog

Department of Applied Mathematics, Faculty of Computer Science and Telecommunications, Cracow University of Technology, 31-155 Cracow, Poland; mherzog@pk.edu.pl

Abstract: In this paper, we study the approximation properties of some exponential and semi-exponential operators. We focus on modifications of these operators in King's sense, examining the rate of convergence for basic and modified operators. The presented line of reasoning emphasizes some symmetry in the modifications of exponential and semi-exponential operators.

Keywords: exponential operators; exponential weighted spaces; modulus of continuity; Laplace transform

MSC: 41A25; 41A35; 41A36; 44A10

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1. Introduction

This paper is a continuation of the previous article [1] devoted to a new class of linear positive operators, the so-called semi-exponential operators.

The idea came mainly from [2,3]. Inspired by the paper [4], Tyliba and Wachnicki proposed a new class of approximation operators W_λ. The main difference between the new semi-exponential operators and the exponential ones is that the semi-exponential operators do not preserve linear functions.

We focus now on examining the rate of convergence for semi-exponential operators comparing it to that of basic exponential operators. As stated in Theorem 2, if we consider the classical modulus of continuity, the rate of convergence of semi-exponential operators is worse than the rate of exponential ones. However, there is a way to improve this result. Motivated by an increasing interest in the study of operators that preserve different test functions in order to obtain a better error of approximation (see, for example, [5–15]), we propose new modifications of semi-exponential operators. First, following King [2], we construct a sequence of operators that preserve e_0 and e_2. In this case, the obtained estimation is not quite satisfactory and rather far from the expected one. Next, we assume that our sequence of operators preserves the test functions e_0 and e^{2at} for fixed $a > 0$. In this case, we obtain the expected estimation for semi-exponential operators.

At the beginning of this article, we recall some basic definitions and theorems connected to exponential and semi-exponential operators. Later on, we show the construction of new operators and their basic properties. In the last two sections, we compare the rate of convergence for exponential- and semi-exponential-type operators. In the Conclusions, we bring up the main differences between the initial operators and their modifications.

2. Preliminaries

Let $C(a,b)$ be the space of real-valued continuous functions on (a,b), where $-\infty \leq a < b \leq +\infty$. Moreover, let $C_B(a,b)$ denote the Banach space of all continuous bounded functions on (a,b) endowed with the sup-norm $\|\cdot\|_\infty$.

We shall also consider the space:

$$C_*(a,b) = \{f \in C(a,b) \,|\, \lim_{x\to g} f(x) = G\},$$

where $g \in \{a, b\}$ and $G \in \mathbb{R}$. For any positive integer q:

$$v_q(x) = e^{-q|x|} \quad \text{for} \quad x \in (a, b)$$

is an exponential weight function. The space:

$$E_q = \{f \in C(a, b) \mid v_q f \in C_B(a, b)\}$$

is a Banach space with the norm:

$$\|f\|_q = \|v_q f\|_\infty \quad \text{for} \quad f \in E_q.$$

Moreover, for every $q \geq 1$, we have:

$$C_*(a, b) \subset C_B(a, b) \subset E_q \subset E_{q+1}$$

and:

$$\| \cdot \|_q \leq \| \cdot \|_\infty \quad \text{on} \quad C_B(a, b),$$
$$\| \cdot \|_{q+1} \leq \| \cdot \|_q \quad \text{on} \quad E_q.$$

Throughout this article, we denote by e_i for $i \in \mathbb{N}$ the monomials $e_i(t) = t^i$.

Now, we shall recall some basic results from the paper of Ismail and May [4], as well as from the paper of Tyliba and Wachnicki [3]. In [4], the authors proposed some generalizations of integral operators as follows:

$$S_\lambda(f; t) = \int_a^b W(\lambda, t, u) f(u) du, \tag{1}$$

with the normalization condition for the kernel $W(\lambda, t, u)$:

$$\int_a^b W(\lambda, t, u) du = 1 \tag{2}$$

which is supposed to satisfy the subsequent homogenous partial differential equation with the operator $D = \frac{\partial}{\partial t}$:

$$D(W(\lambda, t, u)) = \frac{\lambda(u - t)}{p(t)} W(\lambda, t, u) \tag{3}$$

for $\lambda > 0$ and p an analytic and positive function on (a, b).

For example, the Gauss–Weierstrass operator with $p(t) \equiv 1$, the Szász–Mirakjan operator with $p(t) = t$, the Bernstein polynomial operator with $p(t) = t(1 - t)$, the Baskakov operator with $p(t) = t(1 + t)$, the Post–Widder operator with $p(t) = t^2$, and many others are exponential-type operators. Currently, scientists are still attracted to investigating the approximation properties of these operators, for example, in the [16], Gupta examined these operators for $p(t) = t^3$.

In [4], the authors proved that for each function $p = p(t)$, fulfilling the assumptions mentioned above, the conditions (2) and (3) determine the approximation operator S_λ uniquely. They presented these operators as the two-sided Laplace transform:

$$S_\lambda(f; t) = \int_{-\infty}^{\infty} C(\lambda, u) \exp\left(\lambda \int_c^t \frac{u - \theta}{p(\theta)} d\theta\right) f(u)\, du \tag{4}$$

for some $c \in (a, b)$ and proved that there is at most one generalized function $C(\lambda, u)$ satisfying (2). The explicit form of S_λ is not obvious at all, but in [17], Ismail showed that the generalized function $C(\lambda, u)$ is a sum of delta functions:

$$C(\lambda, u) = \sum_{k=0}^{\infty} \phi_k(\lambda) \delta(k - \lambda u), \tag{5}$$

provided that $1/p(z)$ has a simple pole at $z = 0$. The sequence of polynomials $(\phi_k(\lambda))_{k=0}^{\infty}$ in (5) stands for a basic set of the binomial-type. In [3], the authors introduced some new operators, which were obtained after some minor changes of the left-hand side of (3), meaning

$$(D+\beta)K(\lambda,t,u) = \frac{\lambda(u-t)}{p(t)}K(\lambda,t,u) \tag{6}$$

for $\beta > 0$ and with the same normalization condition:

$$\int_a^b K(\lambda,t,u)du = 1$$

as well as the previous requirements concerning λ, a, b, D, and $p = p(t)$.

We call the operators:

$$W_\lambda(f;t) = \int_a^b K(\lambda,t,u)f(u)du \tag{7}$$

semi-exponential operators with the kernel $K(\lambda,t,u)$. Similarly as in (4), we can present them using the two-sided Laplace transform again:

$$W_\lambda(f;t) = \int_{-\infty}^{\infty} C(\lambda,u)\exp\left(\lambda\int_c^t \frac{u-\theta}{p(\theta)}d\theta - \beta t\right)f(u)\,du.$$

In [3], the authors provided us with two examples of semi-exponential operators. For $p(t) \equiv 1$, they obtained the Gauss–Weierstrass semi-exponential operator:

$$W_\lambda(f;t) = \sqrt{\frac{\lambda}{2\pi}}\int_{\mathbb{R}}\exp\left(-\frac{\lambda(u-t-\frac{\beta}{\lambda})^2}{2}\right)f(u)du.$$

For $p(t) = t$, the semi-exponential Szász–Mirakjan operator is achieved:

$$W_\lambda(f;t) = e^{-(\beta+\lambda)t}\sum_{k=0}^{\infty}\frac{(\beta+\lambda)^k t^k}{k!}f\left(\frac{k}{\lambda}\right). \tag{8}$$

Building upon these ideas for $p(t) = t^2$, we obtain in [1] the semi-exponential Post–Widder operator:

$$W_\lambda(f;t) = \frac{\lambda}{t^\lambda\exp(\beta t)}\int_0^{+\infty}\frac{(\frac{\lambda u}{\beta})^{\frac{\lambda-1}{2}}I_{\lambda-1}(2\sqrt{\lambda\beta u})}{\exp(\frac{\lambda u}{t})}f(u)du.$$

As a consequence of the minor changes in Equation (3), the operators W_λ do not preserve linear functions. This follows immediately from Lemma 2 below.

Lemma 1 ([4])**.** *Let $S_\lambda(\cdot,t)$ be an exponential operator. Then, for $t \in (a,b)$, we have:*

(a) $S_\lambda(e_0;t) = 1$,
(b) $S_\lambda(e_1;t) = e_1(t)$,
(c) $S_\lambda(e_2;t) = e_2(t) + \frac{p(t)}{\lambda}$.

In [3], for the semi-exponential operators, the following is obtained:

Lemma 2. *Let $W_\lambda(\cdot,t)$ be a semi-exponential operator. Then, for $t \in (a,b)$, we have:*

(a) $W_\lambda(e_0;t) = e_0(t)$;
(b) $W_\lambda(e_1;t) = e_1(t) + \frac{\beta p(t)}{\lambda}$;
(c) $W_\lambda(e_2;t) = e_2(t) + \frac{p(t)+2\beta t p(t)}{\lambda} + \frac{\beta p(t)p'(t)+\beta^2 p^2(t)}{\lambda^2}$.

Using Lemma 2.3 from [3] and the previous lemma, we have the following:

Corollary 1. *Let $W_\lambda(\cdot, t)$ be a semi-exponential operator. Then, for $t \in (a, b)$, we have:*

(a) $W_\lambda(\psi_t; t) = \frac{\beta p(t)}{\lambda}$;

(b) $W_\lambda(\psi_t^2; t) = \frac{p(t)}{\lambda} + \frac{\beta p(t) p'(t)}{\lambda^2} + \frac{\beta^2 p^2(t)}{\lambda^2}$;

where $\psi_t(x) = x - t$ for $x \in (a, b)$.

In both cases, there are approximation theorems for functions from exponential weighted spaces E_q available; see [3,4].

3. Some Modifications of the Semi-Exponential Operator

At the beginning of this section, we recall some basic constructions connected to King-type operators. By Definition (1), for $\lambda = n \in \mathbb{N}$, we have the operator:

$$S_n(f; t) = \int_a^b W(n, t, u) f(u) du.$$

Let $(r_n(t))$ be a sequence of real-valued continuous functions defined on (a, b), such that $p(r_n(t)) > 0$. According to King's idea, we consider the following operators:

$$S_n^*(f; t) = S_n(f; r_n(t)). \tag{9}$$

and we assume that they fulfil the condition:

$$S_n^*(e_2; t) = t^2. \tag{10}$$

This implies that S_n^* preserve the test functions e_0 and e_2. By Lemma 1, the assumption (10) is equivalent to the following condition:

$$(r_n(t))^2 + \frac{p(r_n(t))}{n} - t^2 = 0 \tag{11}$$

for $t \in (a, b)$ and $n \in \mathbb{N}$. It is obvious that we cannot deduce the form of the sequence $r_n(t)$ if we do not know an explicit formula for $p(t)$. In [13], the authors determined the sequence $(r_n(t))$ for $p(t) = t$ and $t \geq 0$. In this case, we can write (11) as:

$$(r_n(t))^2 + \frac{r_n(t)}{n} - t^2 = 0,$$

so that:

$$r_n(t) = -\frac{1}{2n} + \frac{\sqrt{1 + 4n^2 t^2}}{2n} \tag{12}$$

for $t \geq 0$, $n \in \mathbb{N}$. Thus, we obtain the modified Szász–Mirakjan operator as follows:

$$S_n^*(f; t) = \exp \frac{1 - \sqrt{1 + 4n^2 t^2}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{1 + 4n^2 t^2} - 1)^k}{2^k k!} f\left(\frac{k}{n}\right). \tag{13}$$

These operators, as well as the initial Szász–Mirakjan operators represent the approximation process for functions defined on unbounded intervals. Moreover, the modification of the Szász–Mirakjan operators gives an estimation with a lower error. For example, in [13] (see also [9]), the authors proved the following:

Theorem 1. *For every function $f \in C_B[0, \infty)$, $t \geq 0$, and $n \in \mathbb{N}$, we have:*

$$|S_n^*(f; t) - f(t)| \leq 2\omega(f, \delta_t),$$

where $\delta_t = \sqrt{S_n^*(\psi_t^2;t)} = \sqrt{2t^2 + \frac{t}{n} - \frac{t\sqrt{4n^2t^2+1}}{n}}$ *and* $\omega(f,\delta)$ *denotes the classical modulus of continuity.*

For the basic Szász–Mirakjan operators, the estimation is:

$$|S_n(f;t) - f(t)| \leq 2\omega(f,\alpha_t),$$

where $\alpha_t = \sqrt{S_n(\psi_t^2;t)} = \sqrt{\frac{t}{n}}$, and because $\sqrt{4n^2t^2+1} \geq 2nt$ for $t \geq 0$ and $n \in \mathbb{N}$, we have $\delta_t \leq \alpha_t$ for $t \geq 0$. Hence, the approximation error for the operators S_n^* is at least as good as for the operators S_n.

These facts are the reason for considering the semi-exponential operators and exploring them from the point of view above.

Now, we introduce the new operators $W_n^*(f;t) = W_n(f;z_n(t))$ fulfilling the following relations:

$$W_n^*(e_0;t) = e_0(t), \quad W_n^*(e_2;t) = e_2(t) \tag{14}$$

for $n \in \mathbb{N}$ and $t \in (a,b)$.

We consider a sequence (W_λ) of semi-exponential operators defined by (7), which for $\lambda = n \in \mathbb{N}$ have the following form:

$$W_n(f;t) = \int_a^b K(n,t,u)f(u)du \tag{15}$$

with the kernel $K(n,t,u)$. By Lemma 2 and the relations (14), we have:

$$(z_n(t))^2 + \frac{p(z_n(t)) + 2\beta z_n(t)p(z_n(t))}{n} + \frac{\beta p(z_n(t))p'(z_n(t)) + (\beta)^2 p^2(z_n(t))}{n^2} - t^2 = 0$$

for $t \in (a,b)$ and $n \in \mathbb{N}$. We cannot express $z_n(t)$ in explicit form again, but for the semi-exponential Szász–Mirakjan operator $p(t) = t$ for $t > 0$, we obtain:

$$z_n(t) = -\frac{1}{2(n+\beta)} + \frac{\sqrt{1+4n^2t^2}}{2(n+\beta)}.$$

In this case, we achieve the modified semi-exponential operators as follows:

$$W_n^*(f;t) = W_n(f;z_n(t)) = S_n^*(f;t) \tag{16}$$

for $t \geq 0$ and $n \in \mathbb{N}$. For the semi-exponential Szász–Mirakjan operator W_n, we can estimate a quantitative error of approximation as follows:

Theorem 2. *For every function* $f \in C_B[0,\infty)$*,* $t \geq 0$*, and* $n \in \mathbb{N}$*, we have:*

$$|W_n(f;t) - f(t)| \leq 2\omega(f,\beta_t),$$

where $\beta_t = \sqrt{W_n(\psi_t^2;t)}$*.*

It is easy to conclude by Corollary 1 that $\beta_t = \sqrt{\frac{t}{n} + \frac{\beta t(1+\beta t)}{n^2}}$. Moreover, in the case of the modified semi-exponential Szász–Mirakjan operator W_n^*, which we obtain from (15) and Theorem 1, we have:

$$|W_n^*(f;t) - f(t)| = |S_n^*(f;t) - f(t)| \leq 2\omega(f,\delta_t)$$

for $t \geq 0$ and $n \in \mathbb{N}$. Now, we conclude that $\beta_t \geq \alpha_t \geq \delta_t$ for $t \geq 0$; hence, the semi-exponential Szász–Mirakjan operator W_n has a worse estimation error than the basic Szász–Mirakjan operator S_n, but on the other hand, the modified semi-exponential Szász–Mirakjan operator W_n^* has at least as good an approximation error as the classical one.

4. Quantitative Results in $C_*[0,\infty)$

Now, we focus our attention on the convergence of exponential operators and their modifications for function $f \in C_*[0,\infty)$. In many papers, some modifications of exponential operators are considered, which preserve e_0 and e^{2ax} for fixed $a > 0$. For example, in [18], the authors concentrated on this modification for the basic Szász–Mirakjan operators.

We recall some quantitative results from the paper [18], which are useful from our point of view, but in that paper, there are also many interesting theorems regarding the approximation of functions by linear positive operators in a general sense. Some theorems come from [19–23], and they have an interesting application in the paper [18]. To estimate the rate of convergence, we recall from [20] the definition of the modulus of continuity for $f \in C_*[0,\infty)$ and $\delta \geq 0$:

$$\omega^*(f;\delta) = \sup\{|f(t) - f(x)| : x > 0, t > 0, |e^{-x} - e^{-t}| \leq \delta\}.$$

Then, we have the following theorem:

Theorem 3 ([20])**.** *If $L_n : C_*(a,b) \to C_*(a,b)$ is a sequence of positive linear operators, then for $f \in C_*(a,b)$, we have:*

$$\|L_n f - f\|_\infty \leq 2\omega^*(f; \sqrt{\alpha_n + 2\beta_n + \gamma_n}),$$

where:

$$\alpha_n = \|L_n(e_0) - 1\|_\infty, \quad \beta_n = \|L_n(e^{-t}) - e^{-t}\|_\infty, \quad \gamma_n = \|L_n(e^{-2t}) - e^{-2t}\|_\infty.$$

This theorem has the following application in [18].

Theorem 4. *For $f \in C_*[0,\infty)$, we have:*

$$\|R_n^* f - f\|_\infty \leq 2\omega^*(f; \sqrt{2\beta_n + \gamma_n}),$$

where:

$$\beta_n = \|R_n^*(e^{-t}) - e^{-t}\|_\infty, \quad \gamma_n = \|R_n^*(e^{-2t}) - e^{-2t}\|_\infty.$$

Moreover, $\beta_n \to 0$, $\gamma_n \to 0$ as $n \to \infty$, so that $R_n^* f$ tends to f uniformly on $[0,\infty)$.

Let us explain that operators R_n^* are the modifications of the basic Szász–Mirakjan operators in the sense that we have mentioned above, which means:

$$R_n^*(f;t) = \exp\frac{-2at}{e^{\frac{2a}{n}} - 1} \sum_{k=0}^{\infty} \frac{(2at)^k}{\left(e^{\frac{2a}{n}} - 1\right)^k k!} f\left(\frac{k}{n}\right)$$

and:

$$R_n^*(e_0;t) = e_0(t), \quad R_n^*(e^{2ax};t) = e^{2at},$$

for $t \geq 0$ and $n \in \mathbb{N}$. The same theorem can be applied to the basic Szász–Mirakjan operators to reach the following estimation.

Theorem 5. *For $f \in C_*[0,\infty)$, we have:*

$$\|S_n f - f\|_\infty \leq 2\omega^*(f; \sqrt{2\tilde{\beta}_n + \tilde{\gamma}_n}),$$

where:

$$\tilde{\beta}_n = \|S_n(e^{-t}) - e^{-t}\|_\infty, \quad \tilde{\gamma}_n = \|S_n(e^{-2t}) - e^{-2t}\|_\infty.$$

Moreover, $\tilde{\beta}_n \to 0$*,* $\tilde{\gamma}_n \to 0$ *as* $n \to \infty$*, so that* $S_n f$ *tends to* f *uniformly on* $[0, \infty)$*.*

To express an interesting observation, we need:

Lemma 3. *For* $t \geq 0$*,* $\lambda \geq 0$*, and* $n \in \mathbb{N}$*, we have:*

$$S_n(e^{-\lambda x}; t) = e^{-tv_n},$$

where $v_n = n\left(1 - e^{\frac{-\lambda}{n}}\right)$*.*

In the case of the operators R_n^* in paper [18], the following relation is obtained:

$$R_n^*(e^{-\lambda x}; t) = e^{-tu_n},$$

where $u_n = 2a\frac{1-e^{\frac{-\lambda}{n}}}{e^{\frac{2a}{n}}-1}$.

Now, if we compare the above estimations for both operators, we arrive at a surprising result to the effect that:

$$\sqrt{2\tilde{\beta}_n + \tilde{\gamma}_n} \leq \sqrt{2\beta_n + \gamma_n}$$

because $u_n \leq v_n$ for $n \in \mathbb{N}$.

This means that for $f \in C_*[0, \infty)$, the basic Szász–Mirakjan operators give a better error of approximation in the sense of modulus ω^*.

Now, we consider the new modification of the semi-exponential Szász–Mirakjan operators W_n in the sense that is presented above, which means that we assume that the new operators $W_n^\diamond(f; t) = W_n(f; s_n(t))$ fulfill the following conditions:

$$W_n^\diamond(e_0; t) = e_0(t), \quad W_n^\diamond(e^{2ax}; t) = e^{2at}$$

for $t \geq 0$ and fixed $a > 0$. The conditions above yield the following form of $s_n(t)$:

$$s_n(t) = \frac{2at}{(\beta + n)(e^{\frac{2a}{n}} - 1)}.$$

Hence, we have $W_n^\diamond(f; t) = R_n^*(f; t)$. On the other hand, for the operators W_n, we obtain:

Lemma 4. *For* $t \geq 0$*,* $\lambda \geq 0$*, and* $n \in \mathbb{N}$*, we have:*

$$W_n(e^{-\lambda x}; t) = e^{-tp_n},$$

where $p_n = (\beta + n)\left(1 - e^{\frac{-\lambda}{n}}\right)$*.*

Now, by Theorem 3, we obtain the following estimation for the operators above.

Theorem 6. *For* $f \in C_*[0, \infty)$*, we have:*

$$\|W_n f - f\|_\infty \leq 2\omega^*(f; \sqrt{2\bar{\beta}_n + \bar{\gamma}_n}),$$

where:

$$\bar{\beta}_n = \|W_n(e^{-t}) - e^{-t}\|_\infty, \quad \bar{\gamma}_n = \|W_n(e^{-2t}) - e^{-2t}\|_\infty.$$

Moreover, $\bar{\beta}_n \to 0$*,* $\bar{\gamma}_n \to 0$ *as* $n \to \infty$*, so that* $W_n f$ *tends to* f *uniformly on* $[0, \infty)$*.*

Thus, we achieve the following inequalities:

$$\sqrt{2\bar{\beta}_n + \bar{\gamma}_n} \leq \sqrt{2\tilde{\beta}_n + \tilde{\gamma}_n} \leq \sqrt{2\beta_n + \gamma_n}$$

because $u_n \leq v_n \leq p_n$ for $n \in \mathbb{N}$. For $f \in C_*[0,\infty)$, we deduce that in the sense of the modulus of continuity ω^*, the semi-exponential Szász–Mirakjan operators W_n have an error of approximation at least as good as the basic Szász–Mirakjan operators.

5. Conclusions

This short study underlined some symmetry in the modifications of exponential and semi-exponential operators. We focused on the rate of convergence of these operators in the sense of some moduli of continuity. Often, if we consider a new modification of the basic operator, we expect a better error of approximation, which means the modified operators should be closer to our approximated function in some sense, for example, in the sense of some kind of a modulus of continuity. In Section 3, our modification for the semi-exponential Szász–Mirakjan operator was good enough for $f \in C_B[0,\infty)$, because we achieved $W_n^* = S_n^*$, and the error of approximation of W_n^* was at least as good as in the case of the classical Szász–Mirakjan operators. On the other hand, in Section 4, we explored symmetries in the behavior of the modification of Szász–Mirakjan operators introduced in [18], the basic Szász–Mirakjan operators, and the semi-exponential Szász–Mirakjan operators W_n for functions $f \in C_*[0,\infty)$. In these cases, we argued that the classical Szász–Mirakjan operators S_n give a better error of approximation than R_n^*, and later on, we showed that the semi-exponential Szász–Mirakjan operators W_n are better in the sense of the modulus of continuity ω^*.

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