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Maximal Type Elements for Families

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Abstract: In this paper, we present a variety of existence theorems for maximal type elements in a general setting. We consider multivalued maps with continuous selections and multivalued maps which are admissible with respect to Gorniewicz and our existence theory is based on the author's old and new coincidence theory. Particularly, for the second section we present presents a collectively coincidence coercive type result for different classes of maps. In the third section we consider considers majorized maps and presents a variety of new maximal element type results. Coincidence theory is motivated from real-world physical models where symmetry and asymmetry play a major role.

Keywords: continuous selections; coincidence theory; maximal type elements

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1. Introduction

Using some collectively fixed and coincidence type results of the author [1,2] and also a new general collectively coincidence result in Section 2 of this paper we present some new maximal type element theorems for families of majorized type maps [3,4]. The maps we consider are usually multivalued and either in the class of admissible maps of Gorniewicz [5] or multivalued maps which may have continuous selections (i.e., the $\Phi^\star$ maps [6]). There are a number of papers in the literature which consider collectively coincidence coercive type results for maps in the same class, usually the $\Phi^\star$ classes of maps; see [2,3,7] and the references therein. Our main result in Section 2 is Theorem 2 which considers a collectively coincidence coercive type result between two different classes of maps, namely the $\Phi^\star$ and Ad classes. One of the main difficulties encountered here is to try to set up a strategy so that one could use a coincidence result of the author [2] for the compact case. Now, Theorem 2 will immediately provide a maximal element type result in Section 3. In particular, Section 3 considers a generalization of majorized maps in the literature (see [3,8,9] and the references therein) and using new ideas and results in Section 2 we establish very general and applicable maximal element type results. Note coincidence theory arises naturally in many physical models and one can discuss symmetry and asymmetry together in this general setting. For applications and an overview we refer the reader to [3,4,8,9] and the references therein. In particular, we note that fixed or coincidence points (equilibria) occur in generalized game theory (or abstract economies) so arise naturally in the study or markets. Our theory in Sections 2 and 3 generalizes and improves corresponding results in [9,10]. Finally we note in real-world applications many problems arising in differential and integral equations and many problems arising in variational settings can be rewritten in operator form where the operators are either compact or satisfy some sort of monotonicity type assumption. These are two examples contained within the general corecive setting. For example, consider (steady-state temperature in a rod) the boundary value problem $y''(t) = -e^{y(t)}$, $t \in [0,1]$ with $y(0) = y(1) = 0$. This can be rewritten as $y(t) = \int_0^1 G(t,s)\, e^{y(s)} ds \equiv F\, y(t)$, where

$$G(t,s) = \begin{cases} t\,(1-s), & 0 \le t \le s \le 1 \\ s\,(1-t), & 0 \le s \le t \le 1. \end{cases}$$

One can consider a fixed (coincidence) point problem $y = F\,y$, where $F : X \to X$ with $X = \{u \in C[0,1] : |u|_0 = \sup_{t\in[0,1]} |u(t)| \le 1\}$ (note $\max_{t\in[0,1]} \int_0^1 |G(t,s)|ds = \frac{1}{8}$ and $\frac{1}{8}\,e^1 \le 1$) and the Arzela Ascoli theorem guarantees that $F : X \to X$ is a compact map so our theory below guarantees a fixed (coincidence) point and as a result the boundary value problem has a solution.

We now give a brief description of the main results [4,11] in the literature to date. Our paper was motivated by [11], where the authors' considered some collectively fixed point results in the compact case. Here, we replaced the compactness condition with the less restrictive coercive condition and in addition we established collectively coincidence results for different classes of maps which is a new contribution to the literature. Ding and Tan [4] discussed a particular coercive condition for a single majorized map and presented a fixed point result. In this paper, we generalized majorized maps and considered a collection of maps and presented a collection of collectively fixed point and coincidence point results. These results generate maximal element type results in a very general setting.

Now, we describe the general maps of this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. We say a space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \ge 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions hold:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). p is a perfect map i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i). p is a Vietoris map
and

(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

We are now in a position to define the admissible maps of Gorniewicz [5]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X,Y)$) provided there exists a selected pair (p,q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to Kakutani (and we write $\phi \in Kak(X,Y)$); here $K(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

The following class of maps will also be considered in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in \Phi^\star(Z,W)$ [6] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \ne \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$.

We recall that a point $x \in X$ is a maximal element of a set valued map F from a topological space X to another topological space Y if $F(x) = \emptyset$.

Our paper is arranged as follows. In Section 2, we present a collectively coincidence type result for different classes of maps. The result is then used in Section 3 to examine maximal type elements for a generalization of majorized maps in the literature and as a result we improve the corresponding results in [3,4,8–10].

In [2], the author presented collectively coincidence type results between maps in the same classes and the idea there (see [2] (Theorem 2.15)) was to generate continuous single valued selections for appropriate maps and then use a single valued map with the Brouwder fixed point theorem to conclude the existence of a coincidence. In this paper, in Section 2, we consider collectively coincidence type results between maps in different classes and the idea here is to obtain a continuous selection for an appropriate map from one class so that its composition with an appropriate map from the other class will be a multivalued map which is admissible with respect to Gorniewicz and then we can apply a fixed point theorem of the author to conclude the existence of a coincidence. In [2], the author did not see this connection for maps from different classes in the coercive case. In Section 3 (the main results in this paper), the author uses the results in [2] and the results in this paper to present a variety of new maximal element type results for generalized majorized maps.

2. Coincidence Results

Rcent fixed point and coincidence point results of the author [1,2] will generate some maximal type element results. We will present three results in Section 3 and for recent results in other classes and for other types of maps we refer the reader to [1,2,7]. In this section, we will prove a new coincidence result (which can be considered as the main result) for the $\Phi^\star$ and *Ad* classes. As mentioned in Section 1 this is a collectively coincidence coercive type result between two different classes of maps.

To establish a new coincidence result between the $\Phi^\star$ and *Ad* classes, we need to recall a recent result of the author [2].

Theorem 1 ([2])**.** *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1,...,N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$ and in addition assume there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in \{1,....N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally suppose for each $y \in Y$ there exists a $j \in \{1,...,N\}$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$, a $i_0 \in \{1,...,N\}$ with $y_j \in F_j(x)$ for all $j \in \{1,...,N_0\}$ and $x_{i_0} \in G_{i_0}(y)$.*

Theorem 2. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1,...,N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1,....N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also assume there is a compact subset K of Y; and for each $i \in \{1,..,N\}$, a convex compact subset Z_i of X_i, such that for each $y \in Y\backslash K$, there exists a $i \in \{1,..,N\}$ with $S_i(y) \cap Z_i \neq \emptyset$. Finally, suppose for each $y \in Y$, there exists a $j \in \{1,...,N\}$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$, a $i_0 \in \{1,...,N\}$ with $y_j \in F_j(x)$ for all $j \in \{1,...,N_0\}$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. We begin by noting that $C_i = \{y \in Y : S_i(y) \neq \emptyset\}, i \in \{1,..,N\}$ is an open covering of Y (recall the fibres of S_i are open) so from [12] (Lemma 5.1.6, pp301) there exists a covering $\{D_i\}_{i=1}^N$ of Y where D_i is closed in Y and $D_i \subset C_i$ for all $i \in \{1,...,N\}$. For each $i \in \{1,..,N\}$, let $M_i : Y \to X_i$ and $L_i : Y \to X_i$ be given by

$$M_i(y) = \begin{cases} G_i(y), \ y \in D_i \\ X_i, \ y \in Y\backslash D_i \end{cases} \quad \text{and} \quad L_i(y) = \begin{cases} S_i(y), \ y \in D_i \\ X_i, \ y \in Y\backslash D_i. \end{cases}$$

We begin by showing that for each $i \in \{1,...,N\}$ we have $M_i \in \Phi^\star(Y, X_i)$. Note for $i \in \{1,...,N\}$ that $L_i(y) \neq \emptyset$ for $y \in Y$, since if $y \in D_i$, then $L_i(y) = S_i(y) \neq \emptyset$ since $D_i \subset C_i$, whereas if $y \in Y\backslash D_i$, then $L_i(y) = X_i$. Further, for $y \in Y$ and $i \in \{1,...,N\}$

then, if $y \in D_i$, we have $L_i(y) = S_i(y) \subseteq G_i(y) = M_i(y)$, whereas if $y \in Y \backslash D_i$, we have $L_i(y) = X_i = M_i(y)$. Additionally, if $x \in X_i$, we have

$$\begin{aligned} L_i^{-1}(x) &= \{z \in Y : x \in L_i(z)\} = \{z \in Y \backslash D_i : x \in L_i(z) = X_i\} \cup \{z \in D_i : x \in L_i(z)\} \\ &= (Y \backslash D_i) \cup \{z \in D_i : x \in S_i(z)\} = (Y \backslash D_i) \cup [D_i \cap \{z \in Y : x \in S_i(z)\}] \\ &= (Y \backslash D_i) \cup \left[D_i \cap S_i^{-1}(x)\right] = Y \cap \left[(Y \backslash D_i) \cup S_i^{-1}(x)\right] = (Y \backslash D_i) \cup S_i^{-1}(x) \end{aligned}$$

which is open in Y (note $S_i^{-1}(x)$ is open in Y and D_i is closed in Y). Thus, for each $i \in \{1, ..., N\}$, we have $M_i \in \Phi^\star(Y, X_i)$. Let K be the set as in the statement of Theorem 2 and let $M_i^\star$ (respectively, $L_i^\star$) denote the restriction of M_i (respectively, L_i) to K. We note for $i \in \{1, ..., N\}$ that $M_i^\star \in \Phi^\star(K, X_i)$, since if $x \in X_i$, then we have

$$\begin{aligned} (L_i^\star)^{-1}(x) &= \{z \in K : x \in L_i^\star(z)\} = \{z \in K : x \in L_i(z)\} \\ &= K \cap \{z \in Y : x \in L_i(z)\} = K \cap L_i^{-1}(x) \end{aligned}$$

which is open in $K \cap Y = K$. Thus, for $i \in \{1, ..., N\}$, since $M_i^\star \in \Phi^\star(K, X_i)$, then from [6,11] there exists a single valued continuous selection $g_i : K \to X_i$ of $M_i^\star$ with $g_i(y) \in L_i^\star(y) \subseteq M_i^\star(y)$ for $y \in K$ and there exists a finite subset R_i of X_i with $g_i(K) \subseteq co\,(R_i)$. Let

$$\Omega_i = co\,(co\,(R_i) \cup Z_i) \;\; \text{for} \;\; i \in \{1, ..., N\}$$

which is a convex compact subset of X_i. Let

$$G_i^{\star\star}(y) = G_i(y) \cap \Omega_i \;\; \text{and} \;\; S_i^{\star\star}(y) = S_i(y) \cap \Omega_i \;\; \text{for} \;\; y \in Y \;\; \text{and} \;\; i \in \{1, ..., N\}.$$

Note for $i \in \{1, ..., N\}$ and $y \in Y$ that $S_i^{\star\star}(y) = S_i(y) \cap \Omega_i \subseteq G_i(y) \cap \Omega_i = G_i^{\star\star}(y)$ and also note if $x \in \Omega_i$ then

$$\begin{aligned} (S_i^{\star\star})^{-1}(x) &= \{z \in Y : x \in S_i^{\star\star}(z)\} = \{z \in Y : x \in S_i(z) \cap \Omega_i\} \\ &= \{z \in Y : x \in S_i(z)\} = S_i^{-1}(x) \end{aligned}$$

which is open in Y. Next fix $y \in Y$. We now claim there exists a $j \in \{1, .., N\}$ with $S_j^{\star\star}(y) \neq \emptyset$. This is immediate if $y \in Y \backslash K$ since from one of our assumptions in the statement of Theorem 2 there exists a $j \in \{1, .., N\}$ with $S_j(y) \cap Z_j \neq \emptyset$ so $S_j^{\star\star}(y) = S_j(y) \cap \Omega_j \neq \emptyset$ since $Z_j \subseteq \Omega_j$. It remains to consider the case when $y \in K$. Since $\{D_i\}_{i=1}^N$ is a covering of Y there exists a $j_0 \in \{1, .., N\}$ with $y \in D_{j_0}$, and note $g_{j_0}(y) \in L_{j_0}^\star(y) = S_{j_0}(y)$ since $y \in D_{j_0}$ and $g_{j_0}(y) \in co\,(R_{j_0}) \subseteq \Omega_{j_0}$, so $S_{j_0}^{\star\star}(y) = S_{j_0}(y) \cap \Omega_{j_0} \neq \emptyset$. Combining all the above we see there exists a $j \in \{1, .., N\}$ with $S_j^{\star\star}(y) \neq \emptyset$.

Let $\Omega = \prod_{i=1}^N \Omega_i$ which is a convex comapct subset of X and let $F_i^\star$ denote the restriction of F_i to Ω. Note for $i \in \{1, .., N_0\}$ that $F_i^\star \in Ad(\Omega, Y_i)$ (recall Ad is closed under compositions) so in particular since $F_i^\star$ is upper semicontinuous with compact values then (see [13]) $W_i = F_i^\star(\Omega)$ is a compact subset of Y_i. Now, Theorem 2.5 (with X_i replaced by Ω_i, X replaced by Ω, F_i replaced by $F_i^\star$, K_i replaced by W_i, G_i replaced by $G_i^{\star\star}$, S_i replaced by $S_i^{\star\star}$ and note $G_i^{\star\star} : Y \to \Omega_i$, $S_i^{\star\star} : Y \to \Omega_i$, $F_i^\star : \Omega \to Y$) guarantees a $x \in \Omega$, a $y \in Y$, and a $i_0 \in \{1, ..., N\}$ with $y_j \in F_j^\star(x)$ for all $j \in \{1, ..., N_0\}$ and $x_{i_0} \in G_{i_0}^{\star\star}(y)$. □

3. Maximal Type Element Results

In this section, we will first rewrite collectively fixed and coincidence point results as maximal type element results and from these maximal element results and other ideas we will obtain our general theory.

Theorem 3. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, ..., N\}$, suppose $F_i : X \to X_i$ and in addition* *there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Additionally, assume there is a compact subset K of X; and for each $i \in \{1, .., N\}$, a convex compact subset Y_i of X_i, such that for each $x \in X \backslash K$, there exists a $j \in \{1, .., N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Now, suppose for all $i \in \{1, .., N\}$ that $x_i \notin F_i(x)$ for each $x \in X$. Then there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$, there exists a $i \in \{1, ..., N\}$ with $S_i(x) \neq \emptyset$. Now, [1] guarantees a $x \in X$ and a $i \in \{1, .., N\}$ with $x_i \in F_i(x)$, a contradiction. □

We next discuss a generalization of majorized mappings in the literature (see [3,4,8,9]). Let Z and W be sets in a Hausdorff topological vector space with Z paracompact and W convex. Suppose $H : Z \to W$, $J : Z \to W$ and for each $y \in Z$, assume there exists a map $A_y : Z \to W$ and an open set U_y containing y with $H(z) \subseteq A_y(z)$ for every $z \in U_y$, A_y is convex valued, $(A_y)^{-1}(x)$ is open (in Z) for each $x \in W$ and $J(w) \cap A_y(w) = \emptyset$ for $w \in Z$. We now claim that there exists a map $T : Z \to W$ with $H(z) \subseteq T(z)$ for $z \in Z$, T is convex valued, $T^{-1}(x)$ is open (in Z) for each $x \in W$ and $J(w) \cap T(w) = \emptyset$ for $w \in Z$. To see this note $\{U_y\}_{y \in Z}$ is an open covering of Z and since Z is paracompact there exists [12,14] a locally finite open covering $\{V_y\}_{y \in Z}$ of Z with $y \in V_y$ and $\Omega_y = \overline{V_y} \subseteq U_y$ for each $y \in Z$. Now, for each $y \in Z$, let

$$Q_y(z) = \begin{cases} A_y(z), & z \in \Omega_y \\ W, & z \in Z \backslash \Omega_y. \end{cases}$$

Note, as in Theorem 2, for any $x \in W$, we have

$$(Q_y)^{-1}(x) = (Z \backslash \Omega_y) \cup (A_y)^{-1}(x)$$

which is open in Z, Q_y is convex valued and $H(z) \subseteq Q_y(z)$ for every $z \in Z$ (to see this note if $z \in \Omega_y$, then it is immediate, since $\Omega_y \subseteq U_y$, whereas if $z \in Z \backslash \Omega_y$, then it is immediate since $Q_y(z) = W$). Let $T : Z \to W$ be given by

$$T(z) = \bigcap_{y \in Z} Q_y(z) \quad \text{for} \quad z \in Z.$$

Now T is convex valued, $H(z) \subseteq T(z)$ for every $z \in Z$ and $J(w) \cap T(w) = \emptyset$ for $w \in Z$; to see this let $w \in Z$ and note there exists a $y^\star \in Z$ with $w \in \Omega_{y^\star}$ (recall $\{V_y\}_{y \in Z}$ is a covering of Z) so $T(w) = \bigcap_{y \in Z} Q_y(w) \subseteq Q_{y^\star}(w) = A_{y^\star}(w)$ (since $w \in \Omega_{y^\star}$) and thus $J(w) \cap T(w) = \emptyset$, since $J(w) \cap T(w) \subseteq J(w) \cap A_{y^\star}(w) = \emptyset$. It remains to show $T^{-1}(x)$ is open for each $x \in W$. Fix $x \in W$ and let $u \in T^{-1}(x)$. We now claim there exists an open set W_u containing u with $u \in W_u \subseteq T^{-1}(x)$, so then as a result $T^{-1}(x)$ is open. To prove our claim, note since $\{V_y\}_{y \in Z}$ is locally finite, there exists an open neighborhood N_u of u (in Z) such that $\{y \in Z : N_u \cap V_y \neq \emptyset\} = \{y_1, ..., y_m\}$ (a finite set). Now, if $y \notin \{y_1, ..., y_m\}$, then $\emptyset = V_y \cap N_u = \overline{V_y} \cap N_u = \Omega_y \cap N_u$ so $Q_y(z) = W$ for all $z \in N_u$, and as a result

$$T(z) = \bigcap_{y \in Z} Q_y(z) = \bigcap_{i=1}^{m} Q_{y_i}(z) \quad \text{for all} \quad z \in N_u.$$

Now $T^{-1}(x) = \{z \in Z : x \in T(z)\}$, whereas

$$\{z \in N_u : x \in T(z)\} = \left\{ z \in N_u : x \in \bigcap_{i=1}^{m} Q_{y_i}(z) \right\} = N_u \cap \left[\cap_{i=1}^{m} (Q_{y_i})^{-1}(x) \right]$$

so

$$u \in N_u \cap \left[\cap_{i=1}^{m} (Q_{y_i})^{-1}(x) \right] \subseteq T^{-1}(x)$$

and our claim is true (note $N_u \cap \left[\cap_{i=1}^{m} (Q_{y_i})^{-1}(x) \right]$ is an open neighborhood of u).

The above discussion with Theorem 3 will guarantee our next result.

Theorem 4. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, ..., N\}$, suppose $H_i : X \to X_i$; and for each $x \in X$, assume there exists a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x}$ is convex valued, $(A_{i,x})^{-1}(z)$ is open (in X) for each $z \in X_i$ and $w_i \notin A_{i,x}(w)$ for each $w \in X$. Additionally, assume there is a compact subset K of X; and for each $i \in \{1, .., N\}$, a convex compact subset Y_i of X_i, such that for each $x \in X \backslash K$, there exists a $j \in \{1, .., N\}$ with $H_j(x) \cap Y_j \neq \emptyset$. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. Let $i \in \{1, ..., N\}$. From the discussion after Theorem 3 (with $Z = X$, $W = X_i$, $H = H_i$, $J =$ Projection of X on X_i, $A_y = A_{i,x}$), there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i is convex valued, $(T_i)^{-1}(z)$ is open for each $z \in X_i$ and $w_i \notin T_i(w)$ for each $w \in X$; here

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in \Omega_{i,x} \\ X_i, & z \in X \backslash \Omega_{i,x} \end{cases}$$

and

$$T_i(z) = \bigcap_{x \in X} Q_{i,x}(z) \quad \text{for } z \in X$$

where $\{V_{i,x}\}_{x \in X}$ is a locally finite open covering of X with $x \in V_{i,x}$ and $\Omega_{i,x} = \overline{V_{i,x}} \subseteq U_{i,x}$ for each $x \in X$.

Now, we will apply Theorem 3 with $F_i = S_i = T_i$ (note if for each $x \in X \backslash K$ there exists a $j \in \{1, .., N\}$ with $H_j(x) \cap Y_j \neq \emptyset$ with K and Y_i being in the statement of Theorem 4, then $T_j(x) \cap Y_j \neq \emptyset$ since $H_j(w) \subseteq T_j(w)$ for $w \in X$) and so there exists a $x \in X$ with $T_j(x) = \emptyset$ for all $j \in \{1, ..., N\}$. Now, since $H_j(w) \subseteq T_j(w)$ for $w \in X$ then $H_j(x) = \emptyset$ for all $j \in \{1, ..., N\}$. □

Theorem 5. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^N X_i$ and $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, ..., N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, ...N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. In addition assume there is a compact subset K of Y; and for each $i \in \{1, .., N\}$ a convex compact subset Z_i of X_i, such that for each $y \in Y \backslash K$, there exists a $i \in \{1, .., N\}$ with $S_i(y) \cap Z_i \neq \emptyset$. Now, suppose either for all $j \in \{1, ..., N_0\}$ we have $y_j \notin F_j(x)$ for each $(x, y) \in X \times Y$ or for all $i \in \{1, ..., N\}$ we have $x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$. Then either there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N_0\}$ or there exists a $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$, there exists a $i \in \{1, ..., N_0\}$ with $T_i(x) \neq \emptyset$; and for each $y \in Y$, there exists a $j \in \{1, ..., N\}$ with $S_j(y) \neq \emptyset$. Now, [2] guarantees a $x \in X$, a $y \in Y$, a $j_0 \in \{1, ..., N_0\}$ and a $i_0 \in \{1, ..., N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Theorem 6. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^N X_i$ and $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, ..., N_0\}$ and for each $j \in \{1, ..., N\}$, suppose $H_i : X \to Y_i$ and $\Psi_j : Y \to X_j$, and for each $x \in X$, assume there exists a map $A_{i,x} : X \to Y_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x}$ is convex valued, $(A_{i,x})^{-1}(z)$ is open (in X) for each $z \in Y_i$, and for each $y \in Y$, assume there exists a map $B_{j,y} : Y \to X_j$ and an open set $O_{j,y}$ containing y with $\Psi_j(z) \subseteq B_{j,y}(z)$ for every $z \in O_{j,y}$,*

$B_{j,y}$ is convex valued, $(B_{j,y})^{-1}(z)$ is open (in Y) for each $z \in X_j$ and also assume either for all $i \in \{1, ..., N_0\}$ we have $v_i \notin A_{i,x}(u)$ for each $(u, v) \in X \times Y$ or for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. In addition, assume there is a compact subset K of Y; and for each $i \in \{1, .., N\}$, a convex compact subset Z_i of X_i, such that for each $y \in Y \backslash K$, there exists a $i \in \{1, .., N\}$ with $\Psi_i(y) \cap Z_i \neq \emptyset$. Then either there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, ..., N_0\}$ or there exists a $y \in Y$ with $\Psi_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$.

Proof. We modify slightly the ideas in the discussion after Theorem 3. Fix $i \in \{1, .., N_0\}$ (respectively, $j \in \{1, .., N\}$). Note $\{U_{i,x}\}_{x \in X}$ is an open covering of X (respectively, $\{O_{j,y}\}_{y \in Y}$ is an open covering of Y) so there exists a locally finite open covering $\{V_{i,x}\}_{x \in X}$ of X with $x \in V_{i,x}$ and $\Omega_{i,x} = \overline{V_{i,x}} \subseteq U_{i,x}$ for each $x \in X$ (respectively, a locally finite open covering $\{C_{j,y}\}_{y \in Y}$ of Y with $y \in C_{j,y}$ and $D_{j,y} = \overline{C_{j,y}} \subseteq O_{j,y}$ for each $y \in Y$). Now, for each $x \in X$ (respectively, $y \in Y$), let

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in \Omega_{i,x} \\ Y_i, & z \in X \backslash \Omega_{i,x} \end{cases}$$

(respectively,

$$R_{j,y}(z) = \begin{cases} B_{j,y}(z), & z \in D_{j,y} \\ X_j, & z \in Y \backslash D_{j,y} \end{cases})$$

and let $T_i : X \to Y_i$ (respectively, $S_j : Y \to X_j$) be given by

$$T_i(z) = \bigcap_{x \in X} Q_{i,x}(z), \; z \in X \; \text{(respectively, } S_j(w) = \bigcap_{y \in Y} R_{j,y}(w), \; w \in Y).$$

The argument in the discussion after Theorem 3 guarantees that $H_i(z) \subseteq T_i(z)$ for every $z \in X$ (respectively, $\Psi_j(w) \subseteq S_j(w)$ for $w \in Y$), T_i (respectively, S_j) is convex valued and $T_i^{-1}(w)$ is open for each $w \in Y_i$ (respectively, $S_j^{-1}(z)$ is open for each $z \in X_j$).

There are two cases to consider (see the statement of Theorem 6). Suppose first that for each $x \in X$ for all $i \in \{1, ..., N_0\}$ we have $v_i \notin A_{i,x}(u)$ for each $(u, v) \in X \times Y$. Then for all $i \in \{1, ..., N_0\}$ we have $v_i \notin T_i(u)$ for each $(u, v) \in X \times Y$; to see this fix $i \in \{1, ..., N_0\}$ and $(u, v) \in X \times Y$ and note there exists a $x^\star \in X$ with $u \in \Omega_{i,x^\star}$ so

$$T_i(u) = \bigcap_{x \in X} Q_{i,x}(u) \subseteq Q_{i,x^\star}(u) = A_{i,x^\star}(u)$$

and as a result, $v_i \notin T_i(u)$ since $v_i \notin A_{i,x^\star}(u)$ and $T_i(u) \subseteq A_{i,x^\star}(u)$. Next consider the case that for each $y \in Y$ for all $j \in \{1, ..., N\}$, we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. As in the first case (with $D_{j,y}$ and S_j replacing $\Omega_{i,x}$ and T_i), we obtain for all $j \in \{1, ..., N\}$ we have $u_j \notin S_j(v)$ for each $(u, v) \in X \times Y$.

Now, apply Theorem 5 (with $F_i = T_i$ and $G_j = S_j$) so either there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N_0\}$ or there exists a $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$, Now, since $H_i(z) \subseteq T_i(z)$, $z \in X$ and $\Psi_j(w) \subseteq S_j(w)$, $w \in Y$, the conclusion follows. □

Theorem 7. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, ...N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Additionally, assume there is a compact subset K of Y; and for each $i \in \{1, .., N\}$, a convex compact subset Z_i of X_i, such that for each $y \in Y \backslash K$, there exists a $i \in \{1, .., N\}$ with $S_i(y) \cap Z_i \neq \emptyset$. Now, suppose either for all $i \in \{1, ..., N\}$ we have $x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$ or for each $(x, y) \in X \times Y$ there exists a $j \in \{1, ..., N_0\}$ with $y_j \notin F_j(x)$. Then there exists a $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. Suppose the conclusion is false. Then for each $y \in Y$, there exists a $j \in \{1, ..., N\}$ with $S_j(y) \neq \emptyset$. Now, Theorem 2 guarantees a $x \in X$, a $y \in Y$, a $i_0 \in \{1, ..., N\}$ with $y_j \in F_j(x)$ for all $j \in \{1, ..., N_0\}$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Theorem 8. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, ..., N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, ...N\}$, suppose $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$; and for each $y \in Y$, assume there exists a map $B_{j,y} : Y \to X_j$ and an open set $O_{j,y}$ containing y with $\Psi_j(z) \subseteq B_{j,y}(z)$ for every $z \in O_{j,y}$, $B_{j,y}$ is convex valued and $(B_{j,y})^{-1}(z)$ is open (in Y) for each $z \in X_j$. In addition, assume there is a compact subset K of Y; and for each $i \in \{1, .., N\}$, a convex compact subset Z_i of X_i, such that for each $y \in Y \backslash K$, there exists a $i \in \{1, .., N\}$ with $\Psi_i(y) \cap Z_i \neq \emptyset$. Additionally, suppose either for each $y \in Y$ for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$ or for each $(x, y) \in X \times Y$ there exists a $i \in \{1, ..., N_0\}$ with $y_i \notin F_i(x)$. Then there exists a $y \in Y$ with $\Psi_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$.*

Proof. Let $j \in \{1, ..., N\}$ and create $C_{j,y}$, $D_{j,y}$, $R_{j,y}$ and S_j as in Theorem 6. We now claim that for all $j \in \{1, ..., N\}$, we have $u_j \notin S_j(v)$ for each $(u, v) \in X \times Y$ if in the statement of Theorem 8 we have for each $y \in Y$ for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. Thus, for a fixed $j \in \{1, ..., N\}$ and $(u, v) \times X \times Y$, note there exists a $y^\star \in Y$ with $v \in D_{j,y^\star}$ so

$$S_j(v) = \bigcap_{y \in Y} R_{j,y}(v) \subseteq R_{j,y^\star}(v) = B_{j,y^\star}(v)$$

and as a result $u_j \notin S_j(v)$ since $u_j \notin B_{j,y^\star}(v)$ and $S_j(v) \subseteq B_{j,y^\star}(v)$. Thus, our claim is true. Now, apply Theorem 7 (with $G_j = S_j$) so there exists a $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1, ..., N\}$. The conclusion follows, since $\Psi_j(w) \subseteq S_j(w)$, $w \in Y$. □

4. Conclusions

In Section 2, we present a new collectively coincidence coercive type result between two different classes of maps, namely the $\Phi^\star$ and Ad classes. The coincidence theory in Section 2 is then used to establish some new maximal element type results in Section 3. Further, in Section 3, we consider majorized maps to establish a variety of new maximal element type results. In a future paper, we hope to use some of the ideas, techniques and results in this paper to consider applications in generalized abstract economies (generalized games).

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References

1. O'Regan, D. Collectively fixed point theory in the compact and coercive cases. *Analele Stiintifice Ale Univ. Ovidius Constanta Ser. Math.* **2021**, accepted.
2. O'Regan, D. Collectively coincidence type results and applications. *Appl. Anal.* **2021**. [CrossRef]
3. Deguire, P.; Tan, K.K.; Yaun, X.Z. The study of maximal elements, fixed points for L_S–majorized mappings and their applications to minimax and variational inequalities in product topological spaces. *Nonlinear Anal.* **1999**, *37*, 933–951. [CrossRef]
4. Ding, X.P.; Tan, K.K. On equilibria of non-compact generalized games. *J. Math. Anal. Appl.* **1993**, *177*, 226–238. [CrossRef]
5. Gorniewicz, L. *Topological Fixed Point Theory of Multivalued Mappings*; Kluwer Acad. Publishers: Dordrecht, The Netherlands, 1991.
6. Ben-El-Mechaiekh, H.; Deguire, P.; Granas, A. Points fixes et coincidences pour les applications multivoques II (Applications de type Φ and $\Phi^\star$). *C.R. Acad. Sci.* **1982**, *295*, 381–384.
7. O'Regan, D. A note on collectively fixed and coincidence points. **2021**, submitted.

8. Borglin, A.; Keiding, H. Existence of equilibrium actions and of equilibrium: A note on "new" existence theorems. *J. Math. Econ.* **1976**, *3*, 313–316. [CrossRef]
9. Yanelis, N.C.; Prabhakar, N.D. Existence of maximal elements and equlibria in linear topological spaces. *J. Math. Econ.* **1983**, *12*, 233–245. [CrossRef]
10. Yuan, X.Z. The study of equilibria for abstract economies in topological vector spaces-a unified approach. *Nonlinear Anal.* **1999**, *37*, 409–430. [CrossRef]
11. Ding, X.P.; Kim, W.K.; Tan, K.K. A selection theorem and its applications. *Bull. Aust. Math. Soc.* **1992**, *46*, 205–212. [CrossRef]
12. Engelking, R. *General Topology (Revised and Completed Edition)*; Heldermann Verlag: Berlin, Germany, 1989.
13. Aliprantis, C.D.; Border, K.C. *Infinite Dimensional Analysis*; Springer: Berlin, Germany, 1994.
14. Dugundji, J. *Topology*; Allyn and Bacon: Boston, MA, USA, 1966.