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On the Semi-Local Convergence of an Ostrowski-Type Method for Solving Equations

Christopher I. Argyros [1], Ioannis K. Argyros [2,*], Janak Joshi [2], Samundra Regmi [3] and Santhosh George [4]

[1] Department of Computing and Technology, Cameron University, Lawton, OK 73505, USA; Christopher.Argyros@cameron.edu
[2] Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA; jjoshi@cameron.edu
[3] Learning Commons, University of North Texas at Dallas, Dallas, TX 75201, USA; samundra.regmi@untdallas.edu
[4] Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Mangalore 575025, India; sgeorge@nitk.edu.in
* Correspondence: iargyros@cameron.edu

Abstract: Symmetries play a crucial role in the dynamics of physical systems. As an example, microworld and quantum physics problems are modeled on principles of symmetry. These problems are then formulated as equations defined on suitable abstract spaces. Then, these equations can be solved using iterative methods. In this article, an Ostrowski-type method for solving equations in Banach space is extended. This is achieved by finding a stricter set than before containing the iterates. The convergence analysis becomes finer. Due to the general nature of our technique, it can be utilized to enlarge the utilization of other methods. Examples finish the paper.

Keywords: Ostrowski-type method; Banach space; convergence criterion

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1. Introduction

We are concerned with finding x_* solving

$$F(x) = 0, \tag{1}$$

where $F : D \subset E \longrightarrow E_1$ is an operator acting between Banach spaces E and E_1 with $D \neq \emptyset$.

The famous Ostrowski-type method is defined for $x_0 \in D$ and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ x_{k+1} &= y_k - A_kF(x_k), \end{aligned} \tag{2}$$

where $A_k = 2[y_k, x_k; F]^{-1} - F'(x_k)^{-1}$, with $[.,.,F] : D \times D \to L(E, E_1)$. There are numerous results for the convergence of iterative methods utilizing the information (D, x_0, F, F') and higher order derivatives [1–39]. However, higher order derivatives cannot be found on method (2). Moreover, these results do not give uniqueness ball or estimates on $\|x_k - x_*\|$ or $\|x_{k+1} - x_k\|$. That is why we are motivated to write this paper, where only hypotheses on the derivative and divided differences of order one are used. Notice that only these operators appear on method (2).

The method (2) is shown to be of order four using Taylor expansion and assumptions on the fifth order derivative of F, which is not on these schemes [5]. So, the assumptions on the sixth derivative reduce the applicability of this method.

For example: Let $E = E_1 = \mathbb{R}$, $D = [-0.5, 1.5]$. Define λ on D by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\lambda'''(t) = 6\log t^2 + 60t^2 - 24t + 22.$$

Obviously, $\lambda'''(t)$ is not bounded on D. So, the convergence of method (2) is not guaranteed by the previous analyses in [5].

The rest of the study is organized as follows: Section 2 contains results on majorizing sequences. In Section 3, we develop the semi-local convergence analysis based on majorizing sequences. The local convergence analysis can be found in Section 4. Numerical examples can be found in Section 5. The paper ends with some concluding remarks in Section 6.

2. Majorizing Sequences

We recall the definition of a majorizing sequences.

Definition 1. *Let $\{v_k\}$ be a sequence in a complete normed space. Then, a non-decreasing scalar sequence $\{d_k\}$ is called majorizing for $\{v_k\}$ if*

$$\|v_{k+1} - v_k\| \leq d_{k+1} - d_k \text{ for each } k = 0, 1, 2, \ldots.$$

Then, the convergence of sequence $\{v_k\}$ reduces to studying that of $\{d_k\}$ [40].

Let $\eta \geq 0$ and $L, L_i, i = 0,1,2,3,4$ be positive parameters. Set $M_0 = \frac{L_0 L_2}{2}$, $M = \frac{L}{2}$, $M_1 = \frac{LL_2}{2}$ and $M_2 = \frac{LL_3}{2}$. Define sequences $\{t_k\}$, $\{s_k\}$, $\{\alpha_k\}$ and $\{\beta_k\}$ for each $k = 0,1,2,\ldots$ by $t_0 = 0, s_0 = \eta$

$$\begin{aligned}
t_1 &= s_0 + \frac{M_0 s_0^2}{1 - L_1 s_0} + \frac{M_2 s_0^3}{(1 - L_1 s_0)(1 - L_0 s_0)} + \frac{M s_0^2}{1 - L_0 s_0}, \\
s_1 &= t_1 + \frac{M(t_1 - t_0)^2 + t_1 - s_0}{1 - L_0 t_1} \\
t_{k+1} &= s_k + \alpha_k (s_k - t_k), \\
s_{n+1} &= t_{n+1} + \frac{M(t_{n+1} - t_n)^2 + L_4 t_n (t_{n+1} - s_n)}{1 - L_0 t_{n+1}}, \\
\alpha_k &= \frac{M_1 (s_k - t_k)}{(1 - L_1(s_k + t_k))(1 - L_0 t_k)} \\
&\quad + \frac{M_2 (s_k - t_k)}{(1 - L_1(s_k + t + k))(1 - L_0 s_k)} + \frac{M(s_k - t_k)}{1 - L_0 s_k}, \\
\beta_k &= \frac{M(t_{k+1} - t_k) + L_4 t_k}{1 - L_0 t_{k+1}}.
\end{aligned} \tag{3}$$

Moreover, define quadratic polynomials and functions on the interval $[0, 1]$ for some $b > 1$

$$p_1(t) = t^2 - (1 - L_0 t_1)t + L_4 t_1,$$

$$p_2(t) = t^2 + t - (1 - \frac{2bL_1 t_1}{b - 1}),$$

$$p_3(t) = t^2 + t - (1 - L_0 t_1),$$

$$g_1(t) = (M_1 b + M_2 b + M)t$$

and

$$g_2(t) = Mt((1 + t)t - 1)(1 + t)) + (1 + t)^2 t^2 (L_4 + L_0 t^2 (1 + t)).$$

Denote by γ_0, γ_1 or γ_2 or γ_3, γ_4, the non-negative zeros of p_1, p_2, p_3 if they exist. Furthermore, define sequences of functions on the interval $[0,1]$ for $\delta = \delta(t) = (1+t)t$ by

$$f_n^{(1)}(t) = M_1 bt\delta^{n-1}t_1 + M_2 bt\delta^{n-1}t_1 + Mt\delta^{n-1}t_1 + L_0 t(1+\delta+\ldots+\delta^n)t_1 - t$$

and

$$f_n^{(2)}(t) = M(t^2\delta^{n-1} + t\delta^{n-1})t_1 + L_4(1+\delta+\ldots+\delta^n)t_1 + L_0 t(1+\delta+\ldots+\delta^n)t_1 - t.$$

Next, we present two results on the majorizing sequence for method (2).

Lemma 1. *Suppose that for each $k = 0, 1, 2, \ldots$, items*

$$s_k \le t_{k+1} < \frac{1}{L_0} \tag{4}$$

and

$$s_k + t_k < \frac{1}{L_1} \tag{5}$$

hold. Then, sequences $\{s_k\}$ and $\{t_k\}$ are increasing, bounded from above by $\frac{1}{L_0}$ and converge to their unique least upper bound $s_* \in [0, \frac{1}{L_0}]$.

Proof. It follows from (3)–(5) that sequences $\{s_k\}, \{t_k\}$ are increasing, bounded from above by $\frac{1}{L_0}$ and as such they converge to s_*. □

Remark 1. *Conditions (4) and (5) hold only in some special cases. This is why we present stronger conditions that can be verified more easily.*

We shall use the following set of conditions denoted by (A) in our second result on majorizing sequences for method (2).

Suppose: there exists $\gamma \in S := (0, \frac{\sqrt{5}-1}{2})$, $b > 0, \eta > 0$ satisfying

$$\begin{aligned} 0 &\le \alpha_0 \le \gamma,\ 0 \le \beta_0 \le \gamma, \\ L_0\eta &< 1,\ L_1\eta < 1,\ L_1 t_1 < \frac{b-1}{2b}, \\ \gamma_0 &\le \gamma \le \gamma_1 \text{ if } g_2(t) \ge 0 \text{ for each } t \in S, \end{aligned}$$

or

$$\begin{aligned} f_1^{(2)}(\gamma) &\le 0 \text{ if } g_2(t) \le 0 \text{ for each } t \in S, \\ \gamma &\le \gamma_2 \\ &\text{and} \\ \gamma_3 &\le \gamma \le \gamma_4. \end{aligned}$$

Then, under the preceding notation and conditions (A), we can show.

Lemma 2. *Under conditions (A), the conclusions of Lemma 1 hold for sequences $\{s_k\}, \{t_k\}$. Moreover, the following assertions hold for each $k = 0, 1, 2, \ldots$*

$$0 \le s_k - t_k \le \gamma^k (1+\gamma)^{k-1}(t_1 - t_0), \tag{6}$$

$$0 \le t_{k+1} - s_k \le \gamma^{k+1}(1+\gamma)^{k-1}(t_1 - t_0), \tag{7}$$

$$0 \le s_k \le \frac{1-\delta^{k+1}}{1-\delta} t_1 \tag{8}$$

and

$$0 \le t_{k+1} \le \frac{1-\delta^{n+2}}{1-\delta} t_1. \quad (9)$$

Recall that $\delta = \gamma(\gamma+1)$ *and* $\delta(t) = \delta$.

Proof. We shall show using induction on n that the following hold.

$$0 \le \alpha_n \le \gamma, \quad (10)$$

$$0 \le \beta_n \le \gamma, \quad (11)$$

$$L_0 t_n \le 1,\ L_0 s_n < 1,\ L_1(s_n + t_n) < 1, \quad (12)$$

and

$$t_n \le s_n \le t_{n+1}. \quad (13)$$

Estimates (10)–(13) hold for $n = 0$, by the initial conditions and conditions (A). We also have

$$\begin{aligned}
0 &\le s_0 - t_0 \le \eta,\ 0 \le t_1 - s_0 \le \gamma\eta,\ 0 \le s_1 - t_1 \le \gamma(t_1 - t_0),\\
0 &\le t_2 - s_1 \le \gamma^2(t_1 - t_0),\\
0 &\le s_2 - t_2 \le \gamma^2(1+\gamma)(t_1 - t_0),\\
0 &\le t_3 - s_2 \le \gamma^3(1+\gamma)(t_1 - t_0),\\
&\vdots\\
0 &\le s_n - t_n \le \gamma^n(1+\gamma)^{n-1}(t_1 - t_0), \quad (14)\\
0 &\le t_{n+1} - s_n \le \gamma^{n+1}(1+\gamma)^{n-1}(t_1 - t_0), \quad (15)\\
t_{n+1} &\le s_n + \gamma^{n+1}(1+\gamma)^{n-1}(t_1 - t_0) \le t_n + \gamma^n(1+\gamma)^{n-1}(t_1 - t_0)\\
&\quad + \gamma^{n+1}(1+\gamma)^{n-1}(t_1 - t_0)\\
&\vdots\\
&\le t_1 + \gamma(1+\gamma)^{n-1}(t_1 - t_0) + \gamma^2(1+\gamma)(t_1 - t_0) +\\
&\vdots\\
&\quad + \gamma^n(1+\gamma)^{n-1}(t_1 - t_0) + \gamma^{n+1}(1+\gamma)^{n-1}(t_1 - t_0)\\
&\le (1 + \delta + \ldots + \delta^{n+1})t_1 = \frac{1-\delta^{n+2}}{1-\delta} t_1 \quad (16)
\end{aligned}$$

and

$$s_n \le t_n + \gamma^n(1+\gamma)^{n-1} t_1 \le \ldots \le \frac{1-\delta^{n+1}}{1-\delta} t_1. \quad (17)$$

Suppose these estimates hold for all integers smaller or equal to n. Then, evidently, (10) holds (since $\frac{1}{1-L_1(s_n+t_n)} \le b$), if we show instead using (14)–(17) that

$$\frac{M_1 b(s_n - t_n)}{1 - L_0 t_n} + \frac{M_2 b(s_n - t_n)}{1 - L_0 s_n} + \frac{M(s_n - t_n)}{1 - L_0 s_n} \le \gamma \quad (18)$$

or

$$\begin{aligned}
&M_1 b\gamma\delta^{n-1} t_1 + M_2 b\gamma\delta^{n-1} t_1 + M\gamma\delta^{n-1} t_1\\
&+ \gamma L_0(1 + \delta + \ldots + \delta^n)t_1 - \gamma \le 0. \quad (19)
\end{aligned}$$

Notice that expression (19) is obtained if we replace $s_n - t_n$, t_n, s_n by the right hand sides of (14), (15) and (17), respectively, in (18), remove denominators and move all terms at the right hand side of the inequality.

Estimate (19) motivates us to define functions $f_n^{(1)}$ on the interval $[0,1]$ and show instead of (19)

$$f_n^{(1)}(t) \leq 0 \text{ at } t = \gamma. \tag{20}$$

We shall find a relationship between two consecutive functions $f_n^{(1)}$. We can write in turn that

$$\begin{aligned}
f_{n+1}^{(1)}(t) &= M_1bt\delta^n t_1 + M_2bt\delta^n t_1 + Mt\delta^n t_1 + L_0t(1+\delta+\ldots+\delta^{n+1})t_1 - t \\
&\quad - M_1bt\delta^{n-1}t_1 - M_2bt\delta^{n-1}t_1 - Mt\delta^{n-1}t_1 \\
&\quad - L_0t(1+\delta+\ldots+\delta^n)t_1 + t + f_n^{(1)}(t) \\
&= f_n^{(1)}(t) + (M_1bt\delta^n t_1 - M_1bt\delta^{n-1}t_1) \\
&\quad + (M_2bt\delta^n t_1 - M_2bt\delta^{n-1}t_1) + (Mt\delta^n t_1 - Mt\delta^{n-1}t_1) + tL_0\delta^{n+1}t_1 \\
&= f_n^{(1)}(t) + (\delta-1)t(M_1b + M_2b + M)\delta^{n-1}t_1 \\
&= f_n^{(1)}(t) + (\delta-1)g_1(t)\delta^{n-1}t_1 \\
&\leq f_n^{(1)}(t),
\end{aligned}$$

since $t \in [0, \frac{\sqrt{5}-1}{2}]$, so

$$f_{n+1}^{(1)}(t) \leq f_n^{(1)}(t). \tag{21}$$

Define function

$$f_\infty^{(1)}(t) = \lim_{k \longrightarrow +\infty} f_k^{(1)}(t). \tag{22}$$

By the definition of functions $f_n^{(1)}$ and $f_\infty^{(1)}$, we get

$$f_\infty^{(1)}(t) = \frac{tL_0t_1}{1-\delta} - t. \tag{23}$$

Then, we can show instead of (20) that

$$f_\infty^{(1)}(t) \leq 0 \text{ at } t = \gamma, \tag{24}$$

which is true by the definition of p_3 and $\gamma_3 \leq \gamma \leq \gamma_4$. Similarly, (11) holds if

$$\begin{aligned}
&M(\gamma^2\delta^{n-1} + \gamma\delta^{n-1})t_1 + L_4(1+\delta+\ldots+\delta^n)t_1 \\
&+ L_0\gamma(1+\delta+\ldots+\delta^{n+1})t_1 - \gamma \leq 0
\end{aligned} \tag{25}$$

or

$$f_n^{(2)}(t) \leq 0 \text{ at } t = \gamma. \tag{26}$$

This time, we have

$$\begin{aligned}
f_{n+1}^{(2)}(t) &= M(t^2\delta^n + t\delta^n)t_1 + L_4(1+\delta+\ldots+\delta^{n+1})t_1 \\
&\quad + L_0t((1+\delta+\ldots+\delta^{n+2})t_1 - t \\
&\quad - M(t^2\delta^{n-1} + t\delta^{n-1})t_1 - L_4(1+\delta+\ldots+\delta^n)t_1 \\
&\quad - L_0t(1+\delta+\ldots+\delta^{n+1})t_1 + t + f_n^{(2)}(t) \\
&= f_n^{(2)} + M(t^2\delta^n + t\delta^n - t^2\delta^{n-1} - t\delta^{n-1})t_1 \\
&\quad + L_4\delta^{n+1}t_1 + L_0t\delta^{n+2}t_1 \\
&= f_n^{(2)}(t) + g_2(t)\delta^{n-1}t_1,
\end{aligned}$$

so

$$f_{n+1}^{(2)}(t) = f_n^{(2)}(t) + g_2(t)\delta^{n-1}t_1. \tag{27}$$

Define function

$$f_\infty^{(2)}(t) = \lim_{k \longrightarrow +\infty} f_n^{(2)}(t). \tag{28}$$

Then, we get

$$f_\infty^{(2)}(t) = \frac{L_4 t_1}{1-t} + \frac{L_0 t t_1}{1-t} - t. \tag{29}$$

If $\gamma_0 \leq \gamma \leq \gamma_1$, then $g_2(t) \geq 0$ for each $t \in S$ and $f_\infty^{(2)}(t) \leq 0$ holds at $t = \gamma$. However, if $g_2(t) \leq 0$ for each $t \in S$, then

$$f_{n+1}^{(2)}(t) \leq f_n^{(2)}(t). \tag{30}$$

In this case, (26) holds if $f_1^{(2)}(t) \leq 0$ at $t = \gamma$, which is true. Therefore, the induction for (10)–(13) is completed. Hence, sequences $\{s_k\}, \{t_k\}$ are non-decreasing, bounded from above by $\frac{t_1}{1-\delta}t_1$ and as such they converge to s_* satisfying $s_* \in [\eta, \frac{t_1}{1-\delta}]$. □

3. Semi-Local Convergence

We shall use conditions (H):

Suppose

(H1) There exist $x_0 \in D$, $\eta \geq 0$ such that $F'(x_0)$ is invertible and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta.$$

(H2) For each $u \in D$

$$\|F'(x_0)^{-1}(F'(u) - F'(x_0))\| \leq L_0\|u - x_0\|.$$

Set $D_0 = U(x_*, \frac{1}{L_0}) \cap D$.

(H3) For each $v, w \in D_0$

$$\|F'(x_0)^{-1}(F'(w) - F'(v))\| \leq L\|w - v\|,$$

$$\|F'(x_0*)^{-1}([v,w;F] - F'(x_0))\| \leq L_1(\|v - x_0\| + \|w - x_0\|),$$

$$\|F'(x_0)^{-1}([v,w;F] - F'(w))\| \leq L_2\|v - w\|$$

$$\|F'(x_0)^{-1}([v,w;F] - F'(v))\| \leq L_3\|v - w\|.$$

and

$$\|F'(x_0)^{-1}F'(w)\| \leq L_4\|w - x_0\|.$$

(H4) $U[x_0, s_*] \subset D$

and

(H5) Conditions of Lemma 1 or Lemma 2 hold.

Then, based on conditions (H), we present the semi-local convergence analysis of method (2).

Theorem 1. *Suppose hypotheses (H) hold. Then, sequences $\{x_k\}, \{y_k\}$ generated by method (2) with starter x_0 are well defined in $U[x_0, s_*]$, remain in $U[x_0, s_*]$ for each $n = 0, 1, 2, \ldots$ and converge to a solution $x_* \in U[x_0, s_*]$ of equation $F(x) = 0$. Moreover, the following error estimates hold*

$$\|x_k - x_*\| \leq t_n - s_*. \tag{31}$$

Proof. Mathematical induction is employed to show

$$\|x_{k+1} - y_k\| \leq t_{k+1} - s_k \tag{32}$$

and

$$\|y_k - x_k\| \leq s_k - t_k. \tag{33}$$

Iterate y_0 is well defined by the first substep of method (2) and (H1). We can write

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 = s_0 \leq s_*,$$

so $y_0 \in U[x_0, s_*]$. Using (H3), we get in turn for $v, w \in U(x_0, s_*)$

$$\begin{aligned} \|F'(x_0)^{-1}([v,w;F] - F'(x_0))\| &\leq L_1(\|v - x_0\| + \|w - x_0\|) \\ &\leq L_1(s_* + s_*) = 2L_1 s_* < 1 \end{aligned} \tag{34}$$

by the Lemma on invertible opertors due to Banach [41,42], leading to

$$\|[v,w;F]^{-1}F'(x_0)\| \leq \frac{1}{1 - L_1(\|v - x_0\| + \|w - x_0\|)}. \tag{35}$$

Similarly, iterate x_1 is well defined by the second substep of method (2). We also have by (H2) for $w \in U(x_0, s^*)$

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \leq L_0\|w - x_0\| \leq L_0 s^* < 1,$$

so $F'(w)^{-1} \in L(E_1, E)$ and

$$\|F'(w)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|w - x_0\|}. \tag{36}$$

Hence, by (35) for $v = y_0$ and $w = x_0$ and (36) for $w = x_0$, we have

$$\begin{aligned} x_1 - y_0 &= -[y_0, x_0; F]^{-1}(F'(x_0) - [y_0, x_0; F])F'(x_0)^{-1}F(y_0) \\ &\quad -[y_0, x_0; F]^{-1}(F'(y_0) - [y_0, x_0; F])F'(y_0)^{-1}F(y_0) \\ &\quad -F'(y_0)^{-1}F(y_0). \end{aligned} \tag{37}$$

In view of (H3), (35), (36) (for $v = y_0, w = x_0$), (37) and triangle inequality, we get in turn

$$\begin{aligned} \|x_1 - y_0\| &\leq \frac{LL_2\|y_0 - x_0\|\|y_0 - x_0\|^2}{2(1 - L_1(\|y_0 - x_0\| + \|x_0 - x_0\|))(1 - L_0\|x_0 - x_0\|)} \\ &\quad + \frac{LL_3\|y_0 - x_0\|\|y_0 - x_0\|^2}{2(1 - L_1(\|y_0 - x_0\| + \|x_0 - x_0\|))(1 - L_0\|y_0 - x_0\|)} \\ &\quad + \frac{L\|y_0 - x_0\|^2}{2(1 - L_0\|y_0 - x_0\|)} \leq \alpha_0(s_0 - t_0) = t_1 - s_0, \end{aligned} \tag{38}$$

and

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 \leq s_*,$$

so $x_1 \in U[x_0, s_*]$. Thus, estimates (32) and (33) hold for $n = 0$, where we also used

$$\begin{aligned} \|F'(x_0)^{-1}F(y_0)\| &= \|\int_0^1 F'(x_0)^{-1}(F'(x_0 + \theta(y_0 - x_0)) - F'(x_0))(y_0 - x_0)d\theta\| \\ &\leq \frac{L_0}{2}\|y_0 - x_0\|^2 \leq \frac{L}{2}(s_0 - t_0)^2. \end{aligned} \tag{39}$$

We know that (36) holds for $w = x_1$, so iterate y_1 is well defined by the first substep of method (2) for $n = 1$, and we can write

$$\begin{aligned} F(x_1) &= F(x_1) - F(x_0) - F'(x_0)(x_1 - x_0) \\ &\quad + F'(x_0)(x_1 - y_0) \\ &= \int_0^1 (F'(x_0 + \theta(x_1 - x_0)) - F'(x_0))(x_1 - x_0)d\theta + F'(x_0)(x_1 - y_0). \end{aligned} \tag{40}$$

Then, we obtain by method (2), (36) (for $w = x_1$), (40) and the triangle inequality

$$\begin{aligned} \|y_1 - x_1\| &\leq \frac{\frac{L}{2}\|x_1 - x_0\|^2 + \|x_1 - y_0\|}{1 - L_0\|x_1 - x_0\|} \\ &\leq \frac{M(t_1 - t_0)^2 + t_1 - s_0}{1 - L_0 t_1} = s_1 - t_1. \end{aligned} \tag{41}$$

Then, we have

$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq s_1 - t_1 + t_1 = s_1 \leq s_*,$$

so $y_1 \in U[x_0, s_*]$. Suppose estimates (32) and (33) hold for all integers smaller or equal to $n-1$. Then, simply repeat the preceding calculations with x_0, y_0, x_1 replaced by x_m, y_m, x_{m+1}, respectively, and use the induction hypotheses to terminate the proof for (32) and (33). By the Lemma sequence $\{t_k\}$ is Cauchy in a Banach space E and as such it converges to some $x_* \in U[x_0, s_*]$ since it is a closed set. Finally, using (40), we get

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L}{2}\|x_{k+1} - x_k\|^2 \\ &\quad + L_4\|x_k - x_0\|\|x_{k+1} - x_k\| \\ &\leq \frac{L}{2}(t_{k+1} - t_k)^2 + L_4 t_k(t_{k+1} - s_k) \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow +\infty$ implying $F(x_*) = 0$ (by the continuity of F). □

The point s_* can be replaced by $\frac{1}{L_0}$ or $\frac{t_1}{1-\delta}$, respectively, given in closed form.
Next, a uniqueness of the solution x_* of equation $F(x) = 0$ is presented.

Proposition 1. *Suppose:*
(a) There exists a solution $x_* \in D$ of equation $F(x) = 0$;
(b) There exists $s \geq s_*$ such that

$$\frac{L_0}{2}(s + s_*) < 1. \tag{42}$$

Set $D_1 = U[x_0, s_*] \cap D$. Then, the only solution of equation $F(x) = 0$ in the region D_1 is x_*.

Proof. Let $x^{**} \in D_1$ with $F(x^{**}) = 0$. Set $M = \int_0^1 F'(x^{**} + \theta(x_* - x^{**}))d\theta$. Using (H2) and (42), we obtain in turn that

$$\begin{aligned} \|F'(x_0)^{-1}(M - F'(x_0))\| &\leq L_0 \int_0^1 ((1-\theta)\|x_* - x_0\| + \theta\|x^{**} - x_0\|)d\theta \\ &\leq L_0 \int_0^1 (1-\theta)s d\theta + L_0 \int_0^1 \theta s_* d\theta \\ &\leq \frac{L_0}{2}(s + s_*) < 1, \end{aligned} \tag{43}$$

so $x^{**} = x_*$ follows from the invertability of linear operator M and the identity $M(x_* - x^{**}) = F(x_*) - F(x^{**}) = 0 - 0 = 0$. □

4. Local Convergence

Let $\ell, \ell_j, j = 0,1,2,3,4$ be positive parameters. Define function $\psi_1 : [0, \frac{1}{\ell_0}) \longrightarrow [0,+\infty)$ by

$$\psi_1(t) = \frac{\ell t}{2(1-\ell_0 t)}$$

and set

$$\rho_A = \frac{2}{2\ell_0 + \ell}. \tag{44}$$

Define functions $q : [0, \frac{1}{\ell_0}) \longrightarrow [0,+\infty)$ by

$$q(t) = \ell_4(1+\psi_1(t))t - 1.$$

By this definition, we have $q(0) = -1$ and $q(t) \longrightarrow +\infty$ as $t \longrightarrow \frac{1}{\ell_0}^-$. It then follows from the intermediate value theorem that function q has zeros in $(0, \frac{1}{\ell_0})$. Denote by ρ_q the smallest such zero. Similarly, denote by ρ_p the smallest zero of function $p : [0, \frac{1}{\ell_0}) \longrightarrow [0,+\infty)$ defined by $p(t) = \ell_0\psi_1(t)t - 1$. Set $\bar{\rho} = \min\{\rho_q, \rho_p\}$. Moreover, define function $\psi_2 : [0,\bar{\rho}) \longrightarrow [0,+\infty)$ by

$$\begin{aligned}\psi_2(t) \quad = \quad & \Big[\frac{\ell\psi_1(t)t}{2(1-\ell_0\psi_1(t)t)} \\ & + \frac{\ell_3\ell_4(1+\psi_1(t)t)\psi_1(t)t}{(1-\ell_0\psi_1(t)t)(1-\ell_1(1+\psi_1(t))t)} \\ & + \frac{\ell_2\ell_4(1+\psi_1(t))}{(1-\ell_1(1+\psi_1(t))t))^2}\Big]\psi_1(t)t.\end{aligned}$$

Set

$$\mu(t) = \psi_2(t) - 1.$$

We have again $\mu(0) = -1$ and $\mu(t) \longrightarrow +\infty$ as $t \longrightarrow \bar{\rho}^-$. Denote by ρ_μ the smallest zero of function μ in $(0,\bar{\rho})$. We shall show that

$$\rho_* = \min\{\rho_A, \rho_\mu\} \tag{45}$$

is a convergence radius for method (2). Set $T = [0,\rho_*)$. Then, it follows from these definitions that for each $t \in T$

$$0 \le \ell_0 t < 1, \tag{46}$$

$$0 \le q(t) < 1, \tag{47}$$

$$0 \le p(t) < 1, \tag{48}$$

and

$$0 \le \psi_i(t) < 1, i = 1,2. \tag{49}$$

The conditions (C) shall be used together with the preceding notation provided that x_* is a simple solution of equation $F(x) = 0$.

Suppose:

(C1) For each $u \in D$

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \le \ell_0\|u - x_0\|.$$

Set $D_2 = U(x_*, \frac{1}{\ell_0}) \cap D$.

(C2) For each $v, w \in D_2$

$$\|F'(x_*)^{-1}(F'(w) - F'(v))\| \le \ell\|w - v\|,$$

$$\|F'(x_*)^{-1}F'(v)\| \le \ell_1\|v - x_*\|,$$

$$\|F'(x_*)^{-1}([w,v;F]-F'(v))\| \leq \ell_2(\|w-v\|),$$
$$\|F'(x_*)^{-1}([w,v;F]-F'(w))\| \leq \ell_3\|w-v\|$$

and

$$\|F'(x_*)^{-1}([w,v;F]-F'(x_*))\| \leq \ell_4(\|w-x_*\|+\|v-x_*\|).$$

(C3) $U[x_*,\rho_*] \subset D$.

Next, we present the local convergence analysis of method (2).

Theorem 2. *Under the conditions (C) further suppose that* $x_0 \in U(x_*,\rho_*) - \{x_*\}$. *Then, we have* $\lim_{k\longrightarrow+\infty} x_k = x_*$.

Proof. We shall use mathematical induction to show

$$\|y_n - x_*\| \leq \psi_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < \rho_* \tag{50}$$

and

$$\|x_{n+1} - x_*\| \leq \psi_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|. \tag{51}$$

where functions ψ_i are given previously and radius ρ_* is defined by (45). Let $z \in U(x_*,\rho_*) - \{x_*\}$. Then, using (C1), (45) and (46), we obtain

$$\|F'(x_*)^{-1}(F'(z)-F'(x_*))\| \leq \ell_0\|z-x_*\| \leq \ell_0\rho_* < 1,$$

so $F'(z)$ is invertible with

$$\|F'(z)^{-1}F'(x_*)\| \leq \frac{1}{1-\ell_0\|z-x_*\|}, \tag{52}$$

and iterate y_0 exists by (52) for $z = x_0$. Then, we can write

$$y_0 - x_* = \int_0^1 F'(x_0)^{-1}(F'(x_* + \theta(x_0-x_*)) - F'(x_0))d\theta(x_0-x_*),$$

so by (C1), (C2) and (52) (for $z = x_0$), we get

$$\begin{aligned} \|y_0 - x_*\| &\leq \frac{\ell\|x_0-x_*\|^2}{2(1-\ell_0\|x_0-x_*\|)} \\ &\leq \psi_1(\|x_0-x_*\|)\|x_0-x_*\| \\ &\leq \|x_0-x_*\| < \rho_*, \end{aligned} \tag{53}$$

so $y_0 \in U(x_*,r)$ and (50) hold for $n=0$. As in (52), we also show

$$\|F'(y_0)^{-1}F'(x_*)\| \leq \frac{1}{1-\ell_0\|y_0-x_*\|} \tag{54}$$

and

$$\|[y_0,x_0;F]^{-1}F'(x_*)\| \leq \frac{1}{1-\ell_4(\|y_0-x_*\|+\|x_0-x_*\|)}, \tag{55}$$

so iterate x_1 exists. Then, we can write in turn by the second substep of method (2) that

$$\begin{aligned} x_1 - x_* &= y_0 - x_* - F'(y_0)^{-1}F(y_0) \\ &\quad +(F'(y_0)^{-1} - [y_0, x_0; F]^{-1})F(y_0) \\ &\quad +([y_0, x_0; F]^{-1} - F'(x_0)^{-1})F(y_0) \\ &= y_0 - x_* - F'(y_0)^{-1}F(y_0) \\ &\quad +F'(y_0)^{-1}([y_0, x_0; F] - F'(y_0))[y_0, x_0; F]^{-1}F(y_0) \\ &\quad +[y_0, x_0; F]^{-1}(F'(x_0) - [y_0, x_0; F])F'(x_0)^{-1}F(y_0). \end{aligned} \tag{56}$$

Then, in view of (45), (49) (C2), (52) (for $z = y_0$) and (54)–(56), we get in turn that

$$\begin{aligned} \|x_1 - x_*\| &\leq \frac{\ell\|y_0 - x_*\|^2}{2(1 - \ell_0\|y_0 - x_*\|)} \\ &\quad + \frac{\ell_3\|y_0 - x_0\|\ell_4\|y_0 - x_*\|}{(1 - \ell_0\|y_0 - x_*\|)(1 - \ell_1(\|y_0 - x_*\| + \|x_0 - x_*\|))} \\ &\quad + \frac{\ell_2\|y_0 - x_0\|\ell_4\|y_0 - x_*\|}{(1 - \ell_1(\|y_0 - x_*\| + \|x_0 - x_*\|))^2} \\ &\leq \psi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, \end{aligned} \tag{57}$$

showing (51) for $n = 0$ and $x_1 \in U(x_*, \rho_*)$, where we also used (53) and

$$\begin{aligned} \|y_0 - x_0\| &\leq \|y_0 - x_*\| + \|x_* - x_0\| \\ &\leq \psi_1(\|x_0 - x_*\|)\|x_0 - x_*\| + \|x_0 - x_*\| \\ &= (1 + \psi_1(\|x_0 - x_*\|))\|x_0 - x_*\|. \end{aligned}$$

If we exchange x_0, y_0, x_1 by x_m, y_m, x_{m+1}, respectively, in the previous calculations we complete the induction for (50) and (51). Then, from the estimate

$$\|x_{m+1} - x_*\| \leq \lambda_1\|x_m - x_*\|, \tag{58}$$

where $\lambda_1 = \psi_2(\|x_0 - x_*\|) \in [0, 1)$, we conclude $\lim_{m\longrightarrow+\infty} x_m = x_*$. We also have

$$\|y_m - x_*\| \leq \lambda_2\|x_m - x_*\| < \rho_*, \tag{59}$$

where $\lambda_2 = \psi_1(\|x_0 - x_*\|) \in [0, 1)$, so$\lim_{m\longrightarrow+\infty} y_m = x_*$. □

Next, we present a uniqueness of the solution result.

Proposition 2. *Suppose:*
(a) $x_* \in D$ *is a simple solution of equation* $F(x) = 0$.
(b) There exists $\tilde{s} \geq 0$ *such that*

$$\frac{\ell_0}{2}\tilde{s} < 1. \tag{60}$$

Set $D_4 = D \cap U[x_*, \tilde{s}]$. *Then, the only solution of equation* $F(x) = 0$ *in the region* D_4 *is* x_*.

Proof. Let $x^{**} \in D_4$ with $F(x^{**}) = 0$. Set $Q = \int_0^1 F'(x_* + \theta(x^{**} - x_*))d\theta$. Then, using (C1) and (60), we obtain

$$\begin{aligned} \|F'(x_*)^{-1}(Q - F'(x_*))\| &\leq \frac{\ell_0}{2}\|x^{**} - x_*\| \\ &\leq \frac{\ell_0}{2}\tilde{s} < 1, \end{aligned}$$

so $x^{**} = x_*$, since $Q^{-1} \in L(E_1, E)$ and $Q(x^{**} - x_*) = F(x^{**}) - F(x_*) = 0 - 0 = 0$. □

5. Numerical Experiments

We provide some examples, showing that the old convergence criteria are not verified, but ours are. The divided difference is chosen by

$$[u,v;F] = \int_0^1 F'(v+\theta(u-v))d\theta.$$

Example 1. *Define function*

$$h(t) = c_0 t + c_1 + c_2 \sin c_3 t,\ t_0 = 0,$$

where c_j, $j = 0,1,2,3$ are parameters. Then, clearly for c_3 large and c_2 small, $\frac{L_0}{L}$ can be small (arbitrarily).

Example 2. *Let $E = E_1 = H([0,1])$ the domain of functions given on $[0,1]$, which are continuous. We consider the max-norm. Choose $D = U(0,d)$, $d > 1$. Define G on D be*

$$G(x)(s) = x(s) - w(s) - \epsilon \int_0^1 P(s,t)x^3(t)dt, \tag{61}$$

$x \in E, s \in [0,1], w \in E$ is given, ϵ is a parameter and P is the Green's kernel given by

$$P(\epsilon_2,\epsilon_1) = \begin{cases} (1-\epsilon_2)\epsilon_1, & \epsilon_1 \le \epsilon_2 \\ \epsilon_2(1-\epsilon_1), & \epsilon_2 \le \epsilon_1. \end{cases}$$

By (61), we have

$$(G'(x)(z))(s) = z(s) - 3\epsilon \int_0^1 P(s,t)x^2(t)z(t)dt,$$

$t \in E, s \in [0,1]$. Consider $x_0(s) = w(s) = 1$ and $|\epsilon| < \frac{8}{3}$. We get

$$\|I - G'(x_0)\| < \frac{3}{8}|\epsilon|,\ G'(x_0)^{-1} \in L(E_1,E),$$

$$\|F'(x_0)^{-1}\| \le \frac{8}{8-3|\epsilon|},\ \eta = \frac{|\epsilon|}{8-3|\epsilon|},\ L_0 = \frac{12|\epsilon|}{8-3|\epsilon|},$$

and $L = \frac{6\eta|\epsilon|}{8-3|\epsilon|}$.

Example 3. *Let E, E_1 and D be as in the Example 5.3. It is well known that the boundary value problem [4]*

$$\xi(0) = 0, (1) = 1,$$
$$\xi'' = -\xi - \lambda\xi^2$$

can be given as a Hammerstein-like nonlinear integral equation

$$\xi(s) = s + \int_0^1 K(s,t)(\xi^3(t) + \lambda\xi^2(t))dt$$

where λ is a parameter. Then, define $F : D \longrightarrow E_1$ by

$$[F(x)](s) = x(s) - s - \int_0^1 K(s,t)(x^3(t) + \lambda x^2(t))dt.$$

Choose $\xi_0(s) = s$ *and* $D = U(\xi_0, \rho_0)$. *Then, clearly* $U(\xi_0, \rho_0) \subset U(0, \rho_0 + 1)$, *since* $\|\xi_0\| = 1$. *Suppose* $2\lambda < 5$. *Then, conditions (A) are satisfied for*

$$L_0 = \frac{2\lambda + 3\rho_0 + 6}{8}, \; L = \frac{\lambda + 6\rho_0 + 3}{4}$$

and $\eta = \frac{1+\lambda}{5-2\lambda}$. *Notice that* $L_0 < L$.

Example 4. *Consider the motion system*

$$T_1'(x) = e^x, \; T_2'(y) = (e-1)y + 1, \; T_3'(z) = 1$$

with $T_1(0) = T_2(0) = T_3(0) = 0$. *Let* $T = (T_1, T_2, T_3)$. *Let* $E = E_1 = \mathbb{R}^3, D = B[0,1], x_* = (0,0,0)^T$. *Define function* T *on* D *for* $w = (x,y,z)^T$ *by*

$$T(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, we get

$$T'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so $\ell_0 = e - 1, \ell = e^{\frac{1}{e-1}} = \ell_1, \ell_2 = \ell_3 = \frac{\ell}{2}, \; \ell_4 = \frac{\ell_0}{2}$. *Then, the radii are:*

$$\rho_A = 0.3827 = \rho^*, \rho_\mu = 1.7156.$$

6. Conclusions

A finer convergence analysis is presented for method (2) utilizing generalized conditions. This analysis includes weaker criteria of convergence and computable error bounds not given in earlier papers.

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