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Starlikeness of New General Differential Operators Associated with q-Bessel Functions

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Abstract: Owning to the importance and great interest of differential operators, two generalized differential operators, which may be symmetric or assymetric, are newly introduced in the present paper. Motivated by the familiar Jackson's second and third Bessel functions, we derive necessary and sufficient conditions for which the new generalized operators belong to the class of q-starlike functions of order alpha. Several corollaries and consequences of the main results are also pointed out.

Keywords: analytic functions; univalent functions; q-starlike functions; q-difference operator; differential subordination; q-Bessel functions

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1. Introduction

The area of quantum calculus (q-calculus) has caught the attention of many scientists. The great concentration in numerous branches of mathematics and physics is due to its wide spread applications in various areas of sciences. It is also well known that the time scale calculus includes q-calculus as a special case; see, e.g., the papers [1,2], (which have numerous applications in mathematics and phisics) for more details. In the investigation of multiple subclasses of analytic functions, the versatile applications of the q-derivative operator is fairly obvious from its applications. The concept of q-starlike functions was introduced by Ismail et al. [3] in 1990. At the same time, in the way of Geometric Function Theory, a strong foothold of the use of the q-calculus was fruitfully estabilished. Following that, several mathematicians have performed notable studies, which play an important role in the advancement of geometric function theory. A survey-cum-expository review paper was recently published by Srivastava [4], work that could be helpful for further researchers and scholars working on this subject- matter. In this survey, the mathematical description and implementations of the fractional q-derivative operators and fractional q-calculus in geometric function theory were methodically explored [4]. Particularly, Srivastava et al. [5] also studied some classes of q-starlike functions related with conic region. For other recent contributions on this topic, one may refer to [6,7].

As it is well known, one of the top remarkable special functions is the Bessel function. As a result, the Bessel functions are important for solving many problems in physics, engineering, and mathematics (see [8]). In the past few years, many mathematicians been devoted on determining the varied requirements under which a Bessel function has geometric properties such as convexity, close-to-convexity and starlikeness in the frame of a unit disc.

In the present investigation, we consider some geometric properties including starlikeness of order α of Jackson's second and third q-Bessel functions, which are generalizations of the known classical Bessel function J_ν.

We recall some useful notations and concepts that will be used throughout this article.

Denote by $\mathcal{A}$ the class of functions $f(z)$, normalized by $f(0) = 0 = f^{'}(0) - 1$, that are analytic in the unit disk $U = \{z \in \mathbb{C} \mid |z| < 1\}$. The function $f \in \mathcal{A}$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in U. \tag{1}$$

Principle of Subordination (see [9]): If f and g are two analytic functions in U, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ implies $f(U_r) \subset g(U_r)$, where $U_r = \{z \in \mathbb{C}, |z| < r, 0 < r < 1\}$.

Let B denote the class of Schwarz functions $\Phi(z)$ of the form

$$\Phi(z) = \sum_{n=1}^{\infty} c_n z^n,\ z \in U,$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and satisfy the condition $\Phi(0) = 0$ and $|\Phi(z)| < 1$.

We state the following well-known result for the class B.

Lemma 1. *(Schwarz lemma) If $\Phi(z) \in \mathrm{B}$, then $|\Phi(z)| \leq |z|$ and $|c_1| \leq 1$ are obtained.*

We denote by S the class all functions in $\mathcal{A}$ which are univalent in U. Denote by S^* the subclass of functions $f(z) \in S$ that are starlike with the respect to the origin. Anaytically, it is well-known that $f(z) \in S^*$ if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U.$$

The class $S^*(\alpha)$ of starlike functions of order α consist of $f(z) \in \mathcal{A}$ that satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U,\ 0 \leq \alpha < 1,$$

i.e., f has the subordination property

$$\frac{zf^{'}(z)}{f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z},\ z \in U,\ 0 \leq \alpha < 1.$$

We now provide some notations and basic concepts of q-calculus which will be needed in our further considerations.

The theory of q-extensions or q-analogues of classical formulas and functions is based on the remark that

$$\lim_{q \to 1^-} \frac{1 - q^n}{1 - q} = n,\ q \in (0,1),\ n \in \mathbb{N}, \tag{2}$$

therefore the number $\frac{1-q^n}{1-q}$ is sometimes called the basic number $[n]_q$. The q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} [n]_q \cdot [n-1]_q \cdots [1]_q, \text{ for } n = 1, 2, ...; \\ 1, \text{ for } n = 0. \end{cases} \tag{3}$$

As $q \to 1^-$, $[n]_q \to n$, and this is the bookmark of a q-analogue: the limit as $q \to 1^-$ recovers the classical object.

In [10,11], Jackson introduced the q-difference operator $\left(D_q f\right)(z)$ acting on functions $f(z) \in \mathcal{A}$ defined as follows:

$$\begin{cases} \left(D_q f\right)(z) = \frac{f(z)-f(qz)}{z(1-q)},\ z \neq 0,\ 0 < q < 1; \\ \left(D_q f\right)(z)|_{z=0} = f'(0). \end{cases} \tag{4}$$

It can be noticed that $\left(D_q f\right)(z) \to f^{'}(z)$ as $q \to 1^-$.

The q-difference operator plays a major place in the theory of quantum phisics and hypergeometric series (see [12,13]).

Therefore, for a function $f(z) = z^n$ the q-derivative is given by

$$\left(D_q f\right)(z) = D_q(z^n) = \frac{1-q^n}{1-q} \cdot z^{n-1} = [n]_q z^{n-1}, \tag{5}$$

then $\lim\limits_{q\to 1^-} \left(D_q f\right)(z) = \lim\limits_{q\to 1^-} [n]_q z^{n-1} = nz^{n-1} = f^{'}(z)$, where $f^{'}(z)$ is the ordinary derivative.

From (4) we, have

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} a_n z^n,\ z \neq 0.$$

Under the hypothesis of the definition of q-derivates operator, for $f, g \in \mathcal{A}$ we have the following rules:

$$D_q((af(z)) \pm bg(z)) = aD_q f(z) \pm bD_q g(z),\ a, b \in \mathbb{C}\ ,$$

$$D_q(f(z)g(z)) = g(z)D_q f(z) + f(qz)D_q g(z),$$

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(z)g(qz)},\ g(z)g(qz) \neq 0.$$

In [14], Agrawal and Sahoo introduced the class of q-starlike functions of order α, denoted by $S_q^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to belong to the class $S_q^*(\alpha)$, for $0 \leq \alpha < 1$, if

$$\left|\frac{zD_q(f(z))}{f(z)} - \frac{1-\alpha q}{1-q}\right| \leq \frac{1-\alpha}{1-q},\ z \in U.$$

Particularly, when $\alpha = 0$, the class $S_q^*(\alpha)$ coincides with the class S_q^*, which was initiated by Ismail et al. (see [3]).

Recall that the q-shifted factorial, also called the q-Pochhammer symbol, is defined as

$$(a;q)_n = \prod_{k=1}^{n}\left(1 - aq^{k-1}\right),\ 0 < q < 1,$$

with $(a;q)_0 = 1$. The q-Pochhammer symbol can be extended to an infinite product

$$(a;q)_\infty = \prod_{k\geqslant 1}\left(1 - aq^{k-1}\right),\ 0 < q < 1,$$

with the special case

$$(q;q)_\infty = \prod_{k=1}^{\infty}\left(1 - q^k\right),\ 0 < q < 1,$$

known as Euler's function.

The Jackson's second and Hahn–Exton (or third Jackson) q-Bessel functions are defined by (see [15])

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geqslant 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{n(n+\nu)} \tag{6}$$

and

$$J_{\nu}^{(3)}(z;q) = \frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}} \sum_{n \geqslant 0} \frac{(-1)^n (z)^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{\frac{1}{2}n(n+\nu)}, \tag{7}$$

where $z \in \mathbb{C}$, $\nu > -1$, $0 < q < 1$ and

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=1}^{n}\left(1 - aq^{k-1}\right),\ (a;q)_{\infty} = \prod_{k \geqslant 1}\left(1 - aq^{k-1}\right).$$

These analytic functions are $q-$ extensions of the classical Bessel functions of the first kind J_ν. Properties of the above $q-$ extensions of Bessel functions can be found in [16,17] and in the references therein. Because neither $J_{\nu}^{(2)}(z;q)$, nor $J_{\nu}^{(3)}(z;q)$ belongs to the class $\mathcal{A}$, we consider the following normalized forms (see [18]):

$$h_{\nu}^{(2)}(z;q) = 2^{\nu} c_{\nu}(q) z^{1-\frac{\nu}{2}} J_{\nu}^{(2)}\left(\sqrt{z};q\right) = \sum_{n \geqslant 0} K_n z^{n+1}, \tag{8}$$

and

$$h_{\nu}^{(3)}(z;q) = c_{\nu}(q) z^{1-\frac{\nu}{2}} J_{\nu}^{(3)}\left(\sqrt{z};q\right) = \sum_{n \geqslant 0} T_n z^{n+1}, \tag{9}$$

where $c_{\nu}(q) = \frac{(q;q)_{\infty}}{(q^{\nu+1};q)_{\infty}}$, $K_n = \dfrac{(-1)^n q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n}$, $T_n = \dfrac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q;q)_n (q^{\nu+1};q)_n}$ and $\nu > -1$, $0 < q < 1$, $z \in \mathbb{C}$.

Clearly, the above functions $h_{\nu}^{s}(z;q)$, $s \in \{2,3\}$, belong to the class $\mathcal{A}$.

2. Main Results

We now define the following two differential operators:

$$\begin{aligned}
H_{\nu,\lambda}^{(2),0}(q)f(z) &= f(z) * h_{\nu}^{(2)}(z;q),\\
H_{\nu,\lambda}^{(2),1}(q)f(z) &= (1-\lambda)f(z) * h_{\nu}^{(2)}(z;q) + \lambda z D_q\left(f(z) * h_{\nu}^{(2)}(z;q)\right),\\
&\quad \ldots\ldots\\
H_{\nu,\lambda}^{(2),m}(q)f(z) &= H_{\nu,\lambda}^{(2),1}\left(H_{\nu,\lambda}^{(2),m-1}(q)f(z)\right)\\
&= z + \sum_{k=2}^{\infty}\left[1 + \left([k]_q - 1\right)\lambda\right]^m K_{k-1} a_k z^k,
\end{aligned} \tag{10}$$

for $\lambda \geqslant 0, \nu > -1, 0 < q < 1, z \in \mathbb{C}, m \in \mathbb{N}$, where $*$ denotes the usual Hadamard product of analytic functions,

$$K_{k-1} = \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}$$

and

$$\begin{aligned}
H_{\nu,\lambda}^{(3),0}(q)f(z) &= f(z) * h_{\nu}^{(3)}(z;q),\\
H_{\nu,\lambda}^{(3),1}(q)f(z) &= (1-\lambda)f(z) * h_{\nu}^{(3)}(z;q) + \lambda z D_q\left(f(z) * h_{\nu}^{(3)}(z;q)\right),\\
&\quad \ldots\ldots\\
H_{\nu,\lambda}^{(3),m}(q)f(z) &= H_{\nu,\lambda}^{(3),1}\left(H_{\nu,\lambda}^{(3),m-1}(q)f(z)\right)\\
&= z + \sum_{k=2}^{\infty}\left[1 + \left([k]_q - 1\right)\lambda\right]^m T_{k-1} a_k z^k,
\end{aligned} \tag{11}$$

for $\lambda \geqslant 0, \nu > -1, 0 < q < 1, z \in \mathbb{C}, m \in \mathbb{N}$, where $*$ denotes the usual Hadamard product of analytic functions and

$$T_{k-1} = \frac{(-1)^{k-1} q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}.$$

In this paper, we give some necesssary and sufficient conditions for the functions $H_{\nu,\lambda}^{(2),m}(q)f(z)$ and $H_{\nu,\lambda}^{(3),m}(q)f(z)$ to be in the class of $q-$ starlike of order alpha. Some consequences of the main results are also pointed out.

Theorem 1. *The function $H_{\nu,\lambda}^{(2),m}(q)f(z) \in S_q^*(\alpha)$ if and only if*

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz},\ z \in U,\ 0 \le \alpha < 1,\ 0 < q < 1, \lambda \geqslant 0, \nu > -1.$$

Proof. Assuming that $h_\nu^2(z;q) \in S_q^*(\alpha)$, we have:

$$\left| \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \frac{1-\alpha q}{1-q} \right| \le \frac{1-\alpha}{1-q} \Leftrightarrow$$

$$\left| \frac{1-q}{1-\alpha} \cdot \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \frac{1-\alpha q}{1-\alpha} \right| \le 1.$$

Therefore, the function

$$\varphi(z) = \frac{1-q}{1-\alpha} \cdot \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \frac{1-\alpha q}{1-\alpha}$$

has modulus at most 1 in the unit disk U and $\varphi(0) = -q$.

Let

$$\Phi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)} = \frac{\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - 1}{[1-\alpha(1+q)] + q\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)}}.$$

The function $\Phi(z)$ satisfies the conditions of Schwarz lemma, i.e., $\Phi(0) = 0$, $|\Phi(0)| < 1$, so, we have:

$$|\Phi(z)| \le z.$$

We obtain

$$\Phi(z)[1-\alpha(1+q)] + \Phi(z)q\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} = \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - 1.$$

So,

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} = \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)}.$$

The above equality shows that

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz}.$$

Conversely, let

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz}.$$

Then, we have

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} \prec \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)} \Rightarrow$$

$$\begin{aligned}\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \frac{1-\alpha q}{1-q} &= \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)} - \frac{1-\alpha q}{1-q} \\ &= \frac{\Phi(z)(1-\alpha)-q(1-\alpha)}{(1-q\Phi(z))(1-q)} = \frac{1-\alpha}{1-q}\cdot\frac{-q+\Phi(z)}{1-q\Phi(z)}.\end{aligned}$$

The function $\frac{-q+\Phi(z)}{1-q\Phi(z)}$, with $\Phi(z)=0$, $|\Phi(z)|\le 1$, maps the unit disk into itself, so

$$\left|\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \frac{1-\alpha q}{1-q}\right| = \left|\frac{1-\alpha}{1-q}\cdot\frac{-q+\Phi(z)}{1-q\Phi(z)}\right| < \frac{1-\alpha}{1-q}$$

and the proof is now complete. □

Theorem 2. *The function $H_{\nu,\lambda}^{(3),m}(q)f(z)\in S_q^*(\alpha)$ if and only if*

$$\frac{zD_q\left(H_{\nu,\lambda}^{(3),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(3),m}(q)f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz},$$

$z\in U$, $0\le\alpha<1$, $0<q<1, \lambda\geqslant 0, \nu>-1$.

Proof. The proof is similar to the proof of Theorem 1. □

Corollary 1. *Let $H_{\nu,\lambda}^{(2),m}(q)f(z)\in S_q^*(\alpha)$. Then*

$$\frac{1-r[1-\alpha(1+q)]}{1+qr} \le \left|\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)}\right| \le \frac{1+r[1-\alpha(1+q)]}{1-qr},$$

for $z\in U$, $0\le\alpha<1$, $0<q<1, \lambda\geqslant 0, \nu>-1$.

Proof. The linear transformation

$$\omega(z)=\frac{1+z[1-\alpha(1+q)]}{1-qz}$$

maps $|z|=r$ onto the circle with the center $C(r)=(c,0)$, where

$$c=\frac{\omega(r)+\omega(-r)}{2}=\frac{1}{2}\left[\frac{1+r[1-\alpha(1+q)]}{1-qr}+\frac{1-r[1-\alpha(1+q)]}{1+qr}\right]$$

$$= \frac{1+r^2q[1-\alpha(1+q)]}{1-q^2r^2}$$

and the radius

$$\rho(r) = \frac{1+r[1-\alpha(1+q)]}{1-qr} - \frac{1+r^2q-\alpha r^2q(1+q)}{1-q^2r^2}$$
$$= \frac{r(1-\alpha)(1+q)}{1-q^2r^2}.$$

Using the subordination principle, we get

$$\left|\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \frac{1+qr^2[1-\alpha(1+q)]}{1-q^2r^2}\right| \le \frac{r(1-\alpha)(1+q)}{1-q^2r^2},$$

which readily yields

$$\frac{1-r[1-\alpha(1+q)]}{1+qr} \le \left|\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)}\right| \le \frac{1+r[1-\alpha(1+q)]}{1-qr}.$$

This proves the conclusion of the corollary. □

Corollary 2. *Let $H_{\nu,\lambda}^{(3),m}(q)f(z) \in S_q^*(\alpha)$. Then*

$$\frac{1-r[1-\alpha(1+q)]}{1+qr} \le \left|\frac{zD_q\left(H_{\nu,\lambda}^{(3),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(3),m}(q)f(z)}\right| \le \frac{1+r[1-\alpha(1+q)]}{1-qr}.$$

Proof. The proof is similar to the proof of Corollary 1. □

Next we derive Theorem 3 bellow.

Theorem 3. *Letting $H_{\nu,\lambda}^{(2),m}(q)f(z) \in S_q^*(\alpha)$, then*

$$\sum_{k=2}^{n}\left|\frac{q-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2 \le$$

$$\sum_{k=2}^{n-1}\left|\left(\frac{1-q^{k+1}}{1-q}-\alpha(1+q)\right)\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2,$$

for $n = 2, 3, \ldots$.

Proof. By using the definition of $D_qf(z)$ we get

$$zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right) =$$
$$z + \sum_{k=2}^{\infty}\frac{1-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k. \tag{12}$$

On the other hand,

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} = \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)}.$$

It follows that

$$\begin{aligned} & zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right) - H_{\nu,\lambda}^{(2),m}(q)f(z) \\ = \;& \Phi(z)\left[H_{\nu,\lambda}^{(2),m}(q)f(z)(1-\alpha(1+q)) + qzD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)\right]. \end{aligned} \tag{13}$$

Using (10) and (11), we obtain

$$z + \sum_{k=2}^{\infty} \frac{1-q^k}{1-q}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k -$$

$$z - \sum_{k=2}^{\infty}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k =$$

$$\Phi(z)\left[(1-\alpha(1+q))\left(z + \sum_{k=2}^{\infty}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k\right)\right.$$

$$\left. + qz + \sum_{k=2}^{\infty} \frac{1-q^k}{1-q}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} q a_k z^k\right],$$

or, equivalently

$$\sum_{k=2}^{\infty}\left(\frac{1-q^k}{1-q} - 1\right)\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k$$

$$= \Phi(z)[(1+q)(1-\alpha)z +$$

$$\left.\sum_{k=2}^{\infty}\left(\frac{q-q^{k+1}}{1-q} + 1 - \alpha(1+q)\right)\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k\right].$$

So,

$$\sum_{k=2}^{\infty} \frac{q-q^k}{1-q}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k$$

$$= \Phi(z)[(1+q)(1-\alpha)z +$$

$$\left.\sum_{k=2}^{\infty}\left(\frac{1-q^{k+1}}{1-q} - \alpha(1+q)\right)\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k\right].$$

Thus,

$$\sum_{k=2}^{n} \frac{q-q^k}{1-q}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k$$

$$= \Phi(z)[(1+q)(1-\alpha)z +$$

$$\left.\sum_{k=2}^{n-1}\left(\frac{1-q^{k+1}}{1-q} - \alpha(1+q)\right)\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k\right],$$

where the sum $\sum\limits_{k=n+1}^{\infty} b_k z^k$ is convergent in U. Letting $z = re^{i\theta}$, and since $|\Phi(z)| \le 1$, we deduce that

$$\sum_{k=2}^{n}\left|\frac{q-q^k}{1-q}\left[1+\left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k\right|^2 r^{2k} \le$$

$$\sum_{k=2}^{n-1}\left|\left(\frac{1-q^{k+1}}{1-q}-\alpha(1+q)\right)\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2 r^{2k}. \quad (14)$$

Now passing to the limit in (14), as $r \to 1$, we obtain the required inequality, hence is now complete.

The proof in this theorem is based on Clunie's method (see [19]). □

Theorem 4. *Let $H^{(3),m}_{\nu,\lambda}(q)f(z) \in S^*_q(\alpha)$. Then*

$$\sum_{k=2}^{n}\left|\frac{q-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2 \leq$$

$$\sum_{k=2}^{n-1}\left|\left(\frac{1-q^{k+1}}{1-q}-\alpha(1+q)\right)\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2,$$

for $n = 2, 3, \ldots$.

Proof. The proof is similar to the proof of Theorem 3. □

The next result deals with the famous Bieberbach conjecture problem in analytic univalent function theory. The Bieberbach conjecture for the class S^*_q is proved in [20].

A necessary and sufficient condition for functions $f(z)$ to be in $S^*_q(\alpha)$ was obtained in [14]:

Theorem 5. *A function $f(z) \in S^*_q(\alpha)$ if and only if*

$$\left|\frac{f(qz)}{f(z)}-\alpha q\right| \leq 1-\alpha,\ z \in U.$$

By using this result, we will analyse the Bieberbach - de Branges theorem for the class of q-starlike functions of order α.

Theorem 6. *If $H^{(2),m}_{\nu,\lambda}(q)f(z) \in S^*_q(\alpha)$, then for all $k \geq 2$, we have*

$$|a_k| \leq \frac{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}{\left[1+\left([k]_q-1\right)\lambda\right]^m q^{(k-1)(k+\nu-1)}}\cdot$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\cdot\prod_{l=2}^{k-1}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^l}\right),$$

for $z \in U$, $0 \leq \alpha < 1$, $0 < q < 1, \lambda \geqslant 0, \nu > -1$.

Proof. Let

$$H^{(2),m}_{\nu,\lambda}(q)f(z) \in S^*_q(\alpha) \Leftrightarrow$$

$$\left|\frac{H^{(2),m}_{\nu,\lambda}(q)f(qz)}{H^{(2),m}_{\nu,\lambda}(q)f(z)}-\alpha q\right| \leq 1-\alpha \Leftrightarrow \left|\frac{\dfrac{H^{(2),m}_{\nu,\lambda}(q)f(qz)}{H^{(2),m}_{\nu,\lambda}(q)f(z)}-\alpha q}{1-\alpha}\right| \leq 1.$$

Then there exists $\varpi : U \to \overline{U}$ such that

$$\varpi(z) = \frac{\dfrac{H_{\nu,\lambda}^{(2),m}(q)f(qz)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \alpha q}{1-\alpha}.$$

Clearly, $\varpi(0) = q$.
It follows that

$$\varpi(z)(1-\alpha) = \frac{H_{\nu,\lambda}^{(2),m}(q)f(qz)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - \alpha q,$$

so,

$$H_{\nu,\lambda}^{(2),m}(q)f(qz) = \varpi(z)H_{\nu,\lambda}^{(2),m}(q)f(z)(1-\alpha) + \alpha q H_{\nu,\lambda}^{(2),m}(q)f(z).$$

For $a_1 = 1,\ \varpi_0 = q$ we get

$$\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q^k z^k =$$

$$\left(\sum_{k=0}^{\infty}\varpi_k z^k\right)\left(\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k z^k\right).$$

$$(1-\alpha)+\alpha q\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k z^k =$$

$$\left((1-\alpha)\sum_{k=0}^{\infty}\varpi_k z^k+\alpha q\right)\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k z^k.$$

Comparing the coefficients of z^k $(k \geq 2)$, we get

$$\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q^k =$$

$$\alpha\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q+$$

$$(1-\alpha)\left[1+\left([k]_q k-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q+$$

$$(1-\alpha)\sum_{l=1}^{k-1}\varpi_{k-l}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,$$

thus,

$$\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\left(q^k-q\right) =$$

$$(1-\alpha)\sum_{l=1}^{k-1}\varpi_{k-l}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\ \text{for } k \geq 2.$$

Since $|\varpi_k| \leq 1-|\varpi_0|^2 = 1-q^2$, for $k \geq 1$,

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right| \leq$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\sum_{l=1}^{k-1}\left[1+\left([l]_q-1\right)\lambda\right]^m\frac{(-1)^{l-1}q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\ k\geq 2.$$

Thus, for $k=2$, $\left|[1+\lambda]^m\frac{(-1)q^{1+\nu}}{4^{k-1}(q;q)_1(q^{\nu+1};q)_1}a_2\right|\leq\frac{(1-q^2)(1-\alpha)}{q-q^2}$, and for $k\geq 3$, by applying a similar method to estimate $|a_{k-1}|$, we obtain

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|\leq$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)$$

$$\sum_{l=1}^{k-1}\left[1+\left([l]_q-1\right)\lambda\right]^m\frac{(-1)^{l-1}q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l.$$

Iteratively, we conclude that, for $k\geq 3$,

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|\leq$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)$$

$$\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-2}}\right)\ldots\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^2}\right),$$

and the proof is now completed. □

Theorem 7. *(The Bieberbach - de Branges theorem for $S_q^*(\alpha)$) If $H_{\nu,\lambda}^{(3),m}(q)f(z)\in S_q^*(\alpha)$, then for all $k\geq 2$, we have*

$$|a_k|\leq\frac{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}{\left[1+\left([k]_q-1\right)\lambda\right]^m q^{\frac{1}{2}k(k-1)}}.$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\cdot\prod_{l=2}^{k-1}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^l}\right),$$

for $z\in U,\ 0\leq\alpha<1,\ 0<q<1,\lambda\geqslant 0,\nu>-1$.

Proof. Let

$$H_{\nu,\lambda}^{(3),m}(q)f(z)\in S_q^*(\alpha)\Leftrightarrow$$

$$\left|\frac{H_{\nu,\lambda}^{(3),m}(q)f(qz)}{H_{\nu,\lambda}^{(3),m}(q)f(z)}-\alpha q\right|\leq 1-\alpha\Leftrightarrow\left|\frac{\frac{H_{\nu,\lambda}^{(3),m}(q)f(qz)}{H_{\nu,\lambda}^{(3),m}(q)f(z)}-\alpha q}{1-\alpha}\right|\leq 1.$$

Then there exists $\varpi:U\to\overline{U}$ such that

$$\varpi(z)=\frac{\frac{H_{\nu,\lambda}^{(3),m}(q)f(qz)}{H_{\nu,\lambda}^{(3),m}(q)f(z)}-\alpha q}{1-\alpha}.$$

Clearly, $\varpi(0) = q$.

It follows that

$$\varpi(z)(1-\alpha) = \frac{H_{\nu,\lambda}^{(3),m}(q)f(qz)}{H_{\nu,\lambda}^{(3),m}(q)f(z)} - \alpha q,$$

so,

$$H_{\nu,\lambda}^{(3),m}(q)f(qz) = \varpi(z)H_{\nu,\lambda}^{(3),m}(q)f(z)(1-\alpha) + \alpha q H_{\nu,\lambda}^{(3),m}(q)f(z).$$

For $a_1 = 1,\ \varpi_0 = q$ we get

$$\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q^k z^k =$$

$$\left(\sum_{k=0}^{\infty}\varpi_k z^k\right)\left(\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k z^k\right)(1-\alpha)+$$

$$\alpha q\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k z^k =$$

$$\left((1-\alpha)\sum_{k=0}^{\infty}\varpi_k z^k + \alpha q\right)\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k z^k.$$

Comparing the coefficients of z^k $(k \geq 2)$, we get

$$\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q^k =$$

$$\alpha\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q+$$

$$(1-\alpha)\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k q+$$

$$(1-\alpha)\sum_{l=1}^{k-1}\varpi_{k-l}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,$$

thus,

$$\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\left(q^k-q\right) =$$

$$(1-\alpha)\sum_{l=1}^{k-1}\varpi_{k-l}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\ \text{for } k \geq 2.$$

Since $|\varpi_k| \leq 1 - |\varpi_0|^2 = 1 - q^2$, for $k \geq 1$,

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right| \leq$$

$$\frac{\left(1-q^2\right)(1-\alpha)}{q-q^k}\sum_{l=1}^{k-1}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\ k \geq 2.$$

Thus, for $k = 2$, $\left|[1+\lambda]^m \frac{(-1)q}{(q;q)_1(q^{\nu+1};q)_1} a_2\right| \leq \frac{(1-q^2)(1-\alpha)}{q-q^2}$, and for $k \geq 3$, by applying a similar method to estimate $|a_{k-1}|$, we obtain

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k\right| \leq$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)$$

$$\sum_{l=1}^{k-1}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}} a_l.$$

Iteratively, we conclude that, for $k \geq 3$,

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k\right| \leq$$

$$\frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)$$

$$\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-2}}\right)\ldots\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^2}\right),$$

and the proof is now completed. □

Remark 1. *Our usages in the current investigation potentially own local or non-local symmetric or asymmetric properties. Our purpose for further investigation is to study the local symmetry of $H_{\nu,\lambda}^{(2),m}$ and $H_{\nu,\lambda}^{(3),m}$ and also to introduce and study an extention of them, symmetric under the interchange of q and q^{-1}, motivated by the work of Dattoli and Torre, who introduced (see [21]) a q-analogue of Bessel functions which are symmetric under the interchange of q and q^{-1}.*

3. Discussion

The current study is inspired by the clearly established potential for the applications of the q-calculus in Geometric Function Theory, as detailed in a newly released article by Srivastava [4]. We have considered two new generalized differential operators and motivated by the familiar Jackson's second and third Bessel functions, we obtained necessary and sufficient conditions for which the new generalized operators belong to the class of q-starlike functions of order alpha. Several corollaries and consequences of the main results were also pointed out.

We are hopeful that this research offers a base for further study in investigating several other classes of analytic functions by using the previously two new introduced generalized differential operators and their various geometric properties.

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