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Schur-Convexity for Elementary Symmetric Composite Functions and Their Inverse Problems and Applications

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Abstract: This paper investigates the Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity for the elementary symmetric composite function and its dual form. The inverse problems are also considered. New inequalities on special means are established by using the theory of majorization.

Keywords: symmetric function; Schur-convexity; inequality; special means

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1. Introduction

Throughout the article, the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, and $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i > 0, i = 1, \ldots, n\}$. $\mathbb{R}^1$ is denoted by $\mathbb{R}$ for simplicity.

In 1923, Schur [1] introduced the concept of the Schur-convex function. It can be applied to many aspects, including extended mean values [2–7], isoperimetric inequalities on the polyhedron [8], theory of statistical experiments [9], gamma and digamma functions [10], combinational optimization [11], graphs and matrices [12], reliability [13], information theoretic topics [14], stochastic orderings [15], and other related fields.

Zhang [16] and Chu et al. [17] proposed the notations of Schur-geometric convexity (or "Schur-multiplicative convexity") and Schur-harmonic convexity, respectively. Then the theory of majorization was enriched [18–27].

Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the k-th elementary symmetric function and its dual form, denoted by $E_k(\boldsymbol{x})$ and $E_k^*(\boldsymbol{x})$, respectively, are defined as

$$E_k(\boldsymbol{x}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} x_{i_j}, \quad E_k^*(\boldsymbol{x}) = \prod_{1 \le i_1 < \cdots < i_k \le n} \sum_{j=1}^{k} x_{i_j}, \quad k = 1, 2, \ldots, n.$$

Let $f : I \to \mathbb{R}$ be a function on an interval $I \subseteq \mathbb{R}$. In this paper, the k-th elementary symmetric composite function and its dual form are denoted by

$$E_k(f, \boldsymbol{x}) = E_k(f(x_1), \ldots, f(x_n)), \quad E_k^*(f, \boldsymbol{x}) = E_k^*(f(x_1), \ldots, f(x_n)).$$

Clearly $E_1(f, \boldsymbol{x}) = E_n^*(f, \boldsymbol{x})$, $E_n(f, \boldsymbol{x}) = E_1^*(f, \boldsymbol{x})$.

Schur [1] obtained that $E_k(\boldsymbol{x})$ is Schur-concave, increasing on $\mathbb{R}^n_+$. Shi et al. [21–23] proved that $E_k^*(\boldsymbol{x})$ is increasing Schur-concave on $\mathbb{R}^n_+$, $E_k(\boldsymbol{x})$ and $E_k^*(\boldsymbol{x})$ are increasing Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$. Xia et al. [24], Guan [25], Shi et al. [26], Sun [27], Chu et al. [17] constructed and studied the Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity of various special cases of $E_k(f, \boldsymbol{x})$ and $E_k^*(f, \boldsymbol{x})$; many interesting inequalities were established and proved.

Schur [1], Hardy et al. [28] studied the Schur-convexity of $E_1(f, \boldsymbol{x})$ (or $E_n^*(f, \boldsymbol{x})$) and obtained that:

Theorem 1 ([1,28]). *$E_1(f, \boldsymbol{x})$ (or $E_n^*(f, \boldsymbol{x})$) is Schur-convex on I^n if f is convex on $I \subseteq \mathbb{R}$.*

If f is continuous, the inverse problem of Theorem 1 also holds [29]. That is:

Theorem 2 ([29]). *If f is continuous on I, then f is convex on I if $E_1(f, \boldsymbol{x})$ (or $E_n^*(f, \boldsymbol{x})$) is Schur-convex on I^n.*

In 2010, Rovenţa [30] investigated the Schur-convexity of $E_2(f, \boldsymbol{x})$ and $E_{n-1}(f, \boldsymbol{x})$ and obtained that:

Theorem 3 ([30]). *Let $I \subseteq \mathbb{R}_+$ be an interval. If $f : I \to \mathbb{R}_+$ is differentiable in the interior of I and $\log f$ is convex and continuous on I, then $E_2(f, \boldsymbol{x})$ and $E_{n-1}(f, \boldsymbol{x})$ are Schur-convex functions on I.*

However, Rovenţa did not discuss the case of $3 \leq k \leq n - 2$.
In 2011, Wang et al. [31] proved the following two results.

Theorem 4 ([31]). *Let $I \subseteq \mathbb{R}_+$ be symmetric and convex with non-empty interior, and let $f : I \to \mathbb{R}_+$ be differentiable in the interior of I and continuous on I. If $\log f$ is convex, then $E_k(f, \boldsymbol{x})$ is a Schur-convex function on I^n for any $k = 1, 2, \ldots, n$.*

Theorem 5 ([31]). *Let $I \subseteq \mathbb{R}_+$ be symmetric and convex with non-empty interior, and let $f : I \to \mathbb{R}_+$ be differentiable in the interior of I and continuous on I. If $\log f$ is convex and increasing, then $E_k(f, \boldsymbol{x})$ is a Schur-geometrically convex and Schur-harmonically convex function on I^n for any $k = 1, 2, \ldots, n$.*

In 2013, Zhang and Shi [32] gave a simple proof of Theorems 4 and 5. In 2014, Shi et al. [33] obtained the following two results.

Theorem 6 ([33]). *Let $I \subseteq \mathbb{R}_+$ be symmetric and convex with non-empty interior, and let $f : I \to \mathbb{R}_+$ be differentiable in the interior of I and continuous on I. If $\log f$ is convex, then $E_k^*(f, \boldsymbol{x})$ is a Schur-convex function on I^n for any $k = 1, 2, \ldots, n$.*

Theorem 7 ([33]). *Let $I \subseteq \mathbb{R}_+$ be symmetric and convex with non-empty interior, and let $f : I \to \mathbb{R}_+$ be differentiable in the interior of I and continuous on I. If $\log f$ is convex and increasing, then $E_k^*(f, \boldsymbol{x})$ is a Schur-geometrically convex and Schur-harmonically convex function on I^n for any $k = 1, 2, \ldots, n$.*

Theorem 2 is the inverse problem of Theorem 1. Thus, the first aim of this paper is to study the inverse problems from Theorems 3 to 7. In contrast with these results, our study suggests that the functions that do not have to be monotonous and continuous.

The arithmetic mean of $x, y \in \mathbb{R}$ is defined by

$$A(x, y) = \frac{x + y}{2}.$$

The geometric mean, harmonic mean, identity mean, and logarithmic mean of $x, y > 0$ are respectively defined by

$$G(x,y) = \sqrt{xy}, \quad H(x,y) = \frac{2xy}{x+y},$$

$$I(x,y) = \begin{cases} \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}, & x \neq y, \\ x, & x = y, \end{cases} \quad L(x,y) = \begin{cases} \frac{x-y}{\log x - \log y}, & x \neq y, \\ x, & x = y. \end{cases}$$

It is well known that the following inequalities on special means

$$H(x,y) \leq G(x,y) \leq L(x,y) \leq I(x,y) \leq A(x,y), \quad x,y > 0 \tag{1}$$

have many important applications. Another aim of this paper is to establish new inequalities on special means by use of the Schur-convexity of $E_k(f, \boldsymbol{x})$, $E_k^*(f, \boldsymbol{x})$, and the theory of majorization.

2. Definitions and Lemmas

First, we introduce the concepts of Schur-convex function, Schur-geometrically convex function, and Schur-harmonically convex function.

For positive vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, we denote by

$$\frac{1}{\boldsymbol{x}} := \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right), \quad \log \boldsymbol{x} := (\log x_1, \ldots, \log x_n), \quad e^{\boldsymbol{x}} := (e^{x_1}, \ldots, e^{x_n}).$$

A function $\varphi : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be increasing on Ω if $x_i \leq y_i (1 \leq i \leq n)$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$ for any $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n) \in \Omega$.

Definition 1. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) *([34]) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if*

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for} \quad 1 \leq k \leq n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) *([34]) A function $\varphi : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) on Ω if*

$$\boldsymbol{x} \prec \boldsymbol{y} \Rightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y}), \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \Omega.$$

(iii) *([16]) A function $\varphi : \Omega \subseteq \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be Schur-geometrically convex (Schur-geometrically concave) on Ω if*

$$\log \boldsymbol{x} \prec \log \boldsymbol{y} \Rightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y}), \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \Omega.$$

(iv) *([23]) A function $\varphi : \Omega \subseteq \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be Schur-harmonically convex (Schur-harmonically concave) on Ω if*

$$\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}} \Rightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y}), \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \Omega.$$

Next, we introduce the concepts of convex function, geometrically convex function, and harmonically convex function.

Definition 2 ([22,23])**.** *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a function.*

(i) *f is called a convex (concave) function on I if*

$$f(\lambda x + (1-\lambda)y) \leq (\geq) \lambda f(x) + (1-\lambda) f(y), \ \forall\, x, y \in I, \ 0 \leq \lambda \leq 1.$$

(ii) $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *is called a geometrically convex (geometrically concave) function on I if*

$$f(x^\lambda y^{1-\lambda}) \le (\ge)[f(x)]^\lambda [f(y)]^{1-\lambda}, \ \forall\, x, y \in I,\ 0 \le \lambda \le 1.$$

(iii) $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *is called a harmonically convex (harmonically concave) function on I if*

$$f\left(\left(\frac{\lambda}{x} + \frac{1-\lambda}{y}\right)^{-1}\right) \le (\ge)\left(\frac{\lambda}{f(x)} + \frac{1-\lambda}{f(y)}\right)^{-1}, \ \forall\, x, y \in I,\ 0 \le \lambda \le 1.$$

Lemma 1. *Let* $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\varphi : \Omega \subseteq \mathbb{R}^n_+ \to \mathbb{R}_+$ *be functions.*

(i) ([22]) *f is geometrically convex (geometrically concave) on* $[a, b]$ *if and only if* $\log f(e^x)$ *is convex (concave) on* $[\log a, \log b]$.

(ii) ([23,35]) *f is harmonically convex (harmonically concave) on* $[a, b]$ *if and only if* $\frac{1}{f(\frac{1}{x})}$ *is concave (convex) on* $\left[\frac{1}{b}, \frac{1}{a}\right]$.

(iii) ([22]) φ *is Schur-geometrically convex (Schur-geometrically concave) on* Ω *if and only if* $\varphi(e^{\boldsymbol{x}})$ *is Schur-convex (Schur-concave) on* $\{\log \boldsymbol{x} \mid \boldsymbol{x} \in \Omega\}$.

(iv) ([23]) φ *is Schur-harmonically convex (Schur-harmonically concave) on* Ω *if and only if* $\varphi\left(\frac{1}{\boldsymbol{x}}\right)$ *is Schur-convex (Schur-concave) on* $\left\{\frac{1}{\boldsymbol{x}} \mid \boldsymbol{x} \in \Omega\right\}$.

Lemma 2 ([16,36])**.** *Let* $I \subseteq \mathbb{R}$ *be an interval, and let* $f : I \to \mathbb{R}$ *be a continuous function.*

(i) *f is convex (concave) on I if and only if*

$$f(A(x, y)) \le (\ge)A(f(x), f(y)), \ \forall\, x, y \in I.$$

(ii) $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *is geometrically convex (geometrically concave) on I if and only if*

$$f(G(x, y)) \le (\ge)G(f(x), f(y)), \ \forall\, x, y \in I.$$

(iii) $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *is harmonically convex (harmonically concave) on I if and only if*

$$f(H(x, y)) \le (\ge)H(f(x), f(y)), \ \forall\, x, y \in I.$$

Next, we prove the convexity of some functions involving $I(x, a + x)$ and $L(x, a + x)$.

Lemma 3. *Let* $a > 0$. *Then*

(i) $I(x, a + x)$ *and* $L(x, a + x)$ *are concave on* $\mathbb{R}_+$.

(ii) $I(x, a + x)$, $L(x, a + x)$ *and* $e^{[1/L(x,a+x)]}$ *are geometrically convex on* $\mathbb{R}_+$, $e^{[1/I(x,a+x)]}$ *is geometrically convex on* $[a, +\infty)$.

(iii) $I(x, a + x)$ *and* $L(x, a + x)$ *are harmonically convex on* $\mathbb{R}_+$.

Proof. For simplicity, we denote $f(x) = I(x, a + x)$, $g(x) = L(x, a + x)$.

(i) By a simple calculation, we can obtain that

$$f''(x) = f(x)\left[(\log f(x))'' + (\log f(x))'^2\right] = \frac{f(x)}{a^2}\left[\frac{-a^2}{x(a+x)} + \left(\log(1 + \tfrac{a}{x})\right)^2\right],$$
$$g''(x) = \frac{-a^2[(2x+a)(\log(x+a) - \log x) - 2a]}{x^2(x+a)^2[\log(x+a) - \log x]^3}.$$

Let

$$f_1(t) = -t - \tfrac{1}{t} + (\log t)^2 + 2, \quad t > 1;$$
$$\phi_x(s) = (2x + s)(\log(x + s) - \log x) - 2s, \quad s > 0,\ x > 0,$$

then

$$f''(x) = \frac{f(x)f_1(1+\frac{a}{x})}{a^2}, \quad g''(x) = \frac{-a^2\phi_x(a)}{x^2(x+a)^2[\log(x+a)-\log x]^3},$$

and

$$\begin{aligned} f_1'(t) &= \tfrac{1}{t}(-t+\tfrac{1}{t}+2\log t) < 0, \\ \phi_x'(s) &= -\log\tfrac{x}{x+s} - \tfrac{s}{x+s} > 0. \end{aligned}$$

Note that $f_1(1)=0$, $\phi_x(0)=0$, so $f_1(t)<0 (t>1)$ and $\phi_x(s)>0 (s>0)$. It follows that $f''(x)<0$ and $g''(x)<0$ on $\mathbb{R}_+$. Hence, $f(x)$ and $g(x)$ are concave on $\mathbb{R}_+$.

(ii) Note that

$$\begin{aligned} (\log f(e^x))'' &= \tfrac{e^x}{-a}\left[\tfrac{a}{a+e^x}+\log\tfrac{e^x}{a+e^x}\right] > 0, \ \ x\in\mathbb{R}, \\ (\log g(e^x))'' &= \tfrac{g(e^x)^2 e^x[ae^{-x}-\log(1+ae^{-x})]}{a(a+e^x)^2} > 0, \ \ x\in\mathbb{R}, \\ \left(\tfrac{1}{g(e^x)}\right)'' &= \tfrac{e^x}{(a+e^x)^2} > 0, \ \ x\in\mathbb{R}. \end{aligned}$$

It means that $\log f(e^x)$, $\log g(e^x)$ and $\frac{1}{g(e^x)}$ are convex on $\mathbb{R}$. So $f(x)$, $g(x)$ and $e^{[1/g(x)]}$ are geometrically convex on $\mathbb{R}_+$ by Lemma 1(i).

Next we prove that $e^{[1/f(x)]}$ is geometrically convex on $[a,+\infty)$. Clearly we have

$$\left(\frac{1}{f(e^x)}\right)'' = \frac{e^{2x}}{a^2 f(e^x)}\left[\frac{a}{e^x}\left(\log\frac{e^x}{a+e^x}+1-\frac{e^x}{a+e^x}\right)+\left(\log\frac{e^x}{a+e^x}\right)^2\right], \ \ x\ge\log a.$$

Let

$$p(t) = (1/t-1)(\log t+1-t)+(\log t)^2, \ \ \frac{1}{2}\le t<1.$$

then $\left(\frac{1}{f(e^x)}\right)'' = \frac{e^{2x}p(\frac{e^x}{a+e^x})}{a^2 f(e^x)}$ and

$$p'(t) = \frac{1}{t}\left[(t-1)+\left(2-\frac{1}{t}\right)\log t\right] < 0.$$

Note that $p(1)=0$, so $p(t)>0$ ($\frac{1}{2}\le t<1$). It follows that $\left(\frac{1}{f(e^x)}\right)''>0$ on $[\log a,+\infty)$ and $e^{1/f(x)}$ is geometrically convex on $[a,+\infty)$ by Lemma 1(i).

(iii) Note that

$$\left[\frac{1}{g(\frac{1}{x})}\right]'' = \frac{-a}{(ax+1)^2} < 0, \ \ x>0.$$

So $\frac{1}{g(\frac{1}{x})}$ is concave and $g(x)$ is harmonically convex on $\mathbb{R}_+$ by Lemma 1(ii).

Next, we prove that $f(x)$ is harmonically convex on $\mathbb{R}_+$. Clearly we have

$$\left[\frac{1}{f(\frac{1}{x})}\right]'' = \frac{1}{a^2x^4 f(\frac{1}{x})}\left[-2ax\log(ax+1)+\frac{a^2x^2}{ax+1}+(\log(ax+1))^2\right], \ \ x>0.$$

Let

$$h(t) = -2(t-1)\log t+t+\frac{1}{t}-2+(\log t)^2, \quad t>1,$$

then $\left[\frac{1}{f(\frac{1}{x})}\right]'' = \frac{h(ax+1)}{a^2x^4 f(\frac{1}{x})}$ and

$$h'(t) = -\left(1-\frac{1}{t}\right)\left(1-\frac{1}{t}+2\log t\right) < 0.$$

Note that $h(1) = 0$, so $h(t) < 0 (t > 1)$ and $\left[\frac{1}{f(\frac{1}{x})}\right]'' < 0 (x > 0)$. Hence $\frac{1}{f(\frac{1}{x})}$ is concave and $f(x)$ is harmonically convex on $\mathbb{R}_+$ by Lemma 1(ii).

□

In the following, we introduce some relevant conclusions on the Schur-convexity of the composite function. For further details, please refer to [22,23,29].

Lemma 4 ([29])**.** *Let $I \subseteq \mathbb{R}$ be an interval, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$, $f : I \to \mathbb{R}$ and $\psi(\boldsymbol{x}) = \varphi(f(x_1), \cdots, f(x_n)) : \mathbb{R}^n \to \mathbb{R}$ be functions.*

(i) If f is convex and φ is increasing Schur-convex, then ψ is Schur-convex on I^n.
(ii) If f is concave and φ is increasing Schur-concave, then ψ is Schur-concave on I^n.

Lemma 5 ([22,23])**.** *Let $I \subseteq \mathbb{R}_+$ be an interval, and let $\varphi : \mathbb{R}^n_+ \to \mathbb{R}_+$, $f : I \to \mathbb{R}_+$ and $\psi(\boldsymbol{x}) = \varphi(f(x_1), \cdots, f(x_n)) : \mathbb{R}^n_+ \to \mathbb{R}_+$ be functions.*

(i) If f is geometrically convex and φ is increasing Schur-geometrically convex, then ψ is Schur-geometrically convex on I^n.
(ii) If f is geometrically concave and φ is increasing Schur-geometrically concave, then ψ is Schur-geometrically concave on I^n.
(iii) If φ is increasing and Schur-harmonically convex and f is harmonically convex, then ψ is Schur-harmonically convex on I^n.

Symmetric functions $E_k(\boldsymbol{x})$ and $E^*_k(\boldsymbol{x})$ have the following properties.

Lemma 6 ([1,21–23])**.** *$E_k(\boldsymbol{x})$ and $E^*_k(\boldsymbol{x})$ are increasing Schur-concave, Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$.*

Lemma 7 ([29])**.** *Let $I \subseteq \mathbb{R}_+$ be an interval, and let $\varphi : I^n \to \mathbb{R}$ be a continuous symmetric function. If φ is differentiable on I^n, then φ is Schur-convex (Schur-concave) on I^n if and only if*

$$(x_1 - x_2)\left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - \frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 (\leq 0).$$

Let $E_0(x_3, \cdots, x_n) = 1$, $\sum_{i=1}^{0} x_i = 0$, it is easy to induce that

$$E_1(\boldsymbol{x}) = \sum_{i=1}^{n} x_i, \qquad E^*_1(\boldsymbol{x}) = \prod_{i=1}^{n} x_i,$$
$$E_k(\boldsymbol{x}) = x_1 E_{k-1}(x_3, \cdots, x_n) + x_2 E_{k-1}(x_3, \cdots, x_n) + x_1 x_2 E_{k-2}(x_3, \cdots, x_n) + E_k(x_3, \cdots, x_n), \quad 2 \leq k \leq n,$$
$$\frac{\partial E^*_k(\boldsymbol{x})}{\partial x_1} = \sum_{3 \leq i_1 < \cdots < i_{k-1} \leq n} \frac{E^*_k(\boldsymbol{x})}{x_1 + \sum_{j=1}^{k-1} x_{i_j}} + \sum_{3 \leq i_1 < \cdots < i_{k-2} \leq n} \frac{E^*_k(\boldsymbol{x})}{x_1 + x_2 + \sum_{j=1}^{k-2} x_{i_j}}, \quad 2 \leq k \leq n,$$
$$\frac{\partial E^*_k(\boldsymbol{x})}{\partial x_2} = \sum_{3 \leq i_1 < \cdots < i_{k-1} \leq n} \frac{E^*_k(\boldsymbol{x})}{x_2 + \sum_{j=1}^{k-1} x_{i_j}} + \sum_{3 \leq i_1 < \cdots < i_{k-2} \leq n} \frac{E^*_k(\boldsymbol{x})}{x_1 + x_2 + \sum_{j=1}^{k-2} x_{i_j}}, \quad 2 \leq k \leq n.$$

Hence, by use of Lemma 7, Lemma 1(iii), (iv) and Lemma 6, we have

Lemma 8. *Let $k = 1, 2, \cdots, n$, then*

(i) $E_k(e^{\boldsymbol{x}})$ and $E^*_k(e^{\boldsymbol{x}})$ are increasing and Schur-convex on $\mathbb{R}^n$.
(ii) $E_k(\log \boldsymbol{x})$ and $E^*_k(\log \boldsymbol{x})$ are increasing and Schur-geometrically concave on $\{e^{\boldsymbol{x}} | \boldsymbol{x} \in \mathbb{R}^n_+\}$.
(iii) $E_k(\frac{1}{\boldsymbol{x}})$ and $E^*_k(\frac{1}{\boldsymbol{x}})$ are decreasing and Schur-harmonically concave on $\mathbb{R}^n_+$.

3. Main Results

In this section, we prove our main results. Firstly, we investigate the Schur-convexity of $E_k(f,\boldsymbol{x})$ and $E_k^*(f,\boldsymbol{x})$ and their inverse problems. Note that Theorems 1 and 2 study the cases of $E_1(f,\boldsymbol{x})$ and $E_n^*(f,\boldsymbol{x})$, so we only consider the other cases in the following.

Theorem 8. *Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}_+$ be a function.*

(i) *If $\log f$ is convex, then $E_k(f,\boldsymbol{x})(2 \le k \le n)$ and $E_k^*(f,\boldsymbol{x})(1 \le k \le n-1)$ are Schur-convex on I^n. Conversely, if $E_k(f,\boldsymbol{x})(2 \le k \le n)$ or $E_k^*(f,\boldsymbol{x})(1 \le k \le n-1)$ is Schur-convex on I^n and f is continuous, then f is convex.*

(ii) *If f is concave, then $E_k(f,\boldsymbol{x})(1 \le k \le n)$ and $E_k^*(f,\boldsymbol{x})(1 \le k \le n)$ are Schur-concave on I^n. Conversely, if $E_1(f,\boldsymbol{x})$ or $E_n^*(f,\boldsymbol{x})$ is Schur-concave on I^n and f is continuous, then f is concave. If $E_k(f,\boldsymbol{x})(2 \le k \le n)$ or $E_k^*(f,\boldsymbol{x})(1 \le k \le n-1)$ is Schur-concave on I^n and f is continuous, then $\log f$ is concave.*

Proof. We only prove that the results hold for $E_k(f,\boldsymbol{x})$. A similar argument leads to the proof of the results for $E_k^*(f,\boldsymbol{x})$.

(i) If $\log f$ is convex, then $E_k(f,\boldsymbol{x}) = E_k(e^{\log f},\boldsymbol{x})$ is Schur-convex on I^n by Lemmas 4(i) and 8(i). Conversely, if $2 \le k \le n$ and $E_k(f,\boldsymbol{x})$ is Schur-convex on I^n, note that $E_k(\boldsymbol{x})$ is Schur-concave on $\mathbb{R}_+^n$, so for all $(x_1,\cdots,x_n) \in I^n$, we have

$$\begin{aligned}
&E_k(f(A(x_1,x_2)),f(A(x_1,x_2)),f(x_3),\cdots,f(x_n))\\
\le&E_k(f(x_1),f(x_2),f(x_3),\cdots,f(x_n))\\
\le&E_k\big(A\big(f(x_1),f(x_2)\big),A\big(f(x_1),f(x_2)\big),f(x_3),\cdots,f(x_n)\big).
\end{aligned}$$

Since $E_k(\boldsymbol{x})$ is increasing on $\mathbb{R}_+^n$, then

$$f(A(x_1,x_2)) \le A\big(f(x_1),f(x_2)\big).$$

Since f is continuous, f is convex by Lemma 2(i).

(ii) If f is concave, then $E_k(f,\boldsymbol{x})$ is Schur-concave on I^n by Lemmas 4(ii) and 6. Conversely, if $E_1(f,\boldsymbol{x})$ is Schur-concave on I^n and f is continuous, then $-E_1(f,\boldsymbol{x}) = E_1(-f,\boldsymbol{x})$ is Schur-convex on I^n, so $-f$ is convex on I by Theorem 2. Hence f is concave.
If $2 \le k \le n$ and $E_k(f,\boldsymbol{x})$ is Schur-concave on I^n, note that $E_k(e^{\boldsymbol{x}})$ is Schur-convex by Lemma 8(i), so for all $(x_1,\cdots,x_n) \in I^n$ and $2 \le k \le n$, we have

$$\begin{aligned}
&E_k(f(A(x_1,x_2)),f(A(x_1,x_2)),f(x_3),\cdots,f(x_n))\\
\ge&E_k(f(x_1),f(x_2),f(x_3),\cdots,f(x_n))\\
=&E_k\Big(e^{\log f(x_1)},e^{\log f(x_2)},e^{\log f(x_3)},\cdots,e^{\log f(x_n)}\Big)\\
\ge&E_k\Big(e^{A(\log f(x_1),\log f(x_2))},e^{A(\log f(x_1),\log f(x_2))},e^{\log f(x_3)},\cdots,e^{\log f(x_n)}\Big)\\
=&E_k(G(f(x_1),f(x_2)),G(f(x_1),f(x_2)),f(x_3),\cdots,f(x_n)).
\end{aligned}$$

Since $E_k(\boldsymbol{x})$ is increasing on $\mathbb{R}_+^n$, then

$$f(A(x_1,x_2)) \ge G(f(x_1),f(x_2)).$$

Since f is continuous, $\log f$ is concave by Lemma 2(i).

□

Secondly, we prove the Schur-geometrically convexity of $E_k(f,\boldsymbol{x})$ and $E_k^*(f,\boldsymbol{x})$ and their inverse problems.

Theorem 9. *Let $1 \le k \le n$ and $I \subseteq \mathbb{R}_+$ be an interval, and let $f : I \to \mathbb{R}_+$ be a function.*

(i) If f is geometrically convex, then $E_k(f, \boldsymbol{x})$ and $E_k^*(f, \boldsymbol{x})$ are Schur-geometrically convex on I^n. Conversely, if $E_n(f, \boldsymbol{x})$ or $E_1^*(f, \boldsymbol{x})$ is Schur-geometrically convex on I^n and f is continuous, then f is geometrically convex. If $E_k(f, \boldsymbol{x})(1 \le k \le n-1)$ or $E_k^*(f, \boldsymbol{x})(2 \le k \le n)$ is Schur-geometrically convex on I^n and f is continuous, then e^f is geometrically convex;

(ii) If f is geometrically concave, then $E_n(f, \boldsymbol{x})$ and $E_1^*(f, \boldsymbol{x})$ are Schur-geometrically concave on I^n. If e^f is geometrically concave, then $E_k(f, \boldsymbol{x})(1 \le k \le n-1)$ and $E_k^*(f, \boldsymbol{x})(2 \le k \le n)$ are Schur-geometrically concave on I^n. Conversely, if $E_k(f, \boldsymbol{x})$ or $E_k^*(f, \boldsymbol{x})$ is Schur-geometrically concave on I^n and f is continuous, then f is geometrically concave.

Proof. We only prove that the results hold for $E_k(f, \boldsymbol{x})$. A similar argument leads to the proof of the results for $E_k^*(f, \boldsymbol{x})$.

(i) If f is geometrically convex, then $E_k(f, \boldsymbol{x})$ is Schur-geometrically convex on I^n by Lemmas 5(i) and 6. Conversely, if $E_n(f, \boldsymbol{x})$ is Schur-geometrically convex on I^n, then for all $(x_1, \cdots, x_n) \in I^n$, we have

$$E_n(f(G(x_1, x_2)), f(G(x_1, x_2)), f(x_3), \cdots, f(x_n)) = f^2(G(x_1, x_2)) \prod_{i=3}^{n} f(x_i) \le \prod_{i=1}^{n} f(x_i).$$

So we have

$$f(G(x_1, x_2)) \le G(f(x_1), f(x_2)).$$

Since f is continuous, f is geometrically convex by Lemma 2(ii).
If $E_k(f, \boldsymbol{x})(1 \le k \le n-1)$ is Schur-geometrically convex on I^n, note that $E_k(\log \boldsymbol{x})$ is Schur-geometrically concave by Lemma 8(ii), so for all $(x_1, \cdots, x_n) \in I^n$, we have

$$\begin{aligned}
&E_k(f(G(x_1, x_2)), f(G(x_1, x_2)), f(x_3), \cdots, f(x_n)) \\
\le &E_k(f(x_1), f(x_2), f(x_3), \cdots, f(x_n)) \\
= &E_k\left(\log e^{f(x_1)}, \log e^{f(x_2)}, \log e^{f(x_3)}, \cdots, \log e^{f(x_n)}\right) \\
\le &E_k\left(\log G(e^{f(x_1)}, e^{f(x_2)}), \log G(e^{f(x_1)}, e^{f(x_2)}), \log e^{f(x_3)}, \cdots, \log e^{f(x_n)}\right) \\
= &E_k(A(f(x_1), f(x_2)), A(f(x_1), f(x_2)), f(x_3), \cdots, f(x_n)).
\end{aligned}$$

Which implies that

$$f(G(x_1, x_2)) \le A(f(x_1), f(x_2)).$$

Since f is continuous, e^f is geometrically convex by Lemma 2(ii).

(ii) If f is geometrically concave, then $\frac{1}{f}$ is geometrically convexity, it follows that the function

$$\frac{1}{E_n(f, \boldsymbol{x})} = E_n\left(\frac{1}{f}, \boldsymbol{x}\right)$$

is Schur-geometrically convex on I^n by (i); hence, $E_n(f, \boldsymbol{x})$ is Schur-geometrically concave on I^n.
If e^f is geometrically concave, then for any $1 \le k \le n-1$, $E_k(f, \boldsymbol{x}) = E_k(\log e^f, \boldsymbol{x})$ is Schur-geometrically concave on I^n by Lemmas 5(ii) and 8(ii).
Conversely, if $E_k(f, \boldsymbol{x})$ is Schur-geometrically concave on I^n, note that $E_k(\boldsymbol{x})$ is Schur-geometrically convex on I^n, so for all $(x_1, \cdots, x_n) \in I^n$, we have

$$\begin{aligned}
&E_k(f(G(x_1, x_2)), f(G(x_1, x_2)), f(x_3), \cdots, f(x_n)) \\
\ge &E_k(f(x_1), f(x_2), f(x_3), \cdots, f(x_n)) \\
\ge &E_k(G(f(x_1), f(x_2)), G(f(x_1), f(x_2)), f(x_3), \cdots, f(x_n)).
\end{aligned}$$

Which implies that

$$f(G(x_1,x_2)) \geq G(f(x_1),f(x_2)).$$

Since f is continuous, f is geometrically concave by Lemma 2(ii).
□

Finally, we prove the Schur-harmonically convexity of $E_k(f,\boldsymbol{x})$ and $E_k^*(f,\boldsymbol{x})$ and their inverse problems.

Theorem 10. *Let $1 \leq k \leq n$ and $I \subseteq \mathbb{R}_+$ be an interval, and let $f : I \to \mathbb{R}_+$ be a function.*

(i) If f is harmonically convex, then $E_k(f,\boldsymbol{x})$ and $E_k^*(f,\boldsymbol{x})$ are Schur-harmonically convex on I^n. Conversely, if $E_k(f,\boldsymbol{x})$ or $E_k^*(f,\boldsymbol{x})$ is Schur-harmonically convex on I^n and f is continuous, then $\frac{1}{f}$ is harmonically concave.

(ii) If $\frac{1}{f}$ is harmonically convex, then $E_k(f,\boldsymbol{x})$ and $E_k^*(f,\boldsymbol{x})$ are Schur-harmonically concave on I^n. Conversely, if $E_k(f,\boldsymbol{x})$ or $E_k^*(f,\boldsymbol{x})$ is Schur-harmonically concave on I^n and f is continuous, then f is harmonically concave.

Proof. We only prove that the results hold for $E_k(f,\boldsymbol{x})$. A similar argument leads to the proof of the results for $E_k^*(f,\boldsymbol{x})$.

(i) If f is harmonically convex, then $E_k(f,\boldsymbol{x})$ is Schur-harmonically convex on I^n by Lemmas 5(iii) and 6. Conversely, if $E_k(f,\boldsymbol{x})$ is Schur-harmonically convex on I^n, note that $E_k(\frac{1}{\boldsymbol{x}})$ is Schur-harmonically concave by Lemma 8(iii), so for all $(x_1,\cdots,x_n) \in I^n$, we have

$$\begin{aligned}
&E_k(f(H(x_1,x_2)),f(H(x_1,x_2)),f(x_3),\cdots,f(x_n))\\
\leq &E_k(f(x_1),f(x_2),f(x_3),\cdots,f(x_n))\\
= &E_k\left(\frac{1}{\frac{1}{f(x_1)}},\frac{1}{\frac{1}{f(x_2)}},\frac{1}{\frac{1}{f(x_3)}},\cdots,\frac{1}{\frac{1}{f(x_n)}}\right)\\
\leq &E_k\left(\frac{1}{H\left(\frac{1}{f(x_1)},\frac{1}{f(x_2)}\right)},\frac{1}{H\left(\frac{1}{f(x_1)},\frac{1}{f(x_2)}\right)},\frac{1}{\frac{1}{f(x_3)}},\cdots,\frac{1}{\frac{1}{f(x_n)}}\right).
\end{aligned}$$

Which implies that

$$\frac{1}{f(H(x_1,x_2))} \geq H\left(\frac{1}{f(x_1)},\frac{1}{f(x_2)}\right).$$

Since f is continuous, $\frac{1}{f}$ is harmonically concave by Lemma 2(iii).

(ii) If $\frac{1}{f}$ is harmonically convex, note that $\left[E_k(\frac{1}{\boldsymbol{x}})\right]^{-1}$ is increasing Schur-harmonically convex on $\mathbb{R}_+^n$ by Lemma 8(iii), so the function

$$(E_k(f,\boldsymbol{x}))^{-1} = \left(E_k\left(\frac{1}{\frac{1}{f}},\boldsymbol{x}\right)\right)^{-1}$$

is Schur-harmonically convex on I^n by Lemma 5(iii). It follows that $E_k(f,\boldsymbol{x})$ is Schur-harmonically concave on I^n. Conversely, if $E_k(f,\boldsymbol{x})$ is Schur-harmonically concave on I^n, note that $E_k(\boldsymbol{x})$ is Schur-harmonically convex on I^n by Lemma 6, so for all $(x_1,\cdots,x_n) \in I^n$, we have

$$\begin{aligned}
&E_k(f(H(x_1,x_2)),f(H(x_1,x_2)),f(x_3),\cdots,f(x_n))\\
\geq &E_k(f(x_1),f(x_2),f(x_3),\cdots,f(x_n))\\
\geq &E_k(H(f(x_1),f(x_2)),H(f(x_1),f(x_2)),f(x_3),\cdots,f(x_n)).
\end{aligned}$$

Which implies that

$$f(H(x_1, x_2)) \geq H(f(x_1), f(x_2)).$$

Since f is continuous, f is harmonically concave by Lemma 2(iii).
□

4. Applications to Means

Now, we use Theorems 8–10 to establish new inequalities on special means.

Let $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$,, the arithmetic mean, geometric mean, harmonic mean of $x_1, \cdots, x_n$ are respectively defined by

$$A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i, \ \ G_n(\boldsymbol{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \ \ H_n(\boldsymbol{x}) = n\left(\sum_{i=1}^{n} x_i^{-1}\right)^{-1}.$$

For simplicity, we denote

$$I(\boldsymbol{x}, a+\boldsymbol{x}) = (I(x_1, a+x_1), \cdots, I(x_n, a+x_n)),$$
$$L(\boldsymbol{x}, a+\boldsymbol{x}) = (L(x_1, a+x_1), \cdots, L(x_n, a+x_n)).$$

If we replace $f(x)$ with $I(x, a+x)$ and $L(x, a+x)$, respectively, in Theorem 8(ii), then by Lemma 3(i) and Theorem 8(ii) we can get:

Theorem 11. *Let $a > 0$, $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}_+$, $n \geq 2$, $1 \leq k \leq n$, then*

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} I(x_{i_j}, a + x_{i_j}) \leq \binom{n}{k} I(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x}))^k, \tag{2}$$

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} L(x_{i_j}, a + x_{i_j}) \leq \binom{n}{k} L(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x}))^k, \tag{3}$$

$$\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} I(x_{i_j}, a + x_{i_j}) \leq k^{\binom{n}{k}} I(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x}))^{\binom{n}{k}}, \tag{4}$$

$$\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} L(x_{i_j}, a + x_{i_j}) \leq k^{\binom{n}{k}} L(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x}))^{\binom{n}{k}}. \tag{5}$$

In particular, if we let $k = 1$ in (2) and (3), respectively, then we have

$$A_n(I(\boldsymbol{x}, a+\boldsymbol{x})) \leq I(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x})), \tag{6}$$

$$A_n(L(\boldsymbol{x}, a+\boldsymbol{x})) \leq L(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x})). \tag{7}$$

If we replace $f(x)$ with $I(x, a+x)$, $L(x, a+x)$, $e^{[1/I(x,a+x)]}$ and $e^{[1/L(x,a+x)]}$ respectively in Theorem 9(i), then by Lemma 3(ii) and Theorem 9(i) we have:

Theorem 12. *Let $a > 0$, $\boldsymbol{x} = (x_1, \cdots, x_n)$, $n \geq 2$, $1 \leq k \leq n$, then*

$$\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k} I(x_{i_j},a+x_{i_j}) \ge \binom{n}{k} I(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x}))^k, \qquad \boldsymbol{x}\in\mathbb{R}_+^n, \tag{8}$$

$$\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k} e^{[1/I(x_{i_j},a+x_{i_j})]} \ge \binom{n}{k} e^{[k/I(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x}))]}, \qquad \boldsymbol{x}\in[a,+\infty)^n, \tag{9}$$

$$\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k} L(x_{i_j},a+x_{i_j}) \ge \binom{n}{k} L(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x}))^k, \qquad \boldsymbol{x}\in\mathbb{R}_+^n, \tag{10}$$

$$\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}\left(\frac{a+x_{i_j}}{x_{i_j}}\right)^{\frac{1}{a}} \ge \binom{n}{k}\left(\frac{a+G_n(\boldsymbol{x})}{G_n(\boldsymbol{x})}\right)^{\frac{k}{a}}, \qquad \boldsymbol{x}\in\mathbb{R}_+^n, \tag{11}$$

$$\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k} I(x_{i_j},a+x_{i_j}) \ge k^{\binom{n}{k}} I(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x}))^{\binom{n}{k}}, \qquad \boldsymbol{x}\in\mathbb{R}_+^n, \tag{12}$$

$$\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k} e^{[1/I(x_{i_j},a+x_{i_j})]} \ge k^{\binom{n}{k}} e^{[\binom{n}{k}/I(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x}))]}, \qquad \boldsymbol{x}\in[a,+\infty)^n, \tag{13}$$

$$\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k} L(x_{i_j},a+x_{i_j}) \ge k^{\binom{n}{k}} L(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x}))^{\binom{n}{k}}, \qquad \boldsymbol{x}\in\mathbb{R}_+^n, \tag{14}$$

$$\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\left(\frac{a+x_{i_j}}{x_{i_j}}\right)^{\frac{1}{a}} \ge k^{\binom{n}{k}}\left(\frac{a+G_n(\boldsymbol{x})}{G_n(\boldsymbol{x})}\right)^{\binom{n}{k}/a}, \qquad \boldsymbol{x}\in\mathbb{R}_+^n. \tag{15}$$

In particular, if we let $k=n$ in (8), (9), (10) and (11), respectively, then we have

$$G_n(I(\boldsymbol{x},a+\boldsymbol{x})) \ge I(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x})),\ \boldsymbol{x}\in\mathbb{R}_+^n, \tag{16}$$

$$H_n(I(\boldsymbol{x},a+\boldsymbol{x})) \le I(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x})),\ \boldsymbol{x}\in[a,+\infty)^n, \tag{17}$$

$$G_n(L(\boldsymbol{x},a+\boldsymbol{x})) \ge L(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x})),\ \boldsymbol{x}\in\mathbb{R}_+^n, \tag{18}$$

$$H_n(L(\boldsymbol{x},a+\boldsymbol{x})) \le L(G_n(\boldsymbol{x}),a+G_n(\boldsymbol{x})),\ \boldsymbol{x}\in\mathbb{R}_+^n. \tag{19}$$

If we replace $\frac{1}{f(x)}$ with $I(x,a+x)$ and $L(x,a+x)$, respectively, in Theorem 10(ii), then by Lemma 3(iii) and Theorem 10(ii), we can get:

Theorem 13. *Let $a>0$, $\boldsymbol{x}=(x_1,\cdots,x_n)\in\mathbb{R}_+^n$, $n\ge 2$, $1\le k\le n$, then*

$$\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}\frac{1}{I(x_{i_j},a+x_{i_j})} \le \binom{n}{k}\frac{1}{I(H_n(\boldsymbol{x}),a+H_n(\boldsymbol{x}))^k}, \tag{20}$$

$$\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}\frac{1}{L(x_{i_j},a+x_{i_j})} \le \binom{n}{k}\frac{1}{L(H_n(\boldsymbol{x}),a+H_n(\boldsymbol{x}))^k}, \tag{21}$$

$$\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\frac{1}{I(x_{i_j},a+x_{i_j})} \le k^{\binom{n}{k}}\frac{1}{I(H_n(\boldsymbol{x}),a+H_n(\boldsymbol{x}))^k}, \tag{22}$$

$$\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\frac{1}{L(x_{i_j},a+x_{i_j})} \le k^{\binom{n}{k}}\frac{1}{L(H_n(\boldsymbol{x}),a+H_n(\boldsymbol{x}))^k}. \tag{23}$$

In particular, if we let $k=1$ in (20) and (21), respectively, then we have

$$H_n(I(\boldsymbol{x},a+\boldsymbol{x})) \ge I(H_n(\boldsymbol{x}),a+H_n(\boldsymbol{x})), \tag{24}$$

$$H_n(L(\boldsymbol{x},a+\boldsymbol{x})) \ge L(H_n(\boldsymbol{x}),a+H_n(\boldsymbol{x})). \tag{25}$$

By the inequalities (6), (7), (16)–(19), (24) and (25), we can obtain the following new inequalities.

Theorem 14. *Let* $a > 0$, $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$, $n \geq 2$, *then*

$$\begin{aligned} &I(H_n(\boldsymbol{x}), a + H_n(\boldsymbol{x})) \leq H_n(I(\boldsymbol{x}, a + \boldsymbol{x})) \leq I(G_n(\boldsymbol{x}), a + G_n(\boldsymbol{x})) \\ &\quad \leq G_n(I(\boldsymbol{x}, a + \boldsymbol{x})) \leq A_n(I(\boldsymbol{x}, a + \boldsymbol{x})) \leq I(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x})), \quad \boldsymbol{x} \in [a, +\infty)^n, \end{aligned} \tag{26}$$

$$\begin{aligned} &L(H_n(\boldsymbol{x}), a + H_n(\boldsymbol{x})) \leq H_n(L(\boldsymbol{x}, a + \boldsymbol{x})) \leq L(G_n(\boldsymbol{x}), a + G_n(\boldsymbol{x})) \\ &\quad \leq G_n(L(\boldsymbol{x}, a + \boldsymbol{x})) \leq A_n(L(\boldsymbol{x}, a + \boldsymbol{x})) \leq L(A_n(\boldsymbol{x}), a + A_n(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{R}^n_+. \end{aligned} \tag{27}$$

5. Discussion

In this paper, the Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity and the inverse problem for $E_k(f, \boldsymbol{x})$ and $E^*_k(f, \boldsymbol{x})$ are established in Theorems 8–10, then some results in the papers [1,17,24–33] are generalized.

The inequalities involving special means (arithmetic mean, geometric mean, harmonic mean, identity mean, and logarithmic mean) are very important. In this paper, by use of Theorems 8–10 and the theory of majorization, new inequalities on special means are established in Theorems 11–14.

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