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A Note on the Summation of the Incomplete Gamma Function

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Abstract: We examine the improved infinite sum of the incomplete gamma function for large values of the parameters involved. We also evaluate the infinite sum and equivalent Hurwitz-Lerch zeta function at special values and produce a table of results for easy reading. Almost all Hurwitz-Lerch zeta functions have an asymmetrical zero distribution.

Keywords: Hurwitz-Lerch zeta function; incomplete gamma function; Catalan's constant; Apréy's constant; Cauchy integral; contour integral

MSC: Primary 30E20; 33-01; 33-03; 33-04; 33-33B; 33E20; 33E33C

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Citation: Reynolds, R.; Stauffer, A. A Note on the Summation of the Incomplete Gamma Function. *Symmetry* **2021**, *13*, 2369. https://doi.org/10.3390/sym13122369

Academic Editors: Junesang Choi and Djurdje Cvijović

Received: 8 November 2021
Accepted: 27 November 2021
Published: 9 December 2021

1. Significance Statement

In 1887, Mathias Lerch [1] published the Hurwitz-Lerch zeta function. Series representations of Hurwitz-Lerch zeta function were researched in [1–3]. Our purpose in this paper is to add to the existing literature on special functions [4–7], in particular the Hurwitz-Lerch zeta function series representations, by offering a formal derivation of the Hurwitz-Lerch zeta function defined in terms of the infinite sum of the incomplete gamma function. We believe that researchers will find this new formula valuable in their present and future research activities. All the results in this work are new.

2. Introduction

In this present work, we derive a new expression for the Hurwitz-Lerch zeta function in terms of the infinite sum of the incomplete gamma function given by:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\Bigg(&k(i\log(a))^{-k}\log^k(a)a^{-m-2in-i}(im-2n-1)^{-k-1}\Gamma(k,(-m-2in-i)\log(a))\\&+e^{\frac{i\pi k}{2}}ka^{-m+2in+i}(-im-2n-1)^{-k-1}\Gamma(k,(-m+2in+i)\log(a))\\&-\frac{1}{(2n+1)^2}(i\log(a))^{-k}\log^{k-1}(a)(im-2n-1)^{-k-1}(\log(a)(-m-2in-i))^k\left(m^2\log(a)+k(m+2in+i)\right)\\&+\frac{1}{(2n+1)^2}e^{\frac{i\pi k}{2}}(-im-2n-1)^{-k}(\log(a)(-m+2in+i))^{k-1}\left(im^2\log(a)+ikm+2kn+k\right)\Bigg)\\=&\frac{1}{4}\pi\left(\log^k(a)-2\pi^k e^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi},-k,\frac{\log(a)}{\pi}+\frac{1}{2}\right)\right)\end{aligned} \tag{1}$$

where the variables k, a, m are general complex numbers. The derivations follow the method used by us in [8]. This method involves using a form of the generalized Cauchy's integral formula given by:

$$\frac{y^k}{\Gamma(k+1)}=\frac{1}{2\pi i}\int_C\frac{e^{wy}}{w^{k+1}}dw, \tag{2}$$

where $y, w \in \mathbb{C}$, and C is in general an open contour in the complex plane where the bilinear concomitant [8] has the same value at the end points of the contour.

The Incomplete Gamma Function

The multivalued incomplete gamma functions [9], $\gamma(s,z)$ and $\Gamma(s,z)$, are defined by:

$$\gamma(s,z) = \int_0^z t^{s-1}e^{-t}dt,$$

and:

$$\Gamma(s,z) = \int_z^\infty t^{s-1}e^{-t}dt,$$

where $Re(s) > 0$. The incomplete gamma function has a recurrence relation given by:

$$\gamma(s,z) + \Gamma(s,z) = \Gamma(s),$$

where $a \neq 0, -1, -2, \ldots$ The incomplete gamma function is continued analytically by:

$$\gamma(a, ze^{2m\pi i}) = e^{2\pi m i a}\gamma(a,z),$$

and:

$$\Gamma(s, ze^{2m\pi i}) = e^{2\pi m i s}\Gamma(s,z) + (1 - e^{2\pi m i s})\Gamma(s),$$

where $m \in \mathbb{Z}$. When $z \neq 0$, $\Gamma(s,z)$ is an entire function of s and $\gamma(s,z)$ is meromorphic with simple poles at $s = -n$ for $n = 0, 1, 2, \ldots$ with residue $\frac{(-1)^n}{n!}$. These definitions are listed in Section 8.2 (i) and (ii) in [9].

3. The Hurwitz-Lerch Zeta Function

We use Equation (1.11.3) in [10], where $\Phi(z,s,v)$ is the Hurwitz-Lerch zeta function, which is a generalization of the Hurwitz zeta $\zeta(s,v)$ and polylogarithm functions $Li_n(z)$. The Hurwitz-Lerch zeta function has a series representation given by:

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{3}$$

where $|z| < 1, v \neq 0, -1, -2, -3, \ldots,$ and is continued analytically by its integral representation given by:

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{4}$$

where $Re(v) > 0$ and either $|z| \leq 1, z \neq 1, Re(s) > 0$ or $z = 1, Re(s) > 1$.

4. Hurwitz-Lerch Zeta Function in Terms of the Contour Integral

We use the method in [8]. The cut and contour are in the first quadrant of the complex w-plane with $0 < Re(w+m) < 1$. Using a generalization of Cauchy's integral formula, we first replace $y \to \log(a) + \frac{1}{2}\pi(2y+1)$, then multiply both sides by $\frac{1}{2}\pi(-1)^y e^{\frac{1}{2}\pi m(2y+1)}$ and take the infinite sum over $y \in [0,\infty)$ to obtain:

$$\begin{aligned}\frac{\pi^{k+1}e^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi}, -k, \frac{\log(a)}{\pi} + \frac{1}{2}\right)}{2k!} &= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C \frac{1}{2}\pi(-1)^y a^w w^{-k-1}e^{\frac{1}{2}\pi(2y+1)(m+w)}dw \\ &= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty}\frac{1}{2}\pi(-1)^y a^w w^{-k-1}e^{\frac{1}{2}\pi(2y+1)(m+w)}dw \\ &= \frac{1}{2\pi i}\int_C \frac{1}{4}\pi a^w w^{-k-1}\operatorname{sech}\left(\frac{1}{2}\pi(m+w)\right)dw\end{aligned} \tag{5}$$

from Equation (1.232.2) in [11], where $Im(w+m) > 0$, in order for the sum to converge.

Additional Contour Integral

Using Equation (2), we replace $y \to \log(a)$ and multiply both sides by $-\frac{\pi}{4}$ to obtain:

$$\frac{\pi \log^k(a)}{4k!} = \frac{1}{2\pi i}\int_C \frac{1}{4}\pi a^w w^{-k-1} dw \tag{6}$$

5. Incomplete Gamma Function in Terms of the Contour Integral

Using Equation (2), we replace $y \to iy + \log(a)$, multiply both sides by $e^{y(im+2n+1)}$, and take the definite integral over $y \in [0, \infty)$ to obtain:

$$\begin{aligned} &\frac{(-im-2n-1)^{-k-1} e^{\frac{1}{2}i(\pi k+\log(a)(2im+4n+2))}\Gamma(k+1, -(m-i(2n+1))\log(a))}{\Gamma(k+1)} \\ &= \frac{1}{2\pi i}\int_C \frac{i a^w w^{-k-1}}{m-2in+w-i} dw \end{aligned} \tag{7}$$

from Equation (8.350.2) in [11]. Again using Equation (2), we replace $y \to -iy + \log(a)$, multiply both sides by $e^{y(-im+2n+1)}$, and take the definite integral over $y \in [0, \infty)$ to obtain:

$$\begin{aligned} &\frac{(i\log(a))^{-k}\log^k(a) a^{-m-2in-i}(im-2n-1)^{-k-1}\Gamma(k+1, -(m+2in+i)\log(a))}{\Gamma(k+1)} \\ &= -\frac{1}{2\pi i}\int_C \frac{i a^w w^{-k-1}}{m+2in+w+i} dw \end{aligned} \tag{8}$$

from Equation (8.350.2) in [11]. Next, we add Equations (7) and (8) to obtain:

$$\begin{aligned} &\frac{(-1)^{n-1}}{2k!}\Big((i\log(a))^{-k}\log^k(a) a^{-m-2in-i}(im-2n-1)^{-k-1}\Gamma(k+1,(-m-2in-i)\log(a)) \\ &+ (-im-2n-1)^{-k-1} e^{\frac{1}{2}i(\pi k+\log(a)(2im+4n+2))}\Gamma(k+1,(-m+2in+i)\log(a))\Big) \\ &= \frac{1}{2\pi i}\int_C \frac{(-1)^n(2n+1)a^w w^{-k-1}}{(m+w)^2+(2n+1)^2} dw \end{aligned} \tag{9}$$

Finally, we form three equations by multiplying Equation (9) by $\frac{m^2}{(2n+1)^2}$, $\frac{1}{(2n+1)^2}$, and $\frac{2m}{(2n+1)^2}$ add take the infinite sum over $n \in [0, \infty)$ and simplify to obtain:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{2(2n+1)^2}(-1)^n a^{-m-2in-i}\Bigg(-\frac{m^2}{k!}\Big(e^{\frac{i\pi k}{2}} a^{4in+2i}(-im-2n-1)^{-k-1} \\ &\Gamma(k+1,(-m+2in+i)\log(a)) + (i\log(a))^{-k}\log^k(a)(im-2n-1)^{-k-1}\Gamma(k+1,(-m-2in-i)\log(a))\Big) \\ &\quad + \frac{2m}{\Gamma(k)}\Big(e^{\frac{1}{2}i\pi(k-1)}\Big(-a^{4in+2i}\Big)(-im-2n-1)^{-k}\Gamma(k,(-m+2in+i)\log(a)) - i(i\log(a))^{-k}\log^k(a)(im-2n-1)^{-k}\Gamma(k,(-m-2in-i)\log(a))\Big) \\ &\qquad + \frac{e^{\frac{i\pi k}{2}} a^{4in+2i}}{\Gamma(k-1)}(-im-2n-1)^{1-k}\Gamma(k-1,(-m+2in+i)\log(a)) + (i\log(a))^{-k}\log^k(a)(im-2n-1)^{1-k}\Gamma(k-1,(-m-2in-i)\log(a))\Bigg) \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_C \frac{(-1)^n a^w w^{-k-1}(m+w)^2}{(2n+1)\left((m+w)^2+(2n+1)^2\right)} dw \\ &= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} \frac{(-1)^n a^w w^{-k-1}(m+w)^2}{(2n+1)\left((m+w)^2+(2n+1)^2\right)} dw \\ &= \frac{1}{2\pi i}\int_C a^{m+w}(m+w)^{-k-1}\left(\frac{\pi}{4} - \frac{1}{4}\pi \operatorname{sech}\left(\frac{1}{2}\pi(2m+w)\right)\right) dw \end{aligned} \tag{10}$$

from Equation (5.1.26.11) in [12] where $Re(w+m) > 0$.

Theorem 1. *For all $k, a, m \in \mathbb{C}$,*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\Big(k(i\log(a))^{-k}\log^k(a)a^{-m-2in-i}(im-2n-1)^{-k-1}\Gamma(k,(-m-2in-i)\log(a))\\ +e^{\frac{i\pi k}{2}}ka^{-m+2in+i}(-im-2n-1)^{-k-1}\Gamma(k,(-m+2in+i)\log(a))\\ -\frac{1}{(2n+1)^2}(i\log(a))^{-k}\log^{k-1}(a)(im-2n-1)^{-k-1}(\log(a)(-m-2in-i))^k\left(m^2\log(a)+k(m+2in+i)\right)\\ +\frac{1}{(2n+1)^2}e^{\frac{i\pi k}{2}}(-im-2n-1)^{-k}(\log(a)(-m+2in+i))^{k-1}\left(im^2\log(a)+ikm+2kn+k\right)\Big)\\ =\frac{1}{4}\pi\left(\log^k(a)-2\pi^ke^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi},-k,\frac{\log(a)}{\pi}+\frac{1}{2}\right)\right)\end{aligned} \tag{11}$$

Proof. Observe that the sum of the right-hand side of Equations (5) and (6) is equal to the right-hand side of (10), so we can equate the left-hand sides to yield the stated result. □

6. Special Cases

In the proceeding section, we evaluate Equation (11) and simplify using the following functions and fundamental constants: Catalan's constant C, Glaisher's constant A, Aprèy's constant $\zeta(3)$, exponential integral function $E_n(z)$, Riemann zeta function $\zeta(s)$, polylogarithm function $Li_n(z)$, Hurwitz zeta function $\zeta(s,a)$, digamma function $\psi(z)$, and hypergeometric function ${}_2F_1\,(a,b;c;z)$.

Example 1. *The degenerate case.*

$$\sum_{n=0}^{\infty}\frac{m^2(-1)^n}{(2n+1)\left(m^2+(2n+1)^2\right)}=\frac{1}{4}\left(\pi-\pi\operatorname{sech}\left(\frac{\pi m}{2}\right)\right) \tag{12}$$

Proof. Use Equation (11), and set $k=0$ and simplify using entry (2) in [13]. □

Lemma 1. *For all* $k,m\in\mathbb{C}$,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\Big(-\frac{1}{(2n+1)^2}ie^{-\frac{1}{2}i\pi k}\pi^{k-1}\\ \left((2im-4n-2)^{-k}(-m-2in-i)^{k-1}\left(\pi m^2+2k(m+2in+i)\right)-e^{i\pi k}(-2im-4n-2)^{-k}(-m+2in+i)^{k-1}\left(\pi m^2+2k(m-2in-i)\right)\right)\\ +ke^{\frac{1}{2}i\pi(k+im+2n+1)}(-im-2n-1)^{-k-1}\Gamma\left(k,-\frac{1}{2}(m-i(2n+1))\pi\right)\\ +ke^{-\frac{1}{2}i\pi(k-im+2n+1)}(im-2n-1)^{-k-1}\Gamma\left(k,-\frac{1}{2}(m+2in+i)\pi\right)\Big)=\frac{1}{4}\pi\left(2\pi^ke^{-\frac{\pi m}{2}}\operatorname{Li}_{-k}\left(-e^{m\pi}\right)+\left(\frac{\pi}{2}\right)^k\right)\end{aligned} \tag{13}$$

Proof. Use Equation (11), and set $a=e^{\pi/2}$ and simplify using Equation (64:12:1) in [13]. □

Lemma 2. *For all* $k\in\mathbb{C}$,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{2\pi(2n+1)^2}k(-1)^ne^{-\frac{1}{2}i\pi(k+2n+1)}(-4n-2)^{-k}\Big(-2\pi^ke^{i\pi n}\left((-i(2n+1))^k-e^{i\pi k}(i(2n+1))^k\right)\\ -\pi2^k(2n+1)\left(\Gamma\left(k,-\frac{1}{2}i(2\pi n+\pi)\right)-e^{i\pi(k+2n)}\Gamma\left(k,\frac{1}{2}i(2\pi n+\pi)\right)\right)\Big)\\ =\frac{1}{4}\pi^{k+1}\left(\left(2^{k+2}-2\right)\zeta(-k)+2^{-k}\right)\end{aligned} \tag{14}$$

Proof. Use Equation (13), and set $m=0$ and simplify using entry (4) in the table below (25:12:5) in [13]. □

Lemma 3. *For all* $k,a\in\mathbb{C}$,

$$\sum_{n=0}^{\infty} \frac{1}{2}(-1)^n \left(ka^{-2in-i}(-2n-1)^{-k-1}(i\log(a))^{-k}\log^k(a)\Gamma(k,(-2in-i)\log(a)) + e^{\frac{i\pi k}{2}}ka^{2in+i}(-2n-1)^{-k-1}\Gamma(k,(2in+i)\log(a)) - \frac{k(2in+i)(-2n-1)^{-k-1}(i\log(a))^{-k}\log^{k-1}(a)((-2in-i)\log(a))^k}{(2n+1)^2} + \frac{e^{\frac{i\pi k}{2}}(2kn+k)(-2n-1)^{-k}((2in+i)\log(a))^{k-1}}{(2n+1)^2} \right) = \frac{1}{4}\pi\left(\log^k(a) - 2\pi^k\left(2^k\zeta\left(-k,\frac{1}{2}\left(\frac{\log(a)}{\pi}+\frac{1}{2}\right)\right) - 2^k\zeta\left(-k,\frac{1}{2}\left(\frac{\log(a)}{\pi}+\frac{3}{2}\right)\right)\right)\right) \tag{15}$$

Proof. Use Equation (11), and set $m = 0$ and simplify using entry (4) in the table below (64:12:7) in [13]. □

Example 2. *An example in terms of Catalan's constant C,*

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)(\Gamma(-2,-i(2n+1)\pi) + \Gamma(-2,i(2n+1)\pi)) = \frac{8C-7}{4\pi} \tag{16}$$

Proof. Use Equation (11), take the first partial derivative with respect to k, and set $m = 0$, $k = -2$, $a = e^{\pi}$ and simplify using Equation (7) in [14]. □

Example 3. *An example in terms of Glaisher's constant A,*

$$\sum_{n=0}^{\infty} \frac{\Gamma\left(0,-\frac{1}{2}i(2\pi n+\pi)\right) + \Gamma\left(0,\frac{1}{2}i(2\pi n+\pi)\right)}{(2n+1)^2} = -\pi^2\log\left(\frac{A^3}{2^{7/12}\sqrt[4]{e}}\right) \tag{17}$$

Proof. Use Equation (11), take the first partial derivative with respect to k, and set $m = 0$, $k = -2$, $a = e^{\pi/2}$ and simplify using Equation (8) in [14]. □

Example 4. *An example in terms of Apery's constant $\zeta(3)$,*

$$\sum_{n=0}^{\infty} \frac{i\left(E_1\left(-\frac{1}{2}i(2\pi n+\pi)\right) - E_1\left(\frac{1}{2}i(2\pi n+\pi)\right)\right)}{(2n+1)^3} = \frac{3\pi^3}{32} - \frac{7\pi\zeta(3)}{8} \tag{18}$$

Proof. Use Equation (11), take the first partial derivative with respect to k, and set $m = 0, k = 2, a = e^{\pi/2}$ and simplify using Equation (9) in [14]. □

Example 5. *An example in terms of Catalan's constant C,*

$$\sum_{n=0}^{\infty}(-1)^{n+\frac{1}{4}}e^{-i\pi n}\left(-\frac{16ie^{2i\pi n}E_3\left(\frac{1}{4}i(4n+3)\pi\right)}{\pi^2(4n+3)} - i(4n+1)\Gamma\left(-2,-\frac{1}{4}i(4n+1)\pi\right) + (4n+3)\Gamma\left(-2,-\frac{1}{4}i(4n+3)\pi\right) - (4n+1)\Gamma\left(-2,\frac{1}{4}i(4\pi n+\pi)\right)\right) = \frac{i\left(\pi^2-48C\right)}{24\sqrt{2}\pi} \tag{19}$$

$$\sum_{n=0}^{\infty}(-1)^{n+\frac{1}{4}}e^{-i\pi n}\left(-\frac{16ie^{2i\pi n}E_3\left(\frac{1}{4}i(4n+3)\pi\right)}{\pi^2(4n+3)} - i(4n+1)\Gamma\left(-2,-\frac{1}{4}i(4n+1)\pi\right) + (4n+3)\Gamma\left(-2,-\frac{1}{4}i(4n+3)\pi\right) - (4n+1)\Gamma\left(-2,\frac{1}{4}i(4\pi n+\pi)\right)\right) = \frac{i\left(\pi^2-48C\right)}{24\sqrt{2}\pi} \tag{20}$$

Proof. Use Equation (13), and set $k = -2$, $m = i/2$ and simplify using Equation (2.2.1.2.7) in [15] and Equation (8.19.1) in [9]. □

Example 6. *An example in terms of the fundamental constant* $\log(2)$*,*

$$\sum_{n=0}^{\infty}\left(\Gamma\left(-1,-\frac{1}{2}i(2\pi n+\pi)\right)+\Gamma\left(-1,\frac{1}{2}i(2\pi n+\pi)\right)\right)=\log(2)-1 \tag{21}$$

Proof. Use Equation (14), and apply l'Hopital's rule as $k \to -1$ and simplify. □

Example 7. *An example in terms of* π*,*

$$\sum_{n=0}^{\infty} i(2n+1)\left(\Gamma\left(-2,-\frac{1}{2}i(2\pi n+\pi)\right)-\Gamma\left(-2,\frac{1}{2}i(2\pi n+\pi)\right)\right)=\frac{1}{\pi}-\frac{\pi}{24} \tag{22}$$

Proof. Use Equation (14), and set $k = -2$ and simplify. □

Example 8. *An example in terms of Apery's constant* $\zeta(3)$*,*

$$\sum_{n=0}^{\infty}(2n+1)^2\left(\Gamma\left(-3,-\frac{1}{2}i(2\pi n+\pi)\right)+e^{2i\pi n}\Gamma\left(-3,\frac{1}{2}i(2\pi n+\pi)\right)\right)=\frac{16-3\zeta(3)}{12\pi^2} \tag{23}$$

Proof. Use Equation (14), and set $k = -3$ and simplify. □

Example 9. *An example in terms of* $1/\pi$*,*

$$\sum_{n=0}^{\infty} 2i(2n+1)^3\left(\Gamma\left(-4,-\frac{1}{2}i(2\pi n+\pi)\right)-\Gamma\left(-4,\frac{1}{2}i(2\pi n+\pi)\right)\right)=\frac{7\pi}{1440}-\frac{4}{\pi^3} \tag{24}$$

Proof. Use Equation (14), and set $k = -4$ and simplify. □

Example 10. *An example in terms of* π*,*

$$\sum_{n=0}^{\infty}(2n+1)^4\left(\Gamma\left(-5,-\frac{1}{2}i(2\pi n+\pi)\right)+\Gamma\left(-5,\frac{1}{2}i(2\pi n+\pi)\right)\right)=\frac{15\zeta(5)-256}{80\pi^4} \tag{25}$$

Proof. Use Equation (14), and set $k = -5$ and simplify. □

Example 11. *An example in terms of* π*,*

$$\sum_{n=0}^{\infty}\frac{i\left(\Gamma\left(6,-\frac{1}{2}i(2\pi n+\pi)\right)-\Gamma\left(6,\frac{1}{2}i(2\pi n+\pi)\right)\right)}{(2n+1)^7}=-\frac{\pi^7}{768} \tag{26}$$

Proof. Use Equation (14), and set $k = 6$ and simplify. □

Example 12. *An example in terms of the Polylogarithm function* $Li_k(x)$*,*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\left(\frac{e^{\frac{i\pi}{4}}\sqrt{\frac{2}{\pi}}\left(n+\frac{i\pi}{18}+\left(\frac{1}{2}-\frac{i}{6}\right)\right)}{\sqrt{-2n-\left(1-\frac{i}{3}\right)}\sqrt{2in+\left(\frac{1}{3}+i\right)}(2n+1)^2}\right.\\
&\quad+\frac{(-1)^{3/4}\sqrt{\frac{2}{\pi}}\sqrt{\left(\frac{1}{3}-i\right)-2in}\left(\frac{\pi}{18}+\frac{1}{2}\left(2in-\left(\frac{1}{3}-i\right)\right)\right)}{\left(-2n-\left(1+\frac{i}{3}\right)\right)^{3/2}(2n+1)^2}\\
&\quad-\frac{(-1)^{3/4}e^{\frac{1}{2}\pi\left(\left(\frac{1}{3}-i\right)-2in\right)}\Gamma\left(\frac{1}{2},\frac{1}{2}\left(\left(\frac{1}{3}-i\right)-2in\right)\pi\right)}{2\left(-2n-\left(1+\frac{i}{3}\right)\right)^{3/2}}\\
&\quad\left.+\frac{e^{\frac{1}{2}\pi\left(2in+\left(\frac{1}{3}+i\right)\right)+\frac{i\pi}{4}}\Gamma\left(\frac{1}{2},\frac{1}{2}\left(2in+\left(\frac{1}{3}+i\right)\right)\pi\right)}{2\left(-2n-\left(1-\frac{i}{3}\right)\right)^{3/2}}\right)\\
&=\frac{1}{4}\pi\left(2e^{\pi/6}\sqrt{\pi}\mathrm{Li}_{-\frac{1}{2}}\left(-e^{-\pi/3}\right)+\sqrt{\frac{\pi}{2}}\right)
\end{aligned}
\tag{27}
$$

Proof. Use Equation (11), and set $k = 1/2, a = e^{\pi/2}, m = -1/3$ and simplify using Equation (25.14.2) in [9]. □

Example 13. *An example in terms of the Hurwitz-Lerch zeta function* $\Phi(s,u,v)$*,*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n\left(-\frac{\sqrt[4]{-1}\left(\frac{1}{2}((1-2i)-2in)-2i\pi\right)}{\pi^{3/2}\sqrt{-2n-(2+i)}\sqrt{(1-2i)-2in}(2n+1)^2}+\frac{e^{-\frac{i\pi}{4}}\sqrt{-2n+i}\left(-n+2\pi+\frac{i}{2}\right)}{\pi^{3/2}(1+2in)^{3/2}(2n+1)^2}\right.\\
&\quad\left.-\frac{\sqrt[4]{-1}e^{\pi((1-2i)-2in)}\Gamma\left(-\frac{1}{2},((1-2i)-2in)\pi\right)}{2\sqrt{-2n-(2+i)}}-\frac{e^{\pi(1+2in)-\frac{i\pi}{4}}\Gamma\left(-\frac{1}{2},(2in+1)\pi\right)}{2\sqrt{-2n+i}}\right)\\
&=\frac{1}{2}\pi\left(\frac{1}{\sqrt{\pi}}-\frac{2ie^{-\pi/2}\Phi\left(e^{-\pi},\frac{1}{2},\frac{3}{2}\right)}{\sqrt{\pi}}\right)
\end{aligned}
\tag{28}
$$

Proof. Use Equation (11), and set $k = -1/2, a = e^{\pi}, m = -1+i$ and simplify. □

Example 14. *An example in terms of the hyperbolic functions.*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}e^{-i\pi n}(\Gamma(-1,((1-i)-2in)\pi)-\Gamma(-1,2i\pi n+(1+i)\pi))\\
&\quad=-\frac{1}{2}ie^{-\pi}\left(-4e^{\pi/2}+4e^{\pi}\cot^{-1}\left(e^{\pi/2}\right)+\operatorname{sech}\left(\frac{\pi}{2}\right)\right)
\end{aligned}
\tag{29}
$$

Proof. Use Equation (11), and set $k = -1, a = e^{\pi}, m = -1$ and simplify using entry (3) in the table below (64:12:7) in [13]. □

Example 15. *An example in terms of the exponential function,*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{64n(n+1)+32}(1+(1+i)n)E_{-2}\left(\frac{1}{2}((1-i)-2in)\pi\right)\\
&\quad-e^{2i\pi n}((1+i)n+i)E_{-2}\left(i\pi n+\left(\frac{1}{2}+\frac{i}{2}\right)\pi\right)\\
&\quad=-\frac{\left(\frac{1}{96}-\frac{i}{96}\right)\left(e^{\pi}\left(5+7e^{\pi}\left(e^{\pi}-5\right)\right)-1\right)\pi}{(1+e^{\pi})^4}
\end{aligned}
\tag{30}
$$

Proof. Use Equation (11), and set $k = 3, a = e^{\pi/2}, m = -1$ and simplify using entry (4) in the table below (64:12:7) in [13] and Equation (8.19.1) in [9]. □

Example 16. *An example in terms of the Polylogarithm function* $Li_k(x)$*,*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{2}(-1)^{n+\frac{3}{4}}\Bigg(&-\frac{4\sqrt{-4n+\frac{2i}{\pi}-2}\left(-n+\frac{i}{\pi}-\frac{1}{2}\right)}{(2n+1)^2(1+i\pi(2n+1))^{3/2}}+\frac{2\sqrt{2}(2\pi n+\pi+2i)}{\pi^{3/2}(2n+1)^2\sqrt{-\pi(2n+1)-i\sqrt{1-i\pi(2n+1)}}}\\ &+\frac{ie^{\frac{1}{2}+i\pi n}\Gamma\left(-\frac{1}{2},\frac{1}{2}i(2\pi n+\pi-i)\right)}{\sqrt{-2n+\frac{i}{\pi}-1}}+\frac{\sqrt{\pi}e^{\frac{1}{2}-i\pi n}\Gamma\left(-\frac{1}{2},-\frac{1}{2}i(2\pi n+\pi+i)\right)}{\sqrt{-\pi(2n+1)-i}}\Bigg)\\ =\ &\sqrt{e\pi}\mathrm{Li}_{\frac{1}{2}}\left(-\frac{1}{e}\right)+\sqrt{\frac{\pi}{2}}\end{aligned} \tag{31}$$

Proof. Use Equation (11), and set $k = -1/2, a = e^{\pi/2}, m = -1/\pi$ and simplify using Equation (64:12:2) in [13]. □

Example 17. *An example in terms of the trigamma function* $\psi^{(1)}(x)$*,*

$$\begin{aligned}&\sum_{n=0}^{\infty}e^{\pi(-1+(-2+i)n)}(2n+1)\left(\Gamma(-2,(-2n-1)\pi)+e^{4\pi n+2\pi}\Gamma(-2,2\pi n+\pi)\right)\\ &=\frac{2+\psi^{(1)}\left(\frac{1}{4}+\frac{i}{2}\right)-\psi^{(1)}\left(\frac{3}{4}+\frac{i}{2}\right)}{8\pi}\end{aligned} \tag{32}$$

Proof. Use Equation (15), and set $k = -2, a = -1$ and simplify using Equation (64:4:1) in [13]. □

Example 18. *An example in terms of* π*,*

$$\sum_{n=0}^{\infty}(-1)^n e^{-2i\pi n}\left(\Gamma(-1,-i(2n+1)\pi)-e^{4i\pi n}\Gamma(-1,i(2n+1)\pi)\right)=-\frac{1}{2}i(\pi-3) \tag{33}$$

Proof. Use Equation (15), and apply l'Hopital's rule as $k \to -1$ and simplify using Equation (64:9:2) in [13], then set $a = e^{\pi}$ and simplify. □

Example 19. *An example in terms of* $\log(x)$ *and* π*,*

$$\sum_{n=0}^{\infty}\left(E_1\left(\frac{1}{6}(-6in-(1+3i))\pi\right)+E_1\left(\frac{1}{6}(6in-(1-3i))\pi\right)\right)=-\log\left(1+e^{\pi/3}\right) \tag{34}$$

Proof. Use Equation (11), and set $a = e^{\pi/2}, k = -1, m = 1/3$ and simplify using entry (3) in the table below (64:12:7) in [13] and Equation (8.19.1) in [9]. □

Example 20. *An example in terms of the Hurwitz-Lerch zeta function* $\Phi(s, u, v)$*,*

$$\sum_{n=0}^{\infty}(-1)^n\left(-\frac{1}{9(2n+1)^2}e^{\pi/2}\pi^{-1+i}\left(-2n-\left(1+\frac{i}{3}\right)\right)^{-1-i}\left(\left(\frac{1}{3}-i\right)-2in\right)^{i}\right.$$
$$(-18n+\pi-(9+3i))$$
$$+\frac{ie^{-\pi/2}\pi^{-1+i}\left(-2n-\left(1-\frac{i}{3}\right)\right)^{-i}\left(2in+\left(\frac{1}{3}+i\right)\right)^{-1+i}(18n+\pi+(9-3i))}{9(2n+1)^2}$$
$$-ie^{\pi\left(\frac{5}{6}-2in\right)}\left(-2n-\left(1+\frac{i}{3}\right)\right)^{-1-i}\Gamma\left(i,\left(\left(\frac{1}{3}-i\right)-2in\right)\pi\right)$$
$$\left.+ie^{\frac{1}{6}i\pi(12n+(6+i))}\left(-2n-\left(1-\frac{i}{3}\right)\right)^{-1-i}\Gamma\left(i,\left(2in+\left(\frac{1}{3}+i\right)\right)\pi\right)\right)$$
$$=\frac{1}{2}\pi^{1+i}\left(1-2e^{-\pi/6}\Phi\left(-e^{-\pi/3},-i,\frac{3}{2}\right)\right) \tag{35}$$

Proof. Use Equation (11), and set $a=e^{\pi}, k=i, m=-1/3$ and simplify. □

Example 21. *An example in terms of the Polylogarithm function* $\mathrm{Li}_k(x)$,

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{12(2n+1)^2}ie^{-\pi/2}\pi^{i}\left((-12n-(4-3i))^{-1-i}\left(6in+\left(\frac{3}{2}+2i\right)\right)^{i}\right.\right.$$
$$((144+144i)n+(12+5i)\pi+(84+12i))+e^{\pi}(-12n-(8+3i))^{-1-i}\left(\left(\frac{3}{2}-4i\right)-6in\right)^{i}$$
$$\left.((12+5i)\pi-(12+12i)(12n+(8+3i)))\right)$$
$$+(1-i)e^{-\frac{1}{12}i\pi(12n+(8+9i))}\left(-2n-\left(\frac{4}{3}+\frac{i}{2}\right)\right)^{-2-i}\Gamma\left(1+i,-\frac{1}{12}i(12n+(8+3i))\pi\right)$$
$$\left.-(1-i)e^{\frac{1}{12}i\pi(12n+(4+3i))}\left(-2n-\left(\frac{2}{3}-\frac{i}{2}\right)\right)^{-2-i}\Gamma\left(1+i,i\pi n+\left(\frac{1}{4}+\frac{i}{3}\right)\pi\right)\right)$$
$$=\frac{1}{4}\pi^{2+i}\left(2^{-i}-4(-1)^{\frac{5}{6}-\frac{i}{4}}\mathrm{Li}_{-1-i}\left(-(-1)^{\frac{1}{3}+\frac{i}{2}}\right)\right) \tag{36}$$

Proof. Use Equation (11), and set $a=e^{\pi/2}, k=1+i, m=-1/2+i/3$ and simplify using Equation (25.14.3) in [9]. □

Example 22. *An example in terms of the Hurwitz-Lerch zeta function* $\Phi(s,u,v)$,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2\pi^2(4n+1)}\left(\frac{-32+64e^{\frac{1}{8}i\pi(4n+3)}\left(8n^2+6n+1\right)E_3\left(\frac{1}{8}i(4n+3)\pi\right)}{(2n+1)(4n+3)}\right.$$
$$\left.+64e^{-\frac{1}{8}i\pi(4n+1)}E_3\left(-\frac{1}{8}i(4n+1)\pi\right)\right)$$
$$=\frac{8+(-1)^{3/4}\Phi\left(i,2,\frac{3}{4}\right)}{2\pi} \tag{37}$$

Proof. Use Equation (11), and set $a=e^{\pi/4}, k=-2, m=-i/2$ and simplify using Equation (8.19.1) in [9]. □

Example 23. *An example in terms of the Hypergeometric function,*

$$\sum_{n=0}^{\infty} \frac{4}{\pi(4n+1)(4n+3)}(-1)^n e^{-\frac{1}{8}i\pi(4n+1)} \left((4n+3)E_2\left(-\frac{1}{8}i(4n+1)\pi\right) + ie^{i\pi n}(4n+1)E_2\left(\frac{1}{8}i(4n+3)\pi\right)\right) = \frac{1}{3}\sqrt{2}\left(3-(1-i)\,{}_2F_1\left(\frac{3}{4},1;\frac{7}{4};i\right)\right) \tag{38}$$

Proof. Use Equation (11), and set $a = e^{\pi/4}, k = -1, m = -i/2$ and simplify using Equation (9.559) in [11] and Equation (8.19.1) in [9]. □

Example 24. *An example in terms of the log-gamma function* $\log(\Gamma(x))$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n e^{-\frac{1}{3}i\pi(2n+1)}\left(\Gamma\left(0,-\frac{1}{3}i(2n+1)\pi\right)+e^{\frac{1}{3}\pi(4in+2i)}\Gamma\left(0,\frac{1}{3}i(2n+1)\pi\right)\right)}{2(2n+1)} = \frac{1}{2}\pi\log\left(\frac{\sqrt{6}\Gamma\left(-\frac{1}{12}\right)}{7\Gamma\left(-\frac{7}{12}\right)}\right) \tag{39}$$

Proof. Use Equation (15), take the first partial derivative with respect to k, and set $k = 0$, $a = e^{\pi/3}$ and simplify using Equation (25.11.18) in [9]. □

7. Discussion

The authors constructed an expression for the Hurwitz-Lerch zeta function $\Phi(k,a,m)$ in terms of the infinite sum of the incomplete gamma function $\Gamma(k,\{a,m\})$, where the parameter constraints are wide. The derivations used fundamental constants and special functions, and the infinite sum allowed for a wide range of the parameters. We checked the outcome numerically using Wolfram Mathematica.

Author Contributions: Conceptualization, R.R.; funding acquisition, A.S.; supervision, A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by NSERC Canada under grant 504070.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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