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On Some New Simpson's Formula Type Inequalities for Convex Functions in Post-Quantum Calculus

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Abstract: In this work, we prove a new (p,q)-integral identity involving a (p,q)-derivative and (p,q)-integral. The newly established identity is then used to show some new Simpson's formula type inequalities for (p,q)-differentiable convex functions. Finally, the newly discovered results are shown to be refinements of comparable results in the literature. Analytic inequalities of this type, as well as the techniques used to solve them, have applications in a variety of fields where symmetry is important.

Keywords: Simpson's inequalities; post-quantum calculus; convex functions

MSC: 26D10; 26D15; 26A51

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1. Introduction

During his lifetime, Thomas Simpson created important approaches for numerical integration and the estimation of definite integrals, which became known as Simpson's rule (1710–1761). J. Kepler, who made a comparable calculation roughly a century before Newton, is the inspiration for Kepler's rule. Estimations based exclusively on a three-step quadratic kernel are commonly referred to as Newton-type results because Simpson's technique incorporates the three-point Newton–Cotes quadrature rule.
(1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\pi_1}^{\pi_2} \mathcal{F}(x)dx \approx \frac{\pi_2-\pi_1}{6}\left[\mathcal{F}(\pi_1)+4\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right)+\mathcal{F}(\pi_2)\right].$$

(2) Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule)

$$\int_{\pi_1}^{\pi_2} \mathcal{F}(x)dx \approx \frac{\pi_2-\pi_1}{8}\left[\mathcal{F}(\pi_1)+3\mathcal{F}\left(\frac{2\pi_1+\pi_2}{3}\right)+3\mathcal{F}\left(\frac{\pi_1+2\pi_2}{3}\right)+\mathcal{F}(\pi_2)\right].$$

The following estimation, known as Simpson's inequality, is one of many linked with these quadrature rules in the literature:

Theorem 1. *Suppose that* $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ *is a four-times continuously differentiable mapping on* (π_1, π_2)*, and let* $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{x\in(\pi_1,\pi_2)} \left|\mathcal{F}^{(4)}(x)\right| < \infty$*. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2}+2\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\mathcal{F}(x)dx\right| \leq \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty(\pi_2-\pi_1)^4.$$

Many researchers have focused on Simpson-type inequality in various categories of mappings in recent years. Because convexity theory is an effective and powerful technique to solve a huge number of problems from various disciplines of pure and applied mathematics, some mathematicians have worked on the results of Simpson's and Newton's type in obtaining a convex map. The novel Simpson's inequalities and their applications in numerical integration quadrature formulations were presented by Dragomir et al. [1]. Furthermore, Alomari et al. [2] discovered a number of inequalities in Simpson's kind of s-convex functions. The variance of Simpson-type inequality as a function of convexity was then observed by Sarikaya et al. in [3]. Refs. [4–6] can be consulted for further research on this subject.

On the other hand, quantum and post-quantum integrals for many types of functions have been used to study many integral inequalities. The authors of [7–21] employed left–right q-derivatives and integrals to prove HH integral inequalities and associated left–right estimates for convex and coordinated convex functions. Noor et al. proposed a generalized version of quantum integral inequalities in their paper [22]. In [23], the authors demonstrated some parameterized quantum integral inequalities for generalized quasi-convex functions. In [24], Khan et al. used the green function to prove quantum HH inequality. For convex and coordinated convex functions, the authors of [25–30] constructed new quantum Simpson's and quantum Newton's type inequalities. Consult [31–33] for quantum Ostrowski's inequality for convex and co-ordinated convex functions. Using the left (p,q)-difference operator and integral, the authors of [34] expanded the results of [9] and demonstrated HH-type inequalities and associated left estimates. In [16], the authors discovered the right estimates of HH-type inequalities, as demonstrated in [34]. Vivas-Cortez et al. [35] recently generalized the results of [11] and used the right (p,q)-difference operator and integral to prove HH-type inequalities and associated left estimates.

We use the (p,q)-integral to establish some new post-quantum Simpson's type inequalities for (p,q)-differentiable convex functions, as inspired by recent research. The newly revealed inequalities are also shown to be extensions of previously discovered inequalities.

The structure of this article is as follows. The principles of q-calculus, as well as other relevant topics in this subject, are briefly discussed in Section 2. The basics of (p,q)-calculus, as well as some recent research in this topic, are covered in Section 3. In Section 4, we prove a new (p,q)-integral identity involving a (p,q)-derivative. Section 5 describes the Simpson's type inequalities for (p,q)-differentiable functions via (p,q)-integrals. It is also taken into account the relationship between the findings given here and similar findings in the literature. Section 6 finishes with some research suggestions for the future.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we revisit several previously regarded ideas. In addition, we utilize the following notation here and elsewhere (see [36]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1}, \quad q \in (0,1).$$

In [37], Jackson gave the q-Jackson integral from 0 to π_2 for $0 < q < 1$ as follows:

$$\int_0^{\pi_2} \mathcal{F}(x)\ d_qx\ = (1-q)\pi_2\sum_{n=0}^{\infty} q^n\mathcal{F}(\pi_2 q^n) \tag{1}$$

provided that the sum converges absolutely.

Definition 1 ([38]). *For a function $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$, the left q-derivative of $\mathcal{F}$ at $x \in [\pi_1, \pi_2]$ is characterized by the expression*

$$ {}_{\pi_1}D_q\mathcal{F}(x) = \frac{\mathcal{F}(x) - \mathcal{F}(qx + (1-q)\pi_1)}{(1-q)(x-\pi_1)}, \ x \neq \pi_1. \tag{2}$$

If $x = \pi_1$, we define ${}_{\pi_1}D_q\mathcal{F}(\pi_1) = \lim_{x\to\pi_1} {}_{\pi_1}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 2 ([11]). *For a function $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$, the right q-derivative of $\mathcal{F}$ at $x \in [\pi_1, \pi_2]$ is characterized by the expression*

$$ {}^{\pi_2}D_q\mathcal{F}(x) = \frac{\mathcal{F}(qx + (1-q)\pi_2) - \mathcal{F}(x)}{(1-q)(\pi_2-x)}, \ x \neq \pi_2. \tag{3}$$

If $x = \pi_2$, we define ${}^{\pi_2}D_q\mathcal{F}(\pi_2) = \lim_{x\to\pi_2} {}^{\pi_2}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 3 ([38]). *Let $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ be a function. Then, the left q-definite integral on $[\pi_1, \pi_2]$ is defined as*

$$\begin{aligned} \int_{\pi_1}^{\pi_2} \mathcal{F}(x) \ {}_{\pi_1}d_qx &= (1-q)(\pi_2-\pi_1)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\pi_2 + (1-q^n)\pi_1) \\ &= (\pi_2-\pi_1)\int_0^1 \mathcal{F}((1-t)\pi_1 + t\pi_2) \ d_qt \ . \end{aligned} \tag{4}$$

Definition 4 ([11]). *Let $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ be a function. Then, the right q-definite integral on $[\pi_1, \pi_2]$ is defined as*

$$\begin{aligned} \int_{\pi_1}^{\pi_2} \mathcal{F}(x) \ {}^{\pi_2}d_qx &= (1-q)(\pi_2-\pi_1)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\pi_1 + (1-q^n)\pi_2) \\ &= (\pi_2-\pi_1)\int_0^1 \mathcal{F}(t\pi_1 + (1-t)\pi_2) \ d_qt \ . \end{aligned} \tag{5}$$

Alp et al. [9] proved the following Hermite–Hadamard-type inequalities for convex functions via q-integral.

Theorem 2. *For the convex mapping $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$, the following inequality holds*

$$\mathcal{F}\left(\frac{q\pi_1 + \pi_2}{[2]_q}\right) \leq \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} \mathcal{F}(x) \ {}_{\pi_1}d_qx \ \leq \frac{q\mathcal{F}(\pi_1) + \mathcal{F}(\pi_2)}{[2]_q}.$$

In [11], Bermudo et al. established the following quantum Hermite–Hadamard-type inequalities:

Theorem 3. *For the convex mapping $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$, the following inequality holds*

$$\mathcal{F}\left(\frac{\pi_1 + q\pi_2}{[2]_q}\right) \leq \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} \mathcal{F}(x) \ {}^{\pi_2}d_qx \ \leq \frac{\mathcal{F}(\pi_1) + q\mathcal{F}(\pi_2)}{[2]_q}.$$

$$\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right) \leq \frac{1}{2(\pi_2-\pi_1)}\left[\int_{\pi_1}^{\pi_2}\mathcal{F}(x)\ {}_{\pi_1}d_qx\ +\int_{\pi_1}^{\pi_2}\mathcal{F}(x)\ {}_{\pi_2}d_qx\ \right] \leq \frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2}.$$

Recently, Siricharuanun et al. [29] proved the following Simpson's formula type inequality for convex functions.

Theorem 4. *Let $\mathcal{F}:[\pi_1,\pi_2]\to\mathbb{R}$ be a q^{π_2}-differentiable function on (π_1,π_2) such that ${}^{\pi_2}D_q\mathcal{F}$ is continuous and integrable on $[\pi_1,\pi_2]$. If $\left|{}^{\pi_2}D_q\mathcal{F}\right|$ is convex on $[\pi_1,\pi_2]$, then we have the following inequality for q^{π_2}-integrals:*

$$\begin{aligned}&\left|\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\mathcal{F}(s)\ {}_{\pi_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\pi_1)+q^2[4]_q\mathcal{F}\left(\frac{\pi_1+q\pi_2}{[2]_q}\right)+q\mathcal{F}(\pi_2)\right]\right| \quad (6)\\ \leq\ & q(\pi_2-\pi_1)\{|\,{}^{\pi_2}D_q\mathcal{F}(\pi_1)|[A_1(q)+A_2(q)]+|\,{}^{\pi_2}D_q\mathcal{F}(\pi_2)|[B_1(q)+B_2(q)]\},\end{aligned}$$

where $0<q<1$ and

$$\begin{aligned}A_1(q) &= \frac{2q^2[2]_q^2+[6]_q^2\left([6]_q-[3]_q\right)}{[2]_q^3[3]_q[6]_q^3},\\ B_1(q) &= 2\frac{q[3]_q[6]_q-q^2}{[2]_q[3]_q[6]_q^3}+\frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q}-\frac{q^2+2q}{[6]_q}\right),\\ A_2(q) &= \frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{[6]_q\left(1+[2]_q^3\right)-[3]_q[5]_q\left(1+[2]_q^2\right)}{[2]_q^3[3]_q[6]_q},\\ B_2(q) &= 2\frac{q[5]_q^2[6]_q[3]_q-q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{q^2}{[2]_q[3]_q}-\frac{q[5]_q}{[2]_q[6]_q}\\ &\quad -\frac{1}{[2]_q^3}\left[\frac{[5]_q(2q+q^2)}{[6]_q}-\frac{q+q^2}{[3]_q}\right].\end{aligned}$$

3. Post-Quantum Calculus and Some Inequalities

In this section, we review some fundamental notions and notations of (p,q)-calculus. The $[n]_{p,q}$ is said to be (p,q)-integers and expressed as:

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}$$

with $0<q<p\leq 1$. The $[n]_{p,q}!$ and $\left[\begin{array}{c}n\\k\end{array}\right]!$ are called (p,q)-factorial and (p,q)-binomial, respectively, and expressed as:

$$\begin{aligned}[n]_{p,q}! &= \prod_{k=1}^{n}[k]_{p,q},\ n\geq 1,\ [0]_{p,q}!=1,\\ \left[\begin{array}{c}n\\k\end{array}\right]! &= \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.\end{aligned}$$

Definition 5 ([39]). *The (p,q)-derivative of mapping $\mathcal{F}:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$D_{p,q}\mathcal{F}(x)=\frac{\mathcal{F}(px)-\mathcal{F}(qx)}{(p-q)x},\ x\neq 0$$

with $0 < q < p \leq 1$.

Definition 6 ([40]). *The left* p,q*-derivative of mapping* $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ *is given as:*

$$ {}_{\pi_1}D_{p,q}\mathcal{F}(x) = \frac{\mathcal{F}(px + (1-p)\pi_1) - \mathcal{F}(qx + (1-q)\pi_1)}{(p-q)(x-\pi_1)}, \ x \neq \pi_1 \tag{7} $$

with $0 < q < p \leq 1$. *For* $x = \pi_1$, *we state that* ${}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1) = \lim_{x\to\pi_1} {}_{\pi_1}D_{p,q}\mathcal{F}(x)$ *if it exists and it is finite.*

Definition 7 ([35]). *The right* (p,q)*-derivative of mapping* $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ *is given as:*

$$ {}^{\pi_2}D_{p,q}\mathcal{F}(x) = \frac{\mathcal{F}(qx + (1-q)\pi_2) - \mathcal{F}(px + (1-p)\pi_2)}{(p-q)(\pi_2-x)}, \ x \neq \pi_2. \tag{8} $$

with $0 < q < p \leq 1$. *For* $x = \pi_2$, *we state that* ${}^{\pi_2}D_{p,q}\mathcal{F}(\pi_2) = \lim_{x\to\pi_2} {}^{\pi_2}D_{p,q}\mathcal{F}(x)$ *if it exists and it is finite.*

Remark 1. *It is clear that if we use* $p = 1$ *in (7) and (8), then the equalities (7) and (8) reduce to (2) and (3), respectively.*

Definition 8 ([40]). *The left* (p,q)*-integral of mapping* $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ *on* $[\pi_1, \pi_2]$ *is stated as:*

$$ \int_{\pi_1}^{x} \mathcal{F}(\tau) \, {}_{\pi_1}d_{p,q}\tau = (p-q)(x-\pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \mathcal{F}\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_1 \right) \tag{9} $$

with $0 < q < p \leq 1$.

Definition 9 ([35]). *The right* (p,q)*-integral of mapping* $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ *on* $[\pi_1, \pi_2]$ *is stated as:*

$$ \int_{x}^{\pi_2} \mathcal{F}(\tau) \, {}^{\pi_2}d_{p,q}\tau = (p-q)(\pi_2-x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \mathcal{F}\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_2 \right) \tag{10} $$

with $0 < q < p \leq 1$.

Remark 2. *It is evident that if we select* $p = 1$ *in (9) and (10), then the equalities (9) and (10) change into (4) and (5), respectively.*

Remark 3. *If we take* $\pi_1 = 0$ *and* $x = \pi_2 = 1$ *in (9), then we have*

$$ \int_0^1 \mathcal{F}(\tau) \, {}_0d_{p,q}\tau = (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \mathcal{F}\left(\frac{q^n}{p^{n+1}} \right). $$

Similarly, by taking $x = \pi_1 = 0$ *and* $\pi_2 = 1$ *in (10), then we obtain that*

$$ \int_0^1 \mathcal{F}(\tau) \, {}^1d_{p,q}\tau = (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \mathcal{F}\left(1 - \frac{q^n}{p^{n+1}} \right). $$

Remark 4. *If* f *is a symmetric function—that is,* $\mathcal{F}(s) = \mathcal{F}(\pi_2 + \pi_1 - s)$, *for* $s \in [\pi_1, \pi_2]$*—then we have*

$$ \int_{\pi_1}^{p\pi_2 + (1-p)\pi_1} \mathcal{F}(s) \, {}_{\pi_1}d_{p,q}s = \int_{p\pi_1 + (1-p)\pi_2}^{\pi_2} \mathcal{F}(s) \, {}^{\pi_2}d_{p,q}s. $$

Lemma 1 ([35]). *We have the following equalities*

$$\int_{\pi_1}^{\pi_2}(\pi_2-x)^{\pi_1}\ {}^{\pi_2}d_{p,q}x=\frac{(\pi_2-\pi_1)^{\pi_1+1}}{[\pi_1+1]_{p,q}} \tag{11}$$

$$\int_{\pi_1}^{\pi_2}(x-\pi_1)^{\pi_1}\ {}_{\pi_1}d_{p,q}x=\frac{(\pi_2-\pi_1)^{\pi_1+1}}{[\pi_1+1]_{p,q}}, \tag{12}$$

where $\pi_1\in\mathbb{R}-\{-1\}$.

Recently, M. Vivas-Cortez et al. [35] proved the following HH-type inequalities for convex functions using the $(p,q)^{\pi_2}$-integral.

Theorem 5 ([35]). *For a convex mapping* $\mathcal{F}:[\pi_1,\pi_2]\to\mathbb{R}$, *which is differentiable on* $[\pi_1,\pi_2]$, *the following inequalities hold for the* $(p,q)^{\pi_2}$*-integral:*

$$\mathcal{F}\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\leq\frac{1}{p(\pi_2-\pi_1)}\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}\mathcal{F}(x)\ {}^{\pi_2}d_{p,q}x\leq\frac{p\mathcal{F}(\pi_1)+q\mathcal{F}(\pi_2)}{[2]_{p,q}}, \tag{13}$$

where $0<q<p\leq 1$.

Theorem 6 ([35]). *For a convex function* $\mathcal{F}:[\pi_1,\pi_2]\to\mathbb{R}$, *the following inequality holds:*

$$\begin{aligned}\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right)&\leq\frac{1}{2p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\mathcal{F}(x)\ {}_{\pi_1}d_{p,q}x+\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}\mathcal{F}(x)\ {}^{\pi_2}d_{p,q}x\right]\\&\leq\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2},\end{aligned} \tag{14}$$

where $0<q<p\leq 1$.

4. An Identity

In this section, we deal with an identity that is required to reach our major estimates. In the following lemma, we first build an identity based on a two-stage kernel.

Lemma 2. *Let* $\mathcal{F}:[\pi_1,\pi_2]\to\mathbb{R}$ *be a differentiable function on* (π_1,π_2). *If* ${}_{\pi_1}D_{p,q}\mathcal{F}$ *is continuous and integrable on* $[\pi_1,\pi_2]$, *then one has the identity*

$$\begin{aligned}&\frac{p}{[6]_{p,q}}\left[p^5\mathcal{F}(\pi_2)+q\Big([5]_{p,q}-1\Big)\mathcal{F}\left(\frac{p\pi_2+q\pi_1}{p+q}\right)+q\mathcal{F}(\pi_1)\right]\\&-\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s\\&=pq(\pi_2-\pi_1)\int_0^1\Lambda(s)\ {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1)\quad d_{p,q}s\ ,\end{aligned} \tag{15}$$

where

$$\Lambda(s)=\begin{cases}s-\frac{1}{[6]_{p,q}}, & s\in\left[0,\frac{p}{[2]_{p,q}}\right)\\ s-\frac{[5]_{p,q}}{[6]_{p,q}}, & s\in\left[\frac{p}{[2]_{p,q}},1\right].\end{cases}$$

Proof. Using the fundamental properties of (p,q)-integrals and the definition of function $\Lambda(s)$, we find that

$$\begin{aligned} & \int_0^1 \Lambda(s) \ {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1) \quad d_{p,q}s \qquad (16)\\ = & \frac{[5]_{p,q}-1}{[6]_{p,q}} \int_0^{\frac{p}{[2]_{p,q}}} {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1) \quad d_{p,q}s \\ & + \int_0^1 \left(s-\frac{[5]_{p,q}}{[6]_{p,q}}\right) \ {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1) \quad d_{p,q}s \ . \end{aligned}$$

According to Definition 6, one must also have

$${}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1) = \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(p-q)(\pi_2-\pi_1)s}.$$

Now, if we substitute the above equation into (16), we obtain

$$\begin{aligned} & \int_0^1 \Lambda(s) \ {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1) \quad d_{p,q}s \qquad (17)\\ = & \frac{[5]_{p,q}-1}{[6]_{p,q}} \int_0^{\frac{p}{[2]_{p,q}}} \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(p-q)(\pi_2-\pi_1)s} \ d_{p,q}s \\ & + \int_0^1 \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(p-q)(\pi_2-\pi_1)} \ d_q s \\ & - \frac{[5]_{p,q}}{[6]_{p,q}} \int_0^1 \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(1-q)(\pi_2-\pi_1)s} \ d_{p,q}s \ . \end{aligned}$$

When the first integral on the right-hand side of (17) is calculated using Definition 8, it is discovered that

$$\begin{aligned} & \int_0^{\frac{p}{[2]_{p,q}}} \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(p-q)(\pi_2-\pi_1)s} \ d_{p,q}s \qquad (18)\\ = & \frac{1}{(\pi_2-\pi_1)}\left\{\sum_{k=0}^{\infty}\mathcal{F}\left(\frac{q^k}{p^{k+1}}\frac{p^2}{[2]_{p,q}}\pi_2+\left(1-\frac{q^k}{p^{k+1}}\frac{p^2}{[2]_{p,q}}\right)\pi_1\right)\right. \\ & \left. -\sum_{k=0}^{\infty}\mathcal{F}\left(\frac{q^{k+1}}{p^{k+1}}\frac{p}{[2]_{p,q}}\pi_2+\left(1-\frac{q^{k+1}}{p^{k+1}}\frac{p}{[2]_{p,q}}\right)\pi_1\right)\right\} \\ = & \frac{1}{(\pi_2-\pi_1)}\left\{\mathcal{F}\left(\frac{p\pi_2+q\pi_1}{[2]_{p,q}}\right)-\mathcal{F}(\pi_1)\right\}. \end{aligned}$$

If we look at the other integrals on the right-hand side of (17), we obtain

$$\int_0^1 \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(p-q)(\pi_2-\pi_1)}\ d_{p,q}s \tag{19}$$
$$= \frac{1}{(\pi_2-\pi_1)}\left\{\frac{1}{pq(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s\ +\frac{1}{q}\mathcal{F}(\pi_2)\right\}$$

and

$$\int_0^1 \frac{\mathcal{F}(ps\pi_2+(1-ps)\pi_1)-\mathcal{F}(qs\pi_2+(1-qs)\pi_1)}{(p-q)(\pi_2-\pi_1)s}\ d_{p,q}s \tag{20}$$
$$= \frac{1}{(\pi_2-\pi_1)}\{\mathcal{F}(\pi_2)-\mathcal{F}(\pi_1)\}$$

Substituting the expressions (18)–(20) into (17), and later multiplying both sides of the resulting identity by $pq(\pi_2-\pi_1)$, the equality (15) can be captured. □

5. Main Results

For (p,q)-differentiable convex functions, we prove some new Simpson's formula type inequalities in this section. For the sake of brevity, we start this section with certain notations that will be utilized in our new results.

$$A_1(p,q)=\frac{2[2]_{p,q}^2\left([3]_{p,q}-[2]_{p,q}\right)+p^2[6]_{p,q}^2\left(p[6]_{p,q}-[3]_{p,q}\right)}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}^3}, \tag{21}$$

$$B_1(p,q)=2\frac{[2]_{p,q}[3]_{p,q}[6]_{p,q}-[3]_{p,q}[6]_{p,q}-[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}$$
$$+\left(\frac{p^2[3]_{p,q}[6]_{p,q}-p[2]_{p,q}^2[3]_{p,q}-p^3[6]_{p,q}+p^2[3]_{p,q}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}\right), \tag{22}$$

$$A_2(p,q)=2\frac{[3]_{p,q}[5]_{p,q}^3-[2]_{p,q}[5]_{p,q}^3}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}+\frac{p^3[6]_{p,q}-p^2[3]_{p,q}[5]_{p,q}+[2]_{p,q}^3[6]_{p,q}-[2]_{p,q}^2[3]_{p,q}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}, \tag{23}$$

$$B_2(p,q)=2\frac{[5]_{p,q}^2[2]_{p,q}[3]_{p,q}[6]_{p,q}-[5]_{p,q}^2[3]_{p,q}[6]_{p,q}-[5]_{p,q}^3[3]_{p,q}+[5]_{p,q}^3[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}$$
$$-\left(\frac{p[5]_{p,q}[2]_{p,q}^2[3]_{p,q}-p^2[3]_{p,q}[6]_{p,q}-p^2[5]_{p,q}[3]_{p,q}+p^3[6]_{p,q}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}\right)$$
$$+\left(\frac{[3]_{p,q}[6]_{p,q}-[5]_{p,q}[2]_{p,q}[3]_{p,q}-[2]_{p,q}[6]_{p,q}+[5]_{p,q}[3]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}}\right) \tag{24}$$

Theorem 7. *Assume that the conditions of Lemma 2 hold. If* $\left|{}_{\pi_1}D_{p,q}\mathcal{F}\right|$ *is convex on* $[\pi_1,\pi_2]$, *then we have following inequality for* $(p,q)_{\pi_1}$*-integrals:*

$$\begin{aligned}&\left|\frac{p}{[6]_{p,q}}\left[p^5\mathcal{F}(\pi_2)+q\Big([5]_{p,q}-1\Big)\mathcal{F}\left(\frac{p\pi_2+q\pi_1}{p+q}\right)+q\mathcal{F}(\pi_1)\right]\right. \qquad (25)\\ &\left.-\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2-(1-p)\pi_1}\mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s\right|\\ \leq\ & pq(\pi_2-\pi_1)\{|\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)|[A_1(p,q)+A_2(p,q)]+|\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)|[B_1(p,q)+B_2(p,q)]\},\end{aligned}$$

where $0<q<p\leq 1$ *and* $A_1(p,q)$, $A_2(p,q)$, $B_1(p,q)$, $B_2(p,q)$ *are given as in (21)–(24), respectively.*

Proof. We observe that when we take the modulus in Lemma 2, because of the modulus' characteristics, we have

$$\begin{aligned}&\left|\frac{1}{[6]_{p,q}}\left[p^5\mathcal{F}(\pi_2)+q\Big([5]_{p,q}-1\Big)\mathcal{F}\left(\frac{p\pi_2+q\pi_1}{p+q}\right)+q\mathcal{F}(\pi_1)\right]\right. \qquad (26)\\ &\left.-\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2-(1-p)\pi_1}\mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s\right|\\ \leq\ & pq(\pi_2-\pi_1)\int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|\,|\,{}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1)\,|d_{p,q}s\\ &+pq(\pi_2-\pi_1)\int_{\frac{p}{[2]_{p,q}}}^1\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right||{}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1)|\,d_{p,q}s\end{aligned}$$

Using the convexity of $\left|{}_{\pi_1}D_{p,q}\mathcal{F}\right|$, we may calculate integrals on the right-hand side of (26) as follows:

$$\begin{aligned}&\int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|\,|\,{}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1)\,|d_{p,q}s\\ \leq\ & |\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)|\int_0^{\frac{p}{[2]_{p,q}}}s\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s+|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)|\int_0^{\frac{p}{[2]_{p,q}}}(1-s)\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s.\end{aligned}$$

When we apply the equality (12) idea to the aforementioned post-quantum integrals, we obtain

$$\begin{aligned}\int_0^{\frac{p}{[2]_{p,q}}}s\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s &= \int_0^{\frac{1}{[6]_{p,q}}}s\left(\frac{1}{[6]_{p,q}}-s\right)d_{p,q}s+\int_{\frac{1}{[6]_{p,q}}}^{\frac{p}{[2]_{p,q}}}s\left(s-\frac{1}{[6]_{p,q}}\right)d_{p,q}s\\ &= 2\int_0^{\frac{1}{[6]_{p,q}}}s\left(\frac{1}{[6]_{p,q}}-s\right)d_{p,q}s+\int_0^{\frac{p}{[2]_{p,q}}}s\left(s-\frac{1}{[6]_{p,q}}\right)d_{p,q}s\\ &= \frac{2[2]_{p,q}^2\Big([3]_{p,q}-[2]_{p,q}\Big)+p^2[6]_{p,q}^2\Big(p[6]_{p,q}-[3]_{p,q}\Big)}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}^3}\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{p}{[2]_{p,q}}}(1-s)\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s\\
=&\int_0^{\frac{1}{[6]_{p,q}}}(1-s)\left(\frac{1}{[6]_{p,q}}-s\right)d_{p,q}s+\int_{\frac{1}{[6]_{p,q}}}^{\frac{p}{[2]_{p,q}}}(1-s)\left(s-\frac{1}{[6]_{p,q}}\right)d_{p,q}s\\
=&\;2\int_0^{\frac{1}{[6]_{p,q}}}(1-s)\left(\frac{1}{[6]_{p,q}}-s\right)d_{p,q}s+\int_0^{\frac{p}{[2]_{p,q}}}(1-s)\left(s-\frac{1}{[6]_{p,q}}\right)d_{p,q}s\\
=&\;2\frac{[2]_{p,q}[3]_{p,q}[6]_{p,q}-[3]_{p,q}[6]_{p,q}-[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}\\
&+\left(\frac{p^2[3]_{p,q}[6]_{p,q}-p[2]_{p,q}^2[3]_{p,q}-p^3[6]_{p,q}+p^2[3]_{p,q}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|\left|{}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1)\right|d_{p,q}s \qquad (27)\\
\leq&\;\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|\left(\frac{2[2]_{p,q}^2\left([3]_{p,q}-[2]_{p,q}\right)+p^2[6]_{p,q}^2\left(p[6]_{p,q}-[3]_{p,q}\right)}{[2]_q^3[3]_q[6]_q^3}\right)\\
&+\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|\left(2\frac{[2]_{p,q}[3]_{p,q}[6]_{p,q}-[3]_{p,q}[6]_{p,q}-[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}\right.\\
&\left.+\left(\frac{p^2[3]_{p,q}[6]_{p,q}-p[2]_{p,q}^2[3]_{p,q}-p^3[6]_{p,q}+p^2[6]_{p,q}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}\right)\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\frac{p}{[2]_{p,q}}}^1\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right|\left|{}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2+(1-s)\pi_1)\right|d_{p,q}s \qquad (28)\\
\leq&\;\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|\left(2\frac{[3]_{p,q}[5]_{p,q}^3-[5]_{p,q}^3[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}+\frac{\begin{array}{c}p^3[6]_{p,q}-p^2[5]_{p,q}[3]_{p,q}+[2]_{p,q}^3[6]_{p,q}\\-[5]_{p,q}[2]_{p,q}^2[3]_{p,q}\end{array}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}\right)\\
&+\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|\left(\begin{array}{c}2\frac{[5]_{p,q}^2[2]_{p,q}[3]_{p,q}[6]_{p,q}-[5]_{p,q}^2[3]_{p,q}[6]_{p,q}-[5]_{p,q}^3[3]_{p,q}+[5]_{p,q}^3[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}^3}\\-\left[\frac{p[5]_{p,q}[2]_{p,q}^2[3]_{p,q}-p^2[3]_{p,q}[6]_{p,q}-p^2[5]_{p,q}[3]_{p,q}+p^3[6]_{p,q}}{[2]_{p,q}^3[3]_{p,q}[6]_{p,q}}\right]\\+\left[\frac{[3]_{p,q}[6]_{p,q}-[5]_{p,q}[2]_{p,q}[3]_{p,q}-[2]_{p,q}[6]_{p,q}+[5]_{p,q}[3]_{p,q}}{[2]_{p,q}[3]_{p,q}[6]_{p,q}}\right]\end{array}\right)
\end{aligned}$$

We obtain the inequality (25) by placing (27) and (28) in (26). This completes the proof. □

Corollary 1. *In Theorem 7, if we set $p = 1$, then we have the following new Simpson's type inequality for q-integrals:*

$$\left|\frac{1}{[6]_q}\left[\mathcal{F}(\pi_2)+q\Big([5]_q-1\Big)\mathcal{F}\left(\frac{\pi_2+q\pi_1}{1+q}\right)+q\mathcal{F}(\pi_1)\right]-\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\mathcal{F}(s)\ {}_{\pi_1}d_qs\right|$$
$$\leq q(\pi_2-\pi_1)\{|\,{}_{\pi_1}D_q\mathcal{F}(\pi_2)|[A_1(1,q)+A_2(1,q)]+|\,{}_{\pi_1}D_q\mathcal{F}(\pi_1)|[B_1(1,q)+B_2(1,q)]\}.$$

Remark 5. *In Theorem 7, if we assume $p = 1$ and later take the limit as $q \to 1^-$, then we obtain the following Simpson's type inequality:*

$$\left|\frac{1}{6}\left[\mathcal{F}(\pi_1)+4\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right)+\mathcal{F}(\pi_2)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\mathcal{F}(s)ds\right|$$
$$\leq \frac{5(\pi_2-\pi_1)}{72}[|\mathcal{F}'(\pi_1)|+|\mathcal{F}'(\pi_2)|].$$

This is proven by Alomari et al. in [2].

Now, we can see how the inequalities appear when we utilize maps with convex q^{π_2}-derivative powers in an absolute value.

Theorem 8. *Assume that the conditions of Lemma 2 hold. If $|\,{}_{\pi_1}D_{p,q}\mathcal{F}|^{p_1}$ is convex on $[\pi_1,\pi_2]$ for some $p_1 > 1$, then we have following inequality for $(p,q)_{\pi_1}$-integrals:*

$$\left|\frac{p}{[6]_{p,q}}\left[p^5\mathcal{F}(\pi_2)+q\Big([5]_{p,q}-1\Big)\mathcal{F}\left(\frac{p\pi_2+q\pi_1}{p+q}\right)+q\mathcal{F}(\pi_1)\right]\right. \tag{29}$$
$$\left.-\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s\right|$$

$$\leq pq(\pi_2-\pi_1)\Bigg[\Theta_1^{\frac{1}{r_1}}(p,q)$$
$$\times\left(\frac{p}{[2]_{p,q}^3}|\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)|^{p_1}+\frac{p[2]_{p,q}^2-p^2}{[2]_{p,q}^3}|\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)|^{p_1}\right)^{\frac{1}{p_1}}$$
$$+\Theta_2^{\frac{1}{r_1}}(p,q)$$
$$\times\left(\frac{[2]_{p,q}^2-p^2}{[2]_{p,q}^3}|\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)|^{p_1}+\frac{[2]_{p,q}^3-[2]_{p,q}^2-p[2]_{p,q}^2+p^2}{[2]_{p,q}^3}|\,{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)|^{p_1}\right)^{\frac{1}{p_1}}\Bigg],$$

where $0 < q < q \leq 1$, $\frac{1}{p_1}+\frac{1}{r_1}=1$ and

$$\Theta_1(p,q) = \int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|^{r_1}d_{p,q}s,$$
$$\Theta_2(p,q) = \int_{\frac{p}{[2]_{p,q}}}^1\left|s-\frac{[5]_q}{[6]_q}\right|^{r_1}d_{p,q}s.$$

Proof. When the integrals on the right-hand side of (26) are subjected to the well-known Hölder's inequality for post-quantum integrals, it is discovered that

$$\left| \frac{p}{[6]_{p,q}} \left[p^5 \mathcal{F}(\pi_2) - q\Big([5]_{p,q} - 1\Big) \mathcal{F}\left(\frac{p\pi_2 + q\pi_1}{p+q} \right) + q\mathcal{F}(\pi_1) \right] - \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{p\pi_2+(1-p)\pi_1} \mathcal{F}(s) \ {}_{\pi_1}d_{p,q}s \right|$$

$$\leq pq(\pi_2 - \pi_1) \left[\left(\int_0^{\frac{p}{[2]_{p,q}}} \left| s - \frac{1}{[6]_{p,q}} \right|^{r_1} d_{p,q}s \right)^{\frac{1}{r_1}} \times \left(\int_0^{\frac{p}{[2]_{p,q}}} \left| \ {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2 + (1-s)\pi_1) \right|^{p_1} d_{p,q}s \right)^{\frac{1}{p_1}} \right]$$

$$\left[+ pq(\pi_2 - \pi_1) \left(\int_{\frac{p}{[2]_{p,q}}}^1 \left| s - \frac{[5]_{p,q}}{[6]_{p,q}} \right|^{r_1} d_{p,q}s \right)^{\frac{1}{r_1}} \times \left(\int_{\frac{p}{[2]_{p,q}}}^1 \left| {}_{\pi_1}D_{p,q}\mathcal{F}(s\pi_2 + (1-s)\pi_1) \right|^{p_1} d_{p,q}s \right)^{\frac{1}{p_1}} \right].$$

By using the convexity of $\left| {}_{\pi_1}D_{p,q}\mathcal{F} \right|^{p_1}$, we obtain

$$\begin{aligned} & \left| \frac{p}{[6]_{p,q}} \left[p^5 \mathcal{F}(\pi_2) - q\Big([5]_{p,q} - 1\Big) \mathcal{F}\left(\frac{p\pi_2 + q\pi_1}{p+q} \right) + q\mathcal{F}(\pi_1) \right] \right. \\ & \left. - \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^{p\pi_2+(1-p)\pi_1} \mathcal{F}(s) \ {}_{\pi_1}d_{p,q}s \right| \end{aligned} \tag{30}$$

$$\leq pq(\pi_2 - \pi_1) \left[\left(\int_0^{\frac{p}{[2]_{p,q}}} \left| s - \frac{1}{[6]_{p,q}} \right|^{r_1} d_{p,q}s \right)^{\frac{1}{r_1}} \left(\left| \ {}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2) \right|^{p_1} \int_0^{\frac{p}{[2]_{p,q}}} s d_{p,q}s + \left| \ {}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1) \right|^{p_1} \int_0^{\frac{p}{[2]_{p,q}}} (1-s) d_{p,q}s \right)^{\frac{1}{p_1}} \right]$$

$$+ pq(\pi_2 - \pi_1) \left[\left(\int_{\frac{p}{[2]_{p,q}}}^1 \left| s - \frac{[5]_{p,q}}{[6]_{p,q}} \right|^{r_1} d_{p,q}s \right)^{\frac{1}{r_1}} \left(\left| \ {}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2) \right|^{p_1} \int_{\frac{p}{[2]_{p,q}}}^1 s d_{p,q}s + \left| \ {}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1) \right|^{p_1} \int_{\frac{p}{[2]_{p,q}}}^1 (1-s) d_{p,q}s \right)^{\frac{1}{p_1}} \right].$$

Using equality (12), we see that, for the other integrals on the right-hand side of (30),

$$\int_0^{\frac{p}{[2]_{p,q}}} s d_{p,q}s = \frac{p^2}{[2]_{p,q}^3}, \tag{31}$$

$$\int_0^{\frac{p}{[2]_{p,q}}} (1-s) d_{p,q}s = \frac{p[2]_{p,q}^2 - p^2}{[2]_{p,q}^3}. \tag{32}$$

Similarly, we obtain

$$\int_{\frac{p}{[2]_{p,q}}}^{1} s d_{p,q}s = \frac{[2]_{p,q}^2 - p^2}{\left([2]_{p,q}\right)^3} \tag{33}$$

$$\int_{\frac{p}{[2]_{p,q}}}^{1} (1-s) d_{p,q}s = \frac{[2]_{p,q}^3 - [2]_{p,q}^2 - p[2]_{p,q}^2 + p^2}{\left([2]_{p,q}\right)^3}. \tag{34}$$

We obtain the desired inequality (29) by inserting (31)–(34) into (30), which completes the proof. □

Corollary 2. *In Theorem 8, if we set $p = 1$, then we obtain the following new Simpson's type inequality for q-integrals:*

$$\begin{aligned}
&\left| \frac{1}{[6]_q}\left[\mathcal{F}(\pi_2) + q\left([5]_q - 1\right)\mathcal{F}\left(\frac{\pi_2 + q\pi_1}{1+q}\right) + q\mathcal{F}(\pi_1)\right] - \frac{1}{(\pi_2 - \pi_1)}\int_{\pi_1}^{\pi_2} \mathcal{F}(s)\ {}_{\pi_1}d_q s \right| \\
&\leq q(\pi_2 - \pi_1)\Bigg[\Theta_1^{\frac{1}{r_1}}(1,q) \\
&\quad \times \left(\frac{1}{[2]_q^3}\left|{}_{\pi_1}D_q\mathcal{F}(\pi_2)\right|^{p_1} + \frac{[2]_q^2 - 1}{[2]_q^3}\left|{}_{\pi_1}D_q\mathcal{F}(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \\
&\quad + \Theta_2^{\frac{1}{r_1}}(1,q) \\
&\quad \times \left(\frac{[2]_q^2 - 1}{[2]_q^3}\left|{}_{\pi_1}D_q\mathcal{F}(\pi_2)\right|^{p_1} + \frac{[2]_q^3 - [2]_q^2 - [2]_q^2 + 1}{[2]_q^3}\left|{}_{\pi_1}D_q\mathcal{F}(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\Bigg].
\end{aligned}$$

Theorem 9. *Assume that the conditions of Lemma 2 hold. If $\left|{}_{\pi_1}D_{p,q}\mathcal{F}\right|^{p_1}$ is convex on $[\pi_1, \pi_2]$ for some $p_1 \geq 1$, then we have following inequality for $(p,q)_{\pi_1}$-integrals:*

$$\begin{aligned}
&\left| \frac{p}{[6]_{p,q}}\left[p^5\mathcal{F}(\pi_2) - q\left([5]_{p,q} - 1\right)\mathcal{F}\left(\frac{p\pi_2 + q\pi_1}{p+q}\right) + q\mathcal{F}(\pi_1)\right]\right. \\
&\left. - \frac{1}{(\pi_2 - \pi_1)}\int_{\pi_1}^{p\pi_2 + (1-p)\pi_1} \mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s \right| \\
&\leq pq(\pi_2 - \pi_1)\Bigg[\left(\frac{2\left([2]_{p,q} - 1\right)}{[2]_{p,q}[6]_{p,q}^2} + \frac{p^2[6]_{p,q} - p[2]_{p,q}^2}{[6]_{p,q}[2]_{p,q}^3}\right)^{1-\frac{1}{p_1}} \\
&\quad \times \left(A_1(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|^{p_1} + B_1(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \\
&\quad + \left(2\frac{[2]_{p,q}[5]_{p,q}^2 - [5]_{p,q}^2}{[2]_{p,q}[6]_{p,q}^2} + \frac{1}{[2]_{p,q}} - \frac{[5]_{p,q}}{[6]_{p,q}} - \frac{p[5]_{p,q}[2]_{p,q}^2 - p^2[6]_{p,q}}{[6]_{p,q}[2]_{p,q}^3}\right)^{1-\frac{1}{p_1}} \\
&\quad \times \left(A_2(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|^{p_1} + B_2(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}},
\end{aligned} \tag{35}$$

where $0 < q < p \leq 1$ and $A_1(p,q)$, $A_2(p,q)$, $B_1(p,q)$, $B_2(p,q)$ are given as in (21)–(24), respectively.

Proof. Using the conclusions obtained in the proof of Theorem 7 after applying the well-known power mean inequality to the integrals on the right-hand side of (26), we discover that, due to the convexity of $\left|{}_{\pi_1}D_{p,q}\mathcal{F}\right|^{p_1}$,

$$\begin{aligned}
&\left|\frac{p}{[6]_{p,q}}\left[p^5\mathcal{F}(\pi_2)-q\left([5]_{p,q}-1\right)\mathcal{F}\left(\frac{p\pi_2+q\pi_1}{p+q}\right)+q\mathcal{F}(\pi_1)\right]\right. \qquad (36)\\
&\left.-\frac{1}{(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\mathcal{F}(s)\ {}_{\pi_1}d_{p,q}s\right|\\
\leq\ & pq(\pi_2-\pi_1)\left[\left(\int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s\right)^{1-\frac{1}{p_1}}\right.\\
&\times\left(\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|^{p_1}\int_0^{\frac{p}{[2]_{p,q}}}s\left|s-\frac{1}{[6]_q}\right|d_{p,q}s\right.\\
&\left.\left.+\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|^{p_1}\int_0^{\frac{p}{[2]_{p,q}}}(1-s)\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s\right)^{\frac{1}{p_1}}\right]\\
&+pq(\pi_2-\pi_1)\left[\left(\int_{\frac{p}{[2]_{p,q}}}^1\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right|d_{p,q}s\right)^{1-\frac{1}{p_1}}\right.\\
&\times\left(\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|^{p_1}\int_{\frac{p}{[2]_{p,q}}}^1 s\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right|d_{p,q}s\right.\\
&\left.\left.+\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|^{p_1}\int_{\frac{p}{[2]_{p,q}}}^1(1-s)\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right|d_{p,q}s\right)^{\frac{1}{p_1}}\right]\\
=\ & pq(\pi_2-\pi_1)\left[\left(\int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s\right)^{1-\frac{1}{p_1}}\right.\\
&\left.\times\left(A_1(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|^{p_1}+B_1(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]\\
&+pq\left[(\pi_2-\pi_1)\left(\int_{\frac{p}{[2]_{p,q}}}^1\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right|d_{p,q}s\right)^{1-\frac{1}{p_1}}\right.\\
&\left.\times\left(A_2(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_2)\right|^{p_1}+B_2(p,q)\left|{}_{\pi_1}D_{p,q}\mathcal{F}(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

We also observe that

$$\begin{aligned}
\int_0^{\frac{p}{[2]_{p,q}}}\left|s-\frac{1}{[6]_{p,q}}\right|d_{p,q}s &= 2\int_0^{\frac{1}{[6]_{p,q}}}\left(\frac{1}{[6]_{p,q}}-s\right)d_{p,q}s+\int_0^{\frac{p}{[2]_{p,q}}}\left(s-\frac{1}{[6]_{p,q}}\right)d_{p,q}s \qquad (37)\\
&= 2\frac{\left([2]_{p,q}-1\right)}{[2]_{p,q}[6]_{p,q}^2}+\frac{p^2[6]_{p,q}-p[2]_{p,q}^2}{[6]_{p,q}[2]_{p,q}^3},
\end{aligned}$$

and by using similar operations, we have

$$\int_{\frac{p}{[2]_{p,q}}}^1\left|s-\frac{[5]_{p,q}}{[6]_{p,q}}\right|d_{p,q}s=2\frac{[2]_{p,q}[5]_{p,q}^2-[5]_{p,q}}{[2]_{p,q}[6]_{p,q}^2}+\frac{1}{[2]_{p,q}}-\frac{[5]_{p,q}}{[6]_{p,q}}-\frac{p[5]_{p,q}[2]_{p,q}^2-p^2[6]_{p,q}}{[6]_{p,q}[2]_{p,q}^3}. \qquad (38)$$

We obtain the needed inequality (35) by swapping (37) and (38) in (36). As a result, the proof is complete. □

Corollary 3. *In Theorem 9, if we set $p = 1$, then we obtain the following new Simpson's type inequality for the q-integral:*

$$\begin{aligned}
&\left| \frac{1}{[6]_q}\left[\mathcal{F}(\pi_2) - q\Big([5]_q - 1\Big)\mathcal{F}\left(\frac{\pi_2 + q\pi_1}{1+q}\right) + q\mathcal{F}(\pi_1)\right] - \frac{1}{(\pi_2 - \pi_1)}\int_{\pi_1}^{\pi_2} \mathcal{F}(s)\ {}_{\pi_1}d_q s \right| \\
\leq\ & q(\pi_2 - \pi_1)\left[\left(\frac{2\Big([2]_q - 1\Big)}{[2]_q[6]_q^2} + \frac{[6]_q - [2]_q^2}{[6]_q[2]_q^3}\right)^{1-\frac{1}{p_1}}\right. \\
&\times\Big(A_1(1,q)\big|\ {}_{\pi_1}D_q\mathcal{F}(\pi_2)\big|^{p_1} + B_1(1,q)\big|\ {}_{\pi_1}D_q\mathcal{F}(\pi_1)\big|^{p_1}\Big)^{\frac{1}{p_1}} \\
&+\left(2\frac{[2]_q[5]_q^2 - [5]_q^2}{[2]_q[6]_q^2} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q[2]_q^2 - [6]_q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{p_1}} \\
&\left.\times\Big(A_2(1,q)\big|\ {}_{\pi_1}D_q\mathcal{F}(\pi_2)\big|^{p_1} + B_2(1,q)\big|\ {}_{\pi_1}D_q\mathcal{F}(\pi_1)\big|^{p_1}\Big)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Remark 6. *In Theorem 9, if we set $p = 1$ and later take the limit as $q \to 1^-$, then we have the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\pi_1) + 4\mathcal{F}\left(\frac{\pi_1 + \pi_2}{2}\right) + \mathcal{F}(\pi_2)\right] - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2}\mathcal{F}(s)ds\right| \\
\leq\ & \frac{1}{(1296)^{\frac{1}{p_1}}}\left(\frac{5}{72}\right)^{1-\frac{1}{p_1}}(\pi_2 - \pi_1) \\
&\times\left[61\big|\mathcal{F}'(\pi_1)\big|^{p_1} + 29\big|\mathcal{F}'(\pi_2)\big|^{p_1}\right]^{\frac{1}{p_1}} + \left[29\big|\mathcal{F}'(\pi_1)\big|^{p_1} + 61\big|\mathcal{F}'(\pi_2)\big|^{p_1}\right]^{\frac{1}{p_1}}.
\end{aligned}$$

This is given by Alomari et al. in [2].

6. Conclusions

In this investigation, we have proven different variants of Simpson's formula type inequalities for (p,q)-differentiable convex functions via post-quantum calculus. We conclude that the findings of this research are universal in nature and contribute to inequality theory, as well as applications in quantum boundary value problems, quantum mechanics, and special relativity theory for determining solution uniqueness. The findings of this study can be utilized in symmetry. Results for the case of symmetric functions can be obtained by applying the concept in Remark 4, which will be studied in future work. Future researchers will be able to obtain similar inequalities for different types of convexity and co-ordinated convexity in their future work, which is a new and important problem.

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References

1. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On Simpson's inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [CrossRef]
2. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of Simpson's type for *s*-convex functions with applications. *RGMIA Res Rep Coll.* **2009**, *2*.
3. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson's type for convex functions. *RGMIA Res. Rep. Coll.* **2010**, *13*, 2.
4. Erden, S.; Iftikhar, S.; Delavar, M.R.; Kumam, P.; Thounthong, P.; Kumam, W. On generalizations of some inequalities for convex functions via quantum integrals. *RACSAM* **2020**, *114*, 1–15. [CrossRef]
5. Iftikhar, S.; Erden, S.; Kumam, P.; Awan, M.U. Local fractional Newton's inequalities involving generalized harmonic convex functions. *Adv. Differ. Equ.* **2020**, *2020*, 1–14.
6. Özdemir, M.E.; Akdemir, A.O.; Kavurmaci, H.; Avci, M. On the Simpson's inequality for co-ordinated convex functions. *arXiv* **2010**, arXiv:1101.0075.
7. Ali, M.A.; Budak, H.; Abbas, M.; Chu, Y.-M. Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^{π_2}-derivatives. *Adv. Differ. Equ.* **2021**, *2021*, 1–12. [CrossRef]
8. Ali, M.A.; Alp, N.; Budak, H.; Chu, Y.; Zhang, Z. On some new quantum midpoint type inequalities for twice quantum differentiable convex functions. *Open Math.* **2021**, *19*, 427–439. [CrossRef]
9. Alp, N.; Sarikaya, M.Z.; Kunt, M. *q*-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. -Sci.* **2018**, *30*, 193–203. [CrossRef]
10. Alp, N.; Sarikaya, M.Z. Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral. *Appl. Math. E-Notes* **2020**, *20*, 341–356.
11. Bermudo, S.; Kórus, P.; Valdés, J.N. On *q*-Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
12. Budak, H. Some trapezoid and midpoint type inequalities for newly defined quantum integrals. *Proyecciones* **2021**, *40*, 199–215. [CrossRef]
13. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* **2020**, *186*, 899–910. [CrossRef]
14. Jhanthanam, S.; Tariboon, J.; Ntouyas, S.K.; Nonlapon, K. On *q*-Hermite-Hadamard inequalities for differentiable convex functions. *Mathematics* **2019**, *7*, 632. [CrossRef]
15. Kalsoom, H.; Rashid, S.; Idrees, M.; Safdar, F.; Akram, S.; Baleanu, D.; Chu, Y.-M. Post quantum inequalities of Hermite-Hadamard type associated with co-ordinated higher-order generalized strongly pre-invex and quasi-pre-invex mappings. *Symmetry* **2020**, *12*, 443. [CrossRef]
16. Latif, M.A.; Dragomir, S.S.; Momoniat, E. Some *q*-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane. *J. King Saud-Univ. -Sci.* **2017**, *29*, 263–273. [CrossRef]
17. Liu, W.; Hefeng, Z. Some quantum estimates of Hermite-Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2016**, *7*, 501–522.
18. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [CrossRef]
19. Sial, I.B.; Ali, M.A.; Murtaza, G.; Ntouyas, S.K.; Soontharanon, J.; Thanin, S. On Some New Inequalities of Hermite–Hadamard Midpoint and Trapezoid Type for Preinvex Functions in (p,q)-Calculus. *Symmetry* **2021**, *13*, 1864. [CrossRef]
20. Sitho, S.; Ali, M.A.; Budak, H.; Ntouyas, S.K. Tariboon, J.; Trapezoid and Midpoint Type Inequalities for Preinvex Functions via Quantum Calculus. *Mathematics* **2021**, *9*, 1996. [CrossRef]
21. Wannalookkhee, F.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. On Hermite-Hadamard type inequalities for coordinated convex functions via (p,q)-calculus. *Mathematics* **2021**, *9*, 698. [CrossRef]
22. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **2015**, *269*, 242–251. [CrossRef]
23. Nwaeze, E.R.; Tameru, A.M. New parameterized quantum integral inequalities via η-quasiconvexity. *Adv. Differ. Equ.* **2019**, *2019*, 1–12. [CrossRef]
24. Khan, M.A.; Noor, M.; Nwaeze, E.R.; Chu, Y.-M. Quantum Hermite–Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 1–20.

25. Ali, M.A.; Budak, H.; Zhang, Z.; Yildrim, H. Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus. *Math. Meth. Appl. Sci.* **2021**, *44*, 4515–4540. [CrossRef]
26. Ali, M.A.; Abbas, M.; Budak, H.; Agarwal, P.; Murtaza, G.; Chu, Y. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. *Adv. Differ. Equ.* **2021**, *2021*, 1–21. [CrossRef]
27. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* **2020**, *44*, 378–390. [CrossRef]
28. Budak, H.; Ali, M.A.; Tunç, T. Quantum Ostrowski-type integral inequalities for functions of two variables. *Math. Meth. Appl. Sci.* **2021**, *44*, 5857–5872. [CrossRef]
29. Siricharuanun, P.; Erden, S.; Ali, M.A.; Budak, H.; Chasreechai, S.; Thanin, S. Some New Simpson's and Newton's Formulas Type Inequalities for Convex Functions in Quantum Calculus. *Mathematics* **2021**, *9*, 1992. [CrossRef]
30. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Sial, I.B.; Zhang, Z. Some New Newton's Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus. *Symmetry* **2020**, *12*, 1476. [CrossRef]
31. Ali, M.A.; Chu, Y.-M.; Budak, H.; Akkurt, A.; Yildrim, H. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. *Adv. Differ. Equ.* **2021**, *2021*, 1–26. [CrossRef]
32. Ali, M.A.; Budak, H.; Akkurt, A.; Chu, Y.-M. Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus. *Open Math.* **2021**, *19*, 440–449. [CrossRef]
33. Budak, H.; Ali, M.A.; Alp, N.; Chu, Y.-M. Quantum Ostrowski type integral inequalities. *J. Math. Inequal.* **2021**, in press.
34. Kunt, M.; İşcan, İ.; Alp, N.; Sarikaya, M.Z. $(p,q)-$Hermite-Hadamard inequalities and $(p,q)-$estimates for midpoint inequalities via convex quasi-convex functions. *Rev. R. Acad. Cienc. Exactas F s. Nat. Ser. A Mat. RACSAM* **2018**, *112*, 969–992. [CrossRef]
35. Vivas-Cortez, M.; Ali, M.A.; Budak, H.; Kalsoom, H.; Agarwal, P. Some New Hermite–Hadamard and Related Inequalities for Convex Functions via (p,q)-Integral. *Entropy* **2021**, *23*, 828. [CrossRef]
36. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
37. Jackson, F.H. On a q-definite integrals. *Quarterly J. Pure Appl. Math.* **1910**, *41*, 193–203.
38. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ*. **2013**, *2013*, 1–19. [CrossRef]
39. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *Results Math.* **2018**, *73*, 1–21.
40. Tunç, M.; Göv, E. Some integral inequalities via (p,q)-calculus on finite intervals. *RGMIA Res. Rep. Coll.* **2016**, *19*, 1–12. [CrossRef]