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Quotients of Euler Equations on Space Curves

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Abstract: Quotients of partial differential equations are discussed. The quotient equation for the Euler system describing a one-dimensional gas flow on a space curve is found. An example of using the quotient to solve the Euler system is given. Using virial expansion of the Planck potential, we reduce the quotient equation to a series of systems of ordinary differential equations (ODEs). Possible solutions of the ODE system are discussed.

Keywords: Euler equation; quotient equation

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1. Introduction

In this paper, we continue the study of the Euler equation describing gas flows on space curves in a constant gravity field. Symmetry algebras and differential invariant fields, as well as their dependence on thermodynamic state equations and the form of a space curve, were considered in [1]. Here, we find a quotient PDE for the Euler equation and show its role in solving the original equation.

Recall that the system of PDEs describing such flows is the following:

$$\begin{cases} \rho(u_t + uu_a) = -p_a - \rho g h', \\ \rho_t + (\rho u)_a = 0, \\ \rho\theta(s_t + us_a) - k\theta_{aa} = 0, \end{cases} \tag{1}$$

where $u(t,a)$ is the flow velocity, $p(t,a)$, $\rho(t,a)$, $s(t,a)$, and $\theta(t,a)$ are the pressure, density, specific entropy, and temperature of the fluid, respectively, k is the constant thermal conductivity, g is the gravitational acceleration, and $h(a)$ is the z-component of a naturally parametrized space curve.

System (1) is incomplete, i.e., it has two more unknown functions than equations. In the present paper, we put aside the question of classification of possible thermodynamic relations, since it was described in detail in [1]. We assume that these relations are given either in the forms $p = P(\rho,\theta)$ and $s = S(\rho,\theta)$, or in terms of the Planck potential [2]. In particular, we consider the ideal gas equation.

This paper is organized as follows. In Section 2, the notion of PDE quotients is discussed. In Section 3, we recall the symmetry algebra and differential invariants for the Euler system. In Section 4, we find the quotient for the Euler equation and discuss possible symmetries and solutions.

All calculations for this paper were performed with the DifferentialGeometry package in Maple. The corresponding Maple files can be found on the webpage http://d-omega.org/appendices/.

2. PDE Quotients

2.1. Algebraic Structures in PDE Geometry

Let $\pi\colon E(\pi) \to M$ be a smooth bundle over a manifold M and let $\pi_k\colon J^k(\pi) \to M$, $k = 0, 1, \ldots$, be the k-jet bundles of sections of the bundle π. To simplify the notations, we use $\mathbf{J}^k$ instead of $J^k(\pi)$.

Depending on $\dim \pi$, the jet geometry [3] is defined by the following pseudogroups.

1. If $\dim \pi = 1$, it is defined by the pseudogroup $\mathrm{Cont}(\pi)$ of the local contact transformations of the manifold $\mathbf{J}^1$.
2. For $\dim \pi \geq 2$, the jet geometry is defined by the pseudogroup $\mathrm{Point}(\pi)$ of the local point transformations, i.e., local diffeomorphisms of the manifold $\mathbf{J}^0$.

It is also known that the prolongations of these pseudogroups to the jet bundles exhaust all Lie transformations, i.e., local diffeomorphisms of jet spaces that preserve the Cartan distributions (see, for example, [3]).

Moreover, bundles $\pi_{k,k-1}\colon \mathbf{J}^k \to \mathbf{J}^{k-1}$ ($k \geq 2$ when $\dim \pi \geq 2$, and $k \geq 3$ when $\dim \pi = 1$) have affine structures, which are invariant with respect to the Lie transformations, and prolongations of the pseudogroups $\mathrm{Cont}(\pi)$ or $\mathrm{Point}(\pi)$ are given by rational functions of u^i_σ in the standard jet coordinates $\left(x, u^i_\sigma\right)$.

The last statement means that, in the case of $\dim \pi \geq 2$, the fibers $\mathbf{J}^{k,0}_\theta$ of the projections $\pi_{k,0}\colon \mathbf{J}^k \to \mathbf{J}^0$ at a point $\theta \in \mathbf{J}^0$ are algebraic manifolds, and the stationary subgroup $\mathrm{Point}_\theta(\pi) \subset \mathrm{Point}(\pi)$ gives us birational isomorphisms of the manifold.

In the case of $\dim \pi = 1$, the fibers $\mathbf{J}^{k,1}_\theta$ of the projections $\pi_{k,1}\colon \mathbf{J}^k \to \mathbf{J}^1$ at a point $\theta \in \mathbf{J}^1$ are *algebraic manifolds* too, and the stationary subgroup $\mathrm{Cont}_\theta(\pi) \subset \mathrm{Cont}(\pi)$ gives us birational isomorphisms of the manifold.

Following this picture, we say that a differential equation $\mathcal{E}_k \subset \mathbf{J}^k$ is algebraic if fibers $\mathcal{E}_{k,\theta}$ of the projections $\pi_{k,0}\colon \mathcal{E}_k \to \mathbf{J}^0$ are algebraic manifolds when $\dim \pi \geq 2$, or $\pi_{k,1}\colon \mathcal{E}_k \to \mathbf{J}^1$ when $\dim \pi = 1$.

All differential equations here are assumed to be formally integrable; then, the prolongations $\mathcal{E}_k^{(l)} = \mathcal{E}_{k+l} \subset \mathbf{J}^{k+l}$ of an algebraic equation $\mathcal{E}_k \subset \mathbf{J}^k$ are algebraic, too.

By a symmetry algebra of an algebraic differential equation, we mean the Lie algebra $\mathrm{Sym}(\mathcal{E}_k)$ of point vector fields if $\dim \pi \geq 2$ or contact vector fields if $\dim \pi = 1$ that act transitively on $\mathbf{J}^0$ in the case of $\dim \pi \geq 2$ or $\mathbf{J}^1$ in the case of $\dim \pi = 1$. Moreover, the stationary sub-algebra $\mathrm{Sym}_\theta(\mathcal{E}_k)$ where $\theta \in \mathbf{J}^0$ or $\theta \in \mathbf{J}^1$ produces actions of algebraic Lie algebras on algebraic manifolds $\mathcal{E}_{l,\theta}$ for all $l \geq k$.

2.2. The Rosenlicht Theorem

Let B be an algebraic manifold, i.e., an irreducible variety without singularities over a field of characteristic zero, let G be an algebraic group, and let $G \times B \to B$ be an algebraic action.

Denote by $\mathcal{F}(B)$ the field of rational functions on the manifold B, and, by $\mathcal{F}(B)^G \subset \mathcal{F}(B)$, denote the field of rational G-invariants on B.

We say that an orbit $Gb \subset B$ is regular (as well as a point b itself) if there are $m = \operatorname{codim} Gb$ G-invariants $x_1, \ldots, x_m$ such that their differentials are linearly independent at the points of the orbit.

Let $B_0 = B \setminus \mathrm{Sing}$ be the set of all regular points and let $Q(B) = B_0/G$ be the set of all regular orbits.

The Rosenlicht theorem [4] states that B_0 is open and dense in B.

Moreover, if the above invariants $x_1, \ldots, x_m$ are considered as local coordinates on the quotient $Q(B)$ at the point $Gb \in Q(B)$, then on the intersections of the coordinate charts, the coordinates are connected by rational functions. In other words, $Q(B)$ is an algebraic manifold of the dimension $m = \operatorname{codim} Gb$, and the rational map $\varkappa\colon B_0 \to Q(B)$ of algebraic manifolds gives us the field isomorphism $\mathcal{F}(B)^G = \pi^*(\mathcal{F}(Q(B)))$.

To apply this theorem to algebraic differential equations, we should reformulate it for the case of Lie algebras.

Let B be an algebraic manifold and let $\mathfrak{g}$ be a Lie sub-algebra of the Lie algebra of the vector fields on B.

We say that $\mathfrak{g}$ is an algebraic Lie algebra if there is an algebraic action of an algebraic group G on B such that $\mathfrak{g}$ coincides with the image of Lie algebra $\mathrm{Lie}(G)$ under this action.

By an algebraic closure $\widetilde{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$, we mean the intersection of all algebraic Lie algebras that contain $\mathfrak{g}$.

Example 1. *Let $B = \mathbb{R}$; then, Lie algebra*

$$\mathfrak{g} = \mathfrak{sl}_2 = \langle \partial_x, x\,\partial_x, x^2\partial_x \rangle$$

is algebraic because it corresponds to the projective action of the algebraic group $\mathbf{SL}_2(\mathbb{R})$.

Example 2. *Let $B = S^1 \times S^1$ be a torus and $\mathfrak{g} = \langle \partial_\phi + \lambda \partial_\psi \rangle$, where ϕ and ψ are the angles, $\lambda \in \mathbb{R}$. Then, $\mathfrak{g}$ is algebraic if and only if $\lambda \in \mathbb{Q}$. Otherwise, $\widetilde{\mathfrak{g}} = \langle \partial_\phi, \partial_\psi \rangle$. A similar situation occurs in the case of $B = \mathbb{R}^2$ and*

$$\mathfrak{g} = \langle x\,\partial_x + \lambda y\,\partial_y \rangle,$$

where $\widetilde{\mathfrak{g}} = \mathfrak{g}$ if $\lambda \in \mathbb{Q}$, and $\widetilde{\mathfrak{g}} = \langle x\partial_x, y\partial_y \rangle$ otherwise.

The Rosenlicht theorem is also true for algebraic Lie algebras or for their algebraic closure in the case of general Lie algebras.

Let us be given a Lie algebra $\mathfrak{g}$ of vector fields on an algebraic manifold B and let $\widetilde{\mathfrak{g}} \supset \mathfrak{g}$ be its algebraic closure. Then, the field $\mathcal{F}(B)^{\mathfrak{g}}$ of rational $\mathfrak{g}$-invariants has a transcendence degree equal to the codimension of regular $\widetilde{\mathfrak{g}}$-orbits, and it is also equal to the dimension of the quotient algebraic manifold $Q(B)$.

2.3. Quotients of Algebraic Differential Equations

Let $\mathfrak{g}$ be an algebraic symmetry Lie algebra of an algebraic formally integrable differential equation $\mathcal{E}_k$, and let $\mathcal{E}_l$ be the $(l-k)$-th prolongations of $\mathcal{E}_k$. Then, all equations $\mathcal{E}_l \subset \mathbf{J}^l$ are algebraic, and we have the tower of algebraic bundles:

$$\mathcal{E}_k \longleftarrow \mathcal{E}_{k+1} \longleftarrow \cdots \longleftarrow \mathcal{E}_l \longleftarrow \mathcal{E}_{l+1} \longleftarrow \cdots .$$

Let $\mathcal{E}_l^0 \subset \mathcal{E}_l$ be the set of strongly regular points and let $Q_l(\mathcal{E})$ be the set of all strongly regular $\mathfrak{g}$-orbits, where, by a strongly regular point (and orbit), we mean such points of $\mathcal{E}_l$ that are regular with respect to $\mathfrak{g}$-action and whose projections on $\mathcal{E}_{l-1}$ are regular, too.

Then, as we have seen, $Q_l(\mathcal{E})$ are algebraic manifolds, and the projections $\varkappa_l \colon \mathcal{E}_l^0 \to Q_l(\mathcal{E})$ are rational maps such that the fields $\mathcal{F}(Q_l(\mathcal{E}))$ (the field of rational functions on $Q_l(\mathcal{E})$), and $\mathcal{F}(\mathcal{E}_l^0)^{\mathfrak{g}}$ (the field of rational functions on $\mathcal{E}_l^0$), which are $\mathfrak{g}$-invariants (rational differential invariants), coincide: $\varkappa_l^*(\mathcal{F}(Q_l(\mathcal{E}))) = \mathcal{F}(\mathcal{E}_l^0)^{\mathfrak{g}}$.

The $\mathfrak{g}$-action preserves the Cartan distributions $C(\mathcal{E}_l)$ on the equations, and therefore, projections $\varkappa_l$ define distributions $C(Q_l)$ on the quotients $Q_l(\mathcal{E})$.

Finally, we get the tower of algebraic bundles of the quotients

$$Q_k(\mathcal{E}) \overset{\pi_{k+1,k}}{\longleftarrow} Q_{k+1}(\mathcal{E}) \longleftarrow \cdots \longleftarrow Q_l(\mathcal{E}) \overset{\pi_{l+1,l}}{\longleftarrow} Q_{l+1}(\mathcal{E}) \longleftarrow \cdots$$

such that the projection of the distribution $C(Q_l)$ belongs to $C(Q_{l-1}(\mathcal{E}))$.

2.4. Tresse Derivatives

Let $\omega \in \Omega^1(\mathbf{J}^k)$ be a differential 1-form on a k-jet manifold. Then, the class

$$\omega^h = \pi_{k+1,k}^*(\omega) \mod \operatorname{Ann} C_{k+1},$$

is called a horizontal part of ω.

In the standard jet coordinates (x, u^j_σ), the horizontal part has the following representation:

$$\omega = \sum_i a_i dx_i + \sum_{\substack{|\sigma|\leq k \\ j\leq m}} a^j_\sigma du^j_\sigma \Longrightarrow \omega^h = \sum_i a_i dx_i + \sum_{\substack{|\sigma|\leq k \\ j\leq m, i\leq n}} a^j_\sigma u^j_{\sigma+1_i} dx_i,$$

where $n = \dim M$ and $m = \dim \pi$.

As a particular case of this construction, we get the total differential $f \in C^\infty(\mathbf{J}^k) \Longrightarrow \widehat{d}f = (df)^h$, or, in the standard jet coordinates,

$$\widehat{d}f = \sum_{i\leq n} \frac{df}{dx_i} dx_i,$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma}$$

are the total derivations.

It is important to observe that the operation of taking a horizontal part, as well as the total differential, is invariant with respect to the point and contact transformations.

We say that functions $f_1, \dots, f_n \in C^\infty(\mathbf{J}^k)$ are in general position on a domain D if

$$\widehat{d}f_1 \wedge \cdots \wedge \widehat{d}f_n \neq 0$$

on this domain.

Let f be a smooth function on this domain; then, we get decomposition in D:

$$\widehat{d}f = \sum_{i\leq n} F_i \, \widehat{d}f_i,$$

where F_i are smooth functions on the domain $\pi^{-1}_{k+1,k}(D) \subset \mathbf{J}^{k+1}$.

We call them Tresse derivatives [5] and denote them by

$$\frac{df}{df_i}.$$

As we have seen, the operation of taking a horizontal part, as well as the total differential, is invariant with respect to the point and contact transformations.

Therefore, we have the following.

Proposition 1. *Let $f_1, \dots, f_n$ be $\mathfrak{g}$-invariants of order $\leq k$ that are in general position. Then, for any $\mathfrak{g}$-invariant f of order $\leq k$, the Tresse derivatives $\frac{df}{df_i}$ are $\mathfrak{g}$-invariants of order $\leq k+1$.*

2.5. The Lie–Tresse Theorem

Theorem 1. *[6] Let $\mathcal{E}_k \subset \mathbf{J}^k$ be a formally integrable algebraic differential equation and let $\mathfrak{g}$ be an algebraic symmetry Lie algebra. Then, there are rational differential $\mathfrak{g}$-invariants $a_1, \dots, a_n, b^1, \dots, b^N$ of order $\leq l$ such that the field of all rational differential $\mathfrak{g}$-invariants is generated by rational functions of these invariants and their Tresse derivatives $\frac{d^{|\alpha|}b^j}{da^\alpha}$.*

We call invariants $a_1, \dots, a_n, b^1, \dots, b^N$ Lie–Tresse coordinates.

It is noteworthy that, in contrast to algebraic invariants, for which we have the algebraic operations only, in the case of differential invariants, we have additional operations, i.e., Tresse derivatives, that allow us to get really new invariants.

Syzygies, in the case of differential invariants, provide us with new differential equations that we call quotient equations.

From the geometrical point of view, the above theorem states that there is a level l and a domain $D \subset Q(\mathcal{E})$ where the invariants a_i and b^j can be considered as local coordinates, and the preimage of D in the tower

$$Q_l(\mathcal{E}) \overset{\pi_{l+1,l}}{\longleftarrow} Q_{l+1}(\mathcal{E}) \longleftarrow \cdots \longleftarrow Q_r(\mathcal{E}) \overset{\pi_{r+1,r}}{\longleftarrow} Q_{r+1}(\mathcal{E}) \longleftarrow \cdots \tag{2}$$

is just an infinitely prolonged differential equation given by the syzygy.

For this reason, we call the quotient tower (2) an algebraic diffiety.

2.6. Relations between Differential Equations and Their Quotients

1. Let $u = f(x)$ be a solution of differential equation $\mathcal{E}$ and let $a_i(f)$ and $b^j(f)$ be values of the invariants a_i and b^j on the section f. Then, locally, $b^j(f) = B^j(a(f))$, and therefore, $b^j = B^j(a)$ is the solution of the quotient equation.
2. The above construction is local. In general, the correspondence between solutions is valid on the level of generalized solutions, i.e., on the level of integral manifolds of the Cartan distributions. In addition, the correspondence will lead us to integral manifolds with singularities.
3. Now let $b^j = B^j(a)$ be a solution of the quotient equation. Then, considering equations $b^j - B^j(a) = 0$ as a differential constraint for the equation $\mathcal{E}$, we get a finite-type equation $\mathcal{E} \cap \{b^j - B^j(a) = 0\}$ with a solution that is a $\mathfrak{g}$-orbit of a solution of $\mathcal{E}$.
4. Symmetries of the quotient equation are Bäcklund-type transformations of the original equation $\mathcal{E}$.

Example 3. *The Lie algebra of the projective transformations of the line $M = \mathbb{R}$, $\mathfrak{g} = \mathfrak{sl}_2 = \langle \partial_x, x\partial_x, x^2\partial_x \rangle$ has the following generators in rational differential invariants for the $\mathfrak{sl}_2$-action on functions:*

$$\left\langle a = u_0,\ b = \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4},\ \frac{db}{da} = \frac{u_4}{u_1^4} - 6\frac{u_2u_3}{u_1^5} + 6\frac{u_2^3}{u_1^6}, \dots \right\rangle.$$

Let

$$F\left(u_0, \frac{u_3}{u_1^3} - \frac{3u_2^2}{2u_1^4}, \frac{u_4}{u_1^4} - 6\frac{u_2u_3}{u_1^5} + 6\frac{u_2^3}{u_1^6}\right) = 0$$

be a fourth-order $\mathfrak{sl}_2$-invariant equation.

Then, the quotient equation has the first order:

$$F\left(a, b, \frac{db}{da}\right) = 0.$$

Example 4. *The Lie algebra $\mathfrak{g} = \langle \partial_x, \partial_y \rangle$ of translations of the plane has the following Lie–Tresse coordinates for the $\mathfrak{g}$-action on functions:*

$$a_1 = u_{1,0},\ a_2 = u_{0,1},\ b = u_{0,0},\ c = u_{1,1}.$$

Then,

$$b_{1,0} = \delta^{-1}(u_{1,0}u_{0,2} - u_{0,1}u_{1,1}), \quad b_{0,1} = \delta^{-1}(u_{0,1}u_{2,0} - u_{1,0}u_{1,1}),$$
$$c_{1,0} = \delta^{-1}(u_{0,2}u_{2,1} - u_{1,1}u_{1,2}), \quad c_{0,1} = \delta^{-1}(u_{2,0}u_{1,2} - u_{1,1}u_{2,1}),$$

where $\delta = u_{2,0}u_{0,2} - u_{1,1}^2$ is a Hessian determinant, and the syzygy is

$$c^2(b_{1,0}^2b_{0,2} - 2b_{1,0}b_{0,1}b_{1,1} + b_{0,1}^2b_{2,0}) + c(b_{1,0}b_{0,1} - a_1b_{0,1}b_{2,0} - a_2b_{1,0}b_{0,2} +$$
$$b_{1,1}(a_1b_{1,0} + a_2b_{0,1}) - a_1a_2b_{1,1} - b_{1,0}b_{0,1}(a_1c_{1,0} + a_2c_{0,1}) = 0.$$

Thus, the quotient of an equation $u_{1,1} = C(u_{1,0}, u_{0,1})$ *is the last equation for* $b(u_{1,0}, u_{0,1})$*, where letter* C *stands for* c*.*

3. Euler Equations on a Curve

In this section, we briefly recall the necessary results obtained in [1].

Consideration of flows of an inviscid medium on a space curve $M = \{x = f(a), y = g(a), z = \lambda a\}$ in a field of constant gravitational force leads to the system

$$\begin{cases} \rho(u_t + uu_a) = -p_a - \rho g\lambda, \\ \rho_t + (\rho u)_a = 0, \\ \rho\theta(s_t + us_a) - k\theta_{aa} = 0, \end{cases} \tag{3}$$

where p and s are expressed in terms of Planck potential [2] $\Phi(\rho, \theta)$:

$$p(\rho, \theta) = -R\rho^2\theta\Phi_\rho, \quad s(\rho, \theta) = R(\Phi + \theta\Phi_\theta),$$

where R is the universal gas constant.

To describe this Lie algebra, we consider a Lie algebra $\mathfrak{g}$ of point symmetries of the PDE system (3).

Let $\vartheta\colon \mathfrak{g} \to \mathfrak{h}$ be the following Lie algebra's homomorphism

$$\vartheta\colon X \mapsto X(\rho)\partial_\rho + X(s)\partial_s + X(p)\partial_p + X(\theta)\partial_\theta,$$

where $\mathfrak{h}$ is a Lie algebra generated by vector fields that act on the thermodynamic values p, ρ, s, and θ.

It was demonstrated [1] that if $h(a) = \lambda a$, the Lie algebra $\mathfrak{g}$ of point symmetries of the system (1) is generated by the vector fields

$$\begin{aligned} &X_1 = \partial_t, \quad X_2 = \partial_p, \quad X_3 = \partial_s, \\ &X_4 = \theta\,\partial_\theta, \quad X_5 = p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s, \\ &X_6 = \partial_a, \quad X_7 = t\,\partial_a + \partial_u, \\ &X_8 = t\,\partial_t + 2a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho - s\,\partial_s, \\ &X_9 = \left(\frac{t^2}{2} + \frac{a}{\lambda g}\right)\partial_a + \left(t + \frac{u}{\lambda g}\right)\partial_u - \frac{2\rho}{\lambda g}\,\partial_\rho. \end{aligned}$$

The pure thermodynamic part $\mathfrak{h}_\mathfrak{t}$ of the symmetry algebra is generated by the vector fields

$$\begin{aligned} &Y_1 = \partial_p, \quad Y_2 = \partial_s, \quad Y_3 = \theta\,\partial_\theta, \\ &Y_4 = p\,\partial_p, \quad Y_5 = \rho\,\partial_\rho, \quad Y_6 = s\,\partial_s. \end{aligned}$$

Thus, the Euler system has a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_\mathfrak{t})$.

It has been shown in [1] that, for $h(a) = const$, $h(a) = \lambda a$, and $h(a) = \lambda a^2$, the basis differential invariants are

$$J_1 = \rho, \quad J_2 = \theta, \quad J_3 = u_a, \quad J_4 = \rho_a, \quad J_5 = \theta_a, \quad J_6 = \theta_t + u\theta_a$$

and the basis invariant derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

4. Quotient Equation

Choosing J_1 and J_2 as Lie–Tresse coordinates (x, y) and

$$K(x,y) = J_3, \quad L(x,y) = J_4, \quad M(x,y) = J_5, \quad N(x,y) = J_6$$

as unknown functions, respectively, we get the quotient equation for (3):

$$\begin{cases} Rxy(xK(\Phi_x + y\Phi_{xy}) - N(2\Phi_y + y\Phi_{yy})) + LM_x + MM_y = 0, \\ xKM_x - NM_y + LN_x + M(N_y - K) = 0, \\ xMK_y - xKL_x + xLK_x + 2KL + NL_y = 0, \\ RxyL(\Phi_{xxx}L^2 + 2\Phi_{xxy}ML + \Phi_{xyy}M^2) + \\ RL(xyLL_x + xyML_y + 2xLM + 3yL^2)\Phi_{xx} + \\ RL(xyLM_x + M(xyM_y + 2xM + 3yL))\Phi_{xy} + \\ LR(2yLL_x + 2yML_y + xLM_x + M(xM_y + 3L))\Phi_x + \\ xK^2L_x - KNL_y - (xKM + LN)K_y - 3LK^2 = 0 \end{cases} \tag{4}$$

Direct computations show that the system (4) has no symmetries if the function Φ is arbitrary. Nevertheless, it is possible to find symmetries for some classes of Φ. Some of these cases are listed below.

Proposition 2. *If the system* (4) *admits a symmetry of the form*

$$\alpha_1 x\partial_x + (\alpha_2 y + \alpha_3)\partial_y - \alpha_2 K\partial_K + \frac{1}{2}(3\alpha_1 - 2\alpha_2 - \alpha_4)L\partial_L + \frac{1}{2}(\alpha_1 - \alpha_4)M\partial_M,$$

then the function Φ *is of the form*

$$\Phi(x,y) = C_5 \int \frac{(\alpha_2 y + \alpha_3)^{-\frac{\alpha_4}{\alpha_2}}}{y^2}dy + \frac{C_4 x^{\frac{\alpha_4}{\alpha_1}-1}}{y} + \frac{C_3 y + C_2}{y} + \frac{C_1}{xy},$$

where $C_1, \ldots C_5$ *are constants.*

Proposition 3. *If the system* (4) *admits a symmetry of the form*

$$x\frac{\partial}{\partial x} + \alpha_2 y\frac{\partial}{\partial y} - \alpha_2 K\frac{\partial}{\partial K} + \frac{1}{2}(3 - 2\alpha_2 - \alpha_4)L\frac{\partial}{\partial L} + \frac{1}{2}(1 - \alpha_4)M\frac{\partial}{\partial M},$$

then the function Φ *is the following*

$$\Phi(x,y) = C_5 + \frac{C_4}{y} + C_3 y^{\frac{\alpha_4}{\alpha_2}-1} + \frac{C_2}{xy} + \frac{C_1 x^{\alpha_4 - 1}}{y},$$

where $C_1, \ldots C_5$ *are constants.*

Particular solutions of (4) for some special classes of the function Φ can be found. For example, consider the Planck potential for the ideal gas model:

$$\Phi(x,y) = \frac{n}{2}\ln y - \ln x, \tag{5}$$

where n is the number of freedom degrees of a gas particle.

Then, for simplicity, let $N = K = 0$, then these are some of the solutions for L and M:

1. $L = 0$, $M = f(x)$.
2. $L = \frac{c_1 x}{y}$, $f(M)x^{\frac{M}{c_1}} = y$.
3. $L = \frac{c_3 x}{(\ln x - c_2)^{c_1}}\left(\frac{-c_5 \ln y + c_4}{c_1} y^{-c_1^{-1}}\right)^{c_1}$, $M = \frac{c_3 y(-c_5 \ln y + c_4)}{(\ln x - c_2)^{c_1}(-\ln x + c_2)c_5}\left(\frac{-c_5 \ln y + c_4}{c_1} y^{-1/c_1}\right)^{c_1}$.

Here, $c_1, \ldots, c_5$ are constants and $f(x)$ is an arbitrary function.

Let us illustrate how we can solve the original Euler PDE system using its quotient. To this end, we consider the system (3) for ideal gas together with the solution (for example, $N = K = L = 0$, $M = x$), which is equivalent to a finite-type system:

$$\begin{cases} \theta_{aa} = 0, \quad \rho_t = 0, \quad \rho_a = 0, \quad R\rho\theta_a + \rho(\mathrm{g}\lambda + u_t) = 0, \\ \theta_a = \rho, \quad u_a = 0, \quad \theta_t + u\theta_a = 0. \end{cases}$$

Solving the latter, we get

$$\rho = \rho_0, \quad u = u_0 - (\lambda \mathrm{g} - R\rho_0)t, \quad \theta = \frac{(R\rho_0 + \mathrm{g}\lambda)\rho_0 t^2}{2} + \rho_0(a - u_0 t) + \theta_0,$$

where ρ_0, u_0, θ_0 are arbitrary constants.

Virial Expansion

Another approach we can take is to exploit the fact that it is often possible to consider the Planck potential Φ in the form of virial expansion:

$$\Phi(x, y) = \frac{n}{2} \ln y - \ln x - \sum_{i=1}^{\infty} \frac{x^i}{i} A_i(y).$$

Then, we can find solutions of the system (4) in the form of power series of x:

$$K(x,y) = x^{d_K} \sum_{k=0} K_k(y) x^k, \qquad L(x,y) = x^{d_L} \sum_{k=0} L_k(y) x^k,$$
$$M(x,y) = x^{d_M} \sum_{k=0} M_k(y) x^k, \qquad N(x,y) = x^{d_N} \sum_{k=0} N_k(y) x^k,$$

where d_K, d_L, d_M, and d_N are the integer constants that should be chosen such that (4) can be expanded as a power series of x. It can be shown that $d_K = 1$, $d_L = 2$, $d_M = 1$, and $d_N = 1$. Hence, the zeroth-order term of this expansion is a system of ordinary differential equations:

$$\begin{cases} (K_0 M_0 + N_0 L_0) K_0' + (R y L_0 M_0 + K_0 N_0) L_0' + 2 R L_0 M_0 M_0' + \\ L_0 (R y L_0^2 + 2 R L_0 M_0 + K_0^2) = 0, \\ M_0 K_0' + N_0 L_0' + K_0 L_0 = 0, \\ M_0 N_0' - N_0 M_0' + N_0 L_0 = 0, \\ M_0 M_0' - R y K_0 + M_0 L_0 - \dfrac{Rn}{2} N_0 = 0. \end{cases} \tag{6}$$

The first-order term of the expansion is a system of linear ordinary differential equations:

$$\begin{cases} (M_0M_1)' + M_0L_1 + 2M_1L_0 + RyA_1'(2N_0 - yK_0) - \\ RyK_0A_1 - \dfrac{R}{2}(nN_1 + 2yK_1) + Ry^2N_0A_1'' = 0, \\ M_0N_1' - N_1M_0' + M_1N_0' - N_0M_1' + M_1K_0 + N_0L_1 + 2N_1L_0 = 0, \\ M_0K_1' + M_1K_0' + N_0L_1' + N_1L_0' + 2L_0K_1 = 0, \\ (N_0L_0 + K_0M_0)K_1' + (RyM_0L_0 + K_0N_0)L_1' + RL_0M_0M_1' + \\ (RyL_0L_0' + RL_0M_0' + K_0K_0' + 3RL_0^2)M_1 + \\ (L_0K_0)'N_1 + (M_0K_0' + N_0L_0' + K_0L_0)K_1 + \\ (4RL_0M_0 + 4RyL_0^2 + N_0K_0' + RM_0M_0' + RyL_0M_0)L_1 + \\ 4RyL_0^2(L_0A_1 + A_1'M_0) + RyA_1''L_0M_0^2 + \\ RyA_1'M_0M_0'L_0 + 2RyL_0L_0'M_0A_1 + 4RL_0^2M_0A_1 + \\ 2RA_1'M_0^2L_0 + RL_0M_0M_0'A_1 = 0. \end{cases} \quad (7)$$

The solutions of (6) must be substituted into (7); thus, we obtain more simple differential equations for the functions K_1, L_1, M_1, and N_1. Repeating this process, we can obtain any number of terms in the expansions of the functions K, L, M, and N.

5. Conclusions

In this paper, we gave a brief recollection of the notion of quotient equations. Using previous results regarding invariants of the Euler system in a space, we found its quotient. We found that the quotient has an infinitesimal symmetry for special cases of the thermodynamical state of a medium. We proposed a method for solving the quotient by means of virial expansion of the Planck potential and by reducing it to series of systems of ordinary differential equations.

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