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More Effective Conditions for Oscillatory Properties of Differential Equations

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Abstract: In this work, we present several oscillation criteria for higher-order nonlinear delay differential equation with middle term. Our approach is based on the use of Riccati substitution, the integral averaging technique and the comparison technique. The symmetry contributes to deciding the right way to study oscillation of solutions of this equations. Our results unify and improve some known results for differential equations with middle term. Some illustrative examples are provided.

Keywords: higher-order; delay differential equations; oscillation

1. Introduction

In this manuscript, we consider an higher-order non-linear delay differential equation of the following type:

$$\left[\alpha_1(z) \left(\zeta^{(j-1)}(z) \right)^\ell \right]' + \alpha_2(z) \left(\zeta^{(j-1)}(z) \right)^\ell + \alpha_3(z) \zeta^\ell(\beta(z)) = 0, \tag{1}$$

where $\alpha_1 \in C^1([z_0, \infty), \mathbb{R})$, $\alpha_1'(z) \geq 0$, α_2, α_3, $\beta \in C([z_0, \infty), \mathbb{R})$, $\alpha_3 > 0$, $\beta(z) \leq z$, $\lim_{z \to \infty} \beta(z) = \infty$, ℓ is a quotient of odd positive integers and under the condition

$$\int_{z_0}^\infty \left[\frac{1}{\alpha_1(s)} \exp\left(- \int_{z_0}^s \frac{\alpha_2(x)}{\alpha_1(x)} dx \right) \right]^{1/\ell} ds = \infty. \tag{2}$$

Delay differential equations contribute to many applications such as torsional oscillations which have been observed during earthquakes, see [1]. However, oscillation theory has gained particular attention due to its widespread applications in mechanical oscillations, earthquake structures, clinical applications, frequency measurements and harmonic oscillator which involves symmetrical properties; see [2,3]. In context of oscillation theory, it has been the object of many researchers who have investigated this notion for non-linear neutral differential and difference equations; the reader can refer to [4–11].

The motivation in studying this work is to extend the results obtained by Elabbasy in [12], we will use the following methods:

- Integral averaging technique.
- Riccati transformations technique.
- Method of comparison with first-order differential equations.

In what follows, we provide some background details regarding the study of oscillation of higher-order differential equations which motivated our study.
Bazighifan and Ramos [13] investigated the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term. Liu et al. [14] examined the Oscillation of even-order half-linear functional differential equations with damping and used integral averaging technique. In [12], the authors obtained oscillation criteria for equation

$$\left[\alpha_1(z)\zeta'''(z)\right]' + p(z)\zeta'''(z) + \alpha_3(z)\zeta(\beta(z)) = 0,$$

under the condition

$$\int_{z_0}^{\infty} \frac{1}{\alpha_1(s)} \exp\left(-\int_{z_0}^{s} \frac{p(u)}{\alpha_1(u)} du\right) ds = \infty.$$

Grace et al. [15] discuss the equation

$$\left[\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^r\right]' + \alpha_3(z)\zeta^r(g(z)) = 0,$$

and used the comparison technique. Zhang et al. [16] studied the equation

$$\left[\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^\ell\right]' + \alpha_3(z)\zeta^r(\beta(z)) = 0, \quad z \geq z_0,$$

where ℓ and r are ratios of odd positive integers, $r \leq \ell$ and under

$$\int_{z_0}^{\infty} \alpha_1^{-1/\ell}(s)\mathrm{d}s < \infty,$$

and used the comparison technique.

The purpose of this paper is to extend the results in [12] and establish new oscillation criteria for (1). Our approach is based on the use of Riccati substitution, integral averaging technique and comparison technique. For examining the validity of the proposed criteria, two examples with particular values are constructed.
For the sake of simplification, we use some notations.

$$\eta(z) \quad : \quad = \int_{z}^{\infty} \left[\frac{1}{\alpha_1(s)} \exp\left(-\int_{z_0}^{s} \frac{\alpha_2(x)}{\alpha_1(x)} dx\right)\right]^{1/\ell} ds.$$

$$\alpha_1(z) \quad : \quad = \frac{\int_{z}^{\infty} (\theta - z)^{j-4} \left(\frac{\int_{\theta}^{\infty} \alpha_3(s)\left(\frac{\beta(s)}{s}\right)^\ell ds}{\alpha_1(\theta)}\right)^{1/(\ell)} d\theta}{(j-4)!},$$

and

$$D(s) := \frac{\alpha_1(s)v_1(s)|h(z,s)|^{\ell+1}}{\ell + 1^{\ell+1}\left[H(z,s)A(s)\mu\frac{s^{j-2}}{(j-2)!}\right]^\ell}.$$

2. Lemmas

The following lemmas are essential in the sequel.

Lemma 1 (Agarwal [17]). *Let $\xi(z) \in C^r[z_0, \infty)$, $\xi^{(r)}(z) \neq 0$ on $[z_0, \infty)$ and $\xi(z)\xi^{(r)}(z) \leq 0$. Then*

(I)   There exists a $z_1 \geq z_0$ such that the functions $\xi^{(m)}(z)$, $m = 1, 2, ..., r - 1$ are of constant sign on $[z_0, \infty)$;

(II) There exists a number $a \in \{1, 3, 5, ..., r - 1\}$ when r is even, $a \in \{0, 2, 4, ..., r - 1\}$ when r is odd, such that, for $z \geq z_1$,

$$\xi(z)\xi^{(m)}(z) > 0,$$

for all $m = 0, 1, ..., a$ and

$$(-1)^{r+m+1}\xi(z)\xi^{(m)}(z) > 0.$$

Lemma 2 (Kiguradze [18]).   *Let $\xi^{(r)} > 0$ for all $r = 0, 1, ..., j$, and $\xi^{(j+1)} < 0$, then*

$$\frac{j!}{z^j}\xi(z) - \frac{(j-1)!}{z^{j-1}}\frac{d}{dz}\xi(z) \geq 0.$$

Lemma 3 (Agarwal [19]).   *Let $\xi \in C^j([z_0, \infty), (0, \infty))$ and $\xi^{(j-1)}(z)\xi^{(j)}(z) \leq 0$. If we have $\lim_{z \to \infty} \xi(z) \neq 0$, then*

$$\xi(z) \geq \frac{\epsilon}{(j-1)!}z^{j-1}\left|\xi^{(j-1)}(z)\right|$$

for all $\epsilon \in (0, 1)$ and $z \geq z_\epsilon$.

3. Main Results

Now, we find oscillation conditions for (1) by using the comparing technique with first order equations.

Theorem 1.   *Let $j \geq 2$ be even and the equation*

$$\varsigma(z)\left(\varsigma'(z) + \frac{\alpha_2(z)}{\alpha_1(z)}\varsigma(z) + \frac{\alpha_3(z)}{\alpha_1(\beta(z))}\left(\frac{\epsilon\beta^{j-1}(z)}{(j-1)!}\right)\varsigma(\beta(z))\right) = 0, \tag{3}$$

has no positive solutions. Then Equation (1) is oscillatory.

Proof.   Let ζ be a nonoscillatory solution of Equation (1), then $\zeta(z) > 0$. Hence we have

$$\zeta'(z) > 0, \ \zeta^{(j-1)}(z) > 0 \text{ and } \zeta^{(j)}(z) < 0. \tag{4}$$

From Lemma 3, we obtain

$$\zeta(z) \geq \frac{\epsilon z^{j-1}}{(j-1)!\alpha_1^{1/\ell}(z)}\alpha_1^{1/\ell}(z)\zeta^{(j-1)}(z), \tag{5}$$

for all $\epsilon \in (0, 1)$. Set

$$\varsigma(z) = \alpha_1(z)\left[\zeta^{(j-1)}(z)\right]^\ell.$$

Using (5) in (1), we obtain the inequality

$$\varsigma'(z) + \frac{\alpha_2(z)}{\alpha_1(z)}\varsigma(z) + \frac{\alpha_3(z)}{\alpha_1(\beta(z))}\left(\frac{\epsilon\beta^{j-1}(z)}{(j-1)!}\right)^\ell\varsigma(\beta(z)) \leq 0.$$

That is, ς is a positive solution of inequality (3), which is a contradiction. Thus, the theorem is proved.   □

Corollary 1.   *Let $j \geq 2$ be even. If*

$$\liminf_{z \to \infty} \int_{\beta(z)}^z \frac{\alpha_3(s)}{\alpha_1(\beta(s))}\left(\beta^{j-1}(s)\right)^\ell \exp\left(\int_{\beta(s)}^s \frac{\alpha_2(u)}{\alpha_1(u)}du\right)ds > \frac{((j-1)!)^\ell}{e}, \tag{6}$$

then Equation (1) is oscillatory.

Definition 1. *Let*

$$D = \{(z,s) \in \mathbb{R}^2 : z \geq s \geq z_0\} \text{ and } D_0 = \{(z,s) \in \mathbb{R}^2 : z > s \geq z_0\}.$$

We say that a function $H \in C(D, \mathbb{R})$ belongs to the class ζ if
(I_1) $H(z, z_0) = 0, H_*(z, z_0) = 0$ *for* $z \geq z_0, H(z,s) > 0, H_*(z,s) > 0, (z,s) \in D_0$;
(I_2) H, H_* *have a nonpositive continuous partial derivative* $\partial H/\partial s, \partial H_*/\partial s$ *on D_0 with respect to the second variable, and there exist functions* $v_1, A, v_2, A_* \in C^1([z_0, \infty), (0, \infty))$ *and* $h, h_* \in C(D_0, \mathbb{R})$ *such that*

$$-\frac{\partial}{\partial s}(H(z,s)A(s)) = H(z,s)A(s)\frac{v_1'(z)}{v_1(z)} + h(z,s) \tag{7}$$

and

$$-\frac{\partial}{\partial s}(H_*(z,s)A_*(s)) = H_*(z,s)A_*(s)\frac{v_2'(z)}{v_2(z)} + h_*(z,s). \tag{8}$$

Second, in the following theorem, we find oscillation conditions for (1) by using the integral averaging and Riccati techniques.

Theorem 2. *Let $j \geq 4$ be even. Assume that (7) and (8) hold. If there exist functions $v_1, v_2 \in C^1([z_0, \infty), (0, \infty))$ such that*

$$\limsup_{z\to\infty}\frac{1}{H(z, z_0)}\int_{z_0}^z\left[H(z,s)A(s)v_1(s)\alpha_3(s)\left(\frac{\beta^{j-1}(s)}{s^{j-1}}\right)^{\ell} - D(s)\right]ds = \infty, \tag{9}$$

for some constant $\mu \in (0, 1)$ and

$$\limsup_{z\to\infty}\frac{1}{H_*(z, z_0)}\int_{z_0}^z\left(H_*(z,s)A_*(s)v_2(s)\alpha_1(s) - \frac{v_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*(s)}\right)ds = \infty, \tag{10}$$

then Equation (1) is oscillatory.

Proof. Let ζ be a nonoscillatory solution of Equation (1), then $\zeta(z) > 0$. From Lemma 1, we have two possible cases:

(C_1) $\zeta(z) > 0, \zeta'(z) > 0, \ldots, \zeta^{(j-1)}(z) > 0, \zeta^{(j)}(z) < 0,$
(C_2) $\zeta(z) > 0, \zeta^{(r)}(z) > 0, \zeta^{(r+1)}(z) < 0$ for all odd integers
$r \in \{1, 2, \ldots, j-3\}, \zeta^{(j-1)}(z) > 0, \zeta^{(j)}(z) < 0.$

Let case (C_1) holds. Define the function $\xi_1(z)$ by

$$\xi_1(z) := v_1(z)\left[\frac{\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^{\ell}}{\zeta^{\ell}(z)}\right]. \tag{11}$$

Then $\xi_1(z) > 0$ for $z \geq z_1$ and

$$\xi_1'(z) \leq v_1'(z)\frac{\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^{\ell}}{\zeta^{\ell}(z)} + v_1(z)\frac{\left(\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^{\ell}\right)'}{\zeta^{\ell}(z)}$$
$$-v_1(z)\frac{\ell\zeta'(z)\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^{\ell}}{\zeta^{\ell+1}(z)}.$$

By Lemma 3, we get

$$\zeta'(z) \geq \frac{\mu}{(j-2)!} z^{j-2} \zeta^{(j-1)}(z). \tag{12}$$

Using (12) and (11), we obtain

$$
\begin{aligned}
\xi_1'(z) \quad \leq \quad & \nu_1'(z) \frac{\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^\ell}{\zeta^\ell(z)} + \nu_1(z) \frac{\left(\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^\ell\right)'}{\zeta^\ell(z)} \\
& -\nu_1(z) \frac{\ell \mu z^{j-2}}{(j-2)!} \frac{\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^{\ell+1}}{\zeta^{\ell+1}(z)}.
\end{aligned}
\tag{13}
$$

By Lemma 2, we find

$$\frac{\zeta(z)}{\zeta'(z)} \geq \frac{z}{j-1}.$$

Thus, we obtain that ζ / z^{j-1} is nonincreasing and so

$$\frac{\zeta(\beta(z))}{\zeta(z)} \geq \frac{\beta^{j-1}(z)}{z^{j-1}}. \tag{14}$$

From (1) and (13), we get

$$
\begin{aligned}
\xi_1'(z) \quad \leq \quad & \nu_1'(z) \frac{\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^\ell}{\zeta^\ell(z)} - \nu_1(z) \frac{\alpha_3(z)\left(\zeta^\ell(\beta(z))\right)}{\zeta^\ell(z)} \\
& -\nu_1(z) \frac{\alpha_2(z)\left(\zeta^{(j-1)}(z)\right)^\ell}{\zeta^\ell(z)} - \nu_1(z) \frac{\ell \mu z^{j-2}}{(j-2)!} \frac{\alpha_1(z)\left|\left(\zeta^{(j-1)}(z)\right)\right|^{\ell+1}}{\zeta^{\ell+1}(z)}.
\end{aligned}
\tag{15}
$$

From (14) and (15), we obtain

$$
\begin{aligned}
\xi_1'(z) \quad \leq \quad & \left(\frac{\nu_1'(z)}{\nu_1(z)} - \frac{\alpha_2(z)}{\alpha_1(z)}\right) \xi_1(z) - \nu_1(z)\alpha_3(z)\left(\frac{\beta^{j-1}(z)}{z^{j-1}}\right)^\ell \\
& -\frac{(\ell)\mu z^{j-2}}{(j-2)!(\nu_1(z)\alpha_1(z))^{1/(\ell)}} \xi_1^{(\ell+1)/\ell}(z).
\end{aligned}
\tag{16}
$$

It follows from (16) that

$$\nu_1(z)\alpha_3(z)\left(\frac{\beta^{j-1}(z)}{z^{j-1}}\right)^\ell \leq \left(\frac{\nu_1'(z)}{\nu_1(z)} - \frac{\alpha_2(z)}{\alpha_1(z)}\right) \xi_1(z) - \xi_1'(z) - \frac{\ell \mu z^{j-2}}{(j-2)!(\nu_1(z)\alpha_1(z))^{1/(\ell)}} \xi_1^{(\ell+1)/\ell}(z).$$

Replacing z by s, multiplying two sides by $H(z,s)A(s)$, and integrating the resulting inequality from z_1 to z, we have

$$\int_{z_1}^{z} H(z,s)A(s)v_1(s)\alpha_3(s)\left(\frac{\beta^{j-1}(s)}{s^{j-1}}\right)^{\ell}ds \tag{17}$$

$$\leq -\int_{z_1}^{z} H(z,s)A(s)\xi_1'(s)ds + \int_{z_1}^{z} H(z,s)A(s)\left(\frac{v_1'(s)}{v_1(s)} - \frac{\alpha_2(s)}{\alpha_1(s)}\right)\xi_1(s)ds$$

$$- \int_{z_1}^{z} H(z,s)A(s)\frac{\ell\mu s^{j-2}}{(j-2)!(v_1(s)\alpha_1(s))^{1/(\ell)}}\xi_1^{(\ell+1)/\ell}(s)ds$$

$$= H(z,z_1)A(z_1)\xi_1(z_1) - \int_{z_1}^{z}\left(-\frac{\partial}{\partial s}(H(z,s)A(s)) - H(z,s)A(s)\left(\frac{v_1'(s)}{v_1(s)} - \frac{\alpha_2(s)}{\alpha_1(s)}\right)\right)\xi_1(s)ds$$

$$- \int_{z_1}^{z} H(z,s)A(s)\frac{\ell\mu s^{j-2}}{(j-2)!(v_1(s)\alpha_1(s))^{1/(\ell)}}\xi_1^{(\ell+1)/\ell}(s)ds$$

$$\leq H(z,z_1)A(z_1)\xi_1(z_1) + \int_{z_1}^{z}|h(z,s)|\xi_1(s)d(s)$$

$$- \int_{z_1}^{z} H(z,s)A(s)\frac{\ell\mu s^{j-2}}{(j-2)!(v_1(s)\alpha_1(s))^{1/\ell}}\xi_1^{(\ell+1)/\ell}(s)ds.$$

Note that
$$\varepsilon UV^{\varepsilon-1} - U^{\varepsilon} \leq (\varepsilon-1)V^{\varepsilon}, \quad \varepsilon > 1, \quad U \geq 0, \quad V \geq 0. \tag{18}$$

Here
$$\varepsilon = (\ell+1)/\ell, \quad U = \left(\ell H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\ell/(\ell+1)}\frac{\xi_1(s)}{(v_1(s)\alpha_1(s))^{1/(\ell+1)}}$$

and
$$V = \left(\frac{\ell}{\ell+1}\right)^{\ell}|h(z,s)|^{\ell}\left(\frac{v_1(s)\alpha_1(s)}{\left(\ell H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\ell}}\right)^{\ell/(\ell+1)}.$$

From (18), we get

$$|h(z,s)|\xi_1(s) - H(z,s)A(s)\frac{\ell\mu s^{j-2}}{(j-2)!(v_1(s)\alpha_1(s))^{1/\ell}}\xi_1^{(\ell+1)/\ell}$$

$$\leq \frac{v_1(s)\alpha_1(s)}{\left(H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\ell}}\left(\frac{|h(z,s)|}{\ell+1}\right)^{\ell+1}.$$

Putting the resulting inequality into (17), we obtain

$$\int_{z_1}^{z}\left[H(z,s)A(s)v_1(s)\alpha_3(s)\left(\frac{\beta^{j-1}(s)}{s^{j-1}}\right)^{\ell} - \frac{v_1(s)\alpha_1(s)\left(\frac{|h(z,s)|}{\ell+1}\right)^{\ell+1}}{\left(H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\ell}}\right]ds$$

$$\leq H(z,z_1)A(z_1)\xi_1(z_1)$$

$$\leq H(z,z_0)A(z_1)\xi_1(z_1).$$

Then

$$\frac{1}{H(z,z_0)}\int_{z_0}^{z}\left(H(z,s)A(s)v_1(s)\alpha_3(s)\left(\frac{\beta^{j-1}(s)}{s^{j-1}}\right)^{\ell} - D(s)\right)ds$$

$$\leq A(z_1)\xi_1(z_1) + \int_{z_0}^{z_1} A(s)v_1(s)\alpha_3(s)\left(\frac{\beta^{j-1}(s)}{s^{j-1}}\right)^{\ell}ds < \infty,$$

for some $\mu \in (0, 1)$, which contradicts (9).

Let Case (C_2) hold. By virtue of $\zeta'(z) > 0$ and $\zeta''(z) < z$, from Lemma 2, we obtain

$$\zeta(z) \geq z\zeta'(z).$$

Thus, we obtain that ζ/z is nonincreasing and so

$$\zeta(\beta(z)) \geq \zeta(z)\frac{\beta(z)}{z}. \tag{19}$$

From (19) and integrating (1) from z to ∞, we obtain

$$-\alpha_1(z)\left(\zeta^{(j-1)}(z)\right)^\ell + \int_z^\infty \alpha_3(s)\zeta(s)^\ell \frac{\beta(s)^\ell}{s^\ell}ds \leq 0.$$

It follows from $\zeta'(z) > 0$ that

$$-\zeta^{(j-1)}(z) + \frac{\zeta(z)}{\alpha_1^{1/\ell}(z)}\left(\int_z^\infty \alpha_3(s)\left(\frac{\beta(s)}{s}\right)^\ell ds\right)^{1/\ell} \leq 0. \tag{20}$$

Integrating (20) from z to ∞ for a total of $(j-3)$ times, we obtain

$$\zeta''(z) + \frac{1}{(j-4)!}\int_z^\infty (\theta - z)^{j-4}\left(\frac{\int_\theta^\infty \alpha_3(s)\left(\frac{\beta(s)}{s}\right)^\ell ds}{\alpha_1(\theta)}\right)^{1/\ell} d\theta\zeta(z) \leq 0. \tag{21}$$

Now, define

$$\xi_2(z) := \nu_2(z)\frac{\zeta'(z)}{\zeta(z)}. \tag{22}$$

Then $\xi_1(z) > 0$ for $z \geq z_1$ and

$$\xi_2'(z) = \nu_2'(z)\frac{\zeta'(z)}{\zeta(z)} + \nu_2(z)\frac{\zeta''(z)\zeta(z) - (\zeta'(z))^2}{\zeta^2(z)}.$$

It follows from (21) and (22) that

$$\nu_2(z)\alpha_1(z) \leq -\xi_2'(z) + \frac{\nu_2'(z)}{\nu_2(z)}\xi_2(z) - \frac{1}{\nu_2(z)}\xi_2^2(z).$$

Replacing z by s, multiplying two sides by $H_*(z,s)A_*(s)$, and integrating the resulting inequality from z_1 to z, we have

$$
\begin{aligned}
\int_{z_1}^z H_*(z,s)A_*(s)\nu_2(s)\alpha_1(s)ds \leq\ & -\int_{z_1}^z H_*(z,s)A_*(s)\xi_2'(s)ds \\
& + \int_{z_1}^z H_*(z,s)A_*(s)\frac{\nu_2'(s)}{\nu_2(s)}\xi_2(s)ds \\
& - \int_{z_1}^z \frac{H_*(z,s)A_*(s)}{\nu_2(s)}\xi_2^2(s)ds \\
=\ & H_*(z,z_1)A_*(z_1)\xi_2(z_1) - \int_{z_1}^z \frac{H_*(z,s)A_*(s)}{\nu_2(s)}\xi_2^2(s)ds \\
& - \int_{z_1}^z \left(-\frac{\partial}{\partial s}(H_*(z,s)A_*(s)) - H_*(z,s)A_*(s)\frac{\nu_2'(z)}{\nu_2(z)}\right)\xi_2(s)ds \\
\leq\ & H_*(z,z_1)A_*(z_1)\xi_2(z_1) + \int_{z_1}^z |h_*(z,s)|\xi_2(s)d(s) \\
& - \int_{z_1}^z \frac{H_*(z,s)A_*(s)}{\nu_2(s)}\xi_2^2(s)ds.
\end{aligned}
$$

Hence we have

$$\int_{z_1}^{z}\left[H_*(z,s)A_*(s)v_2(s)\alpha_1(s)-\frac{v_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*}\right]ds$$

$$\leq \quad H_*(z,z_1)A_*(z_1)\xi_2(z_1)$$

$$\leq \quad H_*(z,z_0)A_*(z_1)\xi_2(z_1).$$

Then

$$\frac{1}{H_*(z,z_0)}\int_{z_0}^{z}\left[H_*(z,s)A_*(s)v_2(s)\alpha_1(s)-\frac{v_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*}\right]ds$$

$$\leq \quad A_*(z_1)\xi_2(z_1)+\int_{z_0}^{z}A_*(s)v_2(s)\alpha_1(s)ds<\infty,$$

which contradicts (10). Therefore, the theorem is proved □

4. Applications

This section presents some interesting examples and applications to examine the applicability of theoretical outcomes.

Example 1. *Consider the equation with middle term*

$$\zeta^{(4)}(z)+\frac{1}{z}\zeta^{(3)}(z)+\frac{\varepsilon}{z^4}\zeta\left(\frac{z}{4}\right)=0,\ \varepsilon>0,\ z\geq 1, \tag{23}$$

we see that $j=4$, $\ell=1$, $\alpha_1(z)=1$, $\alpha_2(z)=1/z$, $\beta(z)=z/4$, $\alpha_3(z)=\varepsilon/z^4$ *and*

$$\eta(s)=\int_{z_0}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(u)}{\alpha_1(u)}du\right)\right]^{1/\ell}ds=\infty.$$

Now, we find that

$$\liminf_{z\to\infty}\int_{\beta(z)}^{z}\frac{\alpha_3(s)}{\alpha_1(\beta(s))}\left(\beta^{j-1}(s)\right)^{\ell}\exp\left(\int_{\beta(s)}^{s}\frac{\alpha_2(u)}{\alpha_1(u)}du\right)ds$$

$$= \quad \liminf_{z\to\infty}\int_{\beta(z)}^{z}\frac{\varepsilon}{s^4}\left(\frac{s^3}{64}\right)\exp(\ln 4)ds$$

$$= \quad \liminf_{z\to\infty}\int_{\beta(z)}^{z}\frac{\varepsilon}{16s}ds=\frac{\varepsilon}{16}\ln 4>\frac{6}{e},\quad if\ \ \varepsilon>96/(e\ln 4)=24.$$

Thus, by Corollary 1, Equation (23) is oscillatory if $\varepsilon>24$.

Example 2. *Consider the differential equation*

$$\left(\frac{1}{z}\zeta'''(z)\right)'+\left(1\backslash\left(2z^2\right)\right)\zeta'''(t)+\frac{\varepsilon}{z}\zeta\left(\frac{z}{2}\right)=0,\ z\geq 1, \tag{24}$$

where $\varepsilon>0$ *is a constant. Let* $j=4$, $\ell=1$, $\alpha_1(z)=1/z$, $\alpha_2(z)=1/\left(2z^2\right)$, $\beta(z)=z/2$, $\alpha_3(z)=\varepsilon/z$ *and*

$$\eta(s)=\int_{z_0}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(u)}{\alpha_1(u)}du\right)\right]^{1/\ell}ds=\infty.$$

Now, we find that condition (6) holds. Therefore, by Corollary 1, Equation (24) is oscillatory.

Example 3. *Consider the equation*

$$\zeta^{(4)}(z) + \frac{1}{z^2}\zeta^{(3)}(z) + \frac{\varepsilon}{z^4}\zeta\left(4^{-1/3}z\right) = 0, \quad z \geq 1, \tag{25}$$

where $\varepsilon > 0$ is a constant. Let

$$
\begin{aligned}
j &= 4, \ \alpha_1(z) = 1, \ \alpha_2(z) = 1/z^2, \ \ell = 1, \ \beta(z) = 4^{-1/3}z, \ \alpha_3(z) = \varepsilon/z^4, \\
H(z,s) &= H_*(z,s) = (z-s)^2, \ A(s) = A_*(s) = 1, \\
v_1(s) &= z^3, \ v_2(s) = z, \ h(z,s) = h_*(z,s) = (z-s)\left(5 - s^{-1} + z\left(s^{-2} - 3s^{-1}\right)\right).
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\eta(s) &= \int_{z_0}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(u)}{\alpha_1(u)}du\right)\right]^{1/\ell}ds = \infty, \\
\alpha_1(z) &= \frac{\int_z^{\infty}(\theta - z)^{j-4}\left(\frac{\int_\theta^{\infty}\alpha_3(s)\left(\frac{\beta(s)}{s}\right)^{\ell}ds}{\alpha_1(\theta)}\right)^{1/\ell}d\theta}{(j-4)!} \\
&\geq \varepsilon/\left(12z^2\right).
\end{aligned}
$$

Now, we see that

$$
\begin{aligned}
&\limsup_{z\to\infty}\frac{1}{H(z,z_0)}\int_{z_0}^{z}\left(H(z,s)A(s)v_1(s)\alpha_3(s)\left(\frac{\beta^{j-1}(s)}{s^{j-1}}\right)^{\ell} - D(s)\right)ds \\
&= \limsup_{z\to\infty}\frac{1}{(z-1)^2}\int_1^z[\frac{\varepsilon}{4}z^2 s^{-1} + \frac{\varepsilon}{4}s - \frac{\varepsilon}{2}z - \frac{s}{2\mu}(25 + s^{-2} - 10s^{-1} + z^2 s^{-4} \\
&\quad + 9z^2 s^{-2} - 6z^2 s^{-3} + 16zs^{-2} - 2zs^{-3} - 30zs^{-1})]ds \\
&= \infty, \quad \text{if } \varepsilon > 18/\mu \ \text{ for some } \mu \in (0,1).
\end{aligned}
$$

Set

$$H_*(z,s) = (z-s)^2, \ A_*(s) = 1, \ v_2(s) = z, \ h_*(z,s) = (z-s)\left(3 - zs^{-1}\right).$$

Then we have

$$
\begin{aligned}
&\limsup_{z\to\infty}\frac{1}{H_*(z,z_0)}\int_{z_0}^{z}\left(H_*(z,s)A_*(s)v_2(s)\alpha_1(s) - \frac{v_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*(s)}\right)ds \\
&\geq \limsup_{z\to\infty}\frac{1}{(z-1)^2}\int_1^z\left[\frac{\varepsilon}{12}z^2 s^{-1} + \frac{\varepsilon}{12}s - \frac{\varepsilon}{6}z - \frac{s}{4}\left(9 - 6zs^{-1} + z^2 s^{-2}\right)\right]ds \\
&= \infty, \quad \text{if } \varepsilon > 3.
\end{aligned}
$$

Thus, by Theorem 2, Equation (25) is oscillatory if $\varepsilon \geq 19$.

5. Conclusions

Throughout this article, we establish oscillation conditions for higher-order differential equation with delay. We discussed the oscillation behavior of solutions for Equation (1). We employ different approach based on integral averaging technique, Riccati technique and comparing technique with first order equations. Our results unify and extend some known results for differential equations with middle term. In future work, we will discuss the

oscillatory behavior of these equations by using comparing technique with second-order equations under the condition

$$\int_{z_0}^{\infty} \left[\frac{1}{\alpha_1(s)} \exp\left(- \int_{z_0}^{s} \frac{\alpha_2(x)}{\alpha_1(x)} dx \right) \right]^{1/\ell} ds < \infty.$$

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following cases:

(**S**$_1$) $\zeta(z) > 0,\ \zeta'(z) > 0, \zeta''(z) > 0,\ \zeta^{(j-1)}(z) > 0,\ \zeta^{(j)}(z) < 0,$

(**S**$_2$) $\zeta(z) > 0, \zeta^{(r)}(z) > 0, \zeta^{(r+1)}(z) < 0$ for all odd integers $r \in \{1, 3, ..., j - 3\}, \zeta^{(j-1)}(z) > 0, \zeta^{(j)}(z) < 0.$

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