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# The Stability of Isometries on Restricted Domains

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**Abstract:** We will prove the generalized Hyers–Ulam stability of isometries, with a focus on the stability for restricted domains. More precisely, we prove the generalized Hyers–Ulam stability of the orthogonality equation and we use this result to prove the stability of the equations  $\|f(x) - f(y)\| = \|x - y\|$  and  $\|f(x) - f(y)\|^2 = \|x - y\|^2$  on the restricted domains. As we can easily see, these functional equations are symmetric in the sense that they become the same equations even if the roles of variables  $x$  and  $y$  are exchanged.

**Keywords:** isometry; linear isometry; Hyers-Ulam stability; generalized Hyers-Ulam stability; restricted domain

## 1. Introduction

Throughout this paper, unless otherwise mentioned,  $(E, \|\cdot\|)$  and  $(F, \|\cdot\|)$  are used to denote Hilbert spaces over  $\mathbb{K}$ , where  $\mathbb{K}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ . A mapping  $f : E \rightarrow F$  is said to be an *isometry* if  $f$  satisfies:

$$\|f(x) - f(y)\| = \|x - y\| \tag{1}$$

for any  $x, y \in E$ .

By considering the definition of Hyers and Ulam [1], for each fixed  $\varepsilon \geq 0$ , a function  $f : E \rightarrow F$  is said to be an  $\varepsilon$ -*isometry* if  $f$  satisfies the inequality:

$$\| \|f(x) - f(y)\| - \|x - y\| \| \leq \varepsilon \tag{2}$$

for any  $x, y \in E$ . If there exists a positive constant  $K$  such that for each  $\varepsilon$ -isometry  $f : E \rightarrow F$ , there is an isometry  $U : E \rightarrow F$  satisfying the inequality  $\|f(x) - U(x)\| \leq K\varepsilon$  for every  $x \in E$ , then the functional Equation (1) is said to have (or satisfy) the *Hyers–Ulam stability*.

We can weaken the conditions associated with Hyers–Ulam stability in order to study more diverse topics as follows: Assume that  $\varphi, \Phi : E \times E \rightarrow [0, \infty)$  satisfy some mild conditions. If for each function  $f : E \rightarrow F$  that satisfies the inequality:

$$\| \|f(x) - f(y)\| - \|x - y\| \| \leq \varphi(x, y),$$

there exists a function  $U : E \rightarrow F$  such that  $\|U(x) - U(y)\| = \|x - y\|$  for any  $x, y \in E$  and  $\|f(x) - U(x)\| \leq \Phi(x, x)$  for any  $x \in E$ , then the functional Equation (1) is said to have (or satisfy) the *generalized Hyers–Ulam stability*. We use these terminologies also for other types of functional equations.

To the best of our knowledge, Hyers and Ulam were the first mathematicians to study the Hyers–Ulam stability of isometries (see [1]). Indeed, they were able to prove the stability of isometries based on the properties of the inner product of Hilbert spaces: For each surjective  $\varepsilon$ -isometry  $f : E \rightarrow E$  satisfying  $f(0) = 0$ , there is a surjective isometry  $U : E \rightarrow E$  satisfying  $\|f(x) - U(x)\| \leq 10\varepsilon$  for every  $x \in E$ .



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The theorem of Hyers and Ulam was further generalized by Bourgin [2]: Assume that  $E$  is a real Banach space and that  $F$  belongs to the class of uniformly convex real Banach spaces which includes the spaces  $L_p(0, 1)$  for  $1 < p < \infty$ . For any  $\varepsilon$ -isometry  $f : E \rightarrow F$  with  $f(0) = 0$ , there exists a linear isometry  $U : E \rightarrow F$  such that  $\|f(x) - U(x)\| \leq 12\varepsilon$  for every  $x \in E$ .

Thereafter, Hyers and Ulam [3] investigated the stability problem for the Banach spaces of continuous functions: For each  $i \in \{1, 2\}$ , let  $S_i$  be a compact metric space and let  $C(S_i)$  be the Banach space of real-valued continuous functions on  $S_i$  equipped with the supremum norm  $\|\cdot\|_\infty$ . If a homeomorphism  $T : C(S_1) \rightarrow C(S_2)$  satisfies the inequality:

$$|\|T(f) - T(g)\|_\infty - \|f - g\|_\infty| \leq \varepsilon \quad (3)$$

for some  $\varepsilon > 0$  and for any  $f, g \in C(S_1)$ , then there exists an isometry  $U : C(S_1) \rightarrow C(S_2)$  satisfying  $\|T(f) - U(f)\|_\infty \leq 21\varepsilon$  for any  $f \in C(S_1)$ .

The very result of Hyers and Ulam was profoundly generalized once again by Bourgin [4]: If  $S_i$  is a completely regular Hausdorff space for  $i \in \{1, 2\}$  and  $T : C(S_1) \rightarrow C(S_2)$  is a surjective function satisfying the inequality (3) for some  $\varepsilon > 0$  and for all  $f, g \in C(S_1)$ , then there exists a linear isometry  $U : C(S_1) \rightarrow C(S_2)$  satisfying  $\|T(f) - U(f)\|_\infty \leq 10\varepsilon$  for any  $f \in C(S_1)$ . The study of the stability of isometries was then continued by Bourgin for the finite-dimensional Banach spaces (see [5]).

In 1978, Gruber [6] proved the following theorem: Assume that  $E$  and  $F$  are real normed spaces,  $f : E \rightarrow F$  is a surjective  $\varepsilon$ -isometry, and that  $U : E \rightarrow F$  is an isometry satisfying  $f(p) = U(p)$  for a  $p \in E$ . If  $\|f(x) - U(x)\| = o(\|x\|)$  as  $\|x\| \rightarrow \infty$  uniformly, then  $U$  is a surjective linear isometry and  $\|f(x) - U(x)\| \leq 5\varepsilon$  for all  $x \in E$ . In particular, if  $f$  is continuous, then  $\|f(x) - U(x)\| \leq 3\varepsilon$  for each  $x \in E$ .

Years later, Gevirtz [7] proved the stability of isometries defined between real Banach spaces: Assume that  $E$  and  $F$  are real Banach spaces. For each surjective  $\varepsilon$ -isometry  $f : E \rightarrow F$ , there exists a surjective isometry  $U : E \rightarrow F$  such that  $\|f(x) - U(x)\| \leq 5\varepsilon$  for any  $x \in E$ . After that, the upper bound  $5\varepsilon$  was improved to a sharper  $2\varepsilon$  by Omladič and Šemrl [8].

According to [9,10], a function  $f : E \rightarrow F$  is called an  $(\varepsilon, p)$ -isometry when  $f$  satisfies:

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon \|x - y\|^p$$

for some constants  $p \geq 0$  and  $\varepsilon \geq 0$  and for all  $x, y \in E$ . Dolinar [9] proved the *superstability* of  $(\varepsilon, p)$ -isometries: When  $p > 1$ , every surjective  $(\varepsilon, p)$ -isometry  $f : E \rightarrow F$  between finite-dimensional real Banach spaces is an isometry.

There are many other papers related to the stability of isometries, but it is regrettable that due to the restriction of space, they cannot be quoted one by one. Nevertheless, see [11–25] for more general information on the stability of isometries and related topics.

The following theorem of Fickett provided an important motivation for writing this paper. Fickett [15] proved the following theorem:

**Theorem 1.** For a fixed integer  $n \geq 2$ , let  $S$  be a convex bounded subset of  $\mathbb{R}^n$  and let  $\varepsilon > 0$  be given. If a function  $f : S \rightarrow \mathbb{R}^n$  satisfies the inequality:

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon$$

for all  $x, y \in S$ , then there exists an isometry  $U : S \rightarrow \mathbb{R}^n$  such that:

$$\|f(x) - U(x)\| \leq 27\varepsilon^{1/2^n}$$

for each  $x \in S$ .

The upper bound associated with inequality in Fickett’s theorem becomes very large for any sufficiently small  $\varepsilon$  in comparison to  $\varepsilon$ . This is a big drawback of Fickett’s theorem. Thus, the work of further improving Fickett’s theorem has to be attractive.

In this paper, we prove the generalized Hyers–Ulam stability of isometries with a focus on the stability for restricted domains. More precisely, we prove the generalized Hyers–Ulam stability of the orthogonality equation and we apply this result for proving some theorems about the generalized Hyers–Ulam stability of equations  $\|f(x) - f(y)\| = \|x - y\|$  and  $\|f(x) - f(y)\|^2 = \|x - y\|^2$ , which are equivalent to each other in the sense that their solution sets are the same. As can be easily seen, those functional equations are symmetric in that they become the same equations even if the variables  $x$  and  $y$  are exchanged. For readers interested in the stability and related topics of the orthogonality equation, read the papers [13,14,26–29].

### 2. Preliminaries

In this section,  $E$  and  $F$  denote the Hilbert spaces over  $\mathbb{K}$ , where  $\mathbb{K}$  denotes either  $\mathbb{R}$  or  $\mathbb{C}$ . We note that a function  $f : E \rightarrow F$  satisfies the *orthogonality equation*:

$$\langle f(x), f(y) \rangle = \langle x, y \rangle \tag{4}$$

for all  $x, y \in E$  if and only if it is a linear isometry.

A function  $f : E \rightarrow F$  is called *conjugate-linear* if and only if  $f(\lambda x + \mu y) = \bar{\lambda}f(x) + \bar{\mu}f(y)$  for all  $x, y \in E$  and  $\lambda, \mu \in \mathbb{K}$ . We note that a function  $f : E \rightarrow F$  satisfies the equation:

$$\langle f(x), f(y) \rangle = \langle y, x \rangle$$

for any  $x, y \in E$  if and only if it is a conjugate-linear isometry.

Let  $\mathcal{S} = \{z \in \mathbb{K} : |z| = 1\}$ . Every two functions  $f, g : E \rightarrow F$  are called *phase-equivalent* if and only if there is a function  $\sigma : E \rightarrow \mathcal{S}$  satisfying  $g(x) = \sigma(x)f(x)$  for all  $x \in E$ .

The following equation:

$$|\langle f(x), f(y) \rangle| = |\langle x, y \rangle| \tag{5}$$

is known as the *Wigner equation* or the *generalized orthogonality equation*. Wigner introduced the following theorem in [30].

**Theorem 2.** *If a function  $f : E \rightarrow F$  satisfies the Wigner Equation (5) for all  $x, y \in E$ , then  $f$  is phase-equivalent to a linear isometry or a conjugate-linear isometry.*

Let  $(E, \langle \cdot, \cdot \rangle)$  be a Hilbert space with the associated norm defined as  $\|x\| = \sqrt{|\langle x, x \rangle|}$ . For any fixed  $d \geq 0$ , we define either  $D = \{x \in E : \|x\| \geq d\}$  or  $D = \{x \in E : \|x\| \leq d\}$ . We now select a constant  $c$  satisfying the condition:

$$\begin{cases} 0 < c < 1 & (\text{for } D = \{x \in E : \|x\| \geq d\}), \\ c > 1 & (\text{for } D = \{x \in E : \|x\| \leq d\}). \end{cases} \tag{6}$$

In this paper, we will exclude the trivial case  $D = \{0\}$ . In addition, let us define  $D_0 = D \cup \{0\}$ . We note that we select the constant  $c$  and domain  $D$  with the definition (6) in mind.

The following theorem has been proved in Theorem 2 of [26] which deals with the generalized Hyers–Ulam stability of the orthogonality Equation (4).

**Theorem 3.** *Assume that a function  $\varphi : E \times E \rightarrow [0, \infty)$  satisfies the condition:*

$$\lim_{m+n \rightarrow \infty} c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = 0 \tag{7}$$

for any  $x, y \in D$ . If a function  $f : E \rightarrow F$  satisfies the inequality:

$$|\langle f(x), f(y) \rangle - \langle x, y \rangle| \leq \varphi(x, y)$$

for all  $x, y \in D$ , then there exists a unique linear isometry  $U : E \rightarrow F$  satisfying:

$$\|f(x) - U(x)\| \leq \sqrt{\varphi(x, x)}$$

for any  $x \in D$ .

A normed space  $(E, \|\cdot\|)$  is called *strictly convex* if and only if for any given  $x, y \in E$ ,  $\|x + y\| = \|x\| + \|y\|$  implies that  $x$  and  $y$  are linearly dependent.

Baker demonstrated the following theorem in [11].

**Theorem 4.** *If  $E$  is a normed space over  $\mathbb{R}$  and  $F$  is a strictly convex normed space over  $\mathbb{R}$ , then every isometry  $f : E \rightarrow F$  is affine. In particular, a function  $f : E \rightarrow F$  is an isometry and  $f(0) = 0$  if and only if  $f$  is a linear isometry.*

### 3. Main Result

The condition of the following theorem is weaker than condition (7) of Theorem 3. Indeed, the new condition (8) does not look much different from the old one (7), but we will see from theorems and corollaries in the following section that there are quite significant differences in their utilizations. That is, Theorem 5 is an improved version of Theorem 3 and it will play a central role in this paper.

The proof of the following theorem is based strongly on the well known properties of the inner product of Hilbert spaces and the direct method that was first conceived by Hyers [31].

**Theorem 5.** *Let  $E$  and  $F$  be Hilbert spaces over  $\mathbb{K}$  and assume that a function  $\varphi : E \times E \rightarrow [0, \infty)$  satisfies the condition:*

$$\lim_{m, n \rightarrow \infty} c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = 0 \quad (8)$$

for all  $x, y \in D$ . If a function  $f : E \rightarrow F$  satisfies the inequality:

$$|\langle f(x), f(y) \rangle - \langle x, y \rangle| \leq \varphi(x, y) \quad (9)$$

for all  $x, y \in D$ , then there exists a unique linear isometry  $U : E \rightarrow F$  such that:

$$\|f(x) - U(x)\| \leq \sqrt{\varphi(x, x) + \lim_{n \rightarrow \infty} c^n \varphi\left(\frac{1}{c^n}x, x\right) + \lim_{n \rightarrow \infty} c^n \varphi\left(x, \frac{1}{c^n}x\right)} \quad (10)$$

for all  $x \in D$ .

**Proof.** We select a constant  $c$  satisfying the condition (6). By the definition of norm and the triangle inequality,  $||a| - |b|| \leq |a - b|$  for all  $a, b \in \mathbb{C}$ , and since  $c^n \frac{1}{c^n}x - c^m \frac{1}{c^m}x = 0$  and  $||w||^2 = \langle w, w \rangle$  for all  $w \in F$ , we have:

$$\begin{aligned} & \left\| c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right) \right\|^2 \\ &= \left| \left\| c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right) \right\|^2 - \left\| c^n \frac{1}{c^n}x - c^m \frac{1}{c^m}x \right\|^2 \right| \\ &= \left| \left\langle c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right), c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right) \right\rangle \right. \\ &\quad \left. - \left\langle c^n \frac{1}{c^n}x - c^m \frac{1}{c^m}x, c^n \frac{1}{c^n}x - c^m \frac{1}{c^m}x \right\rangle \right| \\ &\leq \left| \left\langle c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right), c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right) \right\rangle \right. \\ &\quad \left. - \left\langle c^n \frac{1}{c^n}x - c^m \frac{1}{c^m}x, c^n \frac{1}{c^n}x - c^m \frac{1}{c^m}x \right\rangle \right| \end{aligned}$$

for any  $x \in D$  and  $m, n \in \mathbb{N}_0$ . We note that  $\frac{1}{c^m}x \in D$  and  $\frac{1}{c^n}x \in D$  for each  $x \in D$  and any  $m, n \in \mathbb{N}_0$ . Hence, by using (9), the last inequality and triangle inequality, we get:

$$\begin{aligned} & \left\| c^n f\left(\frac{1}{c^n}x\right) - c^m f\left(\frac{1}{c^m}x\right) \right\|^2 \\ &\leq \left| \left\langle c^n f\left(\frac{1}{c^n}x\right), c^n f\left(\frac{1}{c^n}x\right) \right\rangle - \left\langle c^n \frac{1}{c^n}x, c^n \frac{1}{c^n}x \right\rangle \right| \\ &\quad + \left| \left\langle c^n f\left(\frac{1}{c^n}x\right), c^m f\left(\frac{1}{c^m}x\right) \right\rangle - \left\langle c^n \frac{1}{c^n}x, c^m \frac{1}{c^m}x \right\rangle \right| \\ &\quad + \left| \left\langle c^m f\left(\frac{1}{c^m}x\right), c^n f\left(\frac{1}{c^n}x\right) \right\rangle - \left\langle c^m \frac{1}{c^m}x, c^n \frac{1}{c^n}x \right\rangle \right| \\ &\quad + \left| \left\langle c^m f\left(\frac{1}{c^m}x\right), c^m f\left(\frac{1}{c^m}x\right) \right\rangle - \left\langle c^m \frac{1}{c^m}x, c^m \frac{1}{c^m}x \right\rangle \right| \\ &\leq c^{2n} \varphi\left(\frac{1}{c^n}x, \frac{1}{c^n}x\right) + c^{n+m} \varphi\left(\frac{1}{c^n}x, \frac{1}{c^m}x\right) + c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}x\right) + c^{2m} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^m}x\right) \end{aligned} \tag{11}$$

for all  $x \in D$  and  $m, n \in \mathbb{N}_0$ .

On account of (8) and (11), we see that for each  $x \in D$ , the sequence  $\left\{ c^n f\left(\frac{1}{c^n}x\right) \right\}$  is a Cauchy sequence in a Hilbert space  $F$ . Hence, we can define a function  $u : D \rightarrow F$  by:

$$u(x) = \lim_{n \rightarrow \infty} c^n f\left(\frac{1}{c^n}x\right)$$

for every  $x \in D$ .

We will now extend the domain of  $u$  from  $D$  to the whole space  $E$ . For any  $x \in E \setminus \{0\}$ , we set:

$$n(x) = \min \left\{ n \in \mathbb{N}_0 : \frac{1}{c^n}x \in D \right\}$$

and we know that  $c^{-n(x)}x \in D$  for any  $x \in E \setminus \{0\}$ . Thus, we can define a function  $U : E \rightarrow F$  by:

$$U(x) = \begin{cases} c^{n(x)}u(c^{-n(x)}x) & (\text{for } x \neq 0), \\ 0 & (\text{for } x = 0). \end{cases}$$

For  $x = 0$  or  $y = 0$ , it is obvious that  $\langle U(x), U(y) \rangle = 0 = \langle x, y \rangle$  because  $U(x) = x = 0$  or  $U(y) = y = 0$ . When  $x, y \in E \setminus \{0\}$ , we note that  $\frac{1}{c^{n(x)}}x \in D$ ,  $\frac{1}{c^{n(y)}}y \in D$ ,  $\frac{1}{c^{n(x)+n(y)}}x \in D$ , and  $\frac{1}{c^{m+n(y)}}y \in D$  for any  $m, n \in \mathbb{N}_0$ , and it follows from (8) and (9) that:

$$\begin{aligned} & |\langle U(x), U(y) \rangle - \langle x, y \rangle| \\ &= \left| \left\langle c^{n(x)}u\left(\frac{1}{c^{n(x)}}x\right), c^{n(y)}u\left(\frac{1}{c^{n(y)}}y\right) \right\rangle - c^{n(x)+n(y)} \left\langle \frac{1}{c^{n(x)}}x, \frac{1}{c^{n(y)}}y \right\rangle \right| \\ &= \left| \left\langle c^{n(x)} \lim_{n \rightarrow \infty} c^n f\left(\frac{1}{c^{n+n(x)}}x\right), c^{n(y)} \lim_{m \rightarrow \infty} c^m f\left(\frac{1}{c^{m+n(y)}}y\right) \right\rangle \right. \\ &\quad \left. - c^{n(x)+n(y)} \left\langle \frac{1}{c^{n(x)}}x, \frac{1}{c^{n(y)}}y \right\rangle \right| \\ &= c^{n(x)+n(y)} \left| \left\langle \lim_{n \rightarrow \infty} c^n f\left(\frac{1}{c^{n+n(x)}}x\right), \lim_{m \rightarrow \infty} c^m f\left(\frac{1}{c^{m+n(y)}}y\right) \right\rangle - \left\langle \frac{1}{c^{n(x)}}x, \frac{1}{c^{n(y)}}y \right\rangle \right| \\ &= \lim_{m, n \rightarrow \infty} c^{n(x)+n(y)+n+m} \left| \left\langle f\left(\frac{1}{c^{n+n(x)}}x\right), f\left(\frac{1}{c^{m+n(y)}}y\right) \right\rangle - \left\langle \frac{1}{c^{n+n(x)}}x, \frac{1}{c^{m+n(y)}}y \right\rangle \right| \\ &\leq c^{n(x)+n(y)} \lim_{m, n \rightarrow \infty} c^{n+m} \varphi\left(\frac{1}{c^n} \frac{1}{c^{n(x)}}x, \frac{1}{c^m} \frac{1}{c^{n(y)}}y\right) \\ &= 0. \end{aligned}$$

That is,  $U$  is a solution to the orthogonality equation. In other words,  $U$  is a linear isometry.

If we set  $m = 0$  and let  $n$  go to infinity in (11), then we get the inequality (10). It now remains to prove the uniqueness of  $U$ . Let  $V : E \rightarrow F$  be another linear isometry satisfying the inequality (10) for all  $x \in D$ . By the linearity of  $U$  and  $V$ , we know by using (8) that:

$$\begin{aligned} & \|U(x) - V(x)\| \\ &= \left\| c^m U\left(\frac{1}{c^m}x\right) - c^m f\left(\frac{1}{c^m}x\right) \right\| + \left\| c^m f\left(\frac{1}{c^m}x\right) - c^m V\left(\frac{1}{c^m}x\right) \right\| \\ &\leq 2c^m \sqrt{\varphi\left(\frac{1}{c^m}x, \frac{1}{c^m}x\right) + \lim_{n \rightarrow \infty} c^n \varphi\left(\frac{1}{c^{m+n}}x, \frac{1}{c^m}x\right) + \lim_{n \rightarrow \infty} c^n \varphi\left(\frac{1}{c^m}x, \frac{1}{c^{m+n}}x\right)} \\ &\rightarrow 0 \text{ as } m \rightarrow \infty, \end{aligned}$$

for all  $x \in D$ . By using the linearity of  $U$  and  $V$  again, we can conclude that  $U(x) = V(x)$  for all  $x \in E$ .  $\square$

As we can see in the corollaries of the following section, in many application cases of Theorem 5, the limit values in the inequality (10) become zero.

#### 4. Stability of Isometries

Assume temporarily that  $f : E \rightarrow F$  is a function satisfying  $f(0) = 0$ , where  $E$  is a real Hilbert space and  $F$  is a strictly convex real Hilbert space. In view of Theorem 4 with  $f(0) = 0$ ,  $f$  is an isometry if and only if  $f$  is a linear isometry. Therefore, when  $E$  is a real Hilbert space and  $F$  is a strictly convex real Hilbert space, the functional Equation (1) is substantially the same as the orthogonality Equation (4).

The stability problem for the functional Equation (1) will be addressed in the following theorem, which is somewhat closely related to the theorems demonstrated in [16,17,32]. We recall that we select a constant  $c$  satisfying the condition (6), i.e.,

$$\begin{cases} 0 < c < 1 & (\text{for } D = \{x \in E : \|x\| \geq d\}), \\ c > 1 & (\text{for } D = \{x \in E : \|x\| \leq d\}) \end{cases}$$

and  $D_0 = D \cup \{0\}$ , where  $d$  is an arbitrarily given nonnegative constant.

**Theorem 6.** Let  $E$  and  $F$  be Hilbert spaces over  $\mathbb{R}$ . Assume that a function  $\varphi : E \times E \rightarrow [0, \infty)$  satisfies the following conditions:

$$\lim_{m,n \rightarrow \infty} c^{m+n} \left\| \frac{1}{c^m} x - \frac{1}{c^n} y \right\| \varphi \left( \frac{1}{c^m} x, \frac{1}{c^n} y \right) = 0$$

and

$$\lim_{m,n \rightarrow \infty} c^{m+n} \varphi \left( \frac{1}{c^m} x, \frac{1}{c^n} y \right)^2 = 0$$

for all  $x, y \in D_0$ . If a function  $f : E \rightarrow F$  satisfies the inequality:

$$\|f(x) - f(y)\| - \|x - y\| \leq \varphi(x, y) \quad (12)$$

for all  $x, y \in D_0$ , then there exists a unique linear isometry  $U : E \rightarrow F$  such that:

$$\|f(x) - U(x) - f(0)\| \leq \sqrt{\psi(x, x) + \lim_{n \rightarrow \infty} c^n \psi \left( \frac{1}{c^n} x, x \right) + \lim_{n \rightarrow \infty} c^n \psi \left( x, \frac{1}{c^n} x \right)}$$

for all  $x \in D_0$ , where:

$$\psi(x, y) = \|x - y\| \varphi(x, y) + \|x\| \varphi(x, 0) + \|y\| \varphi(0, y) + \frac{1}{2} \varphi(x, y)^2 + \frac{1}{2} \varphi(x, 0)^2 + \frac{1}{2} \varphi(0, y)^2$$

for any  $x, y \in E$ .

**Proof.** If we put  $g(x) = f(x) - f(0)$  in (12), then we see that  $g$  also satisfies the inequality (12) and  $g(0) = 0$ . It follows from (12) with  $g$  in place of  $f$  that:

$$\|g(x) - g(y)\| + \|x - y\| \leq 2\|x - y\| + \varphi(x, y) \quad (13)$$

for any  $x, y \in D_0$ , since  $\|g(x) - g(y)\| \leq \|x - y\| + \varphi(x, y)$  by (12).

Obviously, it holds that:

$$\begin{aligned} & \langle g(x), g(y) \rangle - \langle x, y \rangle \\ &= \langle g(x) - g(y), g(y) - g(x) \rangle + \langle g(x) - g(y), g(x) \rangle + \langle g(y), g(y) \rangle \\ & \quad - \langle x - y, y - x \rangle - \langle x - y, x \rangle - \langle y, y \rangle \\ &= -\|g(x) - g(y)\|^2 + \|g(x)\|^2 - \langle g(y), g(x) \rangle + \|g(y)\|^2 \\ & \quad + \|x - y\|^2 - \|x\|^2 + \langle y, x \rangle - \|y\|^2. \end{aligned}$$

If we move the terms  $\langle g(y), g(x) \rangle$  and  $\langle y, x \rangle$  on the right-hand side to the left-hand side of the above equality and divide both resulting sides by 2, then we get:

$$\begin{aligned} & \langle g(x), g(y) \rangle - \langle x, y \rangle \\ &= -\frac{1}{2} \|g(x) - g(y)\|^2 + \frac{1}{2} \|x - y\|^2 + \frac{1}{2} \|g(x)\|^2 - \frac{1}{2} \|x\|^2 + \frac{1}{2} \|g(y)\|^2 - \frac{1}{2} \|y\|^2. \end{aligned} \quad (14)$$

Since  $F$  is a real Hilbert space, it follows from (12)–(14) that:

$$\begin{aligned}
 & |\langle g(x), g(y) \rangle - \langle x, y \rangle| \\
 & \leq \frac{1}{2} \left| \|g(x) - g(y)\|^2 - \|x - y\|^2 \right| + \frac{1}{2} \left| \|g(x)\|^2 - \|x\|^2 \right| + \frac{1}{2} \left| \|g(y)\|^2 - \|y\|^2 \right| \\
 & = \frac{1}{2} \left| \|g(x) - g(y)\| + \|x - y\| \right| \|g(x) - g(y)\| - \|x - y\| \\
 & \quad + \frac{1}{2} \left| \|g(x)\| + \|x\| \right| \|g(x)\| - \|x\| + \frac{1}{2} \left| \|g(y)\| + \|y\| \right| \|g(y)\| - \|y\| \\
 & \leq \frac{1}{2} (2\|x - y\| + \varphi(x, y)) \|g(x) - g(y)\| - \|x - y\| \\
 & \quad + \frac{1}{2} (2\|x\| + \varphi(x, 0)) \|g(x)\| - \|x\| + \frac{1}{2} (2\|y\| + \varphi(0, y)) \|g(y)\| - \|y\| \\
 & \leq \frac{1}{2} (2\|x - y\| + \varphi(x, y)) \varphi(x, y) + \frac{1}{2} (2\|x\| + \varphi(x, 0)) \varphi(x, 0) \\
 & \quad + \frac{1}{2} (2\|y\| + \varphi(0, y)) \varphi(0, y) \\
 & = \|x - y\| \varphi(x, y) + \|x\| \varphi(x, 0) + \|y\| \varphi(0, y) + \frac{1}{2} \varphi(x, y)^2 + \frac{1}{2} \varphi(x, 0)^2 + \frac{1}{2} \varphi(0, y)^2 \\
 & = \psi(x, y)
 \end{aligned}$$

for all  $x, y \in D_0$ .

Then it follows from the first two conditions for  $\varphi$  that  $\psi$  satisfies the condition (8) for any  $x, y \in D_0$ . Applying Theorem 5 with  $g$  and  $\psi$  instead of  $f$  and  $\varphi$ , we can complete the rest of the proof.  $\square$

When we set  $\varphi(x, y) = 0$  for  $x = 0$  or  $y = 0$ , Theorem 6 will be considerably simplified. The control function  $\varphi(x, y) = \theta \|x\|^p \|y\|^q$  was first used by Rassias [33] for approximate additive functions. The following corollary is an example of applying Theorem 6.

**Corollary 1.** Let  $E$  and  $F$  be Hilbert spaces over  $\mathbb{R}$ . Assume that  $d, p, q$ , and  $\theta$  are constants with  $d \geq 0, p \geq 1, q \geq 1$ , and  $\theta > 0$ . Moreover, let  $D = \{x \in E : \|x\| \leq d\}$ . If a function  $f : E \rightarrow F$  satisfies the following inequality:

$$\left| \|f(x) - f(y)\| - \|x - y\| \right| \leq \theta \|x\|^p \|y\|^q$$

for all  $x, y \in D$ , then there exists a unique linear isometry  $U : E \rightarrow F$  such that:

$$\|f(x) - U(x) - f(0)\| \leq \begin{cases} \frac{\sqrt{2}}{2} \theta \|x\|^{p+q} & (\text{for } p > 1 \text{ and } q > 1), \\ \sqrt{\frac{1}{2} \theta^2 \|x\|^{2q+2} + \theta \|x\|^{q+2}} & (\text{for } p = 1 \text{ and } q > 1), \\ \sqrt{\frac{1}{2} \theta^2 \|x\|^{2p+2} + \theta \|x\|^{p+2}} & (\text{for } p > 1 \text{ and } q = 1), \\ \sqrt{\frac{1}{2} \theta^2 \|x\|^4 + 2\theta \|x\|^3} & (\text{for } p = q = 1) \end{cases}$$

for all  $x \in D$ .

**Proof.** Considering the condition (6), we select a constant  $c$  satisfying  $c > 1$  and we define  $\varphi(x, y) = \theta \|x\|^p \|y\|^q$  for any  $x, y \in E$ . Then, we easily verify that:

$$\lim_{m, n \rightarrow \infty} c^{m+n} \left\| \frac{1}{c^m} x - \frac{1}{c^n} y \right\| \varphi\left(\frac{1}{c^m} x, \frac{1}{c^n} y\right) = \lim_{m, n \rightarrow \infty} \theta \frac{c^{m+n}}{c^{pm+qn}} \left\| \frac{1}{c^m} x - \frac{1}{c^n} y \right\| \|x\|^p \|y\|^q = 0$$



for all  $x, y \in D$ . Moreover, we have:

$$\lim_{m,n \rightarrow \infty} c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right)^2 = \lim_{m,n \rightarrow \infty} \theta^2 \frac{c^{m+n}}{c^{2pm+2qn}} \|x\|^{2p} \|y\|^{2q} = 0$$

for all  $x, y \in D$ . Further, with the definition of  $\psi$  given in Theorem 6, we get:

$$\lim_{n \rightarrow \infty} c^n \psi\left(\frac{1}{c^n}x, x\right) = \begin{cases} 0 & \text{(for } p > 1), \\ \theta \|x\|^{q+2} & \text{(for } p = 1) \end{cases}$$

and

$$\lim_{n \rightarrow \infty} c^n \psi\left(x, \frac{1}{c^n}x\right) = \begin{cases} 0 & \text{(for } q > 1), \\ \theta \|x\|^{p+2} & \text{(for } q = 1) \end{cases}$$

for all  $x \in D$ . Thus, according to Theorem 6, there exists a unique linear isometry  $U : E \rightarrow F$  satisfying the given inequality for all  $x \in D$ .  $\square$

Since  $\|f(x) - f(y)\| \geq 0$  and  $\|x - y\| \geq 0$  for any  $x, y \in E$ , it holds that for all  $x, y \in E$ ,

$$\begin{aligned} \|f(x) - f(y)\|^2 &= \|x - y\|^2 \\ \iff \|f(x) - f(y)\| &= \|x - y\| \quad \text{or} \quad \|f(x) - f(y)\| = -\|x - y\| \\ \iff \|f(x) - f(y)\| &= \|x - y\| \quad \text{or} \quad x = y \\ \iff \|f(x) - f(y)\| &= \|x - y\|. \end{aligned}$$

Hence, the set of solutions to the functional equation  $\|f(x) - f(y)\| = \|x - y\|$  is the same as the set of solutions to the equation  $\|f(x) - f(y)\|^2 = \|x - y\|^2$ , but in both cases stability results are different, as can be seen in Theorems 6 and 7:

**Theorem 7.** Let  $E$  and  $F$  be Hilbert spaces over  $\mathbb{R}$ . Assume that a function  $\varphi : E \times E \rightarrow [0, \infty)$  satisfies the condition (8) for all  $x, y \in D_0$ . If a function  $f : E \rightarrow F$  satisfies the inequality:

$$\left| \|f(x) - f(y)\|^2 - \|x - y\|^2 \right| \leq \varphi(x, y) \tag{15}$$

for all  $x, y \in D$ , then there exists a unique linear isometry  $U : E \rightarrow F$  such that:

$$\|f(x) - U(x) - f(0)\| \leq \sqrt{\psi(x, x) + \lim_{n \rightarrow \infty} c^n \psi\left(\frac{1}{c^n}x, x\right) + \lim_{n \rightarrow \infty} c^n \psi\left(x, \frac{1}{c^n}x\right)}$$

for all  $x \in D$ , where  $\psi(x, y) = \frac{1}{2}\varphi(x, y) + \frac{1}{2}\varphi(x, 0) + \frac{1}{2}\varphi(0, y)$  for all  $x, y \in E$ .

**Proof.** If we put  $g(x) = f(x) - f(0)$  in (15), then  $g$  also satisfies the inequality (15) for all  $x, y \in D$ . By (14) and (15) with  $g$  in place of  $f$ , we have:

$$\begin{aligned} &|\langle g(x), g(y) \rangle - \langle x, y \rangle| \\ &\leq \frac{1}{2} \left| \|g(x) - g(y)\|^2 - \|x - y\|^2 \right| + \frac{1}{2} \left| \|g(x)\|^2 - \|x\|^2 \right| + \frac{1}{2} \left| \|g(y)\|^2 - \|y\|^2 \right| \\ &\leq \frac{1}{2} \varphi(x, y) + \frac{1}{2} \varphi(x, 0) + \frac{1}{2} \varphi(0, y) \\ &= \psi(x, y) \end{aligned}$$

for any  $x, y \in D$ . Moreover, the  $\psi$  satisfies the condition (8) for all  $x, y \in D$ . According to Theorem 5, there exists a unique linear isometry  $U : E \rightarrow F$  satisfying the given inequality for all  $x \in D$ .  $\square$

Finally, the following corollary can be demonstrated by using Theorem 7.

**Corollary 2.** Let  $E$  and  $F$  be Hilbert spaces over  $\mathbb{R}$ . Assume that  $d$ ,  $p$ , and  $\varepsilon$  are constants with  $d \geq 0$ ,  $0 < p \leq 1$ , and  $\varepsilon > 0$ . Moreover, let  $D = \{x \in E : \|x\| \geq d\}$ . If a function  $f : E \rightarrow F$  satisfies the inequality:

$$|\|f(x) - f(y)\|^2 - \|x - y\|^2| \leq \varepsilon \|x - y\|^p$$

for all  $x, y \in D_0$ , then there exists a unique linear isometry  $U : E \rightarrow F$  such that:

$$\|f(x) - U(x) - f(0)\| \leq \begin{cases} \sqrt{\varepsilon \|x\|^p} & (\text{for } 0 < p < 1), \\ \sqrt{3\varepsilon \|x\|} & (\text{for } p = 1) \end{cases}$$

for all  $x \in D_0$ .

**Proof.** Considering the condition (6), we may select a constant  $c$  satisfying  $0 < c < 1$ . If we define  $\varphi(x, y) = \varepsilon \|x - y\|^p$  for all  $x, y \in E$ , then we easily show that:

$$\lim_{m, n \rightarrow \infty} c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = \lim_{m, n \rightarrow \infty} \varepsilon c^{(1-p)(m+n)} \|c^m x - c^n y\|^p = 0$$

for all  $x, y \in D_0$ . Moreover, with the definition of  $\psi$  given in Theorem 7, we obtain:

$$\lim_{n \rightarrow \infty} c^n \psi\left(\frac{1}{c^n}x, x\right) = \begin{cases} 0 & (\text{for } 0 < p < 1), \\ \varepsilon \|x\| & (\text{for } p = 1) \end{cases}$$

and

$$\lim_{n \rightarrow \infty} c^n \psi\left(x, \frac{1}{c^n}x\right) = \begin{cases} 0 & (\text{for } 0 < p < 1), \\ \varepsilon \|x\| & (\text{for } p = 1) \end{cases}$$

for all  $x \in D_0$ . Finally, according to Theorem 7, there exists a unique linear isometry  $U : E \rightarrow F$  such that the given inequality holds for all  $x \in D$ . Obviously, this inequality holds also for  $x = 0$ .  $\square$

We may compare Corollary 2 with Theorem 5 of [32], in which the stability of the Equation (1) was proved under the assumptions that  $D = \{x \in E : \|x\| \leq d\}$  and  $p > 1$ .

## 5. Discussion

This paper focused on stability in the restricted domain of isometries. Most other papers address the stability problem of isometries defined in the whole space. This is the difference between this paper and most other papers.

Fickett's Theorem (Theorem 1) provided an important motivation for writing this paper: The upper bound associated with inequality in Fickett's theorem becomes very large for any sufficiently small  $\varepsilon$  in comparison to  $\varepsilon$ . This is a big drawback of Fickett's theorem. Thus, the work of further improving Fickett's theorem has to be attractive. Although this paper did not achieve that goal completely, we wrote this paper as part of an attempt to further improve the theorem of Fickett.

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