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Topological Indices and f-Polynomials on Some Graph Products

Ricardo Abreu-Blaya [1], Sergio Bermudo [2], José M. Rodríguez [3] and Eva Tourís [4,*]

[1] Facultad de Matemáticas, Universidad Autónoma de Guerrero, Ciudad Universitaria, Avenida Lázaro Cárdenas, S/N, Chilpancingo 39087, Guerrero, Mexico; rabreublaya@yahoo.es
[2] Departamento de Economía, Métodos Cuantitativos e Historia Económica, Universidad Pablo de Olavide, Carretera de Utrera Km. 1, 41013 Sevilla, Spain; sbernav@upo.es
[3] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain; jomaro@math.uc3m.es
[4] Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain
* Correspondence: eva.touris@uam.es; Tel.: +34-497-68-96

Abstract: We obtain inequalities involving many topological indices in classical graph products by using the f-polynomial. In particular, we work with lexicographic product, Cartesian sum and Cartesian product, and with first Zagreb, forgotten, inverse degree and sum lordeg indices.

Keywords: first Zagreb index; forgotten index; inverse degree index; sum lordeg index; lexicographic product; Cartesian sum; Cartesian product; polynomials in graphs

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1. Introduction

Chemical compounds (like hydrocarbons) can be represented by means of graphs. A topological index T is just a number that encapsulates some property of graphs and such that T correlates with a certain molecular characteristic; therefore, it can be employed to grasp chemical properties and physical properties of chemical substances. They play an important role in mathematical chemistry, in particular, in the QSPR/QSAR (quantitative structure-property relationship/quantitative structure-activity relationship) investigations. Computational and mathematical properties of topological indices have been studied in depth for more than 50 years (see, e.g., [1–8] and the references therein). In particular, a main subject on this field is to obtain sharp bounds of topological indices.

See [9] for a review in a dialog manner discussing relevance of topological descriptors to chemical/physical properties.

One of the main topological indices is the following index, called *Randić index*, defined in [1] by

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}},$$

where G is a graph and d_w is the degree of the vertex $w \in V(G)$.

Along the paper, $G = (V(G), E(G))$ will denote a (non-oriented) simple (without loops and multiple edges loops) finite graph without isolated vertices. Hence, every vertex has degree at least 1.

Two of the most popular alternatives to the Randić index are the *first Zagreb and second Zagreb indices*, denoted by M_1 and M_2, respectively, and defined by

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = \sum_{u \in V(G)} d_u^2, \qquad M_2(G) = \sum_{uv \in E(G)} d_u d_v.$$

These two indexes are very useful in mathematical chemistry and so, they have been extensively studied, see [10–13] and the references therein. Further development of Zagreb-type indices deals with the applications to more complex chemical objects, e.g., large carbon-based species with regular structures such as polycyclic aromatic hydrocarbons [14] and carbon nanostructures [15].

Please note that there are topological indices of different types. They may treat only vertices, only edges, or both edges and vertices of the graph to calculate an index. Thus, the first Zagreb index belongs to the first class (every index in the first class, as the first Zagreb index, also belongs to the second class).

Along this paper we obtain results for topological indices in this first class.

The so-called *harmonic index*, is defined in [16] by

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_u + d_v}\,.$$

For more information about the properties of that index we refer to [17–22], and the book [23].

The *inverse degree index* ID is defined as

$$ID(G) = \sum_{u\in V(G)} \frac{1}{d_u} = \sum_{uv\in E(G)} \Big(\frac{1}{d_u^2} + \frac{1}{d_v^2}\Big).$$

This index first attracted attention through many conjectures obtained by the computer programme called Graffiti [16]. Since then, several authors have studied its connections with other parameters of graphs: diameter, matching number, edge-connectivity, Wiener index, etc.; also, its chemical applications have been studied by many researchers (see [24–29]).

In [30–32] the *general first Zagreb and general second Zagreb indices* are defined by

$$M_1^\alpha(G) = \sum_{u\in V(G)} d_u^\alpha, \qquad M_2^\alpha(G) = \sum_{uv\in E(G)} (d_u d_v)^\alpha.$$

Please note that the first Zagreb index M_1 is M_1^2, the inverse degree index ID is M_1^{-1}, the forgotten index F is $M_1^3, \ldots$; moreover, the Randić index R is $M_2^{-1/2}$, the second Zagreb index M_2 is M_2^1, the modified Zagreb index is $M_2^{-1}, \ldots$.

The variable topological indices were introduced as a new way of characterizing heteroatoms (see [33,34]), and to assess the structural differences (see [35]). The idea behind the variable topological indices is that the parameter is determined during the regression in such a way that the error of estimate for a fixed property is minimized.

The *sum lordeg index* was introduced in [36]. It is defined as

$$SL(G) = \sum_{u\in V(G)} d_u \sqrt{\log d_u}\,.$$

This index is interesting from an applied viewpoint since it correlates very well with the octanol-water partition coefficient for octane isomers [36], and so, it appears in numerical packages for the computation of topological indices [37]. For these reasons, in [38] is stated the open problem of finding appropriate bounds for this index.

In [39] the *harmonic polynomial* is introduced as

$$H(G,x) = \sum_{uv\in E(G)} x^{d_u + d_v - 1}.$$

In [40–42] several properties of the harmonic polynomial are obtained. Please note that $2\int_0^1 H(G,x)\,dx = H(G)$.

In [41] the *inverse degree polynomial* is introduced as

$$ID(G,x) = \sum_{u \in V(G)} x^{d_u - 1}.$$

It should be noticed that $\int_0^1 ID(G,x)\,dx = ID(G)$. Thus, the inverse degree polynomial $ID(G,x)$ can be used to obtain information about the inverse degree index $ID(G)$ of a graph G.

Given any function $f : \mathbb{Z}^+ \to \mathbb{R}^+$, let us define the *$f$-index*

$$I_f(G) = \sum_{u \in V(G)} f(d_u),$$

and the *f-polynomial* of G by

$$P_f(G,x) = \sum_{u \in V(G)} x^{1/f(d_u)-1},$$

if $x > 0$. Also, let us define $P_f(G,0) = \lim_{x \to 0^+} P_f(G,x)$. In particular, $P_f(G,x) = ID(G,x)$ if $f(t) = 1/t$. It is clear that $\int_0^1 P_f(G,x)\,dx = I_f(G)$.

Please note that many important indices can be obtained from I_f by choosing appropriate functions f: if $f(t) = t^2$, then I_f is the first Zagreb index; if $f(t) = t^{-1}$, then I_f is the inverse degree index ID; if $f(t) = t^3$, then I_f is the forgotten index F; in general, if $f(t) = t^\alpha$, then I_f is the general first Zagreb index M_1^α; if $f(t) = t\sqrt{\log t}$, then I_f is the sum lordeg index SL. Thus, each theorem in this paper about I_f is a result for each one of these indices.

The f-polynomial of other graph operations (e.g., join, corona product, Mycielskian and line) is studied in [43].

Operations on graphs play an important role in Mathematical Chemistry, see e.g., [44,45], since many chemical structures appear as operations of graphs: The crystal structure of sodium chloride is the Cartesian product of two path graphs. The kernel of the iron crystal structure is the join of the cube graph Q_3 and an isolated vertex. The cyclobutane is the Cartesian product of two path graphs P_2. The alkane C_3H_6 can be represented as the corona product of the path graph P_3 and the null graph N_2. The cyclohexane C_6H_{12} can be represented as the corona product of the cycle graph C_6 and the null graph N_2. The carbon nanotube $TUC_4(m,n)$ can be seen as the Cartesian product of the path graph P_3 and the cycle graph C_5. The fence (respectively, the closed fence) is the lexicographic product of P_5 and P_2 (respectively, C_5 and P_2). The zigzag polyhex nanotube $TUHC_6[2n,2]$ is the generalized hierarchical product of the path graph P_2 and the cycle graph C_{2n}. See [46,47].

Polynomials, in general, have lately proved to be useful in graph theory and, in particular, in Mathematical Chemistry (see [41,43,45,48–55]).

The main goal in this paper is to obtain information of many topological indices (each case of I_f for some particular choice of f) of several graph products, from the information on topological indices of these factors, which are much easier to calculate than the products. Our approach is to obtain information about the corresponding f-polynomials, which are easy to calculate (see for instance Theorems 1, 11 and 17); then, we can deduce information on the I_f index by using the formula $\int_0^1 P_f(G,x)\,dx = I_f(G)$ (see for instance Theorems 4, 13 and 18). This is a good approach since the bounds of the f-polynomial of a product of two graphs allow use of analytic tools to bound the I_f index of such a graph product, simplifying the proofs.

2. Background

The study of the effect of graph operations on topological indices is an active topic of research (see, e.g., [41,54–56]). We study in this section the f-polynomial of several graph products: lexicographic product, Cartesian sum and Cartesian product.

The *Cartesian product* $G_1 \times G_2$ of the graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_l)$ is an edge of $G_1 \times G_2$ if $u_i = u_k$ and $v_j v_l \in E(G_2)$, or $u_i u_k \in E(G_1)$ and $v_j = v_l$.

The *lexicographic product* $G_1 \odot G_2$ of the graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices $(u_i, v_j), (u_k, v_l)$ of $V(G_1 \odot G_2)$ are adjacent if either $u_i u_k \in E(G_1)$, or $u_i = u_k$ and $v_j v_l \in E(G_2)$.

The *Cartesian sum* $G_1 \oplus G_2$ of the graphs G_1 and G_2 has the vertex set $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_l)$ is an edge of $G_1 \oplus G_2$ if $u_i u_k \in E(G_1)$ or $v_j v_l \in E(G_2)$.

In [57] is defined the following *Zagreb polynomial*

$$M_1^*(G, x) := \sum_{u \in V(G)} d_u x^{d_u}.$$

Please note that $x(xID(G, x))' = M_1^*(G, x)$.

In ([43], Propositions 1, 2 and 3) appear the following useful results.

Proposition 1. *If G is a graph of order n and $f : \mathbb{Z}^+ \to \mathbb{R}^+$, then:*

- *$P_f(G, x)$ is a polynomial if and only if $1/f(d_u) \in \mathbb{Z}^+$ for every $u \in V(G)$,*
- *$P_f(G, x)$ is a positive C^∞ function on $(0, \infty)$,*
- *$P_f(G, x)$ is a continuous function on $[0, \infty)$ if and only if $P_f(G, 0) < \infty$,*
- *$P_f(G, x)$ is a continuous function on $[0, \infty)$ if and only if $f(d_u) \le 1$ for every $u \in V(G)$,*
- *$P_f(G, x)$ is an integrable function on $[0, A]$ for every $A > 0$, and $\int_0^1 P_f(G, x)\, dx = I_f(G)$,*
- *$P_f(G, x)$ is increasing on $(0, \infty)$ if and only if $f(d_u) \le 1$ for every $u \in V(G)$,*
- *$P_f(G, x)$ is strictly increasing on $(0, \infty)$ if and only if $f(d_u) \le 1$ for every $u \in V(G)$, and $f(d_v) \neq 1$ for some $v \in V(G)$,*
- *$P_f(G, x)$ is convex on $(0, \infty)$ if $f(d_u) \in (0, 1/2] \cup [1, \infty)$ for every $u \in V(G)$,*
- *$P_f(G, x)$ is strictly convex on $(0, \infty)$ if $f(d_u) \in (0, 1/2] \cup [1, \infty)$ for every $u \in V(G)$, and $f(d_v) \notin \{1/2, 1\}$ for some $v \in V(G)$,*
- *$P_f(G, x)$ is concave on $(0, \infty)$ if $f(d_u) \in [1/2, 1]$ for every $u \in V(G)$,*
- *$P_f(G, x)$ is strictly concave on $(0, \infty)$ if $f(d_u) \in [1/2, 1]$ for every $u \in V(G)$, and $f(d_v) \notin \{1/2, 1\}$ for some $v \in V(G)$,*
- *$P_f(G, 1) = n$.*

If $\alpha \in \mathbb{R}$ and $f(t) = t^\alpha$, then Proposition 1 gives $\int_0^1 P_f(G, x)\, dx = M_1^\alpha(G)$.

Proposition 2. *If G is a k-regular graph with n vertices and $f : \mathbb{Z}^+ \to \mathbb{R}^+$, then $P_f(G, x) = n x^{1/f(k)-1}$.*

Next, we show the polynomial P_f for some important graphs: K_n (complete graph), C_n (cycle graph), Q_n (hypercube graph), K_{n_1,n_2} (complete bipartite graph), S_n (star graph), P_n (path graph), and W_n (wheel graph).

Proposition 3. *If $f : \mathbb{Z}^+ \to \mathbb{R}^+$, then*

$$P_f(K_n, x) = n x^{1/f(n-1)-1}, \qquad P_f(C_n, x) = n x^{1/f(2)-1},$$
$$P_f(Q_n, x) = 2^n x^{1/f(n)-1}, \qquad P_f(K_{n_1,n_2}, x) = n_1 x^{1/f(n_2)-1} + n_2 x^{1/f(n_1)-1},$$
$$P_f(S_n, x) = x^{1/f(n-1)-1} + (n-1) x^{1/f(1)-1}, \qquad P_f(P_n, x) = (n-2) x^{1/f(2)-1} + 2 x^{1/f(1)-1},$$
$$P_f(W_n, x) = x^{1/f(n-1)-1} + (n-1) x^{1/f(3)-1}.$$

Choose $\delta \in \mathbb{Z}^+$ and $f : \mathbb{Z}^+ \to \mathbb{R}^+$. f satisfies the *δ-additive property* 1 ($f \in AP_1(\delta)$) if

$$\frac{1}{f(a+b)} \geq \frac{1}{f(a)} + \frac{1}{f(b)}$$

for every $a, b \in \mathbb{Z}^+$ with $a, b \geq \delta$.

f satisfies the *δ-additive property* 2 ($f \in AP_2(\delta)$) if

$$\frac{1}{f(a+b)} \leq \frac{1}{f(a)} + \frac{1}{f(b)}$$

for every $a, b \in \mathbb{Z}^+$ with $a, b \geq \delta$.

f satisfies the *δ-additive property* 3 ($f \in AP_3(\delta)$) if

$$\frac{1}{f(a+b)} \leq \min\left\{\frac{1}{f(a)}, \frac{1}{f(b)}\right\}$$

for every $a, b \in \mathbb{Z}^+$ with $a, b \geq \delta$.

3. Inequalities for Cartesian Products

First, we prove pointwise inequalities of $P_f(G_1 \times G_2, x)$ involving $P_f(G_1, x)$ and $P_f(G_2, x)$.

Theorem 1. *Let $\delta \in \mathbb{Z}^+$, and let G_1 and G_2 be two graphs with minimum degree at least δ. For $x \in (0,1]$, the f-polynomial of the Cartesian product $G_1 \times G_2$ satisfies.*

(1) *If $f \in AP_1(\delta)$, then*

$$P_f(G_1 \times G_2, x) \leq x\, P_f(G_1, x)\, P_f(G_2, x).$$

(2) *If $f \in AP_2(\delta)$, then*

$$P_f(G_1 \times G_2, x) \geq x\, P_f(G_1, x)\, P_f(G_2, x).$$

(3) *If $f \in AP_3(\delta)$ and G_1 and G_2 have n_1 and n_2 vertices, respectively, then*

$$P_f(G_1 \times G_2, x) \geq \max\left\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\right\}.$$

Proof. If $(u, v) \in V(G_1 \times G_2)$, we have $d_{(u,v)} = d_u + d_v$ and its corresponding monomial of the f-polynomial is

$$x^{1/f(d_u+d_v)-1}.$$

Assume that $f \in AP_1(\delta)$. Since $d_u \geq \delta$ for every $u \in V(G_1) \cup V(G_2)$, $f \in AP_1(\delta)$ and $x \in (0,1]$,

$$\begin{aligned} P_f(G_1 \times G_2, x) &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} x^{1/f(d_u+d_v)-1} \leq \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} x^{1/f(d_u)+1/f(d_v)-1} \\ &= x \sum_{u \in V(G_1)} x^{1/f(d_u)-1} \sum_{v \in V(G_2)} x^{1/f(d_v)-1} = x\, P_f(G_1, x)\, P_f(G_2, x). \end{aligned}$$

If $f \in AP_2(\delta)$, then by a quite similar argument the result is obtained.

If $f \in AP_3(\delta)$, then

$$\begin{aligned} P_f(G_1 \times G_2, x) &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} x^{1/f(d_u+d_v)-1} \geq \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} x^{1/f(d_u)-1} \\ &= \sum_{u \in V(G_1)} x^{1/f(d_u)-1} \sum_{v \in V(G_2)} 1 = n_2 P_f(G_1, x). \end{aligned}$$

The inequality involving $P_f(G_2,x)$ is obtained in a similar way. □

Remark 1. *If $\delta = 1$, then the condition G_1 and G_2 have minimum degree at least δ, is satisfied for every graph G_1, G_2.*

Theorem 1 has the following consequence when we consider $f(t) = t^{\alpha}$.

Theorem 2. *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively, $\alpha \in \mathbb{R}$ and $f(t) = t^{\alpha}$. For $x \in (0,1]$, the f-polynomial of the Cartesian product $G_1 \times G_2$ satisfies.*

(1) *If $\alpha \le -1$, then $f \in AP_1(1)$ and*

$$P_f(G_1 \times G_2, x) \le x\, P_f(G_1,x)\, P_f(G_2,x).$$

(2) *If $\alpha \in [-1,0]$, then $f \in AP_2(1)$ and*

$$P_f(G_1 \times G_2, x) \ge x\, P_f(G_1,x)\, P_f(G_2,x).$$

(3) *If $\alpha \ge 0$, then $f \in AP_3(1)$ and*

$$P_f(G_1 \times G_2, x) \ge \max\left\{ n_2 P_f(G_1,x),\, n_1 P_f(G_2,x) \right\}.$$

Proof. The inequalities are consequences of the following facts and Theorem 1.

If $\alpha \le -1$, then $-\alpha \ge 1$ and

$$\frac{1}{f(x+y)} = (x+y)^{-\alpha} \ge x^{-\alpha} + y^{-\alpha} = \frac{1}{f(x)} + \frac{1}{f(y)}$$

for every $x, y > 0$, and so, $f \in AP_1(1)$.

If $\alpha \in [-1,0]$, then $-\alpha \in [0,1]$ and

$$\frac{1}{f(x+y)} = (x+y)^{-\alpha} \le x^{-\alpha} + y^{-\alpha} = \frac{1}{f(x)} + \frac{1}{f(y)}$$

for every $x, y > 0$, and so, $f \in AP_2(1)$.

If $\alpha \ge 0$, then $1/t^{\alpha}$ is a decreasing function on $(0,\infty)$ and

$$\frac{1}{f(x+y)} = \frac{1}{(x+y)^{\alpha}} \le \min\left\{ \frac{1}{x^{\alpha}}, \frac{1}{y^{\alpha}} \right\} = \min\left\{ \frac{1}{f(x)}, \frac{1}{f(y)} \right\}$$

for every $x, y > 0$, and so, $f \in AP_3(1)$. □

Theorem 2 yields for the inverse degree polynomial:

Corollary 1. *Let G_1 and G_2 be two graphs, the ID polynomial of the Cartesian product $G_1 \times G_2$ is*

$$ID(G_1 \times G_2, x) = x\, ID(G_1,x)\, ID(G_2,x).$$

Since $f(t) = t\sqrt{\log t}$ is an increasing function on $[1,\infty)$, $f \in AP_3(2)$ and Theorem 1 has the following consequence. Recall that a vertex with degree 1 is called *pendant vertex*. Please note that $f(t) = t\sqrt{\log t}$ is a positive function on $\mathbb{Z}^+ \setminus \{1\}$, and so, $P_f(G,x)$ is well-defined for every graph G without pendant vertices.

Please note that a graph has minimum degree at least 2 if and only if it does not have pendant vertices.

Theorem 3. *Let G_1 and G_2 be two graphs with minimum degree at least 2 and of order n_1 and n_2, respectively. If $f(t) = t\sqrt{\log t}$, then $f \in AP_3(2)$ and the f-polynomial of the Cartesian product $G_1 \times G_2$ satisfies for $x \in (0,1]$*

$$P_f(G_1 \times G_2, x) \ge \max\left\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\right\}.$$

Next, we obtain bounds for $I_f(G_1 \times G_2)$ by using the previous inequalities for $P_f(G_1 \times G_2, x)$. This is a good approach since the pointwise bounds of $P_f(G_1 \times G_2, x)$ allow use of analytic tools to bound $I_f(G_1 \times G_2)$. We start with the case $f \in AP_3(\delta)$.

Theorem 4. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of order n_1 and n_2, respectively, and minimum degree at least δ. If $f \in AP_3(\delta)$, then*

$$I_f(G_1 \times G_2) \ge \max\left\{n_2 I_f(G_1),\, n_1 I_f(G_2)\right\}.$$

Proof. Theorem 1 gives

$$P_f(G_1 \times G_2, x) \ge \max\left\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\right\},$$

for every $x \in (0,1]$. Thus, Proposition 1 leads to

$$I_f(G_1 \times G_2) = \int_0^1 P_f(G_1 \times G_2, x)\, dx \ge n_2 \int_0^1 P_f(G_1, x)\, dx = n_2 I_f(G_1).$$

The same argumentation gives $I_f(G_1 \times G_2) \ge n_1 I_f(G_2)$. □

To deal with the cases $f \in AP_1(\delta)$ and $f \in AP_2(\delta)$, we need some technical results.

Lemma 1. *[58] If $f_1, \ldots, f_k$ are non-negative convex functions on $[a,b]$, then*

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^k f_i(x)\, dx \ge \frac{2^k}{k+1} \prod_{i=1}^k \frac{1}{b-a}\int_a^b f_i(x)\, dx.$$

Lemma 1 can be slightly improved as follows.

Proposition 4. *If $f_1, \ldots, f_k$ are convex non-negative functions on (a,b), then*

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^k f_i(x)\, dx \ge \frac{2^k}{k+1} \prod_{i=1}^k \frac{1}{b-a}\int_a^b f_i(x)\, dx.$$

Proof. If $0 < \varepsilon < (b-a)/2$, then $f_1, \ldots, f_k$ are convex non-negative functions on $[a+\varepsilon, b-\varepsilon]$, and Lemma 1 gives

$$\frac{1}{b-a-2\varepsilon}\int_{a+\varepsilon}^{b-\varepsilon} \prod_{i=1}^k f_i(x)\, dx \ge \frac{2^k}{k+1} \prod_{i=1}^k \frac{1}{b-a-2\varepsilon}\int_{a+\varepsilon}^{b-\varepsilon} f_i(x)\, dx.$$

Since $f_1, \ldots, f_k$ are non-negative on (a,b), the functions $F, G : (0, (b-a)/2) \to [0, \infty)$ defined by

$$F(\varepsilon) = \int_{a+\varepsilon}^{b-\varepsilon} \prod_{i=1}^k f_i(x)\, dx, \qquad G(\varepsilon) = \prod_{i=1}^k \int_{a+\varepsilon}^{b-\varepsilon} f_i(x)\, dx,$$

are decreasing, and so, there exist their limits as $\varepsilon \to 0^+$ (although they can be ∞). By taking $\varepsilon \to 0^+$ in the above inequality, we obtain the result. □

Lemma 2. *([59], Corollary 5.2) If $f_1, \dots, f_k$ are convex non-negative functions on $[a,b]$, then*

$$\int_a^b \prod_{i=1}^k f_i(x)\,dx \le \frac{2}{k+1}\left(\prod_{i=1}^k \int_a^b f_i(x)\,dx\right)^{1/k}\left(\prod_{i=1}^k \big(f_i(a)+f_i(b)\big)\right)^{1-1/k}.$$

Since any non-negative concave function on (a,b) has finite lateral limits at a and b, ([59], Corollary 4.3) can be stated as follows.

Lemma 3. *If f_1, f_2 are non-negative concave functions on (a,b), then*

$$\begin{aligned}\frac{2}{3}\,\frac{1}{b-a}\int_a^b f_1(x)\,dx\,\frac{1}{b-a}\int_a^b f_2(x)\,dx &\le \frac{1}{b-a}\int_a^b f_1(x)f_2(x)\,dx\\ &\le \frac{4}{3}\,\frac{1}{b-a}\int_a^b f_1(x)\,dx\,\frac{1}{b-a}\int_a^b f_2(x)\,dx.\end{aligned}$$

With these technical results, we can deal now with the cases $f \in AP_1(\delta)$ and $f \in AP_2(\delta)$.

Theorem 5. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be graphs of orders n_1 and n_2, respectively, and minimum degree at least δ. If the f-polynomials of G_1 and G_2 are convex functions on $(0,1)$, then*
(1) If $f \in AP_1(\delta)$, then

$$I_f(G_1 \times G_2) \le \frac{1}{2}\left(\frac{1}{2}\big(n_1 + P_f(G_1,0)\big)^2\big(n_2 + P_f(G_2,0)\big)^2 I_f(G_1)\, I_f(G_2)\right)^{1/3}.$$

(2) If $f \in AP_2(\delta)$, then

$$I_f(G_1 \times G_2) \ge I_f(G_1)\, I_f(G_2).$$

Proof. Assume first that $f \in AP_1(\delta)$. Theorem 1 gives

$$P_f(G_1 \times G_2, x) \le x\, P_f(G_1,x)\, P_f(G_2,x),$$

for every $x \in (0,1]$.
If $P_f(G_1,0) = \infty$ or $P_f(G_2,0) = \infty$, then (1) trivially holds.
If $P_f(G_1,0) < \infty$ and $P_f(G_2,0) < \infty$, then $P_f(G_1,x)$ and $P_f(G_2,x)$ are continuous functions on $[0,1]$ by Proposition 1, and so, they are convex on $[0,1]$. Proposition 1 and Lemma 2 give

$$\begin{aligned}I_f(G_1 \times G_2) &= \int_0^1 P_f(G_1 \times G_2, x)\,dx \le \int_0^1 x\,P_f(G_1,x)\,P_f(G_2,x)\,dx\\ &\le \frac{1}{2}\left(\int_0^1 x\,dx \int_0^1 P_f(G_1,x)\,dx \int_0^1 P_f(G_2,x)\,dx\right)^{1/3}\\ &\quad\cdot\Big((0+1)\big(P_f(G_1,0)+P_f(G_1,1)\big)\big(P_f(G_2,0)+P_f(G_2,1)\big)\Big)^{2/3}\\ &= \frac{1}{2}\left(\frac{1}{2}\big(n_1 + P_f(G_1,0)\big)^2\big(n_2 + P_f(G_2,0)\big)^2 I_f(G_1)\, I_f(G_2)\right)^{1/3}.\end{aligned}$$

Assume now that $f \in AP_2(\delta)$. Theorem 1 gives

$$P_f(G_1 \times G_2, x) \ge x\, P_f(G_1,x)\, P_f(G_2,x),$$

for every $x \in (0,1]$. Since $P_f(G_1,x)$ and $P_f(G_2,x)$ are convex functions on $(0,1)$, Proposition 1 and Proposition 4 give

$$\begin{aligned} I_f(G_1 \times G_2) &= \int_0^1 P_f(G_1 \times G_2, x)\, dx \geq \int_0^1 x\, P_f(G_1,x)\, P_f(G_2,x)\, dx \\ &\geq 2 \int_0^1 x\, dx \int_0^1 P_f(G_1,x)\, dx \int_0^1 P_f(G_2,x)\, dx \\ &= I_f(G_1)\, I_f(G_2). \end{aligned}$$

□

If the f-polynomials of G_1 and G_2 are concave functions on $(0,1)$, we can obtain simpler bounds for $I_f(G_1 \times G_2)$.

Theorem 6. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs with minimum degree at least δ. If $f \in AP_1(\delta)$ and the f-polynomials of G_1 and G_2 are concave functions on $(0,1)$, then*

$$I_f(G_1 \times G_2) \leq \frac{4}{3}\, I_f(G_1)\, I_f(G_2).$$

Proof. Since $f \in AP_1(\delta)$, Theorem 1 gives

$$P_f(G_1 \times G_2, x) \leq x\, P_f(G_1,x)\, P_f(G_2,x).$$

for every $x \in (0,1]$. Since $P_f(G_1,x)$ and $P_f(G_2,x)$ are concave functions on $(0,1)$, a simple combination of Proposition 1 and Lemma 3 yields

$$\begin{aligned} I_f(G_1 \times G_2) &= \int_0^1 P_f(G_1 \times G_2, x)\, dx \leq \int_0^1 x\, P_f(G_1,x)\, P_f(G_2,x)\, dx \\ &\leq \int_0^1 P_f(G_1,x)\, P_f(G_2,x)\, dx \leq \frac{4}{3} \int_0^1 P_f(G_1,x)\, dx \int_0^1 P_f(G_2,x)\, dx \\ &= \frac{4}{3}\, I_f(G_1)\, I_f(G_2). \end{aligned}$$

□

We can obtain more bounds for $I_f(G_1 \times G_2)$ if the image of f does not intersect $(a/2, a)$ for some constant $a > 0$.

Theorem 7. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ, and $a > 0$. If $f : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, a/2] \cup [a, \infty)$, then*

(1) *If $f \in AP_1(\delta)$, then*

$$I_f(G_1 \times G_2) \leq \frac{1}{2} \Big(\frac{a}{2} \big(n_1 + P_{f/a}(G_1, 0)\big)^2 \big(n_2 + P_{f/a}(G_2, 0)\big)^2 I_f(G_1)\, I_f(G_2) \Big)^{1/3}.$$

(2) *If $f \in AP_2(\delta)$, then*

$$I_f(G_1 \times G_2) \geq \frac{1}{a}\, I_f(G_1)\, I_f(G_2).$$

Proof. Let be the function $g = f/a$. Hence, $g : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, 1/2] \cup [1, \infty)$ and Proposition 1 gives that $P_g(G_1, x)$ and $P_g(G_2, x)$ are convex functions on $(0, \infty)$.

If $f \in AP_1(\delta)$, then $f/a \in AP_1(\delta)$ and Theorem 5 gives

$$\begin{aligned}\frac{1}{a}\, I_f(G_1 \times G_2) &= I_{f/a}(G_1 \times G_2) \\ &\le \frac{1}{2}\Big(\frac{1}{2}\big(n_1 + P_{f/a}(G_1,0)\big)^2\big(n_2 + P_{f/a}(G_2,0)\big)^2 I_{f/a}(G_1)\, I_{f/a}(G_2)\Big)^{1/3} \\ &= \frac{1}{2}\Big(\frac{1}{2}\big(n_1 + P_{f/a}(G_1,0)\big)^2\big(n_2 + P_{f/a}(G_2,0)\big)^2 \frac{1}{a}\, I_f(G_1)\, \frac{1}{a}\, I_f(G_2)\Big)^{1/3}.\end{aligned}$$

If $f \in AP_2(\delta)$, then $f/a \in AP_2(\delta)$ and Theorem 5 gives

$$\frac{1}{a}\, I_f(G_1 \times G_2) = I_{f/a}(G_1 \times G_2) \ge I_{f/a}(G_1)\, I_{f/a}(G_2) = \frac{1}{a}\, I_f(G_1)\, \frac{1}{a}\, I_f(G_2).$$

□

The first inequality in Theorem 7 can be improved as follows.

Theorem 8. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ, and $a > 0$. If $f : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, a/2]$, and $f \in AP_1(\delta)$, then*

$$I_f(G_1 \times G_2) \le \frac{1}{2}\Big(\frac{a}{2}\, n_1^2 n_2^2\, I_f(G_1)\, I_f(G_2)\Big)^{1/3}.$$

Proof. Theorem 7 gives

$$I_f(G_1 \times G_2) \le \frac{1}{2}\Big(\frac{a}{2}\big(n_1 + P_{f/a}(G_1,0)\big)^2\big(n_2 + P_{f/a}(G_2,0)\big)^2 I_f(G_1)\, I_f(G_2)\Big)^{1/3}.$$

Since $f \le a/2$ on $\mathbb{Z}^+ \cap [\delta, \infty)$, we have $a/f - 1 \ge 1$ and $P_{f/a}(G_1,0) = P_{f/a}(G_2,0) = 0$, and so, we obtain the result. □

Theorems 2, 8 (with $a = 2$), 7 (with $a = 2$) and 4 have the following consequence for the general first Zagreb index.

Theorem 9. *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and $\alpha \in \mathbb{R}$.*
(1) *If $\alpha \le -1$, then*

$$M_1^{\alpha}(G_1 \times G_2) \le \frac{1}{2}\big(n_1^2 n_2^2\, M_1^{\alpha}(G_1)\, M_1^{\alpha}(G_2)\big)^{1/3}.$$

(2) *If $\alpha \in [-1, 0]$, then*

$$M_1^{\alpha}(G_1 \times G_2) \ge \frac{1}{2}\, M_1^{\alpha}(G_1)\, M_1^{\alpha}(G_2).$$

(3) *If $\alpha \ge 0$, then*

$$M_1^{\alpha}(G_1 \times G_2) \ge \max\big\{n_2 M_1^{\alpha}(G_1),\, n_1 M_1^{\alpha}(G_2)\big\}.$$

As a rather simple consequence of the above theorem, we have for the first Zagreb, forgotten and inverse degree indices:

Corollary 2. *If G_1 and G_2 are two graphs with n_1 and n_2 vertices, respectively, then*

$$\begin{aligned} M_1(G_1 \times G_2) &\ge \max\big\{n_2 M_1(G_1),\, n_1 M_1(G_2)\big\}, \\ F(G_1 \times G_2) &\ge \max\big\{n_2 F(G_1),\, n_1 F(G_2)\big\}, \\ \frac{1}{2}\, ID(G_1)\, ID(G_2) \le ID(G_1 \times G_2) &\le \frac{1}{2}\big(n_1^2 n_2^2\, ID(G_1)\, ID(G_2)\big)^{1/3}. \end{aligned}$$

Since $f(t) = t\sqrt{\log t} \in AP_3(2)$, Theorem 4 gives the following result for the SL index.

Theorem 10. *If G_1 and G_2 are graphs without pendant vertices and with n_1 and n_2 vertices, respectively, then*

$$SL(G_1 \times G_2) \geq \max\{n_2 SL(G_1),\, n_1 SL(G_2)\}.$$

A particular consequence of Theorems 9 and 10 is the following.

Corollary 3. *If C_{n_1} and C_{n_1} are the cycle graphs with n_1 and n_2 vertices, respectively, then*

$$\begin{aligned} M_1(C_{n_1} \times C_{n_2}) &\geq 4n_1n_2, \\ F(C_{n_1} \times C_{n_2}) &\geq 8n_1n_2, \\ \frac{1}{8}\, n_1n_2 \leq ID(C_{n_1} \times C_{n_2}) &\leq \frac{1}{2^{5/3}}\, n_1n_2, \\ SL(C_{n_1} \times C_{n_2}) &\geq n_1n_2 2\log 2. \end{aligned}$$

4. Inequalities for Lexicographic Products

We start this section by proving pointwise inequalities of $P_f(G_1 \odot G_2, x)$ involving the f-polynomials of G_1 and G_2.

Theorem 11. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ. The f-polynomial of the lexicographic product $G_1 \odot G_2$ satisfies the following inequalities for $x \in (0,1]$.*

(1) *If $f \in AP_1(\delta)$, then*

$$P_f(G_1 \odot G_2, x) \leq x^{n_2} P_f(G_1, x^{n_2}) P_f(G_2, x).$$

(2) *If $f \in AP_2(\delta)$, then*

$$P_f(G_1 \odot G_2, x) \geq x^{n_2} P_f(G_1, x^{n_2}) P_f(G_2, x).$$

(3) *If $f \in AP_3(\delta)$, then*

$$P_f(G_1 \odot G_2, x) \geq \max\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\}.$$

Proof. If $(u, v) \in V(G_1 \odot G_2)$, then $d_{(u,v)} = n_2 d_u + d_v$.

Assume that $f \in AP_1(\delta)$. Since $d_u \geq \delta$ for every $u \in V(G_1) \cup V(G_2)$, one can prove by induction that

$$\frac{1}{f(n_2 d_u + d_v)} \geq \frac{n_2}{f(d_u)} + \frac{1}{f(d_v)}.$$

Since $x \in (0, 1]$,

$$\begin{aligned} P_f(G_1 \odot G_2, x) &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} x^{1/f(n_2 d_u + d_v) - 1} \\ &\leq \sum_{u \in V(G_1)} x^{n_2/f(d_u)} \sum_{v \in V(G_2)} x^{1/f(d_v) - 1} \\ &= \sum_{u \in V(G_1)} \left(x^{n_2}\right)^{1/f(d_u) - 1} x^{n_2} P_f(G_2, x) \\ &= x^{n_2} P_f(G_1, x^{n_2}) P_f(G_2, x). \end{aligned}$$

If $f \in AP_2(\delta)$, then same argument allows obtaining of the result.

If $f \in AP_3(\delta)$, then

$$P_f(G_1 \odot G_2, x) = \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} x^{1/f(n_2 d_u + d_v)-1} \geq \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} x^{1/f(d_u)-1} = n_2 P_f(G_1, x).$$

A similar argument gives $P_f(G_1 \odot G_2, x) \geq n_1 P_f(G_2, x)$. □

Theorems 2 and 11 have the following consequence when $f(t) = t^\alpha$.

Proposition 5. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively, $\alpha \in \mathbb{R}$ and $f(t) = t^\alpha$. The f-polynomial of the lexicographic product $G_1 \odot G_2$ satisfies the following inequalities for $x \in (0,1]$.*

(1) *If $\alpha \leq -1$, then*

$$P_f(G_1 \odot G_2, x) \leq x^{n_2} P_f(G_1, x^{n_2}) P_f(G_2, x).$$

(2) *If $\alpha \in [-1, 0]$, then*

$$P_f(G_1 \odot G_2, x) \geq x^{n_2} P_f(G_1, x^{n_2}) P_f(G_2, x).$$

(3) *If $\alpha \geq 0$, then*

$$P_f(G_1 \odot G_2, x) \geq \max\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\}.$$

Theorem 5 gives the following equality for the inverse degree polynomial.

Corollary 4. *Given two graphs G_1 and G_2, of order n_1 and n_2, respectively, the ID polynomial of the lexicographic product $G_1 \odot G_2$ is*

$$ID(G_1 \odot G_2, x) = x^{n_2} ID(G_1, x^{n_2}) ID(G_2, x).$$

Since $f(t) = t\sqrt{\log t} \in AP_3(2)$, Theorem 11 has the following consequence.

Theorem 12. *Let G_1 and G_2 be two graphs with minimum degree two and of order n_1 and n_2, respectively. If $f(t) = t\sqrt{\log t}$, then the f-polynomial of the lexicographic product $G_1 \odot G_2$ satisfies for $x \in (0,1]$*

$$P_f(G_1 \odot G_2, x) \geq \max\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\}.$$

Next, we obtain bounds for $I_f(G_1 \odot G_2)$ by using the previous inequalities for $P_f(G_1 \odot G_2, x)$. We start with $f \in AP_3(\delta)$.

Theorem 13. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of order n_1 and n_2, respectively, and minimum degree at least δ. If $f \in AP_3(\delta)$, then*

$$I_f(G_1 \odot G_2) \geq \max\{n_2 I_f(G_1),\, n_1 I_f(G_2)\}.$$

Proof. Theorem 11 gives

$$P_f(G_1 \odot G_2, x) \geq n_2 P_f(G_1, x).$$

for every $0 \leq x \leq 1$. Thus, Proposition 1 leads to $I_f(G_1 \odot G_2) \geq n_2 I_f(G_1)$. A similar argument gives the inequality $I_f(G_1 \odot G_2) \geq n_1 I_f(G_2)$. □

We deal now with $f \in AP_1(\delta) \cup AP_2(\delta)$.

Theorem 14. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ, and $a > 0$. If $f : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, a/2]$, then the following inequalities hold.*

(1) *If $f \in AP_1(\delta)$, then*

$$I_f(G_1 \odot G_2) \leq \frac{1}{2}\Big(\frac{n_1^2 n_2 a}{2}\, I_f(G_1) I_f(G_2)\Big)^{1/3}.$$

(2) *If $f \in AP_2(\delta)$, then*

$$I_f(G_1 \odot G_2) \geq \frac{1}{n_2 a}\, I_f(G_1) I_f(G_2).$$

Proof. Consider the function $g = f/a$. We have $g : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, 1/2]$ and so, Proposition 1 implies that $P_g(G_1, x)$ and $P_g(G_2, x)$ are convex functions on the open interval $(0, \infty)$ and continuous on the closed interval $[0, \infty)$; hence, they are convex when $x \in [0, 1]$.

If $f \in AP_1(\delta)$, then $f/a \in AP_1(\delta)$ and Theorem 11 implies

$$P_{f/a}(G_1 \odot G_2, x) \leq x^{n_2} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x).$$

Notice that $f/a \leq 1/2$ implies $a/f - 1 \geq 1$, and thus, $P_{f/a}(G_i, 0) = 0$ and $P_{f/a}(G_i, 1) = n_i$ for $i = 1, 2$. Since x is a convex function when $x \in [0, 1]$, Lemma 2 implies

$$\begin{aligned}
&\frac{1}{a}\, I_f(G_1 \odot G_2) = I_{f/a}(G_1 \odot G_2) = \int_0^1 P_{f/a}(G_1 \odot G_2, x)\, dx \\
&\leq \int_0^1 x^{n_2} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x)\, dx \\
&\leq \frac{1}{2}\Big(\int_0^1 x\, dx \int_0^1 x^{n_2-1} P_{f/a}(G_1, x^{n_2})\, dx \int_0^1 P_{f/a}(G_2, x)\, dx\Big)^{1/3} (1 \cdot n_1 \cdot n_2)^{2/3} \\
&= \frac{1}{2}\Big(\frac{1}{2}\, \frac{1}{n_2} \int_0^1 P_{f/a}(G_1, t)\, dt\, \frac{1}{a}\, I_f(G_2)\, n_1^2 n_2^2\Big)^{1/3} \\
&= \frac{1}{2}\Big(\frac{n_1^2 n_2}{2a^2}\, I_f(G_1) I_f(G_2)\Big)^{1/3}.
\end{aligned}$$

If $f \in AP_2(\delta)$, thus $f/a \in AP_2(\delta)$ and Theorem 11 implies

$$P_{f/a}(G_1 \odot G_2, x) \geq x^{n_2} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x).$$

Thus, Lemma 1 gives

$$\begin{aligned}
\frac{1}{a}\, I_f(G_1 \odot G_2) = I_{f/a}(G_1 \odot G_2) &= \int_0^1 P_{f/a}(G_1 \odot G_2, x)\, dx \\
&\geq \int_0^1 x^{n_2} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x)\, dx \\
&\geq 2 \int_0^1 x\, dx \int_0^1 x^{n_2-1} P_{f/a}(G_1, x^{n_2})\, dx \int_0^1 P_{f/a}(G_2, x)\, dx \\
&= 2\, \frac{1}{2}\, \frac{1}{n_2 a}\, I_f(G_1)\, \frac{1}{a}\, I_f(G_2) = \frac{1}{n_2 a^2}\, I_f(G_1) I_f(G_2).
\end{aligned}$$

□

Now, we deduce several inequalities for many topological indices of lexicographic products.

Theorems 2, 14 (with $a = 2$) and 13 have the following consequence for the variable first Zagreb index.

Theorem 15. *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and $\alpha \in \mathbb{R}$.*
(1) *If $\alpha \le -1$, then*

$$M_1^\alpha(G_1 \odot G_2) \le \frac{1}{2}\big(n_1^2 n_2 M_1^\alpha(G_1) M_1^\alpha(G_2)\big)^{1/3}.$$

(2) *If $\alpha \in [-1, 0]$, then*

$$M_1^\alpha(G_1 \odot G_2) \ge \frac{1}{2n_2}\, M_1^\alpha(G_1) M_1^\alpha(G_2).$$

(3) *If $\alpha \ge 0$, then*

$$M_1^\alpha(G_1 \odot G_2) \ge \max\big\{n_2 M_1^\alpha(G_1),\, n_1 M_1^\alpha(G_2)\big\}.$$

Theorem 15 has the following consequence for the first Zagreb, forgotten and inverse degree.

Corollary 5. *If G_1 and G_2 are two graphs with n_1 and n_2 vertices, respectively, then*

$$M_1(G_1 \odot G_2) \ge \max\big\{n_2 M_1(G_1),\, n_1 M_1(G_2)\big\},$$
$$F(G_1 \odot G_2) \ge \max\big\{n_2 F(G_1),\, n_1 F(G_2)\big\},$$
$$\frac{1}{2n_2}\, ID(G_1) ID(G_2) \le ID(G_1 \odot G_2) \le \frac{1}{2}\big(n_1^2 n_2 ID(G_1) ID(G_2)\big)^{1/3}.$$

Since $f(t) = t\sqrt{\log t} \in AP_3(2)$, Theorem 13 allows deduction of the following result for the SL index.

Theorem 16. *If G_1 and G_2 are graphs without pendant vertices and with n_1 and n_2 vertices, respectively, then*

$$SL(G_1 \odot G_2) \ge \max\big\{n_2 SL(G_1),\, n_1 SL(G_2)\big\}.$$

5. Inequalities for Cartesian Sums

We start this last section by proving pointwise inequalities of $P_f(G_1 \oplus G_2, x)$ involving the f-polynomials of G_1 and G_2.

Theorem 17. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ. For $x \in (0, 1]$, the f-polynomial of the Cartesian sum $G_1 \oplus G_2$ satisfies.*
(1) *If $f \in AP_1(\delta)$, then*

$$P_f(G_1 \oplus G_2, x) \le x^{n_1+n_2-1} P_f(G_1, x^{n_2}) P_f(G_2, x^{n_1}).$$

(2) *If $f \in AP_2(\delta)$, then*

$$P_f(G_1 \oplus G_2, x) \ge x^{n_1+n_2-1} P_f(G_1, x^{n_2}) P_f(G_2, x^{n_1}).$$

(3) *If $f \in AP_3(\delta)$, then*

$$P_f(G_1 \oplus G_2, x) \ge \max\big\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\big\}.$$

Proof. If $(u, v) \in V(G_1 \oplus G_2)$, then $d_{(u,v)} = n_2 d_u + n_1 d_v$.

Suppose that $f \in AP_1(\delta)$. Since $d_u \ge \delta$ for every $u \in V(G_1) \cup V(G_2)$, one can prove by induction that

$$\frac{1}{f(n_2 d_u + n_1 d_v)} \ge \frac{n_2}{f(d_u)} + \frac{n_1}{f(d_v)}\,.$$

Since $x \in (0,1]$,

$$\begin{aligned} P_f(G_1 \oplus G_2, x) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} x^{1/f(n_2 d_u + d_v)-1} \\ &\le \sum_{u\in V(G_1)} x^{n_2/f(d_u)} \sum_{v\in V(G_2)} x^{n_1/f(d_v)} x^{-1} \\ &= \sum_{u\in V(G_1)} \left(x^{n_2}\right)^{1/f(d_u)-1} x^{n_2} \sum_{v\in V(G_2)} \left(x^{n_1}\right)^{1/f(d_v)-1} x^{n_1} x^{-1} \\ &= x^{n_1+n_2-1} P_f(G_1, x^{n_2}) P_f(G_2, x^{n_1}). \end{aligned}$$

If $f \in AP_2(\delta)$, then a similar argument allows obtaining of the corresponding inequality.

Suppose that $f \in AP_3(\delta)$. We deduce

$$\begin{aligned} P_f(G_1 \oplus G_2, x) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} x^{1/f(n_2 d_u + n_1 d_v)-1} \\ &\ge \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} x^{1/f(d_u)-1} = n_2 P_f(G_1, x). \end{aligned}$$

A similar argument gives $P_f(G_1 \oplus G_2, x) \ge n_1 P_f(G_2, x)$. □

Theorems 2 and 17 have the following consequence for $f(t) = t^\alpha$.

Proposition 6. *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, $\alpha \in \mathbb{R}$ and $f(t) = t^\alpha$. For $x \in (0,1]$, the f-polynomial of the Cartesian sum $G_1 \oplus G_2$ satisfies the following inequalities for .*

(1) *If $\alpha \le -1$, then*

$$P_f(G_1 \oplus G_2, x) \le x^{n_1+n_2-1} P_f(G_1, x^{n_2}) P_f(G_2, x^{n_1}).$$

(2) *If $\alpha \in [-1, 0]$, then*

$$P_f(G_1 \oplus G_2, x) \ge x^{n_1+n_2-1} P_f(G_1, x^{n_2}) P_f(G_2, x^{n_1}).$$

(3) *If $\alpha \ge 0$, then*

$$P_f(G_1 \oplus G_2, x) \ge \max\left\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\right\}.$$

Proposition 6 allows deduction of the following equality for the inverse degree polynomial.

Corollary 6. *Given two graphs G_1 and G_2, or order n_1 and n_2, respectively, the ID polynomial of the Cartesian sum $G_1 \oplus G_2$ is*

$$ID(G_1 \oplus G_2, x) = x^{n_1+n_2-1} ID(G_1, x^{n_2}) ID(G_2, x^{n_1}).$$

Since $f(t) = t\sqrt{\log t} \in AP_3(2)$, Theorem 17 has the following consequence.

Corollary 7. *Let G_1 and G_2 be two graphs with minimum degree two and of order n_1 and n_2, respectively. If $f(t) = t\sqrt{\log t}$, then the f-polynomial of the Cartesian sum $G_1 \oplus G_2$ satisfies for $x \in (0,1]$*

$$P_f(G_1 \oplus G_2, x) \ge \max\left\{n_2 P_f(G_1, x),\, n_1 P_f(G_2, x)\right\}.$$

Next, we obtain bounds for $I_f(G_1 \oplus G_2)$ by using the previous inequalities for $P_f(G_1 \oplus G_2, x)$. We start when $f \in AP_3(\delta)$.

Theorem 18. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ. If $f \in AP_3(\delta)$, then*

$$I_f(G_1 \oplus G_2) \geq \max\left\{n_2 I_f(G_1),\, n_1 I_f(G_2)\right\}.$$

Proof. Theorem 17 gives

$$P_f(G_1 \oplus G_2, x) \geq n_2 P_f(G_1, x).$$

for every $0 < x \leq 1$. Hence, $I_f(G_1 \oplus G_2) \geq n_2 I_f(G_1)$ by Proposition 1. A similar argument gives the inequality $I_f(G_1 \oplus G_2) \geq n_1 I_f(G_2)$. □

We consider now the case $f \in AP_1(\delta) \cup AP_2(\delta)$.

Theorem 19. *Let $\delta \in \mathbb{Z}^+$, G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and minimum degree at least δ, and $a > 0$. If $f : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, a/2]$, then the following inequalities hold.*

(1) *If $f \in AP_1(\delta)$, then*

$$I_f(G_1 \oplus G_2) \leq \frac{1}{2}\Big(\frac{n_1 n_2 a}{2}\, I_f(G_1) I_f(G_2)\Big)^{1/3}.$$

(2) *If $f \in AP_2(\delta)$, then*

$$I_f(G_1 \oplus G_2) \geq \frac{1}{n_1 n_2 a}\, I_f(G_1) I_f(G_2).$$

Proof. Consider the function $g = f/a$. Therefore, $g : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, 1/2]$ and Proposition 1 implies that $P_g(G_1, x)$ and $P_g(G_2, x)$ are convex functions on the open interval $(0, \infty)$ and continuous on the closed interval$[0, \infty)$; hence, they are convex when $0 \leq x \leq 1$.

If $f \in AP_1(\delta)$, then the function f/a belongs to $AP_1(\delta)$ and Theorem 17 implies

$$P_{f/a}(G_1 \oplus G_2, x) \leq x^{n_1+n_2-1} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x^{n_1}).$$

Notice that $f/a \leq 1/2$ implies $a/f - 1 \geq 1$, and thus, $P_{f/a}(G_i, 0) = 0$ and $P_{f/a}(G_i, 1) = n_i$ for $i = 1, 2$. Since x is a convex function on the interval $[0, 1]$, Lemma 2 implies

$$\begin{aligned}
&\frac{1}{a}\, I_f(G_1 \oplus G_2) = I_{f/a}(G_1 \oplus G_2) = \int_0^1 P_{f/a}(G_1 \oplus G_2, x)\, dx \\
&\leq \int_0^1 x^{n_1+n_2-1} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x^{n_1})\, dx \\
&\leq \frac{1}{2}\Big(\int_0^1 x\, dx \int_0^1 x^{n_2-1} P_{f/a}(G_1, x^{n_2})\, dx \int_0^1 x^{n_1-1} P_{f/a}(G_2, x^{n_1})\, dx\Big)^{1/3} (1 \cdot n_1 \cdot n_2)^{2/3} \\
&= \frac{1}{2}\Big(\frac{1}{2}\,\frac{1}{n_2} \int_0^1 P_{f/a}(G_1, t)\, dt\, \frac{1}{n_1} \int_0^1 P_{f/a}(G_2, t)\, dt\, n_1^2 n_2^2\Big)^{1/3} \\
&= \frac{1}{2}\Big(\frac{n_1 n_2}{2a^2}\, I_f(G_1) I_f(G_2)\Big)^{1/3}.
\end{aligned}$$

If $f \in AP_2(\delta)$, thus f/a belongs to $AP_2(\delta)$ and Theorem 17 implies

$$P_{f/a}(G_1 \oplus G_2, x) \geq x^{n_1+n_2-1} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x^{n_1}).$$

Thus, Lemma 1 gives

$$\begin{aligned}
\frac{1}{a}\, I_f(G_1 \oplus G_2) = I_{f/a}(G_1 \oplus G_2) &= \int_0^1 P_{f/a}(G_1 \oplus G_2, x)\, dx \\
&\geq \int_0^1 x^{n_1+n_2-1} P_{f/a}(G_1, x^{n_2}) P_{f/a}(G_2, x^{n_1}).\, dx \\
&\geq 2 \int_0^1 x\, dx \int_0^1 x^{n_2-1} P_{f/a}(G_1, x^{n_2})\, dx \int_0^1 x^{n_1-1} P_{f/a}(G_2, x^{n_1})\, dx \\
&= 2\, \frac{1}{2}\, \frac{1}{n_2 a}\, I_f(G_1)\, \frac{1}{n_1 a}\, I_f(G_2) = \frac{1}{n_1 n_2 a^2}\, I_f(G_1) I_f(G_2).
\end{aligned}$$

□

Theorems 2, 19 (with $a = 2$) and 18 have the following consequence for the general first Zagreb index.

Theorem 20. *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively, and $\alpha \in \mathbb{R}$.*
(1) *If $\alpha \leq -1$, then*

$$M_1^\alpha(G_1 \oplus G_2) \leq \frac{1}{2}\big(n_1 n_2 M_1^\alpha(G_1) M_1^\alpha(G_2)\big)^{1/3}.$$

(2) *If $\alpha \in [-1, 0]$, then*

$$M_1^\alpha(G_1 \oplus G_2) \geq \frac{1}{2n_1 n_2}\, M_1^\alpha(G_1) M_1^\alpha(G_2).$$

(3) *If $\alpha \geq 0$, then*

$$M_1^\alpha(G_1 \oplus G_2) \geq \max\big\{n_2 M_1^\alpha(G_1),\, n_1 M_1^\alpha(G_2)\big\}.$$

Theorem 20 has the following consequence for the first Zagreb, forgotten and inverse degree indices.

Corollary 8. *If G_1 and G_2 are two graphs of orders n_1 and n_2, respectively, then*

$$\begin{aligned}
M_1(G_1 \oplus G_2) &\geq \max\big\{n_2 M_1(G_1),\, n_1 M_1(G_2)\big\}. \\
F(G_1 \oplus G_2) &\geq \max\big\{n_2 F(G_1),\, n_1 F(G_2)\big\}. \\
\frac{1}{2n_1 n_2}\, ID(G_1) ID(G_2) \leq ID(G_1 \oplus G_2) &\leq \frac{1}{2}\big(n_1 n_2 ID(G_1) ID(G_2)\big)^{1/3}.
\end{aligned}$$

Since $f(t) = t\sqrt{\log t} \in AP_3(2)$, Theorem 18 implies the following result for the SL index.

Theorem 21. *If G_1 and G_2 are graphs without pendant vertices and of order n_1 and n_2, respectively, then*

$$SL(G_1 \oplus G_2) \geq \max\big\{n_2 SL(G_1),\, n_1 SL(G_2)\big\}.$$

We obtain the following result by using the previous ideas.

Lemma 4. *Let G be a graph and Γ a subgraph of G, $f : \mathbb{Z}^+ \to \mathbb{R}^+$ an increasing function, and $x \in (0, 1]$. Then*

$$P_f(\Gamma, x) \leq P_f(G, x), \qquad I_f(\Gamma) \leq I_f(G).$$

If f is a decreasing function and $V(\Gamma) = V(G)$, then we obtain the converse inequalities.

Lemma 4 has the following consequence, relating the polynomials and indices of Cartesian products, lexicographic products and Cartesian sums.

Proposition 7. *Let G_1 and G_2 be two graphs, $f : \mathbb{Z}^+ \to \mathbb{R}^+$ an increasing function, and $x \in (0,1]$. Then*

$$P_f(G_1 \times G_2, x) \le P_f(G_1 \odot G_2, x) \le P_f(G_1 \oplus G_2, x),$$
$$I_f(G_1 \times G_2) \le I_f(G_1 \odot G_2) \le I_f(G_1 \oplus G_2).$$

If f is a decreasing function, then we obtain the converse inequalities.

6. Conclusions

There are several graph products which play an important role in graph theory. Three of these products are Cartesian product, lexicographic product and Cartesian sum. Many topological indices can be written as $I_f(G) = \sum_{u \in V(G)} f(d_u)$, for an appropriate choice of the function f (e.g., first Zagreb, inverse degree, forgotten, general first Zagreb and sum lordeg indices). By using the f-polynomial $P_f(G,x)$ introduced in [43], we obtain in this paper several inequalities of every topological index which can be written as I_f for a function f in these classical graph products, from the information on topological indices of their factors, which are much easier to calculate than the products. These results are interesting from the theoretical viewpoint, and also from the point of view of applications since many chemical compounds can be represented by graph products (see the introduction). Our approach is to obtain information about the corresponding f-polynomials, which are easy to calculate (as in Theorems 1, 11 and 17); thus, we can deduce information on the I_f index by using the formula $\int_0^1 P_f(G,x)\,dx = I_f(G)$ (as in Theorems 4, 13 and 18). This is a good approach since the bounds of the f-polynomial of a product of two graphs allow the use of analytic tools to bound the I_f index of such a product, simplifying the proofs.

In [43] appear similar results for corona product and join. Consequently, two natural open questions are to study this problem for strong product and tensor product of graphs.

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