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Second-Order Non-Canonical Neutral Differential Equations with Mixed Type: Oscillatory Behavior

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Abstract: In this paper, we establish new sufficient conditions for the oscillation of solutions of a class of second-order delay differential equations with a mixed neutral term, which are under the non-canonical condition. The results obtained complement and simplify some known results in the relevant literature. Example illustrating the results is included.

Keywords: non-canonical differential equations; second-order; neutral delay; mixed type; oscillation criteria

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1. Introduction

This paper discusses the oscillatory behavior of solutions of second-order functional differential equation with a mixed neutral term of the form

$$\left(r(l)\left[\left(y(l)+p_1(l)y(\rho_1(l))+p_2(l)y(\rho_2(l))\right)'\right]^{\gamma}\right)'+q(l)y^{\gamma}(\sigma(l))=0, \qquad (1)$$

where $l \geq l_0$. Throughout this paper, we assume the following:

(C1) $\gamma \in Q^{+}_{odd} := \{a/b : \ a,b \in \mathbb{Z}^{+} \text{ are odd}\}$ and $r \in C([l_0,\infty),(0,\infty))$;
(C2) $\rho_1,\rho_2,\sigma \in C([l_0,\infty),\mathbb{R})$, $\rho_1(l) \leq l \leq \rho_2(l)$, $\sigma(l) \leq l$ and $\rho_1,\rho_2,\sigma \to \infty$ as $l \to \infty$;
(C3) $p_1,p_2,q \in C([l_0,\infty),[0,\infty))$ and $q(l)$ is not identically zero for large l.

Let y be a real-valued function defined for all l in a real interval $[l_y,\infty)$, $l_y \geq l_0$, and having a second derivative for all $l \in [l_y,\infty)$. The function y is called a *solution* of the differential Equation (1) on $[l_y,\infty)$ if y satisfies (1) on $[l_y,\infty)$. A nontrivial solution y of any differential equation is said to be *oscillatory* if it has arbitrary large zeros; otherwise, it is said to be *nonoscillatory*. We will consider only those solutions of (1) which exist on some half-line $[l_b,\infty)$ for $l_b \geq l_0$ and satisfy the condition $\sup\{|y(l)| : l_c \leq l < \infty\} > 0$ for any $l_c \geq l_b$.

A delay differential equation of neutral type is an equation in which the highest order derivative of the unknown function appears both with and without delay. During the last decades, there is a great interest in studying the oscillation of solutions of neutral differential equations. This is due to the fact that such equations arise from a variety of applications including population dynamics, automatic control, mixing liquids, and vibrating masses attached to an elastic bar, biology in explaining self-balancing of the human body, and in robotics in constructing biped robots, it is easy to notice the emergence of models of the neutral delay differential equations, see [1,2].

In the following, we review some of the related works that dealt with the oscillation of the neutral differential equations of mixed-type.

Grammatikopouls et al. [3] established oscillation criteria for the equation

$$\left(r(l)\psi'(l)\right)' + q(l)y(\sigma(l)) = 0, \tag{2}$$

where

$$z(l) = y(l) + p_1(l)y(l - \sigma_1) + p_2(l)y(l + \sigma_2),$$

$r(l) = 1$, $p_2(l) = 0$, $0 \le p_1 \le 1$, and $q(l) \ge 0$. Ruan [4] obtained some oscillation criteria for the Equation (2) by employing Riccati technique and averaging function method, when $p_2(l) = 0$ and $\sigma(l) = l - \sigma$. Arul and Shobha [5] studied the oscillatory behavior of solution of (2), when $0 \le p_1(l) \le p_1 < \infty$ and $0 \le p_2(l) \le p_2 < \infty$.

Dzurina et al. [6] presented some sufficient conditions for the oscillation of the second-order equation

$$\left(\frac{1}{r(l)}y'(l)\right)' + p(l)y(\tau(l)) + q(l)y(\sigma(l)) = 0.$$

Li [7] and Li et al. [8] studied the oscillation of solutions of the second-order equation with constant mixed arguments:

$$\left(r(l)z'(l)\right)' + q_1(l)y(l - \sigma_3) + q_2(l)y(l + \sigma_4) = 0. \tag{3}$$

Arul and Shobha [5] established some sufficient conditions for the oscillation of all solutions of Equation (3) in the canonical case, that is,

$$\int_{l_0}^{\infty} r^{-1}(\vartheta)\mathrm{d}\vartheta = \infty,$$

Thandapani et al. [9] studied the oscillation criteria for the differential equation of the form

$$(z^{\alpha}(l))'' + q(l)y^{\beta}(l - \tau_1) + p(l)y^{\gamma}(l + \tau_1) = 0.$$

Grace et al. [10] studied the oscillatory behavior of solutions of the equation

$$\left(r(l)\left(\left(y(l) + p_1(l)y^{\beta_1}(\sigma_1(l)) + p_2(l)y^{\beta_2}(\sigma_2(l))\right)'\right)^{\gamma}\right)' + q(l)y^{\gamma}(\tau(l)) = 0,$$

and considered the two cases

$$\int_{l_0}^{\infty} r^{-1/\gamma}(\vartheta)\mathrm{d}\vartheta = \infty, \tag{4}$$

and

$$\int_{l_0}^{\infty} r^{-1/\gamma}(\vartheta)\mathrm{d}\vartheta < \infty. \tag{5}$$

In [11], Tunc et al. studied the oscillatory behavior of the differential Equation (1) under the condition (4). Moreover, they considered the two following cases: $p_1(l) \ge 0$, $p_2(l) \ge 1$, and $p_2(l) \ne 1$ eventually; $p_2(l) \ge 0$, $p_1(l) \ge 1$, and $p_2(l) \ne 1$ eventually.

For the third-order equations, Han et al. [12] studied the oscillation and asymptotic properties of the third-order equation

$$\left(a(l)z''(l)\right)' + q_1(l)y(l - \tau_3) + q_2(l)y(l + \tau_4) = 0,$$

and established two theorems which guarantee that the above equation oscillates or tends to zero. Moaaz et al. [13] discussed the oscillation and asymptotic behavior of solutions of the third-order equation

$$\left(r(l)\left(x''(l)\right)^{\alpha}\right)' + q_1(l)f_1(y(\sigma_1(l))) + q_2(l)f_2(y(\sigma_2(l))) = 0,$$

where $x(l) = y(l) + p_1(l)y(\tau_1(l)) + p_2(l)y(\tau_2(l))$. For further results, techniques, and approaches in studying oscillation of the delay differential equations, see in [14–24].

In this paper, we study the oscillatory behavior of solutions of the second-order differential equation with a mixed neutral term (1) under condition (5). We follow a new approach based on deducing a new relationship between the solution and the corresponding function. Using this new relationship, we first obtain one condition that ensures oscillation of (1). Moreover, by introducing a generalized Riccati substitution, we get a new criterion for oscillation of (1). Often these types of equations (such as (1), (2), and (3)) are studied under condition (4). On the other hand, the works that studied these equations under the condition (5) obtained two conditions to ensure the oscillation. Therefore, our results are an extension and simplification as well as improvement of previous results in [3–5,8,11].

2. Main Results

We adopt the following notation for a compact presentation of our results:

$$\psi(l) := y(l) + p_1(l)y(\rho_1(l)) + p_2(l)y(\rho_2(l)),$$

$$\kappa(u,v) := \int_u^v r^{-1/\gamma}(\delta)\mathrm{d}\delta,$$

$$B_1(l) := 1 - p_1(l)\frac{\kappa(\rho_1(l),\infty)}{\kappa(l,\infty)} - p_2(l)$$

and

$$B_2(l) := 1 - p_1(l) - p_2(l)\frac{\kappa(l_1,\rho_2(l))}{\kappa(l_1,l)}.$$

Lemma 1. *Assume that* $\Theta(\vartheta) := A\vartheta - B(\vartheta - C)^{(\gamma+1)/\gamma}$, *where* A, B *and* C *are real constants;* $B > 0$*; and* $\gamma \in Q^{+}_{odd}$. *Then,*

$$\Theta(\vartheta^*) \leq \max_{u\in\mathbb{R}}\Theta(\vartheta) = AC + \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}A^{\gamma+1}B^{-\gamma}.$$

Lemma 2. *Assume that* y *is a positive solution of (1) on* $[l_0,\infty)$. *If* ψ *is a decreasing positive function for* $l \geq l_1$ *large enough, then*

$$\left(\frac{\psi(l)}{\kappa(l,\infty)}\right)' \geq 0, \text{ for } l \geq l_1. \tag{6}$$

While if ψ *is a increasing positive function for* $l \geq l_1$*, then*

$$\left(\frac{\psi(l)}{\kappa(l_1,l)}\right)' \leq 0, \text{ for } l \geq l_1. \tag{7}$$

Proof. Assume that (1) has a positive solution y on $[l_0,\infty)$. Therefore, there exists a $l_1 \geq l_0$ such that, for all $l \geq l_1$, $\psi(l) \geq y(l) > 0$ and $\left(r(l)(\psi'(l))^\gamma\right) \leq 0$. From (1), we see that

$$\left(r(l)(\psi'(l))^\gamma\right)' = -q(l)y^\gamma(\sigma(l)) \leq 0.$$

Obviously, ψ is either eventually decreasing or eventually increasing.
Let ψ be a decreasing function on $[l_1,\infty)$. Then, $\lim_{l\to\infty}\psi(l) < \infty$, and so

$$\psi(l) \geq -\int_l^\infty r^{-1/\gamma}(\vartheta)r^{1/\gamma}(\vartheta)\psi'(\vartheta)\mathrm{d}\vartheta \geq -\kappa(l,\infty)r^{1/\gamma}(l)\psi'(l). \tag{8}$$

Thus,

$$\left(\frac{\psi(l)}{\kappa(l,\infty)}\right)' = \frac{\kappa(l,\infty)\psi'(l) + r^{-1/\gamma}(l)\psi(l)}{(\kappa(l,\infty))^2} \geq 0.$$

Let ψ be a increasing function on $[l_1,\infty)$. Then, we obtain

$$\psi(l) \geq \int_{l_1}^{l} r^{-1/\gamma}(\vartheta) r^{1/\gamma}(\vartheta)\psi'(\vartheta)\mathrm{d}\vartheta \geq \kappa(l_1,l) r^{1/\gamma}(l)\psi'(l),$$

and so

$$\left(\frac{\psi(l)}{\kappa(l_1,l)}\right)' = \frac{\kappa(l_1,l)\psi'(l) - r^{-1/\gamma}(l)\psi(l)}{\kappa^2(l_1,l)} \leq 0.$$

Thus, the proof is complete. □

Theorem 1. *Assume that* $B_2(l) \geq B_1(l) > 0$. *If*

$$\limsup_{l\to\infty} \int_{l_1}^{l} \frac{1}{r^{1/\gamma}(\beta)} \left(\int_{l_1}^{\beta} q(\delta) B_1^{\gamma}(\sigma(\delta))\kappa^{\gamma}(\sigma(\delta),\infty)\mathrm{d}\delta\right)^{1/\gamma} \mathrm{d}\beta = \infty, \tag{9}$$

then, all solutions of (1) are oscillatory.

Proof. Assume the contrary that Equation (1) has a positive solution y on $[l_0,\infty)$. Then, $y(\rho_1(l))$, $y(\rho_2(l))$ and $y(\sigma(l))$ are positive for all $l \geq l_1$, where l_1 is large enough. Thus, from (1) and the definition of ψ, we note that $\psi(l) \geq y(l) > 0$ and $r(l)(\psi'(l))^{\gamma}$ is nonincreasing. Therefore, ψ' is either eventually negative or eventually positive.
Let $\psi'(l) < 0$ on $[l_1,\infty)$. By using Lemma 2, we have

$$\psi(\rho_1(l)) \leq \frac{\kappa(\rho_1(l),\infty)}{\kappa(l,\infty)}\psi(l),$$

based on the fact that $\rho_1(l) \leq l$. Therefore,

$$\begin{aligned} y(l) &= \psi(l) - p_1(l)y(\rho_1(l)) - p_2(l)y(\rho_2(l)) \\ &\geq \psi(l) - p_1(l)\psi(\rho_1(l)) - p_2(l)\psi(\rho_2(l)) \\ &\geq \left(1 - p_1(l)\frac{\kappa(\rho_1(l),\infty)}{\kappa(l,\infty)} - p_2(l)\right)\psi(l) \\ &= B_1(l)\psi(l). \end{aligned}$$

Therefore, (1) becomes

$$\left(r(l)\left(\psi'(l)\right)^{\gamma}\right)' \leq -q(l)B_1^{\gamma}(\sigma(l))\psi^{\gamma}(\sigma(l)). \tag{10}$$

As $\left(r(l)(\psi'(l))^{\gamma}\right)' \leq 0$, we have

$$r(l)\left(\psi'(l)\right)^{\gamma} \leq r(l_1)\left(\psi'(l_1)\right)^{\gamma} := -L < 0, \tag{11}$$

for all $l \geq l_1$, from (8) and (11), we have

$$\psi^{\gamma}(l) \geq L\kappa^{\gamma}(l,\infty) \text{ for all } l \geq l_1. \tag{12}$$

Combining (10) with (12) yields

$$\left(r(l)\left(\psi'(l)\right)^{\gamma}\right)' \leq -Lq(l)B_1^{\gamma}(\sigma(l))\kappa^{\gamma}(\sigma(l),\infty), \tag{13}$$

for all $l \geq l_1$. Integrating (13) from l_1 to l, we obtain

$$\begin{aligned} r(l)\left(\psi'(l)\right)^{\gamma} &\leq r(l_1)\left(\psi'(l_1)\right)^{\gamma} - L\int_{l_1}^{l} q(\delta)B_1^{\gamma}(\sigma(\delta))\kappa^{\gamma}(\sigma(\delta),\infty)\mathrm{d}\delta \\ &\leq -L\int_{l_1}^{l} q(\delta)B_1^{\gamma}(\sigma(\delta))\kappa^{\gamma}(\sigma(\delta),\infty)\mathrm{d}\delta. \end{aligned}$$

Integrating the last inequality from l_1 to l, we get

$$\psi(l) \leq \psi(l_1) - L^{1/\gamma}\int_{l_1}^{l} \frac{1}{r^{1/\gamma}(\beta)}\left(\int_{l_1}^{\beta} q(\delta)B_1^{\gamma}(\sigma(\delta))\kappa^{\gamma}(\sigma(\delta),\infty)\mathrm{d}\delta\right)^{1/\gamma}\mathrm{d}\beta.$$

At $l \to \infty$, we get a contradiction with (9).

Let $\psi'(l) > 0$ on $[l_1,\infty)$. From Lemma 2, we arrive at

$$\psi(\rho_2(l)) \leq \frac{\kappa(l_1,\rho_2(l))}{\kappa(l_1,l)}\psi(l). \tag{14}$$

From the definition of ψ, we obtain

$$\begin{aligned} y(l) &= \psi(l) - p_1(l)y(\rho_1(l)) - p_2(l)y(\rho_2(l)) \\ &\geq \psi(l) - p_1(l)\psi(\rho_1(l)) - p_2(l)\psi(\rho_2(l)). \end{aligned} \tag{15}$$

Using that (14) and $\psi(\rho_1(l)) \leq \psi(l)$ where $\rho_1(l) < l$ in (15), we obtain

$$\begin{aligned} y(l) &\geq \psi(l)\left(1 - p_1(l) - p_2(l)\frac{\kappa(l_1,\rho_2(l))}{\kappa(l_1,l)}\right) \\ &\geq B_2(l)\psi(l). \end{aligned} \tag{16}$$

Thus, (1) becomes

$$\left(r(l)\left(\psi'(l)\right)^{\gamma}\right)' \leq -q(l)B_2^{\gamma}(\sigma(l))\psi^{\gamma}(\sigma(l)). \tag{17}$$

Now, from (9) and (C2), we have that $\int_{l_1}^{l} q(\vartheta)B_1^{\gamma}(\sigma(\vartheta))\kappa^{\gamma}(\sigma(\vartheta),\infty)\mathrm{d}\vartheta$ is unbounded. Therefore, as $\kappa'(l,\infty) < 0$, we obtain that

$$\int_{l_1}^{l} q(\vartheta)B_1^{\gamma}(\sigma(\vartheta))\mathrm{d}\vartheta \to \infty \text{ as } l \to \infty. \tag{18}$$

Integrating (17) from l_2 to l, we get

$$\begin{aligned} r(l)\left(\psi'(l)\right)^{\gamma} &\leq r(l_2)\left(\psi'(l_2)\right)^{\gamma} - \int_{l_2}^{l} q(\vartheta)B_2^{\gamma}(\sigma(\vartheta))\psi^{\gamma}(\sigma(\vartheta))\mathrm{d}\vartheta \\ &\leq r(l_2)\left(\psi'(l_2)\right)^{\gamma} - \psi^{\gamma}(\sigma(l_2))\int_{l_2}^{l} q(\vartheta)B_2^{\gamma}(\sigma(\vartheta))\mathrm{d}\vartheta. \end{aligned}$$

As $B_2(l) > B_1(l)$, we get

$$r(l)\left(\psi'(l)\right)^{\gamma} \leq r(l_2)\left(\psi'(l_2)\right)^{\gamma} - \psi^{\gamma}(\sigma(l_2))\int_{l_2}^{l} q(\vartheta)B_1^{\gamma}(\sigma(\vartheta))\mathrm{d}\vartheta. \tag{19}$$

From (18) and (19), we get a contradiction with the positivity of $\psi'(l)$. Therefore, the proof is complete. □

Theorem 2. *Assume that* $B_2(l) \geq B_1(l) > 0$. *If*

$$\limsup_{l\to\infty} \kappa^{\gamma}(l,\infty) \int_{l_1}^{l} q(\vartheta) B_1^{\gamma}(\vartheta) \mathrm{d}\vartheta > 1, \tag{20}$$

then, all solutions of (1) are oscillatory.

Proof. Assume the contrary that Equation (1) has a positive solution y on $[l_0,\infty)$. Then, $y(\rho_1(l))$, $y(\rho_2(l))$ and $y(\sigma(l))$ are positive for all $l \geq l_1$, where l_1 is large enough. Thus, from (1) and the definition of ψ, we note that $\psi(l) \geq y(l) > 0$ and $r(l)(\psi'(l))^{\gamma}$ is nonincreasing. Therefore, ψ' is either eventually negative or eventually positive.
Let $\psi'(l) < 0$ on $[l_1,\infty)$. Integrating (10) from l_1 to l, we get

$$\begin{aligned} r(l)(\psi'(l))^{\gamma} &\leq r(l_1)(\psi'(l_1))^{\gamma} - \int_{l_1}^{l} q(\vartheta) B_1^{\gamma}(\sigma(\vartheta)) \psi^{\gamma}(\sigma(\vartheta)) \mathrm{d}\vartheta \\ &\leq -\psi^{\gamma}(\sigma(l)) \int_{l_1}^{l} q(\vartheta) B_1^{\gamma}(\vartheta) \mathrm{d}\vartheta. \end{aligned} \tag{21}$$

Using $\psi(\sigma(l)) \geq \psi(l)$ and (8) in (21), we obtain

$$-r(l)(\psi'(l))^{\gamma} \geq -r(l)(\psi'(l))^{\gamma} \kappa^{\gamma}(l,\infty) \int_{l_1}^{l} q(\vartheta) B_1^{\gamma}(\vartheta) \mathrm{d}\vartheta. \tag{22}$$

Divide both sides of inequality (22) by $-r(l)(\psi'(l))^{\gamma}$ and taking the limsup, we get

$$\limsup_{l\to\infty} \kappa^{\gamma}(l,\infty) \int_{l_1}^{l} q(\vartheta) B_1^{\gamma}(\vartheta) \mathrm{d}\vartheta \leq 1.$$

Thus, we get a contradiction with (20).

Let $\psi' > 0$ on $[l_1,\infty)$. From (20) and the fact that $\kappa(l,\infty) < \infty$, we have that (18) holds. Then, this part of proof is similar to that of Theorem 1. Therefore, the proof is complete. □

Theorem 3. *Assume that* $B_2(l) > 0$, $B_1(l) > 0$ *and* $r' > 0$. *If there exist positive functions* $\mu, \delta \in C^1([l_0,\infty))$ *and* $l_1 \in [l_0,\infty)$ *such that*

$$\limsup_{l\to\infty} \left\{ \frac{\kappa^{\gamma}(l,\infty)}{\delta(l)} \int_{l_1}^{l} \left(\delta(\vartheta) q(\vartheta) B_1^{\gamma}(\sigma(\vartheta)) - \frac{r(\vartheta)}{(\gamma+1)^{\gamma+1}} \frac{(\delta'(\vartheta))^{\gamma+1}}{(\delta(\vartheta))^{\gamma}} \right) \mathrm{d}\vartheta \right\} > 1 \tag{23}$$

and

$$\limsup_{l\to\infty} \int_{l_1}^{l} \left(\mu(\vartheta) q(\vartheta) B_2^{\gamma}(\sigma(\vartheta)) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(\vartheta)(\mu'(\vartheta))^{\gamma+1}}{\mu^{\gamma}(\vartheta)(\sigma'(\vartheta))^{\gamma}} \right) \mathrm{d}\vartheta = \infty, \tag{24}$$

then, all solutions of (1) are oscillatory.

Proof. Assume the contrary that Equation (1) has a positive solution y on $[l_0,\infty)$. Then, $y(\rho_1(l))$, $y(\rho_2(l))$ and $y(\sigma(l))$ are positive for all $l \geq l_1$, where l_1 is large enough. Thus, from (1) and the definition of ψ, we note that $\psi(l) \geq y(l) > 0$ and $r(l)(\psi'(l))^{\gamma}$ is nonincreasing. Therefore, ψ' is either eventually negative or eventually positive.
Let $\psi' < 0$ on $[l_1,\infty)$. As in proof of Theorem 1, we arrive at (10). Now, we define the function

$$\omega(l) = \delta(l) \left(\frac{r(l)(\psi'(l))^{\gamma}}{\psi^{\gamma}(l)} + \frac{1}{\kappa^{\gamma}(l,\infty)} \right) \text{ on } [l_1,\infty). \tag{25}$$

From (8), we have that $\omega \geq 0$ on $[l_1, \infty)$. Differentiating (25), we get

$$\begin{aligned}
\omega'(l) &= \frac{\delta'(l)}{\delta(l)}\omega(l) + \delta(l)\frac{\left(r(l)(\psi'(l))^{\gamma}\right)'}{\psi^{\gamma}(l)} - \gamma\delta(l)r(l)\left(\frac{\psi'(l)}{\psi(l)}\right)^{\gamma+1} \\
&\quad + \frac{\gamma\delta(l)}{r^{1/\gamma}(l)\kappa^{\gamma+1}(l,\infty)} \\
&\leq \frac{\delta'(l)}{\delta(l)}\omega(l) + \delta(l)\frac{\left(r(l)(\psi'(l))^{\gamma}\right)'}{\psi^{\gamma}(l)} - \frac{\gamma}{(\delta(l)r(l))^{1/\gamma}}\left(\omega(l) - \frac{\delta(l)}{\kappa^{\gamma}(l,\infty)}\right)^{(\gamma+1)/\gamma} \\
&\quad + \frac{\gamma\delta(l)}{r^{1/\gamma}(l)\kappa^{\gamma+1}(l,\infty)}. \qquad (26)
\end{aligned}$$

Combining (10) and (26), we have

$$\begin{aligned}
\omega'(l) &\leq -\frac{\gamma}{(\delta(l)r(l))^{1/\gamma}}\left(\omega(l) - \frac{\delta(l)}{\kappa^{\gamma}(l,\infty)}\right)^{(\gamma+1)/\gamma} - \delta(l)q(l)B_1^{\gamma}(\sigma(l))\frac{\psi^{\gamma}(\sigma(l))}{\psi^{\gamma}(l)} \\
&\quad + \frac{\gamma\delta(l)}{r^{1/\gamma}(l)\kappa^{\gamma+1}(l,\infty)} + \frac{\delta'(l)}{\delta(l)}\omega(l). \qquad (27)
\end{aligned}$$

Using Lemma 1 with $A := \delta'(l)/\delta(l)$, $B := \gamma(\delta(l)r(l))^{-1/\gamma}$, $C := \delta(l)/\kappa^{\gamma}(l,\infty)$ and $\vartheta := \omega$, we get

$$\begin{aligned}
\frac{\delta'(l)}{\delta(l)}\omega(l) - \frac{\gamma}{(\delta(l)r(l))^{1/\gamma}}\left(\omega(l) - \frac{\delta(l)}{\kappa^{\gamma}(l,\infty)}\right)^{(\gamma+1)/\gamma} &\leq \frac{1}{(\gamma+1)^{\gamma+1}}r(l)\frac{(\delta'(l))^{\gamma+1}}{(\delta(l))^{\gamma}} \\
&\quad + \frac{\delta'(l)}{\kappa^{\gamma}(l,\infty)}.
\end{aligned}$$

As $l \geq \sigma(l)$, we arrive at

$$\psi(\sigma(l)) \geq \psi(l), \qquad (28)$$

which, in view of (27) and (28), gives

$$\begin{aligned}
\omega'(l) &\leq \frac{\delta'(l)}{\kappa^{\gamma}(l,\infty)} + \frac{1}{(\gamma+1)^{\gamma+1}}r(l)\frac{(\delta'(l))^{\gamma+1}}{(\delta(l))^{\gamma}} - \delta(l)q(l)B_1^{\gamma}(\sigma(l))\frac{\psi^{\gamma}(\sigma(l))}{\psi^{\gamma}(l)} \\
&\quad + \frac{\gamma\delta(l)}{r^{1/\gamma}(l)\kappa^{\gamma+1}(l,\infty)} \\
&\leq -\delta(l)q(l)B_1^{\gamma}(\sigma(l)) + \left(\frac{\delta(l)}{\kappa^{\gamma}(l,\infty)}\right)' + \frac{r(l)}{(\gamma+1)^{\gamma+1}}\frac{(\delta'(l))^{\gamma+1}}{(\delta(l))^{\gamma}}. \qquad (29)
\end{aligned}$$

Integrating (29) from l_2 to l, we arrive at

$$\begin{aligned}
\int_{l_2}^{l}\left(\delta(\vartheta)q(\vartheta)B_1^{\gamma}(\sigma(\vartheta)) - \frac{r(\vartheta)}{(\gamma+1)^{\gamma+1}}\frac{(\delta'(\vartheta))^{\gamma+1}}{(\delta(\vartheta))^{\gamma}}\right)\mathrm{d}\vartheta &\leq \left(\frac{\delta(l)}{\kappa^{\gamma}(l,\infty)} - \omega(l)\right)_{l_2}^{l} \\
&\leq -\left(\delta(l)\frac{r(l)(\psi'(l))^{\gamma}}{\psi^{\gamma}(l)}\right)_{l_2}^{l}. \qquad (30)
\end{aligned}$$

From (8), we have

$$-\frac{r^{1/\gamma}(l)\psi'(l)}{\psi(l)} \leq \frac{1}{\kappa(l,\infty)},$$

which, in view of (30), implies

$$\frac{\kappa^{\gamma}(l,\infty)}{\delta(l)}\int_{l_2}^{l}\left(\delta(\vartheta)q(\vartheta)B_1^{\gamma}(\sigma(\vartheta)) - \frac{r(\vartheta)}{(\gamma+1)^{\gamma+1}}\frac{(\delta'(\vartheta))^{\gamma+1}}{(\delta(\vartheta))^{\gamma}}\right)\mathrm{d}\vartheta \leq 1.$$

Thus, we get a contradiction with (23).

Let $\psi'(l) > 0$ on $[l_1, \infty)$. As in proof of Theorem 1, we arrive at (17). Now, we define the function

$$\varphi(l) = \mu(l)\frac{r(l)(\psi'(l))^\gamma}{\psi^\gamma(\sigma(l))}. \tag{31}$$

Therefore, we have that $\omega \geq 0$ on $[l_1, \infty)$. Differentiating (31), we find

$$\varphi'(l) = \frac{\mu'(l)}{\mu(l)}\varphi(l) + \mu(l)\frac{\left(r(l)(\psi'(l))^\gamma\right)'}{\psi^\gamma(\sigma(l))} - \gamma\mu(l)r(l)\frac{(\psi'(l))^\gamma\psi'(\sigma(l))\sigma'(l)}{\psi^{\gamma+1}(\sigma(l))}. \tag{32}$$

Combining (17) and (32), we have

$$\varphi'(l) \leq \frac{\mu'(l)}{\mu(l)}\varphi(l) - \mu(l)q(l)B_2^\gamma(\sigma(l)) - \gamma\mu(l)r(l)\frac{(\psi'(l))^\gamma\psi'(\sigma(l))\sigma'(l)}{\psi^{\gamma+1}(\sigma(l))}.$$

As $\left(r(l)(\psi'(l))^\gamma\right)' < 0$ and $\sigma(l) \leq l$, we arrive at

$$\varphi'(l) \leq \frac{\mu'(l)}{\mu(l)}\varphi(l) - \mu(l)q(l)B_2^\gamma(\sigma(l)) - \gamma\mu(l)r(l)\sigma'(l)\frac{(\psi'(l))^{\gamma+1}}{\psi^{\gamma+1}(\sigma(l))}.$$

From (31), we have

$$\varphi'(l) \leq \frac{\mu'(l)}{\mu(l)}\varphi(l) - \mu(l)q(l)B_2^\gamma(\sigma(l)) - \frac{\gamma\sigma'(l)}{\mu^{1/\gamma}(l)r^{1/\gamma}(l)}\phi^{(\gamma+1)/\gamma}(l).$$

Using the inequality

$$Kv - Lv^{(\gamma+1)/\gamma} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{K^{\gamma+1}}{L^\gamma},\ L > 0, \tag{33}$$

with $K = \mu'(l)/\mu(l)$, $L = \gamma\sigma'(l)/\mu^{1/\gamma}(l)r^{1/\gamma}(l)$ and $v = \varphi$, we have

$$\varphi'(l) \leq -\mu(l)q(l)B_2^\gamma(\sigma(l)) + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(l)(\mu'(l))^{\gamma+1}}{\mu^\gamma(l)(\sigma'(l))^\gamma}. \tag{34}$$

Integrating (34) from l_2 to l, we arrive at

$$\int_{l_2}^{l}\left(\mu(\vartheta)q(\vartheta)B_2^\gamma(\sigma(\vartheta)) - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(\vartheta)(\mu'(\vartheta))^{\gamma+1}}{\mu^\gamma(\vartheta)(\sigma'(\vartheta))^\gamma}\right)\mathrm{d}\vartheta \leq \varphi(l_2).$$

Taking the lim sup on both sides of this inequality, we have a contradiction with (24). The proof of the theorem is complete. □

Example 1. *Consider the second-order neutral differential equation*

$$\left(l^2\left(\left(y(l) + p_0 y\left(\frac{l}{\lambda}\right) + p_* y(\lambda l)\right)'\right)\right)' + q_0 l y(\sigma_0 l) = 0, \tag{35}$$

where $\lambda > 1$, $\sigma_0 \in (0,1)$ *and* $(\lambda p_0 + p_*) \in (0,1)$. *We note that* $r(l) = l^2$, $p_1(l) = p_0$, $p_2(l) = p_*$, $\rho_1(l) = l/\lambda$, $\rho_2(l) = \lambda l$, $q(l) = q_0 l$ *and* $\sigma(l) = \sigma_0 l$. *It is easy to verify that*

$$B_1(l) = 1 - \lambda p_0 - p_*,$$

and

$$B_2(l) = 1 - p_0 - p_*\left(\frac{l-\frac{1}{\lambda}}{l-1}\right),$$

and so $B_2 > B_1 > 0$. *Now, we see that*

$$\limsup_{l\to\infty}\int_{l_1}^{l}\frac{1}{r^{1/\gamma}(\beta)}\left(\int_{l_1}^{\beta}q(\delta)B_1^{\gamma}(\sigma(\delta))\kappa^{\gamma}(\sigma(\delta),\infty)\mathrm{d}\delta\right)^{1/\gamma}\mathrm{d}\beta$$
$$= \limsup_{l\to\infty}\int_{l_1}^{l}\frac{1}{\beta^2}\left(\int_{l_1}^{\beta}q_0\delta(1-\lambda p_0 - p_*)\frac{1}{\sigma_0\delta}\mathrm{d}\delta\right)\mathrm{d}\beta = \infty.$$

Then, by Theorem 1, we have that (35) is oscillatory.

3. Conclusions

In this work, new criteria to test the oscillation of the solutions of second-order non-canonical neutral differential equations with mixed type were presented. These criteria are to further complement and simplify relevant results in the literature.

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