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An Inertial Algorithm for Solving Hammerstein Equations

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Abstract: An inertial algorithm for solving Hammerstein equations is presented. This algorithm is obtained as a consequence of a new inertial algorithm proposed and studied for solving nonlinear equations involving operators that are *m*-accretive. Some strong convergence theorems are proved in real Banach spaces that are uniformly smooth. Furthermore, comparisons of the numerical performance of our algorithms with the numerical performance of some recent important algorithms are presented.

Keywords: nonlinear equations; accretive maps; zeros; strong convergence

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1. Introduction

An algorithm of inertial-type is an iterative procedure in which subsequent terms are obtained using the preceding *two* terms. Inertial-type algorithm was proposed by Polyak [1]. Consider the dynamical system:

$$u''(t) + \gamma u'(t) + \nabla f(u(t)) = 0, \tag{1}$$

where $\gamma > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. The system (1) is discretized such that, having the terms z_{n-1} and z_n, the next term z_{n+1}, can be determined using

$$\frac{z_{n+1} - 2z_n + z_{n-1}}{h^2} + \gamma \frac{z_n - z_{n-1}}{h} + \nabla f(z_n) = 0, \;\; n \geq 1, \tag{2}$$

where h is the step size. Equation (2) yields the following iterative algorithm:

$$z_{n+1} = z_n + \beta(z_n - z_{n-1}) - \alpha \nabla f(z_n), \;\; n \geq 1, \tag{3}$$

where $\beta = 1 - \gamma h$, $\alpha = h^2$ and $\beta(z_n - z_{n-1})$ is called the inertial extrapolation term, which is intended to speed up the convergence of the sequence generated by Equation (3). Our interest in this paper is to propose inertial algorithms for solving nonlinear equations involving accretive operators. Accretive operators were introduced during the late 1960s by Browder [2] and Kato [3]. A motivation for the study of accretive maps is the fact that they appear in evolution equations in Banach spaces. Accretive operators appear also in partial differential equations. For example, consider the equation:

$$\begin{cases} \frac{\partial f}{\partial t}(t,u) \; - \; \triangle f(t,u) \; = \; g(f(t,u)), \;\; t \geq 0, \;\; u \in \Omega, \\ f(t,u) \; = \; 0, \;\; t \geq 0, \;\; u \in \partial\Omega, \\ f(0,u) \; = \; f_0(u), \;\; f_0 \in L^2(\Omega), \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ is smooth and open. Setting $v(t) = f(t,.)$, where

$$v : [0,\infty) \longrightarrow L^2(\Omega)$$

is given by $v(t)(u) = f(t,u)$ and setting $h(\varphi)(u) = g(\varphi(u))$, where

$$h : L^2(\Omega) \longrightarrow L^2(\Omega),$$

we see that Equation (4) reduces to the evolution equation:

$$\begin{cases} v'(t) & + \; Av(t) = h(v(t)), \;\; t \geq 0, \\ v(0) & = \; f_0, \end{cases}$$

where $A := -\triangle$ is accretive.

In real Hilbert spaces, the concepts of accretivity and monotonicity coincide. However, as has been noted by Hezewinkel, Series Editor of *Mathematics and Its Applications*, Kluwer Academic Publishers,

> " It is probably impossible to overestimate the importance of the inner product for the study of problems and phenomena which take place in a Hilbert space. However, many, and probably most, mathematical objects and models do not live in Hilbert spaces" (Cioranescu [4], viii).

Since monotone operators have been studied extensively in Hilbert spaces, we shall concentrate our study of accretive operators on real Banach spaces. Let $A : E \to 2^E$ be a set-valued accretive operator on a real Banach space E. The Cauchy Problem for the following evolution inclusion:

$$0 \in \frac{du}{dt} + Au, \; u(0) = x, \; x \in D(A) \tag{5}$$

has been of interest to many authors (see, e.g., [5,6]).

A map $u : \mathbb{R}^+ \to E$ is a solution of (5) if on any bounded subinterval of $\mathbb{R}^+$, it is absolutely continuous and in addition, it is differentiable a.e. on $\mathbb{R}^+$ with $u(0) = x$, and satisfies the inclusion (5) a.e. on $\mathbb{R}^+$.

With this understanding of a solution of (5), it is known that (5) has at most one solution (see e.g., Cioranescu [4], Proposition 4.2, p. 210). At equilibrium state $\frac{du}{dt} \equiv 0$, thus, we deduce the following from (5):

$$0 \in Au, \tag{6}$$

whose solutions correspond to the equilibrium points of the dynamical system given in (5). Consequently, a problem of interest in the study of accretive operators is:

$$\text{find} \quad u \in D(A) \subset E \quad \text{with} \quad 0 \in Au. \tag{7}$$

The inclusion (7) has been considered in real Hilbert spaces and more general real Banach spaces by many authors. The well-known proximal point algorithm (PPA) of Martinet [7] has been employed for finding solutions of problem (7) involving maximal monotone operators in Hilbert spaces and the algorithm is given by:

$$\text{(PPA)} \qquad \begin{cases} x_1 \in H \\ x_{n+1} = J^A_{\lambda_n} x_n + e_n, \;\; n \geq 1, \end{cases} \tag{8}$$

where $J^A_{\lambda_n} = \left(I + \frac{1}{\lambda_n} A\right)^{-1}$, I is the identity mapping on H, $\lambda_n > 0$ is a regularizing parameter and e_n is an error vector. The algorithm (8) has been studied extensively by Rockafellar [8] who proved weak convergence of the sequence generated by (8) to a solution of (7). Since then, several modifications and alternatives of the PPA have been proposed by many authors to guarantee strong convergence to a solution of the inclusion problem (7) (see, e.g., [9,10] for the progress over the years).

Motivated by the use of the resolvent operator in algorithms for solving equations involving monotone operators, some authors have introduced the resolvent operator in iterative algorithms for solving equations involving m-accretive operators. The following theorems are two of the most general results now known for approximating solutions of (7) in more general real Banach spaces.

Theorem 1 (Xu, [11])**.** *Let E be a reflexive Banach space that has a weakly continuous duality map J with gauge φ and let A be an m-accretive operator on X such that $C = D(A)$ is convex. Assume (i) $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$ (ii) $\lim \lambda_n = \infty$. Given $u, x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J^A_{\lambda_n} x_n,\ n \geq 1. \tag{9}$$

Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Theorem 2 (Qin and Su, [12])**.** *Let E be a uniformly smooth real Banach space and A be an m-accretive operator in E such that $A^{-1}(0) \neq \emptyset$. Given a point $u \in C$ and given $\{\alpha_n\}$ in $(0,1)$ and $\{\beta_n\}$ in $[0,1]$, suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions:*

(i) $\lim \alpha_n = 0$ *and* $\sum \alpha_n = \infty$;

(ii) $\lim \lambda_n \geq \epsilon$, $\forall n$ *and* $\beta_n \in [0, a)$, *for some* $\epsilon > 0$ *and* $a \in (0,1)$;

(iii) $\sum |\alpha_{n-1} - \alpha_n| < \infty$, $\sum |\beta_{n-1} - \beta_n| < \infty$ *and* $\sum |\lambda_{n-1} - \lambda_n| < \infty$.

Let $\{x_n\}$ be the composite process defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) J^A_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n. \end{cases} \tag{10}$$

Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Remark 1.

1. *Examples of spaces that possess the weak sequential continuity of the duality mapping are l_p spaces, $1 < p < \infty$. However, for $p \neq 2$, L_p spaces, $1 < p < \infty$, do not possess this property.*
2. *The recurrence relation of Theorem 1 contains the resolvent operator $J^A_{\lambda_n}$ and the recurrence relation of Theorem 2 as well contains this resolvent operator.*

In line with this, the following question posed by Chidume [13] is of interest:

> "Can an iteration process be developed which will not involve the computation of $J^A_{\lambda_n} x_n$ at each step of the iteration process and which will guarantee strong convergence to a solution of $0 \in Au$?"

A partial answer in the affirmative to this question was given by Chidume and Djitte [14]. They introduced a resolvent free iterative algorithm in real Banach spaces that are 2-uniformly smooth and proved a strong convergence theorem for the class of m-accretive operators which are bounded. Hence, the following question became of interest:

Question 1. Can the requirement that the operator A be bounded imposed in the theorem of Chidume and Djitte [14] be dispensed with?

Recently, Chidume et al. [9] gave a positive answer to Question 1. They first proved a new and important result concerning accretive operators which is of independent interest: every accretive operator A on a real normed space with $0 \in int(D(A))$ is *quasi-bounded*. Combining this result with an incessive construction in some real Banach space E, they were able to dispense with the boundedness requirement on A in the Theorem of Chidume and Djitte [14]. Below is their theorem:

Theorem 3 (Chidume, et al., [9]). *Let E be a uniformly smooth real Banach space let $A : E \to 2^E$ be a set-valued m-accretive mapping such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$, define inductively a sequence $\{u_n\}$ by*

$$u_{n+1} = (1 - \alpha_n\beta_n)u_n - \alpha_n\psi_n,\ \psi_n \in Au_n,\ n \geq 1, \tag{11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0,\ \{\beta_n\}$ *is decreasing,*

(ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty,\ \sum_{n=1}^{\infty} \rho_E(\alpha_n M_0) < \infty$, *for some constant $M_0 > 0$,*

(iii) $\lim_{n\to\infty} \frac{\left(\frac{\beta_{n-1}-\beta_n}{\beta_n}\right)}{\alpha_n\beta_n} = 0$. *Assume that there exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\alpha_n)}{\alpha_n} \leq \gamma_0\beta_n$, then, $\{u_n\}$ converges strongly to a solution of (7).*

The objective of this paper is to introduce an inertial algorithm for solving Hammerstein equations involving accretive operators in certain Banach spaces. To do this, we first introduce a new inertial algorithm for solving nonlinear equations involving m-accretive operators and prove strong convergence theorems in real Banach spaces that are uniformly smooth. Finally, comparisons of the numerical performance of our algorithms with the performance of some recent important algorithms are presented.

2. Preliminaries

We shall make use of the lemmas below in the proof of our main results.

Lemma 1 (Xu and Roach, [15]). *Let E be a uniformly smooth real Banach space. Then, there exist constants D and C such that for all $x,\ y \in E,\ j(x) \in J(x)$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + D \max\left\{\|x\| + \|y\|, \frac{1}{2}C\right\}\rho_E(\|y\|),$$

where ρ_E denotes the modulus of smoothness of E.

Lemma 2 (see e.g., Chidume, [16]). *Let E be a normed real linear space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall \quad j(x + y) \in J(x + y), \quad \forall\ x,\ y \in E.$$

Lemma 3 (Xu, [17]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n b_n + c_n, \quad n \geq 1,$$

where $\{\sigma_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the conditions:

$(i) \quad \{\sigma_n\} \subset [0, 1],\ \sum_{n=1}^{\infty} \sigma_n = \infty; \qquad (ii) \limsup_{n\to\infty} b_n \leq 0; \qquad (iii)\ c_n \geq 0,\ \sum_{n=1}^{\infty} c_n < \infty.$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (Reich, [18]). *Let E be a uniformly smooth real Banach space, and let $A : E \to 2^E$ be m-accretive. Let $J_t x := (I + tA)^{-1}x,\ t > 0$ be the resolvent of A, and assume that $A^{-1}(0)$ is not empty. Then, for each $x \in E$, $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}(0)$.*

Lemma 5 (Fitzpatrick, Hess and Kato, [19]). *Let E be a real reflexive Banach space, $A : D(A) \subset E \to E$ be an accretive mapping. Then, A is locally bounded at any interior point of $D(A)$.*

Lemma 6 (Chidume et al., [9]). *Let E be a smooth and reflexive real Banach space and $A : E \to 2^E$ be an accretive map with $0 \in intD(A)$ ($intD(A)$ means interior of the domain of A). Then, given $M > 0$, there exists a constant $C > 0$ such that:*

(i) $(y, v) \in G(A)$ *($G(A)$ means the graph of A);*
(ii) $\langle v, j(x) - j(x-y) \rangle \leq M(2\|x\| + \|y\|)$;
(iii) $\|y\| \leq M, \quad \|x\| \leq M \quad$ *implies* $\quad \|v\| \leq C$.

3. Main Results

The following assumptions on our control sequences, $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ are central in what follows.

(i) $\lim_{n\to\infty} \theta_n = 0, \quad \{\theta_n\}$ is decreasing;
(ii) $\sum \lambda_n \theta_n = \infty$;
(iii) $\beta_n \leq \lambda_n^4 \theta_n \gamma_0$;
(iv) $\lim_{n\to\infty} \dfrac{\frac{\theta_{n-1}-\theta_n}{\theta_n}}{\lambda_n \theta_n} = 0$,
(v) $\dfrac{\rho_E(\lambda_n M)}{\lambda_n M} \leq \theta_n^2 \gamma_0$,
for some constants $\gamma_0 > 0$ and $M > 0$.

Prototypes. Take $\lambda_n = (n+1)^{-\frac{1}{4}}$, $\theta_n = (n+1)^{-\frac{1}{8}}$ and $\beta_n = (n+1)^{-\frac{9}{8}}$, $n \geq 1$, for L_p spaces, $2 \leq p < \infty$, and $\lambda_n = (n+1)^{-\frac{1}{4}}$, $\theta_n = (n+1)^{-\frac{p}{8}}$ and $\beta_n = (n+1)^{-\frac{(8+p)}{8}}$, $n \geq 1$, for L_p spaces, $1 < p < 2$ (see e.g., [16], for estimates of ρ_E in L_p spaces, $1 < p < \infty$).

One can easily verify assumptions (i)–(v) using these prototypes.

The settings for Lemma 7 and Theorem 4 are:

(1) The space E is a real Banach space which is uniformly smooth.
(2) The operator $A : E \to 2^E$ is set-valued m-accretive.
(3) The set of zeros of A is nonempty and the control sequences satisfy assumptions (i)–(v).

Lemma 7. *Given $z_0, z_1 \in E$, define iteratively a sequence $\{z_n\}$ in E by*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = w_n - \lambda_n \mu_n - \lambda_n \theta_n w_n, \quad \mu_n \in Aw_n, \quad n \geq 1. \end{cases} \tag{12}$$

Then, $\{z_n\}$ is bounded.

Proof. Given z^* in the set of solutions of the inclusion $0 \in Az$, and $z_1 \in E$, there exists $r \geq 2||z^*||$ such that $z_1 \in B(z^*, \frac{r}{2}) := \{z \in E : \|z - z^*\| \leq \frac{r}{2}\}$. Now, define the following constants:

$$M_0 := \sup\{\|\mu + \theta w\| : w \in B,\, 0 < \theta < 1,\, \mu \in Aw\} + 1,$$
$$M_1 := \sup\left\{D \max\left\{\|w - z^*\| + \lambda M_0, \frac{C}{2}\right\} : w \in B, \lambda \in (0,1)\right\},$$
$$M := \max\{M_0, M_1\}, \quad \gamma_0 := \frac{1}{2}\min\left\{1, \frac{r^2}{4M^2}\right\},$$

where C and D are the constants appearing in Lemma 1 (see [9] for a proof that these sups are well-defined).

Claim: $\{z_n\} \subset B$.

We prove this claim by induction. Observe that $z_1 \in B$, by construction. Now, assume $z_n \in B$, for some $n \geq 1$. Then, using the relation (12), Lemmas 1 and 2, definition of w_n and the fact that z^* is a solution, we compute as follows:

$$\begin{aligned} ||z_{n+1} - z^*||^2 &= ||w_n - z^* - \lambda_n(\mu_n + \theta_n w_n)||^2 \\ &\leq ||w_n - z^*||^2 - 2\lambda_n\langle \mu_n + \theta_n w_n, j(w_n - z^*)\rangle \\ &\quad + D\max\left\{||w_n - z^*|| + \lambda_n||\mu_n + \theta_n w_n||, \frac{C}{2}\right\}\rho_E(\lambda_n||\mu_n + \theta_n w_n||) \\ &\leq ||w_n - z^*||^2 - 2\lambda_n\langle \mu_n, j(w_n - z^*)\rangle - 2\lambda_n\theta_n\langle w_n, j(w_n - z^*)\rangle \\ &\quad + M\rho_E(\lambda_n M_0) \\ &\leq ||w_n - z^*||^2 - 2\lambda_n\theta_n\langle w_n - z^*, j(w_n - z^*)\rangle - 2\lambda_n\theta_n\langle z^*, j(w_n - z^*)\rangle \\ &\quad + M\rho_E(\lambda_n M) \\ &\leq (1 - \lambda_n\theta_n)||w_n - z^*||^2 + \lambda_n\theta_n||z^*||^2 + \frac{M\rho_E(\lambda_n M)}{\lambda_n M}\lambda_n M. \end{aligned}$$

$$\begin{aligned} ||z_{n+1} - z^*||^2 &\leq (1 - \lambda_n\theta_n)||z_n - z^*||^2 + 2M\beta_n + \lambda_n\theta_n||z^*||^2 + \frac{M\rho_E(\lambda_n M)}{\lambda_n}\lambda_n \\ &\leq (1 - \lambda_n\theta_n)r^2 + 2M\lambda_n\theta_n\gamma_0 + \frac{\lambda_n\theta_n r^2}{4} + M^2\lambda_n\theta_n\gamma_0 \\ &\leq r^2 - \frac{\lambda_n\theta_n r^2}{4}. \end{aligned} \tag{13}$$

Thus, by induction, $\{z_n\} \subseteq B$. Hence, $\{z_n\}$ and $\{w_n\}$ are bounded. □

Based on the setting above, we now give our main theorem.

Theorem 4. *Given $z_0, z_1 \in E$, define iteratively a sequence $\{z_n\}$ in E by*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = w_n - \lambda_n\mu_n - \lambda_n\theta_n w_n, \ \mu_n \in Aw_n, \quad n \geq 1. \end{cases} \tag{14}$$

Then $\{z_n\}$ converges strongly to a solution of the inclusion $0 \in Az$.

Proof. The proof basically follows as in the proof of Theorem 3.2 of [13]. However, for completeness, we sketch the details. Set $y_n := J_{t_n}z_1$, where z_1 is an arbitrary fixed vector in E, $t_n = \theta_n^{-1}$, $\forall\, n \geq 1$ in Lemma 4 and observe that with $\{t_n\}$, the sequence $\{y_n\}$ satisfies the following conditions:

$$\theta_n(y_n - z_1) + v_n = 0,\ v_n \in Ay_n \ \forall\, n \geq 1, \ \text{ and } \ y_n \to y^* \in A^{-1}0. \tag{15}$$

We now prove that $||z_{n+1} - y_n|| \to 0$, as $n \to \infty$. Using Lemma 1, we get

$$\begin{aligned} ||z_{n+1} - y_n||^2 &= ||w_n - y_n - \lambda_n(\mu_n + \theta_n w_n)||^2 \\ &\leq ||w_n - y_n||^2 - 2\lambda_n\langle \mu_n + \theta_n w_n, j(w_n - y_n)\rangle + M\rho_E(\lambda_n M_1) \\ &\leq ||w_n - y_n||^2 - 2\lambda_n\langle \mu_n - v_n, j(w_n - y_n)\rangle - 2\lambda_n\theta_n||w_n - y_n||^2 \\ &\quad - 2\lambda_n\langle v_n + \theta_n y_n, j(w_n - y_n)\rangle + M\rho_E(\lambda_n M) \\ &\leq (1 - \lambda_n\theta_n)||w_n - y_n||^2 + M\rho_E(\lambda_n M) \\ &\leq (1 - \lambda_n\theta_n)||z_n - y_n||^2 + 2M\beta_n + M\rho_E(\lambda_n M). \end{aligned} \tag{16}$$

Estimating $||y_{n-1} - y_n||$ and $||z_n - y_n||^2$ (see Theorem 3.2 of [13]), we get

$$||y_{n-1}-y_n|| \leq \left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)||y_{n-1}-z_1||. \quad (17)$$

$$||z_n-y_n||^2 \leq ||z_n-y_{n-1}||^2+2||y_{n-1}-y_n||||z_n-y_n||. \quad (18)$$

From inequalities (16)–(18), we have that

$$\begin{aligned}||z_{n+1}-y_n||^2 &\leq (1-\lambda_n\theta_n)||z_n-y_{n-1}||^2+M(\theta_{n-1}-\theta_n)\theta_n^{-1}+ M\rho_E(\lambda_n M)+ 2M\beta_n \\ &= (1-\lambda_n\theta_n)||z_n-y_{n-1}||^2+M(\lambda_n\theta_n)\delta_n+ M\frac{\rho_E(\lambda_n M)}{\lambda_n}\lambda_n+2M\beta_n, \\ &\leq (1-\lambda_n\theta_n)||z_n-y_{n-1}||^2+\lambda_n\theta_n M(\delta_n+ \theta_n\gamma_0)+M\lambda_n^4\theta_n\gamma_0,\end{aligned}$$

where $\sigma_n := \lambda_n\theta_n$, $\delta_n := \frac{\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)}{\lambda_n\theta_n}$, $a_n := ||z_n-y_{n-1}||^2$, $b_n := M(\delta_n+ \theta_n\gamma_0)$ and $c_n := M\lambda_n^4\theta_n\gamma_0$. Hence, by Lemma 3, it follows that $\lim_{n\to\infty}||z_n-y_{n-1}|| = 0$. Using Equation (15), we conclude that $\lim_{n\to\infty} z_n = y^*$. This proof is complete. □

4. Approximating Solutions of Hammerstein Equations

Definition 1. *Let Ω be a bounded subset of $\mathbb{R}^n$. Let $k : \Omega\times\Omega \to \mathbb{R}$ and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be real-valued functions that are measurable. An integral equation of the form*

$$u(x)+\int_\Omega k(x,y)f(y,u(y))dy = w(x), \quad (19)$$

where u and w are real-valued functions defined on Ω and are measurable is said to be of Hammerstein-type.

A motivation for the study of Hammerstein-type integral equations arise from their connection with differential equations, in particular, elliptic boundary value problems see, e.g., [20,21] for concentrate examples.

Let K be defined by $K(v) := \int_\Omega \kappa(x,y)v(y)dy;\ \ x\in\Omega$, and let F be defined by $Fu(y) := f(y,u)$, then, Equation (19) can put in the form

$$u+KFu = 0. \quad (20)$$

Equation (20) is called a Hammerstein equation. See, for example, Refs. [22–25] concerning existence and uniqueness results for the Hammerstein Equation (20) involving monotone mappings. Recently, Chidume et al. [10] established existence result for (20) involving accretive maps and concerning approximation of solutions of the Hammerstein Equation (20), see, e.g., [22,26–32] and the references therein.

Now, we use Theorem 4 to approximate solutions of Equation (20). The lemma below will play a crucial role in the proof of Theorem 5.

Lemma 8 (Chidume and Zegeye [33])**.** *For $q>1$, let E be a q-uniformly smooth real Banach space and let $F : E\to E$ be a continuous α-strongly accretive mapping and $K : E\to E$ be a continuous β-strongly accretive mapping such that $\alpha > \frac{d_q-1}{q}$ and $\beta > \frac{1}{q}$, for some $d_q > 1$. Then, $A : E\times E\to E\times E$ be defined by $A[u,v] := [Fu-v, Kv+u]$, is continuous γ-strongly accretive, where $\gamma = \min\{\alpha-\frac{d_q-1}{q}, \beta-\frac{1}{q}\}$.*

Remark 2. *We remark that a zero ($[u^*,v^*]$) of this A in Lemma 8 solves (20) with $v^* = Fu^*$.*

The setting for Theorem 5

(1) The space X is a real Banach spaces that is q-uniformly smooth, $q>1$.

(2) The operators F and K are as defined in Lemma 8.
(3) The set of solutions of (20) is nonempty and the control sequences satisfy assumptions (i)–(v) above.

Theorem 5. *For arbitrary (u_0, v_0), $(u_1, v_1) \in X \times X$, define the sequences $\{u_n\}$ and $\{v_n\}$ in X, by*

$$\begin{cases} c_n = u_n + \beta_n(u_n - u_{n-1}), \\ d_n = v_n + \beta_n(v_n - v_{n-1}), \\ u_{n+1} = c_n - \lambda_n(Fc_n - d_n) - \lambda_n\theta_n c_n, \quad n \geq 1, \\ v_{n+1} = d_n - \lambda_n(Kd_n + c_n) - \lambda_n\theta_n d_n, \quad n \geq 1. \end{cases} \tag{21}$$

Then the sequences $\{u_n\}$ and $\{v_n\}$ generated by (21) converge strongly to u^* and v^*, respectively, where u^* is a solution of (20), with $v^* = Fu^*$.

Proof. Clearly, $E := X \times X$ is uniformly smooth, by Lemma 8, $A := [Fu - v, Kv + u]$ is m-accretive. Therefore, the conclusion follows from Theorems 4 and Remark 2. □

Remark 3. *For the purpose of numerical illustration, we shall compare our Algorithm (21) with Algorithm (22) of Chidume et al. [10]. We give the theorem for completeness.*

Theorem 6 (Chidume et al. [10])**.** *Let E be a uniformly convex real Banach space and and $F, K : E \to E$ be m-accretive maps. For $(u_1, v_1) \in E \times E$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E, respectively by*

$$\begin{cases} u_{n+1} = u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} = v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1, \end{cases} \tag{22}$$

where λ_n and θ_n are sequences in (0,1). Suppose the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of (20) with $v^* = Fu^*$.

5. Numerical Experiments

In this section, we give comparisons of the numerical performance of our algorithms with the performance of some recent important algorithms.

Example 1. *(Zeros of m-accretive operator in $L_3([0,1])$, see Table 1 and Figures 1 and 2)*

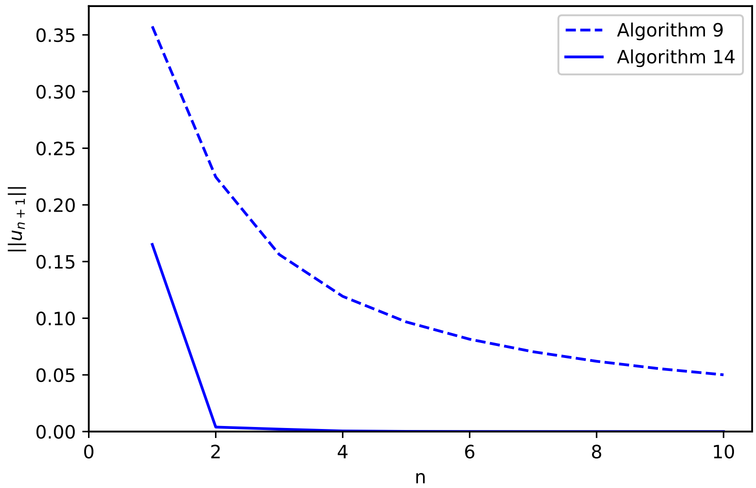
In Theorems 1, 2 and 4, set $E = L_3([0,1])$. Let $A : E \to E$ be defined by

$$(Au)(t) := u(t).$$

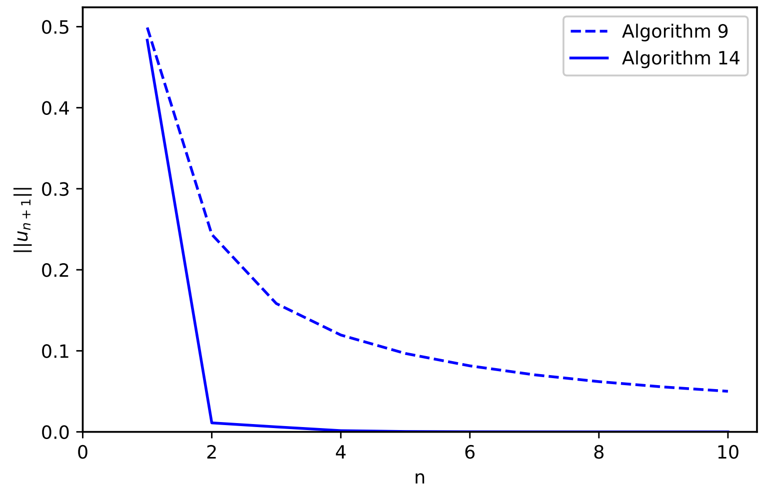
Clearly, A is m-accretive and the function $u(t) = 0, \forall\, t \in [0,1]$ is in the solutions set. In Theorem 4, take $\lambda_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \cdots$, as parameters. Also, in Theorem 1, we take $\alpha_n = \frac{1}{n+1}$ and $\lambda_n = n$, and in Theorem 2, we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{4}$ and $\lambda_n = 5$. Obviously, these parameters satisfy the assumptions of the theorems, respectively. Using a tolerance of 10^{-6} and setting maximum number of iterations $n = 10$, we get the following:

Table 1. Numerical experiment for Example 1 (for zeros of m-accretive map).

Algorithm (9)				Algorithm (10)				Algorithm (14)			
IP	**n**	$\|\|u_{n+1}\|\|$	**T (s)**	**IP**	**n**	$\|\|u_{n+1}\|\|$	**T (s)**	**IP**	**n**	$\|\|u_{n+1}\|\|$	**T (s)**
$u_1(t) = \sin t$	10	0.0501	0.029	$u_1(t) = \sin t$	10	0.0602	1.431	$u_0(t) = t$ $u_1(t) = \sin t$	10	3.0630×10^{-6}	17.252
$u_1(t) = t^2 + 1$	10	0.0501	0.020	$u_1(t) = t^2 + 1$	10	0.0602	0.040	$u_0(t) = t$ $u_1(t) = t^2 + 1$	10	8.887×10^{-6}	12.308
$u_1(t) = \frac{1}{2t+\cos t}$	10	0.0501	0.024	$u_1(t) = \frac{1}{2t+\cos t}$	10	0.0602	0.062	$u_0(t) = t$ $u_1(t) = \frac{1}{2t+\cos t}$	10	3.942×10^{-6}	19.140

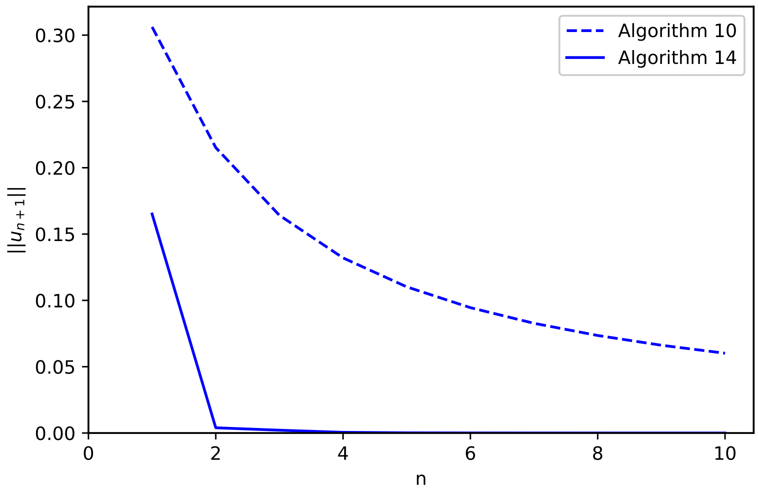


(**a**) Graph of some iterates of Algorithms (9) and (14) with $u_1(t) = \sin(t)$.

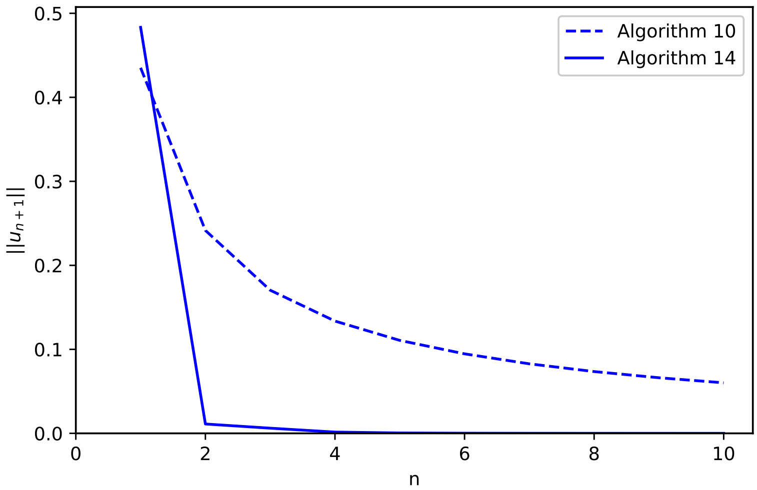


(**b**) Graph of some iterates of Algorithms (9) and (14) with $u_1(t) = t^2 + 1$.

Figure 1. Graph of some iterates of Algorithms (9) and (14).



(**a**) Graph of some iterates of Algorithms (10) and (14) with $u_1(t) = \sin(t)$.



(**b**) Graph of some iterates of Algorithms (10) and (14) with $u_1(t) = t^2 + 1$.

Figure 2. Graph of some iterates of Algorithms (10) and (14).

Example 2. *(Solutions of Hammerstein equation in* $L_3([0,1])$*, see Table 2 and Figure 3)*

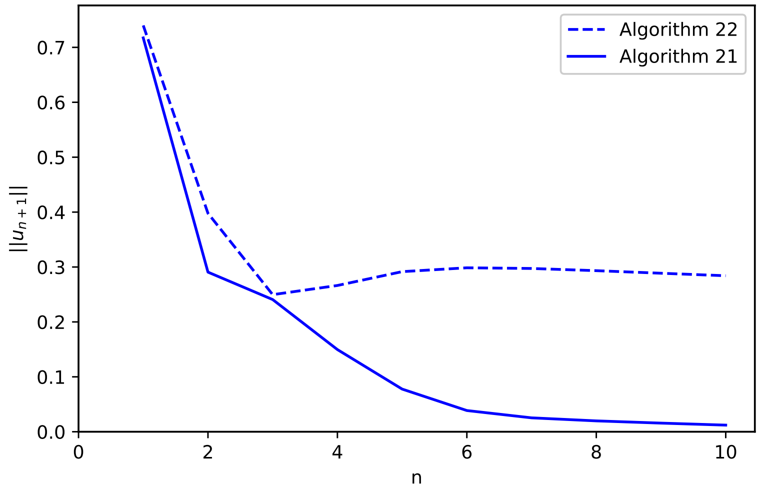
In Theorems 5 and 6, set $E = L_3([0,1])$. Set $F, K : E \to E$

$$Fu(t) = tu(t), \qquad Kv(t) = v(t).$$

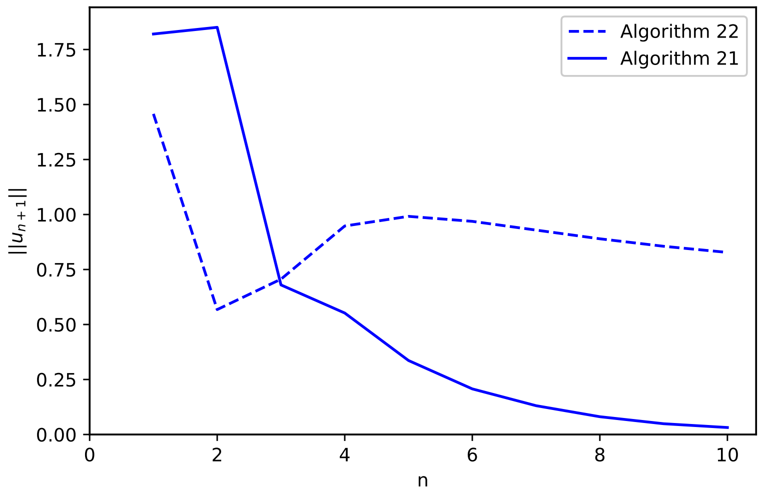
Clearly, K and F are m-accretive and $u^* = (0,0)^T$ is in the solutions set. In Theorem 6, we take $\lambda_n = \theta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, and in Theorem 5, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \cdots$, as our control sequences and fixed $u_0 = (-2,1)^T$ and $v_0 = (1,1)^T$. Obviously, these sequences satisfy the assumptions of the theorems respectively. Using a tolerance of 10^{-6} and setting maximum number of iterations $n = 10$, we get the following iterates:

Table 2. Numerical experiment for Example 2 (For Hammerstein equation).

Algorithm (22)				Algorithm (21)			
IP	**n**	$\|\|u_{n+1}\|\|$	**T (s)**	**IP**	**n**	$\|\|u_{n+1}\|\|$	**T (s)**
$u_1(t) = t\sin t$	10	0.2841	0.711	$u_1(t) = t\sin t$ $v_1(t) = \cos t$	10	0.0121	759.29
$u_1(t) = t - 4$	10	0.8277	1.564	$u_1(t) = t - 4$ $v_1(t) = 2$	10	0.0317	646.93
$u_1(t) = -4$	10	0.9526	0.637	$u_1(t) = -4$ $v_1(t) = e^t + 2t$	10	0.1455	21.39



(**a**) Graph of some iterates of Algorithms (21) and (22) with $u_1(t) = t\sin t$, $v_1(t) = \cos t$.



(**b**) Graph of some iterates of Algorithms (21) and (22) with $u_1(t) = t - 4$, $v_1(t) = 2$.

Figure 3. Graph of some iterates of Algorithms (21) and (22).

6. Conclusions

An inertial algorithm for approximating solutions of nonlinear Hammerstein equations is presented. This algorithm is obtained as a consequence of a new inertial algorithm proposed and studied for approximating zeros of m-accretive operators in uniformly smooth real Banach spaces. Strong convergence theorems are proved. Finally, using test examples, comparisons of the numerical performance of our algorithms with the performance of the algorithms of Theorems 6, 9 and 10 are presented. From the experiments (see Tables 1 and 2 and Figures 1–3) our proposed method appears to be competitive and promising.

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