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Oscillation Results for Nonlinear Higher-Order Differential Equations with Delay Term

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Abstract: The aim of this work is to investigate the oscillation of solutions of higher-order nonlinear differential equations with a middle term. By using the integral averaging technique, Riccati transformation technique and comparison technique, several oscillatory properties are presented that unify the results obtained in the literature. Some examples are presented to demonstrate the main results.

Keywords: delay; oscillation; higher-order

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1. Introduction

Nowadays, analysis of the oscillation properties of partial differential equations is attracting considerable attention from the scientific community due to numerous applications in several contexts such as biology, physics, chemistry, and dynamical systems (see [1–3]). For some details related to recent studies on the oscillation properties of the equations under consideration, we refer the reader to [4,5]. Moreover, the oscillation of partial equations contributes to many applications in economics, medicine, engineering, and biology.

In 2011, Run et al. [6] established new oscillation criteria for second-order partial differential equations with a damping term. Agarwal et al. [7] obtained some oscillation criteria for solutions of second-order neutral partial functional differential equations.

Over the past few years, the oscillation of Emden–Fowler-type neutral delay differential equations has attracted a lot of attention, see [8–15].

In this article, we investigate the oscillation of the higher-order delay differential equations

$$\left(\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}\right)' + \alpha_2(z)\left(w^{(j-1)}(z)\right)^{\gamma} + \sum_{i=1}^{n}\sigma_i(z)w^{\gamma}(\beta_i(z)) = 0,\ z \geq z_0 > 0. \tag{1}$$

Our novel outcomes are obtained by considering the following suppositions:

$$\begin{cases} \alpha_1 \in C^1([z_0,\infty),\mathbb{R}), \alpha_1'(z) \geq 0,\ \alpha_2,\ \sigma_i,\ \beta_i \in C([z_0,\infty),\mathbb{R}),\ \sigma_i > 0, \\ \beta_i \in C([z_0,\infty),\mathbb{R}),\ \beta_i(z) \leq z,\ \lim_{z\to\infty}\beta_i(z) = \infty,\ i = 1,2,..,n, \\ j \text{ is even},\ \gamma \text{ is a quotient of odd positive integers.} \end{cases}$$

The following condition is satisfied:

$$\int_{z_0}^{\infty}\left(\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(x)}{\alpha_1(x)}dx\right)\right)^{1/\gamma} ds = \infty. \tag{2}$$

Our main purpose for studying this work is to extend the results in [16]. We will use different methods to obtain these results.

In [16] the authors obtained oscillation criteria for fourth-order delay differential equations with middle term

$$\left[\alpha_1(z)w'''(z)\right]' + p(z)w'''(z) + \sigma(z)w(\beta(z)) = 0$$

under the condition

$$\int_{z_0}^{\infty} \frac{1}{\alpha_1(s)} \exp\left(-\int_{z_0}^{s} \frac{p(u)}{\alpha_1(u)} du\right) ds = \infty.$$

Bazighifan et al. [17,18] obtained some oscillation conditions for the equation

$$\begin{cases} \left(\alpha_1(z)\Phi_p[w^{(j-1)}(z)]\right)' + \alpha_2(z)\Phi_p[f\left(w^{(j-1)}(z)\right)] + \sum_{i=1}^{j} \sigma(z)\Phi_p[g(w(\beta_i(z)))] = 0, \\ \Phi_p[s] = |s|^{p-2}s,\ j \geq 1,\ z \geq z_0 > 0, \end{cases}$$

Zhang et al. in [19] investigated some oscillation properties of the equation

$$\begin{cases} L_w' + \alpha_2(z)\left|w^{(j-1)}(z)\right|^{p-2} w^{(j-1)}(z) + \sigma(z)|w(\beta(z))|^{p-2}w(\beta(x)) = 0, \\ 1 < p < \infty,\ z \geq z_0 > 0,\ L_w = \left|w^{(j-1)}(z)\right|^{p-2} w^{(j-1)}(z). \end{cases}$$

Bazighifan and Ramos [20] studied the following delay differential equations:

$$\begin{cases} \left(\alpha_1(z)\left(w^{(j-1)}(z)\right)^{p-1}\right)' + \alpha_2(z)\left(w^{(j-1)}(z)\right)^{p-1} + \sigma(z)w(\beta(z)) = 0, \\ z \geq z_0 > 0, \end{cases}$$

where $1 < p < \infty$.

Liu et al. [21] derived oscillation theorems for the equations

$$\begin{cases} \left(\alpha_1(z)\Phi\left(w^{(j-1)}(z)\right)\right)' + \alpha_2(z)\Phi\left(w^{(j-1)}(z)\right) + \sigma(z)\Phi(w(\beta(z))) = 0, \\ \Phi = |s|^{p-2}s,\ z \geq z_0 > 0, \end{cases}$$

where n is even and used the integral averaging technique.

Grace et al. [22] discussed the equation

$$\left[\alpha_1(z)\left(w^{(j-1)}(z)\right)^{r}\right]' + \sigma(z)w^{r}(g(z)) = 0$$

Zhang et al. [23] considered the even-order equation

$$\left[\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}\right]' + \sigma(z)w^{r}(\beta(z)) = 0, \quad z \geq z_0,$$

under condition

$$\int_{z_0}^{\infty} \alpha_1^{-1/\gamma}(s)\mathrm{d}s < \infty$$

and used the comparison technique.

The aim of this paper is to give several oscillatory properties of Equation (1). New criteria extend the results in [16].

In the following, we mention some notations.

$$\eta(z) \quad : \quad = \int_{z}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(x)}{\alpha_1(x)}dx\right)\right]^{1/\gamma}ds.$$

$$B(z) \quad : \quad = \frac{\int_z^{\infty}(\theta-z)^{j-4}\left(\frac{\int_\theta^{\infty}\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i(s)}{s}\right)^{\gamma}ds}{\alpha_1(\theta)}\right)^{1/\gamma}d\theta}{(j-4)!}$$

and

$$D(s) := \frac{\alpha_1(s)\delta_1(s)|h(z,s)|^{\gamma+1}}{\gamma+1^{\gamma+1}\left[H(z,s)A(s)\mu\frac{s^{j-2}}{(j-2)!}\right]^{\gamma}}.$$

2. Main Results

Here we present the following lemmas.

Lemma 1 ([24])**.** *Let $y^{(r)} > 0$ for all $r = 0, 1, ..., j$, and $y^{(j+1)} < 0$, then*

$$\frac{j!}{z^j}y(z) - \frac{(j-1)!}{z^{j-1}}\frac{\mathrm{d}}{\mathrm{d}z}y(z) \geq 0.$$

Lemma 2 ([25])**.** *Let $y \in C^j([z_0,\infty),(0,\infty))$ and $y^{(j-1)}(z)y^{(j)}(z) \leq 0$.If we have $\lim_{z\to\infty} y(z) \neq 0$, then*

$$y(z) \geq \frac{\epsilon}{(j-1)!}z^{j-1}\left|y^{(j-1)}(z)\right|$$

for all $\epsilon \in (0,1)$ and $z \geq z_\epsilon$.

Lemma 3 ([26])**.** *Let $y(z) \in C^r[z_0,\infty)$, $y^{(r)}(z) \neq 0$ on $[z_0,\infty)$ and $y(z)y^{(r)}(z) \leq 0$. Then*

(I) there exists a $z_1 \geq z_0$ such that the functions $y^{(m)}(z)$, $m = 1, 2, ..., r-1$ are of constant sign on $[z_0,\infty)$;

(II) there exists a number $a \in \{1, 3, 5, ..., r-1\}$ when r is even, $a \in \{0, 2, 4, ..., r-1\}$ when r is odd, such that, for $z \geq z_1$,

$$y(z)y^{(m)}(z) > 0,$$

for all $m = 0, 1, ..., a$ and

$$(-1)^{r+m+1}y(z)y^{(m)}(z) > 0.$$

Definition 1. *Let*

$$D = \{(z,s) \in \mathbb{R}^2 : z \geq s \geq z_0\} \text{ and } D_0 = \{(z,s) \in \mathbb{R}^2 : z > s \geq z_0\}.$$

We say that a function $H \in C(D,\mathbb{R})$ belongs to the class w if

(I_1) $H(z,z_0) = 0, H_*(z,z_0) = 0$ for $z \geq z_0, H(z,s) > 0, H_*(z,s) > 0,\ (z,s) \in D_0$;

$(I_2) H, H_*$ have a nonpositive continuous partial derivative $\partial H/\partial s, \partial H_*/\partial s$ on D_0 with respect to the second variable, and there exist functions $\delta_1, A, \delta_2,\ A_* \in C^1([z_0,\infty),(0,\infty))$ and $h, h_* \in C(D_0,\mathbb{R})$ such that

$$-\frac{\partial}{\partial s}(H(z,s)A(s)) = H(z,s)A(s)\frac{\delta_1'(z)}{\delta_1(z)} + h(z,s) \tag{3}$$

and

$$-\frac{\partial}{\partial s}(H_*(z,s)A_*(s)) = H_*(z,s)A_*(s)\frac{\delta_2'(z)}{\delta_2(z)} + h_*(z,s). \tag{4}$$

Theorem 1. *Let $j \geq 4$ be even. Let Equations (3) and (4) hold. If there exist functions $\delta_1, \delta_2 \in C^1([z_0, \infty), (0, \infty))$ such that*

$$\limsup_{z\to\infty} \frac{1}{H(z,z_0)} \int_{z_0}^{z} \left[H(z,s)A(s)\delta_1(s) \sum_{i=1}^{n} \sigma_i(s) \left(\frac{\beta_i^{j-1}(s)}{s^{j-1}} \right)^{\gamma} - D(s) \right] ds = \infty, \tag{5}$$

for some constant $\mu \in (0,1)$ and

$$\limsup_{z\to\infty} \frac{1}{H_*(z,z_0)} \int_{z_0}^{z} \left(H_*(z,s)A_*(s)\delta_2(s)B(s) - \frac{\delta_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*(s)} \right) ds = \infty, \tag{6}$$

then Equation (1) is oscillatory.

Proof. Let w be a nonoscillatory solution of Equation (1), then $w(z) > 0$. From Lemma 3, we have two possible cases:

$$\begin{array}{ll} (C_1) & w(z) > 0,\ w'(z) > 0, \dots,\ w^{(j-1)}(z) > 0,\ w^{(j)}(z) < 0, \\ (C_2) & w(z) > 0, w^{(r)}(z) > 0, w^{(r+1)}(z) < 0 \text{ for all odd integers} \\ & r \in \{1,2,...,j-3\}, w^{(j-1)}(z) > 0, w^{(j)}(z) < 0. \end{array}$$

Let case (C_1) hold. Define the function $y_1(z)$ by

$$y_1(z) := \delta_1(z) \left[\frac{\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}}{w^{\gamma}(z)} \right]. \tag{7}$$

Then $y_1(z) > 0$ for $z \geq z_1$ and

$$\begin{aligned} y_1'(z) \quad &\leq \quad \delta_1'(z) \frac{\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}}{w^{\gamma}(z)} + \delta_1(z) \frac{\left(\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}\right)'}{w^{\gamma}(z)} \\ &\quad -\delta_1(z) \frac{\gamma w'(z)\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}}{w^{\gamma+1}(z)}. \end{aligned}$$

By Lemma 2, we get

$$w'(z) \geq \frac{\mu}{(j-2)!} z^{j-2} w^{(j-1)}(z). \tag{8}$$

Using Equations (7) and (8), we obtain

$$\begin{aligned} y_1'(z) \quad &\leq \quad \delta_1'(z) \frac{\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}}{w^{\gamma}(z)} + \delta_1(z) \frac{\left(\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}\right)'}{w^{\gamma}(z)} \\ &\quad -\delta_1(z) \frac{\gamma\mu z^{j-2}}{(j-2)!} \frac{\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma+1}}{w^{\gamma+1}(z)}. \end{aligned} \tag{9}$$

By Lemma 1, we find

$$\frac{w(z)}{w'(z)} \geq \frac{z}{j-1}.$$

Thus we obtain that w/z^{j-1} is nonincreasing and so

$$\frac{w(\beta_i(z))}{w(z)} \geq \frac{\beta_i^{j-1}(z)}{z^{j-1}}. \tag{10}$$

From Equations (1) and (9), we get

$$y_1'(z) \;\le\; \delta_1'(z)\frac{\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma}}{w^{\gamma}(z)} - \delta_1(z)\frac{\sum_{i=1}^{n}\sigma_i(z)(w^{\gamma}(\beta_i(z)))}{w^{\gamma}(z)} \tag{11}$$
$$-\delta_1(z)\frac{\alpha_2(z)\left(w^{(j-1)}(z)\right)^{\gamma}}{w^{\gamma}(z)} - \delta_1(z)\frac{\gamma\mu z^{j-2}}{(j-2)!}\frac{\alpha_1(z)\left|\left(w^{(j-1)}(z)\right)\right|^{\gamma+1}}{w^{\gamma+1}(z)}.$$

From Equations (10) and (11), we obtain

$$y_1'(z) \;\le\; \left(\frac{\delta_1'(z)}{\delta_1(z)} - \frac{\alpha_2(z)}{\alpha_1(z)}\right)y_1(z) - \delta_1(z)\sum_{i=1}^{n}\sigma_i(z)\left(\frac{\beta_i^{j-1}(z)}{z^{j-1}}\right)^{\gamma} \tag{12}$$
$$-\frac{(\gamma)\mu z^{j-2}}{(j-2)!(\delta_1(z)\alpha_1(z))^{1/(\gamma)}}y_1^{(\gamma+1)/\gamma}(z).$$

It follows from Equation (12) that

$$\delta_1(z)\sum_{i=1}^{n}\sigma_i(z)\left(\frac{\beta_i^{j-1}(z)}{z^{j-1}}\right)^{\gamma} \le \left(\frac{\delta_1'(z)}{\delta_1(z)} - \frac{\alpha_2(z)}{\alpha_1(z)}\right)y_1(z) - y_1'(z) - \frac{\gamma\mu z^{j-2}}{(j-2)!(\delta_1(z)\alpha_1(z))^{1/(\gamma)}}y_1^{(\gamma+1)/\gamma}(z).$$

Replacing z by s, multiplying two sides by $H(z,s)A(s)$, and integrating the resulting inequality from z_1 to z, we have

$$\int_{z_1}^{z} H(z,s)A(s)\delta_1(s)\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i^{j-1}(s)}{s^{j-1}}\right)^{\gamma}ds \tag{13}$$
$$\le\; -\int_{z_1}^{z} H(z,s)A(s)y_1'(s)ds + \int_{z_1}^{z} H(z,s)A(s)\left(\frac{\delta_1'(s)}{\delta_1(s)} - \frac{\alpha_2(s)}{\alpha_1(s)}\right)y_1(s)ds$$
$$-\int_{z_1}^{z} H(z,s)A(s)\frac{\gamma\mu s^{j-2}}{(j-2)!(\delta_1(s)\alpha_1(s))^{1/(\gamma)}}y_1^{(\gamma+1)/\gamma}(s)ds$$
$$=\; H(z,z_1)A(z_1)y_1(z_1) - \int_{z_1}^{z}\left(-\frac{\partial}{\partial s}(H(z,s)A(s)) - H(z,s)A(s)\left(\frac{\delta_1'(s)}{\delta_1(s)} - \frac{\alpha_2(s)}{\alpha_1(s)}\right)\right)y_1(s)ds$$
$$-\int_{z_1}^{z} H(z,s)A(s)\frac{\gamma\mu s^{j-2}}{(j-2)!(\delta_1(s)\alpha_1(s))^{1/(\gamma)}}y_1^{(\gamma+1)/\gamma}(s)ds$$
$$\le\; H(z,z_1)A(z_1)y_1(z_1) + \int_{z_1}^{z}|h(z,s)|y_1(s)d(s)$$
$$-\int_{z_1}^{z} H(z,s)A(s)\frac{\gamma\mu s^{j-2}}{(j-2)!(\delta_1(s)\alpha_1(s))^{1/\gamma}}y_1^{(\gamma+1)/\gamma}(s)ds.$$

Note that

$$\varepsilon U V^{\varepsilon-1} - U^{\varepsilon} \le (\varepsilon-1)V^{\varepsilon}, \quad \varepsilon > 1, \;\; U \ge 0, \;\; V \ge 0. \tag{14}$$

Here

$$\varepsilon = (\gamma+1)/\gamma, \;\; U = \left(\gamma H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\gamma/(\gamma+1)}\frac{y_1(s)}{(\delta_1(s)\alpha_1(s))^{1/(\gamma+1)}}$$

and

$$V = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma}|h(z,s)|^{\gamma}\left(\frac{\delta_1(s)\alpha_1(s)}{\left(\gamma H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\gamma}}\right)^{\gamma/(\gamma+1)}.$$

From Equation (14), we get

$$\begin{aligned} & |h(z,s)|y_1(s) - H(z,s)A(s)\frac{\gamma\mu s^{j-2}}{(j-2)!(\delta_1(s)\alpha_1(s))^{1/\gamma}}y_1^{(\gamma+1)/\gamma} \\ \leq & \frac{\delta_1(s)\alpha_1(s)}{\left(H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\gamma}}\left(\frac{|h(z,s)|}{\gamma+1}\right)^{\gamma+1}. \end{aligned}$$

Putting the resulting inequality into Equation (13), we obtain

$$\begin{aligned} & \int_{z_1}^{z}\left[H(z,s)A(s)\delta_1(s)\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i^{j-1}(s)}{s^{j-1}}\right)^{\gamma} - \frac{\delta_1(s)\alpha_1(s)\left(\frac{|h(z,s)|}{\gamma+1}\right)^{\gamma+1}}{\left(H(z,s)A(s)\frac{\mu s^{j-2}}{(j-2)!}\right)^{\gamma}}\right]ds \\ \leq & H(z,z_1)A(z_1)y_1(z_1) \\ \leq & H(z,z_0)A(z_1)y_1(z_1). \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{H(z,z_0)}\int_{z_0}^{z}\left(H(z,s)A(s)\delta_1(s)\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i^{j-1}(s)}{s^{j-1}}\right)^{\gamma} - D(s)\right)ds \\ \leq & A(z_1)y_1(z_1) + \int_{z_0}^{z_1}A(s)\delta_1(s)\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i^{j-1}(s)}{s^{j-1}}\right)^{\gamma}ds < \infty, \end{aligned}$$

for some $\mu \in (0,\ 1)$, which contradicts Equation (5).

Let Case (C_2) hold. By virtue of $w'(z) > 0$ and $w''(z) < z$, from Lemma 1, we obtain

$$w(z) \geq ty'(z).$$

Thus we obtain that w/z is nonincreasing and so

$$w(\beta_i(z)) \geq w(z)\frac{\beta_i(z)}{z}. \tag{15}$$

From Equation (15) and integrating Equation (1) from z to ∞, we obtain

$$-\alpha_1(z)\left(w^{(j-1)}(z)\right)^{\gamma} + \int_z^{\infty}\sum_{i=1}^{n}\sigma_i(s)w(s)^{\gamma}\frac{\beta_i(s)^{\gamma}}{s^{\gamma}}ds \leq 0.$$

It follows from $w'(z) > 0$ that

$$-w^{(j-1)}(z) + \frac{w(z)}{\alpha_1^{1/\gamma}(z)}\left(\int_z^{\infty}\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i(s)}{s}\right)^{\gamma}ds\right)^{1/\gamma} \leq 0. \tag{16}$$

Integrating Equation (16) from z to ∞ for a total of $(j-3)$ times, we obtain

$$w''(z) + \frac{1}{(j-4)!}\int_z^{\infty}(\theta - z)^{j-4}\left(\frac{\int_{\theta}^{\infty}\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i(s)}{s}\right)^{\gamma}ds}{\alpha_1(\theta)}\right)^{1/\gamma}d\theta w(z) \leq 0. \tag{17}$$

Now, define

$$y_2(z) := \delta_2(z)\frac{w'(z)}{w(z)}. \tag{18}$$

Then $y_1(z) > 0$ for $z \geq z_1$ and

$$y_2'(z) = \delta_2'(z)\frac{w'(z)}{w(z)} + \delta_2(z)\frac{w''(z)w(z) - (w'(z))^2}{w^2(z)}.$$

It follows from Equations (17) and (18) that

$$\delta_2(z)B(z) \leq -y_2'(z) + \frac{\delta_2'(z)}{\delta_2(z)}y_2(z) - \frac{1}{\delta_2(z)}y_2^2(z).$$

Replacing z by s, multiplying two sides by $H_*(z,s)A_*(s)$, and integrating the resulting inequality from z_1 to z, we have

$$\begin{aligned}
\int_{z_1}^{z} H_*(z,s)A_*(s)\delta_2(s)B(s)ds \leq\; & -\int_{z_1}^{z} H_*(z,s)A_*(s)y_2'(s)ds \\
& + \int_{z_1}^{z} H_*(z,s)A_*(s)\frac{\delta_2'(s)}{\delta_2(s)}y_2(s)ds \\
& - \int_{z_1}^{z} \frac{H_*(z,s)A_*(s)}{\delta_2(s)}y_2^2(s)ds \\
=\; & H_*(z,z_1)A_*(z_1)y_2(z_1) - \int_{z_1}^{z} \frac{H_*(z,s)A_*(s)}{\delta_2(s)}y_2^2(s)ds \\
& - \int_{z_1}^{z} \left(-\frac{\partial}{\partial s}(H_*(z,s)A_*(s)) - H_*(z,s)A_*(s)\frac{\delta_2'(z)}{\delta_2(z)}\right)y_2(s)ds \\
\leq\; & H_*(z,z_1)A_*(z_1)y_2(z_1) + \int_{z_1}^{z} |h_*(z,s)|y_2(s)d(s) \\
& - \int_{z_1}^{z} \frac{H_*(z,s)A_*(s)}{\delta_2(s)}y_2^2(s)ds.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \int_{z_1}^{z}\left[H_*(z,s)A_*(s)\delta_2(s)\alpha_1(s) - \frac{\delta_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*}\right]ds \\
\leq\; & H_*(z,z_1)A_*(z_1)y_2(z_1) \\
\leq\; & H_*(z,z_0)A_*(z_1)y_2(z_1).
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{1}{H_*(z,z_0)}\int_{z_0}^{z}\left[H_*(z,s)A_*(s)\delta_2(s)B(s) - \frac{\delta_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*}\right]ds \\
\leq\; & A_*(z_1)y_2(z_1) + \int_{z_0}^{z} A_*(s)\delta_2(s)B(s)ds < \infty,
\end{aligned}$$

which contradicts Equation (6). Therefore, the theorem is proved. □

Theorem 2. *Let $j \geq 2$ be even and the equation*

$$x(z)\left(x'(z) + \frac{\alpha_2(z)}{\alpha_1(z)}x(z) + \frac{\sum_{i=1}^{n}\sigma_i(z)}{\alpha_1(\beta_i(z))}\left(\frac{\epsilon\beta_i^{j-1}(z)}{(j-1)!}\right)x(\beta_i(z))\right) = 0, \tag{19}$$

has no positive solutions. Then Equation (1) is oscillatory.

Proof. Let w be a nonoscillatory solution of Equation (1), then $w(z) > 0$. Hence we have

$$w'(z) > 0,\ w^{(j-1)}(z) > 0 \text{ and } w^{(j)}(z) < 0. \tag{20}$$

From Lemma 2, we obtain

$$w(z) \geq \frac{\epsilon z^{j-1}}{(j-1)!\alpha_1^{1/\gamma}(z)} \alpha_1^{1/\gamma}(z) w^{(j-1)}(z), \tag{21}$$

for all $\epsilon \in (0,1)$. Set

$$x(z) = \alpha_1(z) \left[w^{(j-1)}(z) \right]^{\gamma}.$$

Using Equation (21) in Equation (1), we obtain the inequality

$$x'(z) + \frac{\alpha_2(z)}{\alpha_1(z)} x(z) + \frac{\sum_{i=1}^{n} \sigma_i(z)}{\alpha_1(\beta_i(z))} \left(\frac{\epsilon \beta_i^{j-1}(z)}{(j-1)!} \right)^{\gamma} x(\beta_i(z)) \leq 0.$$

That is, x is a positive solution of the inequality in Equation (19), which is a contradiction. Thus the theorem is proved. □

Corollary 1. *Let $j \geq 2$ be even. If*

$$\liminf_{z \to \infty} \int_{\beta_i(z)}^{z} \frac{\sum_{i=1}^{n} \sigma_i(s)}{\alpha_1(\beta_i(s))} \left(\beta_i^{j-1}(s) \right)^{\gamma} \exp\left(\int_{\beta_i(s)}^{s} \frac{\alpha_2(u)}{\alpha_1(u)} du \right) ds > \frac{((j-1)!)^{\gamma}}{e}, \tag{22}$$

then Equation (1) is oscillatory.

3. Applications

As a matter of fact, the natural of the half-linear/Emden–Fowler differential equation appears in the study of several real-world problems such as biological systems, population dynamics, pharmacokinetics, theoretical physics, biotechnology processes, chemistry, engineering, and control (see [27–29]). In the context of these applications, we provide some examples below in this section.

Example 1. *Consider the delay equation*

$$w^{(4)}(z) + \frac{1}{z} w^{(3)}(z) + \frac{\varepsilon}{z^4} w\left(\frac{z}{4}\right) = 0, \ \varepsilon > 0, \ z \geq 1, \tag{23}$$

we see that $j = 4,\ \gamma = 1,\ \alpha_1(z) = 1,\ \alpha_2(z) = 1/z,\ \beta(z) = z/4,\ \sigma(z) = \varepsilon/z^4$ and

$$\eta(s) = \int_{z_0}^{\infty} \left[\frac{1}{\alpha_1(s)} \exp\left(- \int_{z_0}^{s} \frac{\alpha_2(u)}{\alpha_1(u)} du \right) \right]^{1/\gamma} ds = \infty.$$

Now, we find that

$$\begin{aligned} & \liminf_{z \to \infty} \int_{\beta_i(z)}^{z} \frac{\sum_{i=1}^{n} \sigma_i(s)}{\alpha_1(\beta_i(s))} \left(\beta_i^{j-1}(s) \right)^{\gamma} \exp\left(\int_{\beta_i(s)}^{s} \frac{\alpha_2(u)}{\alpha_1(u)} du \right) ds \\ = \ & \liminf_{z \to \infty} \int_{\beta_i(z)}^{z} \frac{\varepsilon}{s^4} \left(\frac{s^3}{64} \right) \exp(\ln 4) ds \\ = \ & \liminf_{z \to \infty} \int_{\beta_i(z)}^{z} \frac{\varepsilon}{16 s} ds = \frac{\varepsilon}{16} \ln 4 > \frac{6}{e}, \quad \text{if} \ \ \varepsilon > 96/(e \ln 4) = 24. \end{aligned}$$

Thus, using Corollary 1, Equation (23) is oscillatory if $\varepsilon > 24$.

Example 2. *Consider the delay equation*

$$\left(\frac{1}{z} w'''(z) \right)' + \left(1 \backslash \left(2z^2 \right) \right) w'''(z) + \frac{\varepsilon}{z} w\left(\frac{z}{2} \right) = 0, \ z \geq 1, \tag{24}$$

where $\varepsilon > 0$. *Let* $j = 4,\ \gamma = 1,\ \alpha_1(z) = 1/z,\ \alpha_2(z) = 1/\left(2z^2\right),\ \beta(z) = z/2,\ \sigma(z) = \varepsilon/z$ *and*

$$\eta(s) = \int_{z_0}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(u)}{\alpha_1(u)}du\right)\right]^{1/\gamma}ds = \infty.$$

Now, we see that Equation (22) holds. Thus, by Corollary 1, Equation (24) is oscillatory.

Example 3. *Consider the equation*

$$w^{(4)}(z) + \frac{1}{z^2}w^{(3)}(z) + \frac{\varepsilon}{z^4}w\left(4^{-1/3}z\right) = 0, \qquad z \geq 1, \tag{25}$$

where $\varepsilon > 0$ *is a constant. Let*

$$\begin{aligned} j &= 4,\ \alpha_1(z) = 1,\ \alpha_2(z) = 1/z^2,\ \gamma = 1,\ \beta(z) = 4^{-1/3}z,\ \sigma(z) = \varepsilon/z^4, \\ H(z,s) &= H_*(z,s) = (z-s)^2,\ A(s) = A_*(s) = 1, \\ \delta_1(s) &= z^3,\ \delta_2(s) = z,\ h(z,s) = h_*(z,s) = (z-s)\left(5 - s^{-1} + z\left(s^{-2} - 3s^{-1}\right)\right). \end{aligned}$$

Then we get

$$\begin{aligned} \eta(s) &= \int_{z_0}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(u)}{\alpha_1(u)}du\right)\right]^{1/\gamma}ds = \infty, \\ B(z) &= \frac{\int_z^{\infty}(\theta - z)^{j-4}\left(\frac{\int_\theta^\infty \sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i(s)}{s}\right)^{\gamma}ds}{\alpha_1(\theta)}\right)^{1/\gamma}d\theta}{(j-4)!} \\ &\geq \varepsilon/\left(12z^2\right). \end{aligned}$$

Now, we see that

$$\begin{aligned} &\limsup_{z\to\infty}\frac{1}{H(z,z_0)}\int_{z_0}^{z}\left(H(z,s)A(s)\delta_1(s)\sum_{i=1}^{n}\sigma_i(s)\left(\frac{\beta_i^{j-1}(s)}{s^{j-1}}\right)^{\gamma} - D(s)\right)ds \\ = &\limsup_{z\to\infty}\frac{1}{(z-1)^2}\int_1^z[\frac{\varepsilon}{4}z^2s^{-1} + \frac{\varepsilon}{4}s - \frac{\varepsilon}{2}z - \frac{s}{2\mu}(25 + s^{-2} - 10s^{-1} + z^2s^{-4} \\ &+9z^2s^{-2} - 6z^2s^{-3} + 16ts^{-2} - 2ts^{-3} - 30ts^{-1})]ds \\ = &\ \infty, \quad \text{if } \varepsilon > 18/\mu \quad \text{for some } \mu \in (0,1). \end{aligned}$$

Set

$$H_*(z,s) = (z-s)^2,\ A_*(s) = 1,\ \delta_2(s) = z,\ h_*(z,s) = (z-s)\left(3 - ts^{-1}\right).$$

Then we have

$$\begin{aligned} &\limsup_{z\to\infty}\frac{1}{H_*(z,z_0)}\int_{z_0}^{z}\left(H_*(z,s)A_*(s)\delta_2(s)\alpha_1(s) - \frac{\delta_2(s)|h_*(z,s)|^2}{4H_*(z,s)A_*(s)}\right)ds \\ \geq &\limsup_{z\to\infty}\frac{1}{(z-1)^2}\int_1^z\left[\frac{\varepsilon}{12}z^2s^{-1} + \frac{\varepsilon}{12}s - \frac{\varepsilon}{6}z - \frac{s}{4}\left(9 - 6ts^{-1} + z^2s^{-2}\right)\right]ds \\ = &\ \infty, \quad \text{if } \varepsilon > 3. \end{aligned}$$

Thus, by Theorem 1, Equation (25) is oscillatory if $\varepsilon \geq 19$.

4. Conclusions

In this article, we give several oscillation criteria of even-order differential equations with damped. These criteria that we obtained complement some oscillation theorems for delay differential equations with damping. In future work, we will discuss the oscillatory behavior of these equations by using a comparing technique with second-order equations under the condition

$$\int_{z_0}^{\infty}\left[\frac{1}{\alpha_1(s)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(x)}{\alpha_1(x)}dx\right)\right]^{1/\gamma}ds<\infty.$$

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