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The Flexible Burr X-G Family: Properties, Inference, and Applications in Engineering Science

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Abstract: In this paper, we introduce a new flexible generator of continuous distributions called the transmuted Burr X-G (TBX-G) family to extend and increase the flexibility of the Burr X generator. The general statistical properties of the TBX-G family are calculated. One special sub-model, TBX-exponential distribution, is studied in detail. We discuss eight estimation approaches to estimating the TBX-exponential parameters, and numerical simulations are conducted to compare the suggested approaches based on partial and overall ranks. Based on our study, the Anderson–Darling estimators are recommended to estimate the TBX-exponential parameters. Using two skewed real data sets from the engineering sciences, we illustrate the importance and flexibility of the TBX-exponential model compared with other existing competing distributions.

Keywords: Anderson–Darling estimation; Burr X-G family; skewed engineering data; maximum likelihood estimation; moments; transmuted class

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1. Introduction

Recently, several generalized families (also known as generators) of univariate distribution have been constructed based on classical distributions. These generators provide greater flexibility by adding one or more parameters to a baseline model. For example, Marshall-Olkin-G [1], transmuted-G [2], odd Lomax-G [3], Marshall-Olkin alpha-power-G [4], and odd Dagum-G [5], amongst others. The flexibility of these generalized models can be expressed in terms of their ability to model various real-life data encountered in different applied fields, in particular, reliability engineering, medicine, survival analysis, agriculture, actuarial sciences, demography, and others. The flexibility of generalized models is important to model several shapes of aging and failure criteria.

Shaw and Buckley [2] introduced a useful technique of adding a new parameter to an existing distribution called the transmuted-G (T-G) family, which is adopted to propose generalized forms of classical distributions. The T-G family has received widespread recognition in the literature and more than 70 generalized models have been proposed based on the T-G class. For example, the transmuted log-logistic [6], transmuted Marshall-Olkin Fréchet [7], and transmuted Burr XII [8], amongst others. Tahir and Cordeiro [9] listed more than 50 generalized models that were extended using the T-G family.

Several authors have constructed extended forms of the T-G class. Some notable examples are: transmuted exponentiated generalized-G [10], transmuted geometric-G [11], Kumaraswamy transmuted-G [12], generalized transmuted-G [13], and transmuted transmuted-G [14], and complementary generalized transmuted Poisson-G families [15].

The cumulative distribution function (CDF) of the T-G class has the form

$$F(x;\rho,\boldsymbol{\Phi}) = W(x;\boldsymbol{\Phi})[1+\rho-\rho W(x;\boldsymbol{\Phi})],\ x>0,\ |\rho|\leq 1. \tag{1}$$

Its probability density function (PDF) reduces to

$$f(x;\rho,\boldsymbol{\Phi}) = w(x;\boldsymbol{\Phi})[1+\rho-2\rho W(x;\boldsymbol{\Phi})],\ x>0,\ |\rho|\leq 1, \tag{2}$$

where ρ is a shape parameter; $W(x;\boldsymbol{\Phi})$ and $w(x;\boldsymbol{\Phi})$ are the baseline CDF and PDF, respectively; with parameter vector $\boldsymbol{\Phi}$. The T-G density is a mixture of a baseline density and an exponentiated-G (Exp-G) density with power parameter 2. For $\rho = 0$, the T-G class reduces to the baseline distribution.

We were motivated to adopt the T-G family to extend another class of distributions called the Burr X-G (BX-G) class [16] and provide a wider family that can be used to effectively model various real-life data. Hence, the aim of this study was two-fold: First, we wanted to propose a new extended form of the BX-G class based on the T-G class, called the transmuted Burr X-G (TBX-G) family. Various general properties of the TBX-G class are derived. Secondly, we discussed eight estimation methods of the TBX-exponential parameters: maximum likelihood (MLE), Anderson–Darling (ADE), right-tail Anderson–Darling (RADE), Cramér–von Mises minimum distance (CVME), ordinary least squares (OLSE), weighted least-squares (WLSE), maximum product of spacings (MPSE), and percentile (PCE) estimators, and compared them, in terms of their absolute value of biases (*Bias*), mean squared error (MSE), and mean relative error (MRE), using extensive simulations to develop a guideline for choosing the best estimation approach that produces more accurate estimates for the model parameters. The estimation methods were compared based on partial and overall ranks to choose the best estimation method, which will be of important interest to applied statisticians, practitioners, and engineers.

Yousof et al. [16] proposed a new generator for the construction of new extended and flexible versions from classical models, which is known as BX-G. Consider the PDF and CDF of a baseline distribution with a parameter vector $\boldsymbol{\Phi}$, $g(x;\boldsymbol{\Phi})$, and $G(x;\boldsymbol{\Phi})$; then, the CDF of the BX-G class, with a positive-shape parameter φ, takes the form

$$W_{BX}(x;\varphi,\boldsymbol{\Phi}) = \left\{1-e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^2}\right\}^{\varphi},\ x>0,\ \varphi>0. \tag{3}$$

The PDF of the BX-G family reduces to

$$w_{BX}(x;\varphi,\boldsymbol{\Phi}) = 2\varphi\frac{G(x;\boldsymbol{\Phi})\,g(x;\boldsymbol{\Phi})}{[1-G(x;\boldsymbol{\Phi})]^3}\left\{1-e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^2}\right\}^{\varphi-1} e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^2},\ x>0,\ \varphi>0. \tag{4}$$

To this end, we define the CDF and PDF of the proposed TBX-G family. By inserting Equation (3) into Equation (1), the TBX-G family can be specified by the following CDF (for $x>0,\ |\rho|\leq 1,\ \varphi>0$)

$$F\ (x;\rho,\varphi,\boldsymbol{\Phi}) = \left\{1-e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^2}\right\}^{\varphi}\left(1+\rho-\rho\left\{1-e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^2}\right\}^{\varphi}\right). \tag{5}$$

The PDF of the TBX-G class has the form

$$f\,(x;\rho,\varphi,\mathbf{\Phi}) = 2\varphi\frac{G(x;\mathbf{\Phi})\,g(x;\mathbf{\Phi})}{[1-G(x;\mathbf{\Phi})]^3}\left\{1-e^{-\left[\frac{G(x;\mathbf{\Phi})}{\bar{G}(x;\mathbf{\Phi})}\right]^2}\right\}^{\varphi-1} e^{-\left[\frac{G(x;\mathbf{\Phi})}{\bar{G}(x;\mathbf{\Phi})}\right]^2} \times\left(1+\rho-2\rho\left\{1-e^{-\left[\frac{G(x;\mathbf{\Phi})}{\bar{G}(x;\mathbf{\Phi})}\right]^2}\right\}^{\varphi}\right). \tag{6}$$

Hereafter, a random variable (rv) with PDF (6) is denoted by $X \backsim$TBX-G(ρ,φ,δ). The TBX-G class reduces to the BX-G family with $\rho = 0$. The TBX-G is a wider family of continuous distributions. It includes the BX-G family and provides greater flexibility in modeling real life data.

Finally, we summarize the findings of the proposed TBX-G class as follows: (1) Its sub-models provide unimodal, symmetrical, left-skewed, right-skewed, and reversed-J densities. They have decreasing, increasing, bathtub, upside-down bathtub, J-shaped, and reversed-J shaped hazard rates, which are frequently encountered in real-life applied areas. (2) Its special sub-models perform very well compared with other competing models, which are generated by well-known families under the same baseline model. (3) The sub-models generated by the TBX-G class are capable of modeling different shapes of ageing and failure criteria. Hence, the TBX-G can be a useful alternative to several classes for modeling skewed data in application(s).

The reminder of the paper is organized as follows: Two sub-models called the TBX-exponential (TBXE) and TBX-log-logistic (TBXLL) distributions are presented in Section 2. Various general properties of the TBX-G family are explored in Section 3. Properties of the TBXE model are discussed in Section 4. The maximum likelihood estimation for the TBX-G parameters are derived in Section 5. In Section 6, we estimate the TBXE parameters using eight estimation approaches. Section 7 is devoted to studying the behavior of these estimators via simulation results. In Section 8, we illustrate the flexibility of the new class using two real-life applications, showing that the TBXE model can provide a better fit than other competing models. Some concluding remarks are provided in Section 9.

2. Two Sub-Models

In this subsection, we introduce two special models of the TBX-G family based on the exponential and log-logistic models to generate the TBXE and TBXLL distributions, which can provide unimodal, symmetrical, left-skewed, right-skewed, and reversed-J-shaped densities; and decreasing, increasing, bathtub, upside-down bathtub, J-shaped, and reversed-J shaped hazard rates (Figures 1 and 2).

2.1. The TBXE Distribution

Consider the exponential model with PDF, $g(x;\beta) = \beta e^{-\beta x}$, $\beta, x > 0$; hence, CDF, PDF, and the hazard rate function (HRF) of TBXE are, respectively, given (for $\beta,\varphi > 0$, and $|\rho| \le 1$) by

$$F(x;\mathbf{\Theta}) = \left[1-e^{-\left(e^{\beta x}-1\right)^2}\right]^{\varphi}\left\{1+\rho-\rho\left[1-e^{-\left(e^{\beta x}-1\right)^2}\right]^{\varphi}\right\},$$

$$f(x;\mathbf{\Theta}) = 2\beta\varphi e^{\beta x}(e^{\beta x}-1)e^{-\left(e^{\beta x}-1\right)^2}\left[1-e^{-\left(e^{\beta x}-1\right)^2}\right]^{\varphi-1} \times\left\{1+\rho-2\rho\left[1-e^{-\left(e^{\beta x}-1\right)^2}\right]^{\varphi}\right\},\ x>0$$

and

$$h(x;\boldsymbol{\Theta}) = 2\beta\varphi e^{\beta x}(e^{\beta x}-1)\frac{e^{-(e^{\beta x}-1)^2}\left[1-e^{-(e^{\beta x}-1)^2}\right]^{\varphi-1}\left\{1+\rho-2\rho\left[1-e^{-(e^{\beta x}-1)^2}\right]^{\varphi}\right\}}{1-\left[1-e^{-(e^{\beta x}-1)^2}\right]^{\varphi}\left\{1+\rho-\rho\left[1-e^{-(e^{\beta x}-1)^2}\right]^{\varphi}\right\}},$$

where $\boldsymbol{\Theta} = (\beta, \varphi, \rho)^{\intercal}$.

Figure 1 illustrates some possible shapes of the PDF and HRF of the TBXE distribution for several selected values of β, φ, and ρ.

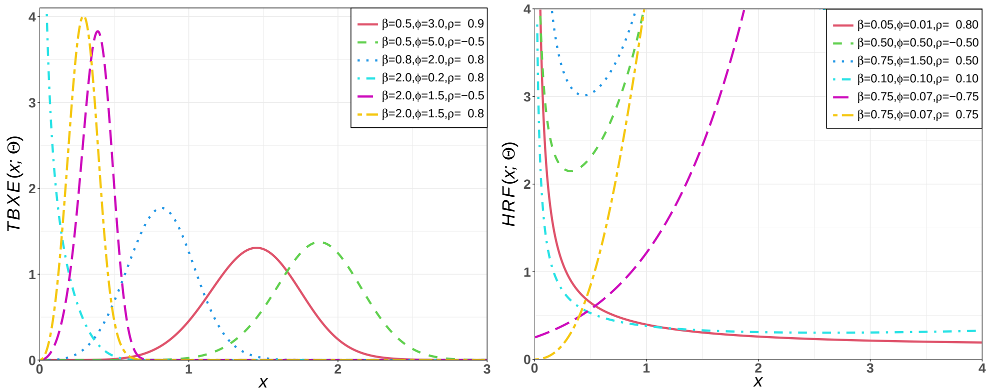


Figure 1. Plots of the probability density function (PDF) and hazard rate function (HRF) of the exponential transmuted Burr X-G (TBXE) distribution for some parameter values.

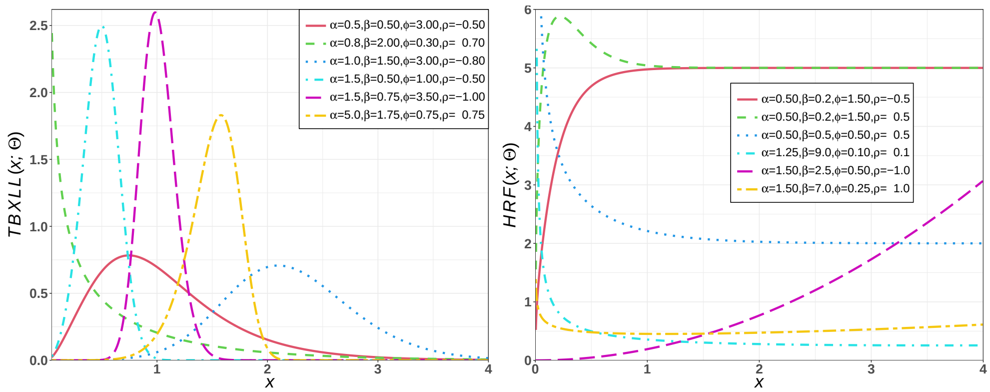


Figure 2. Plots of the PDF and HRF of the TBX-log-logistic (TBXLL) distribution for some parameter values.

2.2. The TBXLL Distribution

Using the PDF and CDF of the log-logistic distribution, $g(x) = \alpha\beta^{-\alpha}x^{\alpha-1}\left[\left(\frac{x}{\beta}\right)^{\alpha}+1\right]^{-2}$, $\alpha, \beta > 0,\ x > 0$ and $G(x) = 1-\left[\left(\frac{x}{\beta}\right)^{\alpha}+1\right]^{-1}$; hence, the CDF, PDF, and the HRF of TBXLL are, respectively, given (for $\alpha, \beta, \varphi > 0$, and $|\rho| \leq 1$) by

$$\begin{aligned} F(x;\boldsymbol{\Theta}) &= \left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi}\left\{1+\rho-\rho\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi}\right\}, \\ f(x;\boldsymbol{\Theta}) &= \frac{2\varphi\alpha}{x}\left(\frac{x}{\beta}\right)^{2\alpha}e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi-1}\left\{1+\rho-2\rho\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi}\right\},\ x>0 \end{aligned}$$

and

$$h(x;\boldsymbol{\Theta})=\frac{2\varphi\alpha}{x}\left(\frac{x}{\beta}\right)^{2\alpha}e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi-1}\left\{1+\rho-2\rho\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi}\right\}}{1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi}\left\{1+\rho-\rho\left[1-e^{-\left(\frac{x}{\beta}\right)^{2\alpha}}\right]^{\varphi}\right\}},\ x>0,$$

where $\boldsymbol{\Theta}=(\alpha,\beta,\varphi,\rho)^{\intercal}$.

Figure 2 illustrates some possible shapes of the PDF and HRF of the TBXLL distribution for several selected values of α, β, φ, and ρ.

3. Properties of the TBX-G Class

This section deals with the general properties of the TBX-G family, such as expansion for the TBX-G density, quantile function, moments, residual and reversed residual life functions, and order statistics.

3.1. Useful Expansion for the TBX-G Density

Some mathematical properties of the TBX-G family can be confirmed through an algebraic expansion in terms of exponential-G (Exp-G) distribution, which is more efficient than directly computing those by numerical integration of its PDF. The TBX-G PDF can be reformulated as

$$\begin{aligned}f(x;\rho,\varphi,\boldsymbol{\Phi})=&\,2(1+\rho)\,\varphi\frac{G(x;\boldsymbol{\Phi})\,g(x;\boldsymbol{\Phi})}{[1-G(x;\boldsymbol{\Phi})]^{3}}\,e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^{2}}\left\{1-e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{1-G(x;\boldsymbol{\Phi})}\right]^{2}}\right\}^{\varphi-1}\\&-4\rho\,\varphi\frac{G(x;\boldsymbol{\Phi})\,g(x;\boldsymbol{\Phi})}{[1-G(x;\boldsymbol{\Phi})]^{3}}\,e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{\bar{G}(x;\boldsymbol{\Phi})}\right]^{2}}\left\{1-e^{-\left[\frac{G(x;\boldsymbol{\Phi})}{1-G(x;\boldsymbol{\Phi})}\right]^{2}}\right\}^{2\varphi-1}.\end{aligned}\tag{7}$$

For a positive real non-integer, the binomial series holds

$$(1-s)^{\varphi-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\varphi-1}{j}s^{j},\ \ \varphi>0,\ \mid s\mid<1.\tag{8}$$

Applying the binomial series in (8) to (7), the TBX-G PDF reduces to

$$\begin{aligned}f\,(x;\rho,\varphi,\boldsymbol{\Phi})=&\,2(1+\rho)\varphi\frac{G(x;\boldsymbol{\Phi})\,g(x;\boldsymbol{\Phi})}{[1-G(x;\boldsymbol{\Phi})]^{3}}\sum_{i=0}^{\infty}(-1)^{i}\binom{\varphi-1}{i}\,e^{-(i+1)\left[\frac{G(x;\boldsymbol{\Phi})}{1-G(x;\boldsymbol{\Phi})}\right]^{2}}\\&-4\varphi\rho\frac{G(x;\boldsymbol{\Phi})\,g(x;\boldsymbol{\Phi})}{[1-G(x;\boldsymbol{\Phi})]^{3}}\sum_{i=0}^{\infty}(-1)^{i}\binom{2\varphi-1}{i}\,e^{-(i+1)\left[\frac{G(x;\boldsymbol{\Phi})}{1-G(x;\boldsymbol{\Phi})}\right]^{2}}.\end{aligned}\tag{9}$$

Applying the exponential series to the PDF (9), we write

$$e^{-(i+1)\left[\frac{G(x;\boldsymbol{\Phi})}{1-G(x;\boldsymbol{\Phi})}\right]^{2}}=\sum_{j=0}^{\infty}\frac{(-1)^{j}(i+1)^{j}}{j!}\frac{G(x;\boldsymbol{\Phi})^{2j}}{[1-G(x;\boldsymbol{\Phi})]^{2j}}.\tag{10}$$

Combining Equations (9) and (10), the PDF of the TBX-G family follows as

$$\begin{aligned}f\,(x;\rho,\varphi,\boldsymbol{\Phi})=&\,g(x;\boldsymbol{\Phi})\sum_{i,j=0}^{\infty}\frac{2\varphi(1+\rho)(-1)^{i+j}(i+1)^{j}\binom{\varphi-1}{i}}{j!}\frac{G(x;\boldsymbol{\Phi})^{2j+1}}{[1-G(x;\boldsymbol{\Phi})]^{2j+3}}\\&-g(x;\boldsymbol{\Phi})\sum_{i,j=0}^{\infty}\frac{4\varphi\rho(-1)^{i+j}(i+1)^{j}\binom{2\varphi-1}{i}}{j!}\frac{G(x;\boldsymbol{\Phi})^{2j+1}}{[1-G(x;\boldsymbol{\Phi})]^{2j+3}}.\end{aligned}\tag{11}$$

Using the general binomial expansion in (10) (for $(\varphi)_m = \varphi\,(\varphi+1)\ldots(\varphi+m-1)$)

$$(1-s)^{-\varphi} = \sum_{m=0}^{\infty} \frac{(\varphi)_m}{m!}\, s^m,\ \mid s \mid < 1, \tag{12}$$

where $(\varphi)_m$ is the ascending factorial, the TBX-G PDF can be formulated as a linear combination of Exp-G PDFs

$$f(x;\rho,\varphi,\mathbf{\Phi}) = \sum_{j,k=0}^{\infty} \xi_{j,k}\ \pi_{(2(j+1)+k)}(x), \tag{13}$$

where $\pi_\nu(x) = \nu\, g(x;\mathbf{\Phi})\ G(x;\delta)^{\nu-1}$ denotes the Exp-G PDF with power parameter ν and

$$\xi_{j,k} = \sum_{i=0}^{\infty} \frac{2\varphi(-1)^{i+j}(i+1)^j\ (2j+3)_k}{j!k!(2(j+1)+k)} \left[(1+\rho)\ \binom{\varphi-1}{i} - 2\rho\ \binom{2\varphi-1}{i}\right].$$

Equation (13) of the TBX-G PDF can be used to derive some mathematical properties of the TBX-G family directly from those of Exp-G distribution.

3.2. Quantile Function

The TBX-G quantile function (QF) takes the form

$$F^{-1}(u) = Q_G(u) = G^{-1}\left(\frac{\{-\log[1-A(\rho,\varphi,u)]\}^{\frac{1}{2}}}{1+\{-\log[1-A(\rho,\varphi,u)]\}^{\frac{1}{2}}}\right),\ 0 < u < 1, \tag{14}$$

where

$$A(\rho,\varphi,u) = \left[\frac{(\rho+1)+\sqrt{(\rho+1)-4\rho u}}{2\rho}\right]^{\frac{1}{\varphi}}$$

and u is a uniform random variable.

3.3. Moments

Let *r.v.* $Z_{(2(j+1)+k)}$ have an Exp-G PDF, $\pi_{(2(j+1)+k)}$, with power parameter $[2(j+1)+k]$. The rth moment of TBX-G family is obtained from (13) as

$$\mu'_r = E(X^r) = \sum_{j,k=0}^{\infty} \xi_{j,k}\ E(Z^r_{(2(j+1)+k)}). \tag{15}$$

The moment-generating function (MGF) of the TBX-G family can be calculated from (13) as

$$M_X(t) = E(e^{tX}) = \sum_{j,k=0}^{\infty} \xi_{j,k}\ M_{(2(j+1)+k)}(t), \tag{16}$$

where $M_{(2(j+1)+k)}(t)$ is the MGF of the *rv* $Z_{(2(j+1)+k)}$.

The sth incomplete moments for the TBX-G class are

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f\,(x)dx = \sum_{j,k=0}^{\infty} \xi_{j,k} \int_{-\infty}^{t} x^s\ \pi_{(2(j+1)+k)}(x)dx, \tag{17}$$

where the above integral denotes the sth incomplete moment of the *rv* $Z_{(2(j+1)+k)}$.

Now, we determine two formulae for the TBX-G first incomplete moment (FIM), which follows from (16) with $s = 1$. The first formula of FIM reduces to

$$\varphi_1(t) = \sum_{j,k=0}^{\infty} \xi_{j,k} I_{(2(j+1)+k)}(t), \tag{18}$$

where

$$I_{(2(j+1)+k)}(t) = \int_{-\infty}^{t} x\ \pi_{(2(j+1)+k)}(x)dx,$$

is the FIM of $Z_{(2(j+1)+k)}$. The second formula of the TBX-G FIM is

$$\varphi_1(t) = \sum_{j,k=0}^{\infty} \xi_{j,k}\ \Lambda_{(2(j+1)+k)}(t), \tag{19}$$

where

$$\Lambda_{(2(j+1)+k)}(t) = (2(j+1)+k) \int_{0}^{G(t)} Q_G(u)\ u^{2j+k+1}du,$$

can be calculated numerically.

3.4. Residual and Reversed Residual Life Functions

The rth-order moment of the residual life has the following formula

$$\begin{aligned} \mu_r(t) = E((X-t)^r \mid X > t) &= \frac{1}{\overline{F}(t)} \int_t^{\infty} (x-t)^r f(x)dx, r \geq 1 \\ &= \frac{1}{\overline{F}(t)} \sum_{j,k=0}^{\infty} \xi^*_{j,k} \int_t^{\infty} x^r\ \pi_{(2(j+1)+k)}(x)dx, \end{aligned}$$

where $\xi^*_{j,k} = \xi_{j,k} \sum_{h=0}^{r} \binom{r}{h} (-t)^{r-h}$. The mean remaining life (life expectancy at age t) of the TBX-G family follows from last formula with $r = 1$.

The rth-order moment of reversed residual life has the formula

$$\begin{aligned} m_r(t) = E((t-X)^r \mid X \leq t) &= \frac{1}{F(t)} \int_0^{t} (t-x)^r f\ (x)dx, r \geq 1 \\ &= \frac{1}{F(t)} \sum_{j,k=0}^{\infty} \xi^*_{j,k} \int_0^{t} x^r\ \pi_{(2(j+1)+k)}(x)dx. \end{aligned}$$

The mean past lifetime refers to the waiting time that elapses since an item fails, on the condition that the failure occurred in $(0, t)$. The mean inactivity time of the TBX-G class is obtained from the above formula with $r = 1$.

3.5. Order Statistics

The ordered statistics of a random sample from the TBX-G family are denoted by $X_{(1)}$, $X_{(2)}$,..., $X_{(n)}$, and they have some applications in reliability and survival analysis. The PDF of the ith-order statistic is defined by the following formula

$$f_{i;n}(x) = \frac{f\ (x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{i+j-1}(x), \tag{20}$$

where $B(.,.)$ refers to the beta function. Using the PDF and CDF of the TBX-G class, we can write

$$f\ (x)F^{i+j-1}(x) = \sum_{h,m=0}^{\infty} \phi_{h,m}\ \pi_{(2(m+1)+h)}(x), \tag{21}$$

where

$$\begin{aligned} \phi_{h,m} = 2\varphi & \sum_{k=0}^{i+j-1} \sum_{d=0}^{\infty} \frac{(-1)^{d+m}\ (d+1)^m\ \Gamma(2m+h+3)}{m!\ h!\ (2(m+1)+h)\ \Gamma(2m+3)} \binom{i+j-1}{k} \\ & \times \left[(\rho+1) \binom{\varphi(i+j)+k-1}{d} - 2\rho \binom{\varphi(i+j+1)+k-1}{d} \right]. \end{aligned}$$

Combining (19) and (20) gives

$$f_{i;n}(x) = \frac{1}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{h,m=0}^{\infty} \phi_{h,m}\, \pi_{(2(m+1)+h)}(x),$$

where $\pi_{(2(m+1)+h)}(x)$ is the Exp-G density with parameter $2(m+1)+h$. Then, the PDF of the TBX-G order statistics is a mixture of Exp-G PDFs.

The moments of $X_{i:n}$ reduce to

$$E(X_{i:n}^s) = \frac{1}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{h,m=0}^{\infty} \phi_{h,m}\, E(Z_{(2(m+1)+h)}^s).$$

4. Properties of TBXE Distribution

4.1. Linear Representation

Using Equation (13) and by applying the binomial expansion $[1-\exp(-\beta\, x)]^{2j+k+1}$, the PDF of TBXE distribution can be rewritten as

$$f(x;\beta,\varphi,\rho) = \sum_{j,k=0}^{\infty} [2(j+1)+k]\, \xi_{j,k}\, \beta \sum_{m=0}^{\infty} (-1)^m \binom{2j+k+1}{m} \exp[-(m+1)\,\beta\, x], \tag{22}$$

which can be expressed as

$$f(x;\beta,\varphi,\rho) = \sum_{m=0}^{\infty} \nu_m\, g_{m+1}(x), \tag{23}$$

where

$$\nu_m = \sum_{j,k=0}^{\infty} \frac{(-1)^m\, [2(j+1)+k]\, \xi_{j,k}}{m+1} \binom{2j+k+1}{m},$$

$$\xi_{j,k} = \sum_{i=0}^{\infty} \frac{2\varphi(-1)^{i+j}(i+1)^j\, (2j+3)_k}{j!k!(2(j+1)+k)} \left[(1+\rho)\, \binom{\varphi-1}{i} - 2\rho\, \binom{2\varphi-1}{i}\right]$$

and $g_{m+1}(x) = (m+1)\,\beta\, \exp[-(m+1)\,\beta\, x]$ denotes the exponential density with scale parameter $(m+1)\,\beta$. Then, the TBXE PDF can be expressed as a single linear combination of exponential densities. Let X be a random variable having an exponential distribution with PDF $g(x) = \beta\, \exp(-\beta\, x), x > 0,\ \beta > 0$. Then, the rth ordinary and incomplete moments and the MGF of X are

$$\mu'_r = \beta^{-r}\, r!,\ \phi_r(t) = \beta^{-r}\, \gamma(r+1,\beta t),\ M(t) = \frac{\beta}{\beta - t},\ t \neq 0,$$

respectively, where $\gamma(r+1,\beta\, t)$ is the lower incomplete gamma function.

4.2. QF, Moments, and MGF

The QF of the TBXE distribution is given by

$$Q_E(u) = \frac{1}{\beta} \log\left[\sqrt{-\log\left(1 - \left(\frac{1}{2\rho}\left[(\rho+1) - \sqrt{(\rho+1)^2 - 4\rho u}\right]\right)^{1/\phi}\right)} + 1\right]. \tag{24}$$

The rth moment of the TBXE distribution follows from Equation (23) as

$$\mu'_r = r! \sum_{m=0}^{\infty} \nu_m\, [(m+1)\,\beta]^{-r}. \tag{25}$$

Based on Equation (23), the rth incomplete moment of the TBXE distribution takes the form

$$\phi_r(t) = \sum_{m=0}^{\infty} \nu_m \, [(m+1)\,\beta]^{-r} \, \gamma(r+1, (m+1)\,\beta\, t). \tag{26}$$

The FIM of X follows from the last equation as

$$\phi_1(t) = \sum_{m=0}^{\infty} \nu_m \, [(m+1)\,\beta]^{-1} \, \gamma(2, (m+1)\,\beta\, t). \tag{27}$$

The MGF of the TBXE distribution is given by

$$M(t) = \sum_{m=0}^{\infty} \nu_m \, \frac{(m+1)\,\beta}{(m+1)\,\beta - t}. \tag{28}$$

4.3. Mean Residual Life and Mean Inactivity Time

The mean residual life (MRL), or life expectancy at age t, represents the expected additional life length for a unit, which is alive at age t.

The MRL of X is

$$m(t) = \frac{1-\phi_1(t)}{S(t)} - t, \tag{29}$$

where $\phi_1(t)$ is given in Equation (27) and $S(t)$ is the survival function of the TBXE distribution. Then, we obtain

$$m(t) = \frac{1}{S(t)} \left\{ 1 - \sum_{m=0}^{\infty} \nu_m \, [(m+1)\,\beta]^{-1} \, \gamma(2, (m+1)\,\beta\, t) \right\} - t. \tag{30}$$

The mean inactivity time (MIT) represents the waiting time elapsed since the failure of an item on the condition that this failure occurred in $(0, t)$.

The MIT of TBXE distribution is given by

$$MI(t) = t - \frac{\phi_1(t)}{F(t)} = t - \frac{1}{F(t)} \sum_{m=0}^{\infty} \nu_m \, [(m+1)\,\beta]^{-1} \, \gamma(2, (m+1)\,\beta\, t). \tag{31}$$

5. Maximum Likelihood Estimation

This section discusses the MLE of the TBX-G parameters. Let $x_1, \ldots, x_n$ be a random sample from the TBX-G class with $\boldsymbol{\psi} = (\rho, \varphi, \boldsymbol{\Phi}^T)^T$ being a parameter vector. The log-likelihood for $\boldsymbol{\psi}$ has the form

$$\begin{aligned} \ell(\boldsymbol{\psi}) = {} & n\log(2) + n\log(\varphi) + \textstyle\sum_{i=1}^{n} \log g(x_i;\boldsymbol{\Phi}) - 3\sum_{i=1}^{n} \log \overline{G}(x_i;\boldsymbol{\Phi}) \\ & + \textstyle\sum_{i=1}^{n} \log G(x_i;\boldsymbol{\Phi}) - \sum_{i=1}^{n} t_i^2 + (\varphi-1)\sum_{i=1}^{n} \log\left(1-e^{-t_i^2}\right) \\ & + \textstyle\sum_{i=1}^{n} \log\left[1+\rho-2\rho\left(1-e^{-t_i^2}\right)^{\varphi}\right], \end{aligned} \tag{32}$$

where $t_i = \frac{G(x_i;\boldsymbol{\Phi})}{[1-G(x_i;\boldsymbol{\Phi})]}$. The components of the score function are

$$U_n(\rho) = \sum_{i=1}^{n} \frac{1-2\left(1-e^{-t_i^2}\right)^{\varphi}}{1+\rho-2\rho\left(1-e^{-t_i^2}\right)^{\varphi}},$$

$$U_n(\varphi) = \frac{n}{\varphi} + \sum_{i=1}^{n} \log\left(1-e^{-t_i^2}\right) - 2\rho\sum_{i=1}^{n} \frac{\left(1-e^{-t_i^2}\right)^{\varphi} \log\left(1-e^{-t_i^2}\right)}{1+\rho-2\rho\left(1-e^{-t_i^2}\right)^{\varphi}}$$

and

$$U_n(\mathbf{\Phi}_k) = \sum_{i=1}^n \frac{g'(x_i;\mathbf{\Phi})}{g(x_i;\mathbf{\Phi})} + 3\sum_{i=1}^n \frac{G'(x_i;\mathbf{\Phi})}{G(x_i;\mathbf{\Phi})} + \sum_{i=1}^n \frac{G'(x_i;\mathbf{\Phi})}{G(x_i;\mathbf{\Phi})}$$
$$- 2\sum_{i=1}^n t_i \frac{G'(x_i;\mathbf{\Phi})}{\overline{G}^2(x_i;\mathbf{\Phi})} + 2(\varphi-1)\sum_{i=1}^n \frac{d_i\, t_i e^{-t_i^2}}{1-e^{-t_i^2}}$$
$$+ \sum_{i=1}^n \log \frac{2\rho\varphi\left(1-e^{-t_i^2}\right)^{\varphi-1} d_i\, t_i e^{-t_i^2}}{1+\rho-2\rho\left(1-e^{-t_i^2}\right)^{\varphi}},$$

where $g'(x_i;\mathbf{\Phi}) = \frac{\partial g(x_i;\mathbf{\Phi})}{\partial \mathbf{\Phi}}$, $G'(x_i;\mathbf{\Phi}) = \frac{\partial G(x_i;\mathbf{\Phi})}{\partial \mathbf{\Phi}}$ and $d_i = \frac{\partial}{\partial \mathbf{\Phi}} \frac{G(x_i;\mathbf{\Phi})}{[1-G(x_i;\mathbf{\Phi})]}$.

The above nonlinear system of equations can be solved using any statistical software.

6. Eight Estimation Methods for TBXE Parameters

We discuss eight estimation approaches of the TBXE parameters: MLE, ADE, RADE, CVME, OLSE, WLSE, MPSE, and PCE.

6.1. Maximum Likelihood

The maximum likelihood approach is used to estimate the unknown parameters of the TBXE model. The log-likelihood of the TBXE distribution reduces to

$$\begin{aligned}
\ell(\mathbf{\Theta}) \quad &= \quad n\sum_{i=1}^n \log(2\beta\varphi) + \log\left[-2\rho\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi} + \rho + 1\right] \\
&\quad +(\varphi-1)\sum_{i=1}^n \log\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right) \\
&\quad -\sum_{i=1}^n \left(e^{\beta x_i}-1\right)^2 + \beta\sum_{i=1}^n x_i + \sum_{i=1}^n \log\left(e^{\beta x_i}-1\right),
\end{aligned}$$

where $\mathbf{\Theta} = (\beta,\varphi,\rho)^\intercal$. From previous equation, we obtain

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} \quad &= \quad \frac{n}{\beta} + \sum_{i=1}^n x_i - \sum_{i=1}^n \frac{4\varphi x_i\rho\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi-1}\left[e^{2\beta x_i-\left(e^{\beta x_i}-1\right)^2} - e^{\beta x_i-\left(e^{\beta x_i}-1\right)^2}\right]}{\rho+1-2\rho\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi}} \\
&\quad + \sum_{i=1}^n \frac{x_i e^{\beta x_i}}{e^{\beta x_i}-1} + 2(\varphi-1)\sum_{i=1}^n \frac{x_i\left[e^{2\beta x_i-\left(e^{\beta x_i}-1\right)^2} - e^{\beta x_i-\left(e^{\beta x_i}-1\right)^2}\right]}{1-e^{-\left(e^{\beta x_i}-1\right)^2}} \\
&\quad - \sum_{i=1}^n \left(2x_i e^{2\beta x_i} - 2x_i e^{\beta x_i}\right),
\end{aligned}$$

$$\frac{\partial \ell}{\partial \varphi} \quad = \quad \frac{n}{\varphi} - \sum_{i=1}^n \frac{2\rho\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi}\log\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)}{\rho+1-2\rho\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi}} + \sum_{i=1}^n \log\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)$$

and

$$\frac{\partial \ell}{\partial \rho} = \sum_{i=1}^{n} \frac{1-2\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi}}{\rho+1-2\rho\left(1-e^{-\left(e^{\beta x_i}-1\right)^2}\right)^{\varphi}}.$$

Solving the previous equations mathematically is complicated, so the equations are solved by the numerical method.

6.2. Anderson–Darling and Right-Tail Anderson–Darling

The ADEs are a type of minimum distance estimators. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the order statistics of a random sample from $F(x)$ of the TBXE model. The ADEs of the TBXE parameters are obtained by minimizing

$$A(\beta,\varphi,\rho) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_i) + \log S(x_i)].$$

These ADEs can also be derived by solving the non-linear equation

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s(x_i)}{F(x_i)} - \frac{\Delta_i(x_{n+1-i})}{S(x_{n+1-i})}\right] = 0, \quad s = 1,2,3,$$

where

$$\Delta_1(x_{(i)}\beta,\varphi,\rho) = \frac{\partial}{\partial\beta}F(x_{(i)}\beta,\varphi,\rho),\ \Delta_2(x_{(i)}\beta,\varphi,\rho) = \frac{\partial}{\partial\phi}F(x_{(i)}\beta,\varphi,\rho)$$

and

$$\Delta_3(x_{(i)}\beta,\varphi,\rho) = \frac{\partial}{\partial\rho}F(x_{(i)}\beta,\varphi,\rho). \tag{33}$$

The RADEs of the TBXE parameters are obtained by minimizing

$$R(\beta,\varphi,\rho) = \frac{n}{2} - 2\sum_{i=1}^{n}F(x_{i:n}\beta,\varphi,\rho) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{n+1-i:n}\beta,\varphi,\rho),$$

with respect to β, φ and ρ. The RADEs can also be obtained by solving the non-linear equations

$$-2\sum_{i=1}^{n}\Delta_s(x_{i:n}\beta,\varphi,\rho) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s(x_{n+1-i:n}\beta,\varphi,\rho)}{S(x_{n+1-i:n}\beta,\varphi,\rho)} = 0,\ \ s = 1,2,3,$$

where $\Delta_1(\cdot\beta,\varphi,\rho)$, $\Delta_2(\cdot\beta,\varphi,\rho)$ and $\Delta_3(\cdot\beta,\varphi,\rho)$ are defined in Equation (33).

6.3. Cramér-Von Mises

The CVME is obtained based on the difference between the estimates of the CDF and the empirical CDF. The CVME of the TBXE parameters are obtained by minimizing

$$C(\beta,\varphi,\rho) = -\frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_i\beta,\varphi,\rho) - \frac{2i-1}{2n}\right]^2,$$

with respect to β, φ and ρ. The CVME can be calculated by solving the non-linear equation

$$\sum_{i=1}^{n}\left[F(x_i\beta,\varphi,\rho) - \frac{2i-1}{2n}\right]\Delta_s(x_i\beta,\varphi,\rho) = 0, \quad s = 1,2,3,$$

where $\Delta_1(\cdot\beta,\varphi,\rho)$, $\Delta_2(\cdot\beta,\varphi,\rho)$ and $\Delta_3(\cdot\beta,\varphi,\rho)$ are defined in Equation (33).

6.4. Ordinary and Weighted Least-Squares

The OLSEs of the parameters of the TBXE model are obtained by minimizing the following function with respect to β, φ, and ρ,

$$V(\beta,\varphi,\rho) = \sum_{i=1}^{n}\left[F(x_i\beta,\varphi,\rho) - \frac{i}{n+1}\right]^2.$$

The OLSE can be obtained by solving the non-linear equation

$$\sum_{i=1}^{n}\left[F(x_i\beta,\varphi,\rho) - \frac{i}{n+1}\right]\Delta_s(x_i\beta,\varphi,\rho) = 0, \quad s = 1,2,3,$$

where $\Delta_1(\cdot\beta,\varphi,\rho)$, $\Delta_2(\cdot\beta,\varphi,\rho)$ and $\Delta_3(\cdot\beta,\varphi,\rho)$ are defined in Equation (33).

The WLSEs of the parameters of the TBXE model are obtained by minimizing the following function

$$W(\beta,\varphi,\rho) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i\beta,\varphi,\rho) - \frac{i}{n+1}\right]^2,$$

with respect to β, φ, and ρ. The WLSE can be obtained by solving the non-linear equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i\beta,\varphi,\rho) - \frac{i}{n+1}\right]\Delta_s(x_i) = 0, \quad s = 1,2,3,$$

where $\Delta_1(\cdot\beta,\varphi,\rho)$, $\Delta_2(\cdot\beta,\varphi,\rho)$ and $\Delta_3(\cdot\beta,\varphi,\rho)$ are defined in Equation (33).

6.5. Maximum Product of Spacing

The maximum product of the spacings method, as an approximation of the Kullback–Leibler information measure, is a suitable alternative to the maximum likelihood method. Let $D_i(\beta,\varphi,\rho) = F\left(x_{(i)}\beta,\varphi,\rho\right) - F\left(x_{(i-1)}\beta,\varphi,\rho\right)$, for $i = 1,2,\ldots,n+1$, be the uniform spacing of a random sample from the TBXE model, where $F\left(x_{(0)}\beta,\varphi,\rho\right) = 0$, $F\left(x_{(n+1)}\beta,\varphi,\rho\right) = 1$, and $\sum_{i=1}^{n+1} D_i(\beta,\varphi,\rho) = 1$. The MPSE for $\widehat{\beta}_{MPSE}$, $\widehat{\varphi}_{MPSE}$, and $\widehat{\rho}_{MPSE}$ can be obtained by maximizing the geometric mean of the spacing

$$G(\beta,\varphi,\rho) = \left[\prod_{i=1}^{n+1} D_i(\beta,\varphi,\rho)\right]^{\frac{1}{n+1}},$$

with respect to β, φ and ρ or, equivalently, by maximizing the logarithm of the geometric mean of sample spacings

$$H(\beta,\varphi,\rho) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\beta,\varphi,\rho).$$

The MPSE of the TBXE parameters can be obtained by solving the nonlinear equations defined by

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\beta,\varphi,\rho)}\left[\Delta_s(x_{(i)}\beta,\varphi,\rho) - \Delta_s(x_{(i-1)}\beta,\varphi,\rho)\right] = 0, \quad s = 1,2,3,$$

where $\Delta_1(\cdot\beta,\varphi,\rho)$, $\Delta_2(\cdot\beta,\varphi,\rho)$ and $\Delta_3(\cdot\beta,\varphi,\rho)$ are defined in Equation (33).

6.6. Percentile

Let $u_i = i/(n+1)$ be an unbiased estimator of $F\left(x_{(i)}\beta, \varphi, \rho\right)$. Then, the PCEs of the parameters of TBXE model are obtained by minimizing the following function

$$P(\beta, \varphi, \rho) = \sum_{i=1}^{n}\left(x_{(i)} - \frac{1}{\beta}\log\left[\sqrt{-\log\left(1 - \left(\frac{1}{2\rho}\left[(\rho+1) - \sqrt{\delta(\rho, u_i)}\right]\right)^{1/\phi}\right)} + 1\right]\right)^2,$$

with respect to β,φ, and ρ, where $\delta(\rho, u_i) = (\rho+1)^2 - 4\rho\, u_i$.

7. Simulations

Here, we compare the eight estimation methods: WLSE, OLSE, MLE, MPSE, CVME, ADE, RADE, and PCE, using numerical simulations in terms of the average of absolute value of biases ($Bias(\widehat{\boldsymbol{\Theta}})$), $Bias(\widehat{\boldsymbol{\Theta}}) = \frac{1}{N}\sum_{i=1}^{N} \widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}$, the average of mean squared errors (MSEs), $MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})^2$, and the average of mean relative errors (MREs), $MREs = \frac{1}{N}\sum_{i=1}^{N} \widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}/\boldsymbol{\Theta}$. The simulation results can be used to develop a guideline for choosing the best estimation approach that provides more accurate estimates for the TBXE parameters. The R software (version 4.0.3) [17] was used to generate 5000 random samples from the TBXE distribution for sample sizes $n = 50$, 150, 300, and 400, along with different parameter values.

The simulation results, including absolute value of bias, MSE, and MRE, for different estimators and eight parameters combinations are reported in Tables A1–A8 in Appendix A. These tables show the rank of each of the estimators among all the estimators in each row, the superscripts are the indicators, and the $\sum Ranks$ is the partial sum of the ranks for each column in a certain sample size. Table 1 lists the partial and overall ranks of the estimators.

We observed that the behavior of the estimates of the TBXE parameters obtained using the eight methods of estimation are quite reliable. The bias decreased as n increased, showing that these estimates are asymptotically unbiased estimators. The MSE and MRE decreased as n increased, showing that these estimators are consistent.

- All estimator methods showed consistency, except the MLE estimator method, which showed consistency for all parameter combinations except for combinations $(\beta = 0.75, \varphi = 1.00, \rho = 0.30)^\intercal$ and $(\beta = 0.75, \varphi = 1.00, \rho = 0.75)^\intercal$.
- Form Table 1 and for the parameter combinations, we conclude that the ADE method outperformed all the other estimator methods (overall score of 71.5). Therefore, based on our study, we can consider the ADE method as the best.

Table 1. Partial and overall ranks of all estimation methods for various combinations of $\boldsymbol{\Theta}$. OLSE, ordinary least squares estimator; WLSE, weighted least squares estimator; MLE, maximum likelihood estimator; ADE, Anderson–Darling estimator; RADe, right-tail ADE; CVME, Cramér–von Mises minimum distance estimator; MPSE, maximum product of spacings estimator; PCE, percentile estimator.

$\boldsymbol{\Theta}^\intercal$	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
$(\beta = 3.00, \varphi = 1.50, \rho = 0.75)$	50	5	7	5	2	8	3	5	1
	150	5	7	6	2	8	3	4	1
	300	6	7.5	4.5	1	7.5	3	4.5	2
	400	6	7	5	1	8	4	3	2
$(\beta = 0.75, \varphi = 1.00, \rho = 0.30)$	50	5	7	4	8	6	2	1	3
	150	2	7	6	8	5	1	4	3
	300	2	8	6.5	6.5	5	1	4	3
	400	2	8	6	7	5	1	4	3

Table 1. *Cont.*

$\boldsymbol{\Theta}^{\top}$	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
$(\beta = 1.00, \varphi = 1.50, \rho = -0.50)$	50	2	5.5	3	7	4	1	8	5.5
	150	2	5.5	5.5	1	4	3	7	8
	300	3	5	4	1	6	2	7	8
	400	2	5	3	1	7	4	8	6
$(\beta = 4.00, \varphi = 3.00, \rho = 0.50)$	50	2	6	7	8	5	1	4	3
	150	2	5	7	3	6	1	4	8
	300	2	7	6	1	5	3	4	8
	400	1.5	4	7	1.5	6	3	5	8
$(\beta = 3.00, \varphi = 1.50, \rho = 0.30)$	50	5	6	1	8	4	2	3	7
	150	3	7	1	8	4	2	5	6
	300	3	8	1	7	6	2	5	4
	400	3	8	1	6	7	2	5	4
$(\beta = 0.75, \varphi = 1.00, \rho = 0.75)$	50	5	6	7	2	8	3	4	1
	150	5	6	8	1	7	4	3	2
	300	5	6	8	1	7	3	4	2
	400	5	6	8	1	7	3	4	2
$(\beta = 4.00, \varphi = 1.50, \rho = -0.50)$	50	2	6	3	7	5	1	8	4
	150	1	5	6	3.5	3.5	2	7	8
	300	2	5	4	1	6	3	8	7
	400	2	6.5	4	1	6.5	3	8	5
$(\beta = 1.00, \varphi = 3.00, \rho = 0.50)$	50	2	4.5	7	6	4.5	1	3	8
	150	1.5	4	7	6	5	1.5	3	8
	300	2	4	6.5	3	5	1	6.5	8
	400	3	7	5	1	6	2	4	8
$\sum$ *Ranks*		99	196.5	163	121.5	187	71.5	157	156.5
Overall Rank		2	8	6	3	7	1	5	4

8. Modeling Two Real Data

This section provides a discussion on the flexibility of TBXE distribution in fitting two real-world data sets from engineering science and comparing it with other competing distributions. The discrimination criteria, including minus maximized log-likelihood ($-\widehat{\ell}$), Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan information criterion (HQIC), Cramér–Von Mises (W), Anderson–Darling (A), and Kolmogorov–Smirnov (K-S) statistics, with their corresponding p-values, are used to compare the fitted competitive distributions.

Data set I represents the data of the breaking stress of carbon fibers, which consist of 100 observations, and was introduced by Nichols and Padgett [18]. This data set was analyzed by [19,20].

Data set II refers to time-to-failure (10^3 h) of the turbocharger of one type of engine, as reported in [21]. The data consist of 40 observations and are used to show the flexibility of TBXE model compared with the same distributions for the first data set.

The two analyzed data sets are used to show the flexibility of the TBXE model compared with some well-known distributions such as Marshall–Olkin logistic exponential (MOLE) [22], gamma (Ga), beta exponential (BE) [23], generalized transmuted Poisson exponential (GTPE) [24], alpha power exponential (APE) [25], transmuted generalized exponential (TGE) [26], exponential (TEGE) [10], exponentiated exponential (EE) [27], transmuted exponentiated generalized Fréchet–Weibull mixture exponential (FWME) [28], and exponential (E) distributions.

Some descriptive statistics of data sets I and II are reported in Tables 2 and 3, respectively. Tables 4 and 5 report parameters estimates with their corresponding standard errors,

$-\widehat{\ell}$, AIC, CAIC, BIC, HQIC, W, A, and (K-S) statistics (K-S (stat)) with their corresponding p-value (K-S (p-value)), for the given data sets using ADE (as recommended in the Simulations section). The figures in these tables reveal that the new TBXE model provides a close fit to the modeled data sets compared with other competing distributions.

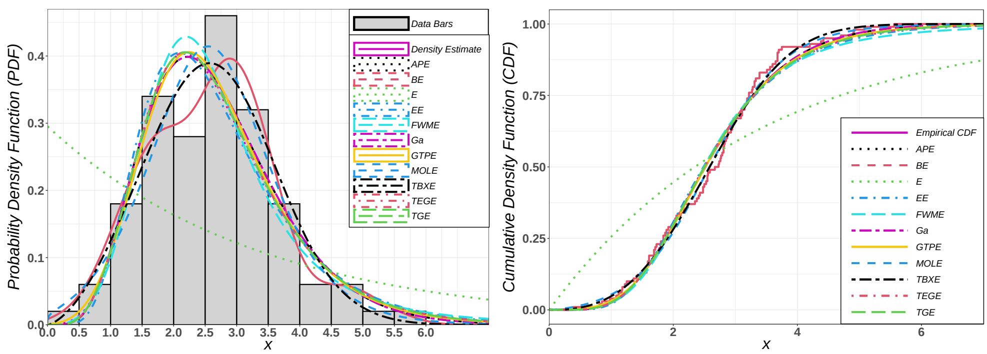
Table 2. Descriptive statistics for the carbon data set.

n	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
100	0.390	1.840	2.7	2.621	3.22	5.56	0.3682	3.10494

Table 3. Descriptive statistics for time-to-failure data set.

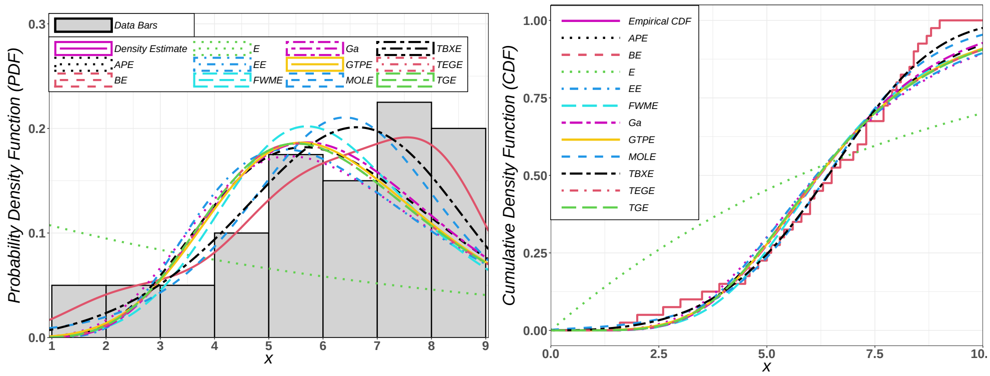
n	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
40	1.6	5.075	6.5	6.253	7.825	9	−0.6626	2.641

The fitted PDF and CDF plots for the TBXE model and other fitted models for the first and second data sets are depicted in Figures 3 and 4, respectively. The HRF plot of the TBXE and total time on test (TTT) plot for the first and second data sets are depicted in Figures 5 and 6, respectively. The TTT plot can be use for identifying the behavior of the HRF of the data. Figures 7 and 8 provide the plots of the probability–probability (PP) of the TBXE model and other fitted models for the two data sets, respectively.



(**a**) PDF plots (**b**) CDF plots

Figure 3. Fitted densities and distribution functions of the competing models for the first data set.

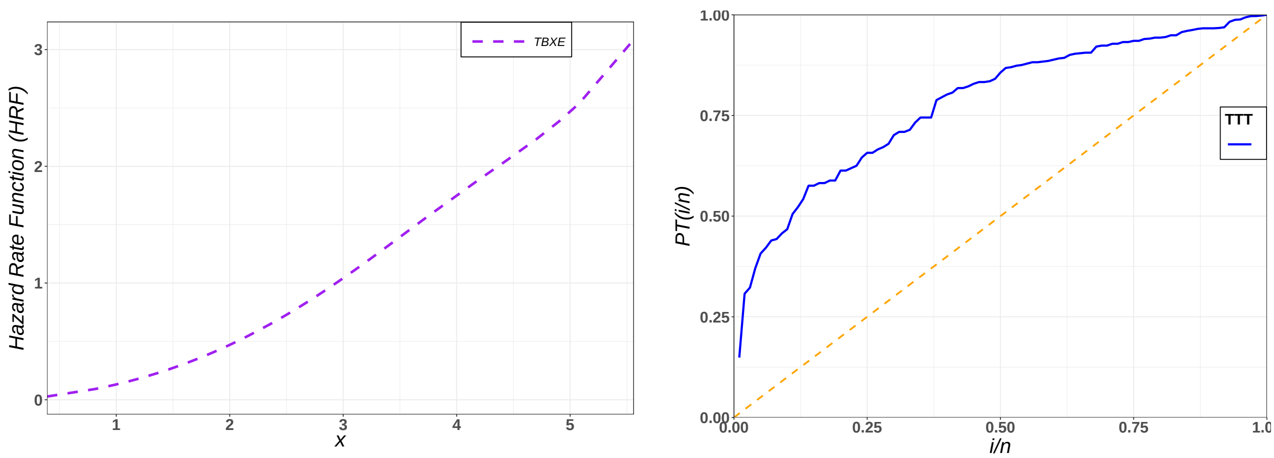


(**a**) PDF plots (**b**) CDF plots

Figure 4. Fitted densities and distribution functions of the competing models for the second data set.

Table 4. Estimates, standard errors (SEs), minus maximized log-likelihood ($-\widehat{\ell}$), Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan information criterion (HQIC), Cramér–Von Mises (W), Anderson–Darling (A), and Kolmogorov–Smirnov (K-S) statistics, with their corresponding p-value for the first data set.

Distribution	Estimates	SEs	$-\ell$	AIC	CAIC	BIC	HQIC	W	A	K-S (stat)	K-S (p-Value)
TBXE	$\hat{\varphi} = 0.2103$ $\hat{\rho} = 1.2244$ $\hat{\beta} = 0.7533$	0.0504 0.2481 0.8499	141.4421	288.8843	289.1343	296.6998	292.0473	0.0558	0.3942	0.0556	0.9167
MOLE	$\hat{\rho} = 1.3181$ $\hat{\alpha} = 1.2146$ $\hat{b} = 60.0294$	2.7204 2.1449 110.6220	141.8820	289.7640	290.0140	297.5795	292.9270	0.0646	0.3974	0.0602	0.8611
Ga	$\hat{a} = 6.1097$ $\hat{b} = 2.2957$	1.6289 0.6393	143.3214	290.6428	290.7665	295.8531	292.7515	0.1483	0.7586	0.0786	0.5676
BE	$\hat{a} = 6.1697$ $\hat{b} = 12.8069$ $\hat{\rho} = 0.1526$	1.6852 35.3987 0.3655	143.3920	292.7840	293.0340	300.5995	295.9471	0.1503	0.7692	0.0788	0.5642
GTPE	$\hat{a} = 1.1667$ $\hat{\rho} = -0.5609$ $\hat{\varphi} = 8.6836$	0.1614 0.8771 6.7468	143.3149	292.6298	292.8798	300.4453	295.7929	0.1543	0.7775	0.0800	0.5448
APE	$\hat{\alpha} = 48{,}068.1765$ $\hat{\rho} = 1.0969$	17,441.3621 0.0491	144.2753	292.5505	292.6743	297.7609	294.6593	0.1857	0.9472	0.0830	0.4967
TGE	$\hat{\alpha} = 7.0857$ $\hat{\rho} = 1.1123$ $\hat{\varphi} = -0.5909$	5.2757 0.1741 0.7526	144.5828	295.1655	295.4155	302.9810	298.3286	0.1867	0.9565	0.0830	0.4959
TEGE	$\hat{a} = 2.4724$ $\hat{b} = 7.0858$ $\hat{\rho} = -0.5909$ $\hat{\varphi} = 0.4499$	28.4292 5.2757 0.7527 5.1732	144.5828	297.1656	297.5867	307.5863	301.3831	0.1867	0.9565	0.0830	0.4959
EE	$\hat{\alpha} = 8.7779$ $\hat{\rho} = 1.0367$	3.4978 0.1598	146.5126	297.0252	297.1489	302.2355	299.1339	0.2349	1.2320	0.0875	0.4285
FWME	$\hat{a} = 1.8003$ $\hat{\alpha} = 2.7086$ $\hat{\rho} = 2.8893$ $\hat{k} = 1.4849$	277.2217 420.0028 110.6217 230.2469	146.3190	300.6381	301.0592	311.0588	304.8555	0.2395	1.2437	0.0890	0.4064
E	$\hat{\rho} = 0.2954$	0.0307	199.3820	400.7641	400.8049	403.3693	401.8184	0.1542	0.7904	0.2504	7.1922×10^{-6}

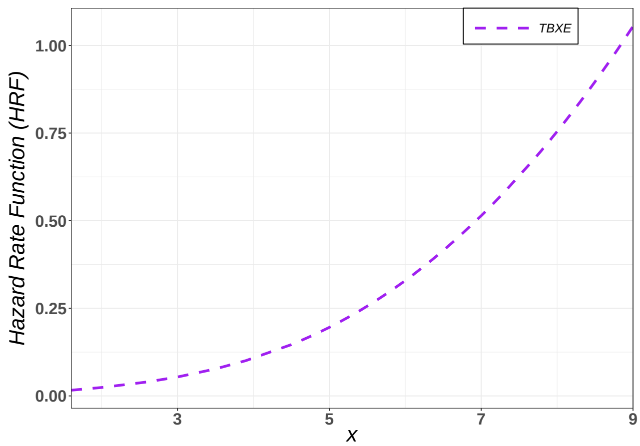


(**a**) HRF plot (**b**) TTT plot

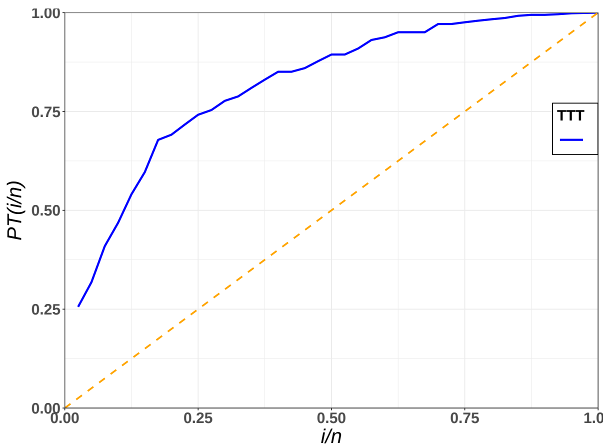
Figure 5. The hazard rate function (HRF) plot of the TBXE model and total time on test (TTT) plot for the first data set.

Table 5. Estimates, SEs, $-\widehat{\ell}$, AIC, CAIC, BIC, HQIC, W, A, and K-S (stat) with their corresponding p-values for the second data set.

Distribution	Estimates	SEs	$-\ell$	AIC	CAIC	BIC	HQIC	W	A	K-S(stat)	K-S(p-Value)
TBXE	$\hat{\varphi} = 0.1124$ $\hat{\rho} = 1.1469$ $\hat{\beta} = -0.5819$	0.0108 1.0054 1.0565	81.9019	169.8038	170.4704	174.8704	171.6357	0.0633	0.4753	0.0832	0.9445
MOLE	$\hat{\rho} = 2.2554$ $\hat{\alpha} = 0.3733$ $\hat{b} = 218.7027$	21.8544 3.6142 289.7554	83.8204	173.6409	174.3075	178.7075	175.4728	0.0838	0.6036	0.1007	0.8117
Ga	$\hat{a} = 8.2700$ $\hat{b} = 1.2760$	3.7431 0.5896	87.6709	179.3418	179.6661	182.7196	180.5631	0.2058	1.3647	0.1331	0.4777
BE	$\hat{a} = 7.8265$ $\hat{b} = 65.7294$ $\hat{\rho} = 0.0175$	3.5522 53.7494 0.0119	87.6205	181.2410	181.9077	186.3076	183.0729	0.2059	1.3655	0.1387	0.4247
GTPE	$\hat{a} = 0.5313$ $\hat{\rho} = -0.6703$ $\hat{\varphi} = 11.5652$	0.1148 0.7033 10.4039	87.9930	181.9860	182.6527	187.0526	183.8179	0.2057	1.3605	0.1486	0.3399
APE	$\hat{\alpha} = 318{,}294.3557$ $\hat{\rho} = 0.4713$	16,781.3618 0.0295	89.0895	182.1790	182.5033	185.5567	183.4003	0.2336	1.5244	0.1666	0.2170
TGE	$\hat{\alpha} = 9.8630$ $\hat{\rho} = 0.5123$ $\hat{\varphi} = -0.6837$	9.1154 0.1225 0.6411	88.8872	183.7743	184.4410	188.8410	185.6063	0.2363	1.5353	0.1520	0.3136
TEGE	$\hat{a} = 0.1939$ $\hat{b} = 9.8643$ $\hat{\rho} = -0.6836$ $\hat{\varphi} = 2.6424$	1.3161 9.1169 0.6412 17.9404	88.8872	185.7745	186.9174	192.5300	188.2171	0.2363	1.5353	0.1520	0.3137
EE	$\hat{\alpha} = 11.9635$ $\hat{\rho} = 0.4683$	8.3549 0.1118	90.6454	185.2909	185.6152	188.6686	186.5122	0.2882	1.8287	0.1631	0.2377
FWME	$\hat{a} = 0.1117$ $\hat{\alpha} = 2.9224$ $\hat{\rho} = 3.9471$ $\hat{k} = 1.6474$	1.8998 378.1227 13.9570 213.1556	88.7075	185.4150	186.5579	192.1705	187.8576	0.2147	1.4101	0.1448	0.3715
E	$\hat{\rho} = 0.1207$	0.0197	114.7675	231.5350	231.6402	233.2238	232.1456	0.2111	1.3954	0.3376	0.0002



(**a**) HRF plot



(**b**) TTT plot

Figure 6. The HRF plot of the TBXE model and TTT plot for the second data set.

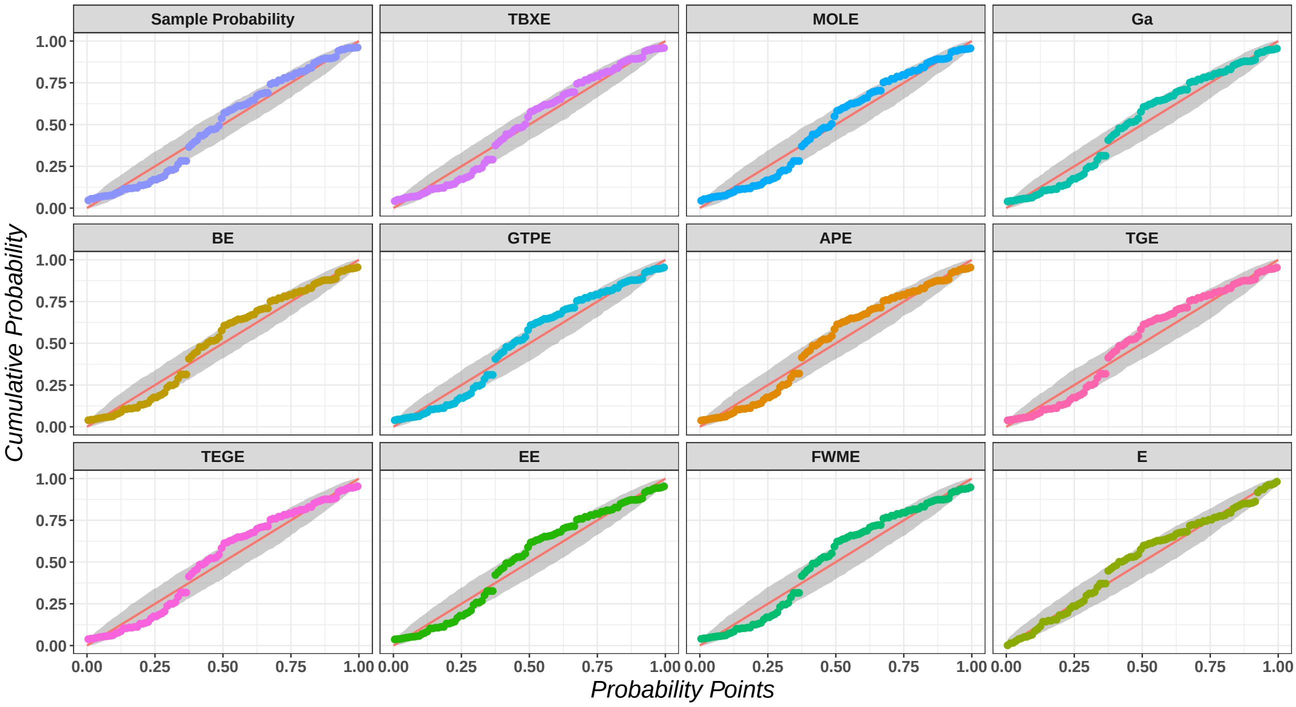


Figure 7. The probability–probability (PP) plot of the TBXE distribution and other fitted distributions for the first data set.

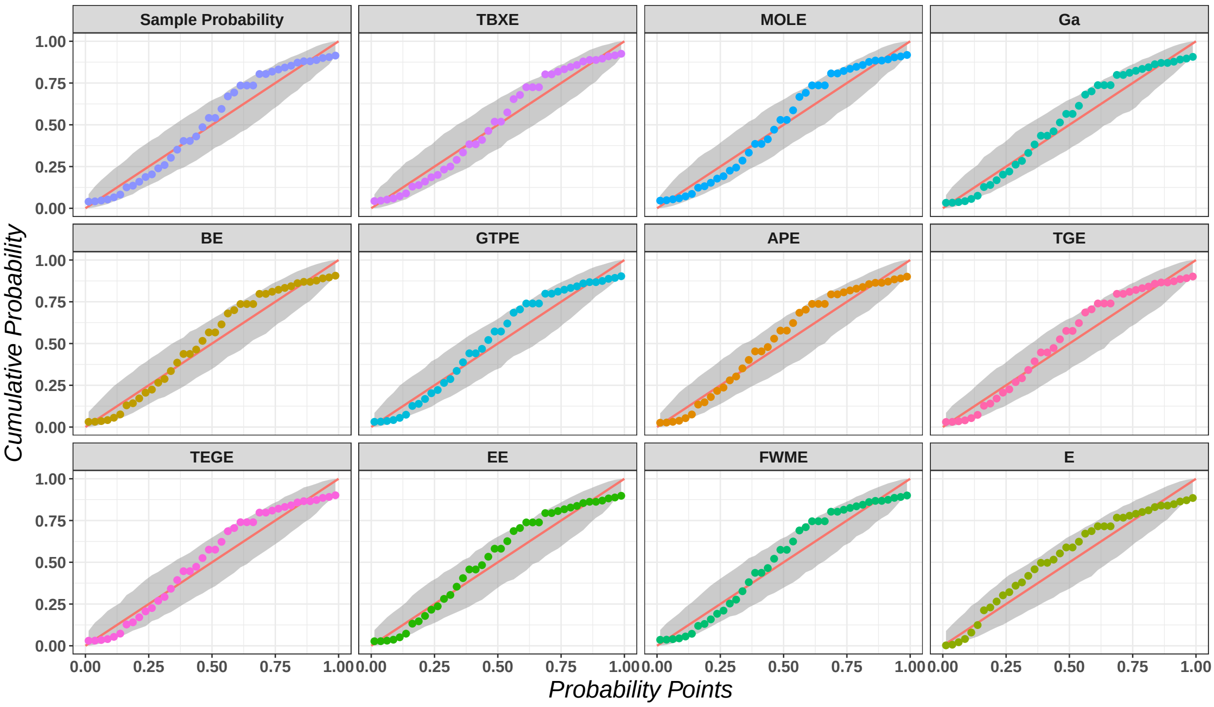


Figure 8. The probability–probability (PP) plot of the TBXE distribution and other fitted distributions for the second data set.

9. Conclusions

In this article, a new two-parameter generator, called transmuted Burr X generator (TBX-G), was proposed to extend the Burr X-G class and provide more flexibility in fitting real-life data. Two special sub-models, TBX-exponential (TBXE) and TBX-log-logistic (TBXLL) distributions, were studied. The structural properties of the TBX-G family were provided. We discussed eight estimation approaches to estimating the TBXE parameters.

Numerical simulations were conducted to compare the eight estimation methods to determine the more accurate estimation methods based on partial and overall ranks. Our study showed that the Anderson–Darling estimation method is the best approach for estimating TBXE parameters. Using two real-life engineering data sets, we illustrated the importance and flexibility of the TBXE distribution compared with their competing existing models. The analyzed data showed that the special cases of the new family are capable of modeling right- and left-skewed data and outperform the existing competing models. The research in the current paper can be extended in several ways. For example, Bayesian estimation based on complete and censored samples could be explored to estimate the parameters of the special models of the TBX-G family. A bivariate family of distributions can be proposed to extended the univariate TBX-G family; a regression model based on the TBXE model can be developed with extensive study for estimating its parameters along with detailed sensitivity analysis.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. Simulation results for $\mathbf{\Theta} = (\beta = 4.00, \varphi = 3.00, \rho = 0.50)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	$BIAS$	$\hat{\beta}$	$0.19636^{\{6\}}$	$0.21275^{\{7\}}$	$0.18543^{\{4\}}$	$0.17595^{\{3\}}$	$0.21916^{\{8\}}$	$0.19164^{\{5\}}$	$0.17161^{\{1\}}$	$0.17453^{\{2\}}$
		$\hat{\varphi}$	$0.21998^{\{5\}}$	$0.23951^{\{6\}}$	$0.20825^{\{2\}}$	$0.21213^{\{4\}}$	$0.24984^{\{7\}}$	$0.21078^{\{3\}}$	$0.25254^{\{8\}}$	$0.20804^{\{1\}}$
		$\hat{\rho}$	$0.21331^{\{4\}}$	$0.23100^{\{6\}}$	$0.26063^{\{8\}}$	$0.17950^{\{1\}}$	$0.23851^{\{7\}}$	$0.20903^{\{3\}}$	$0.22077^{\{5\}}$	$0.19923^{\{2\}}$
	MSE	$\hat{\beta}$	$0.05914^{\{6\}}$	$0.06887^{\{7\}}$	$0.05158^{\{4\}}$	$0.04692^{\{3\}}$	$0.07376^{\{8\}}$	$0.05552^{\{5\}}$	$0.04517^{\{1\}}$	$0.04519^{\{2\}}$
		$\hat{\varphi}$	$0.08234^{\{5\}}$	$0.10155^{\{6\}}$	$0.07567^{\{3\}}$	$0.07208^{\{2\}}$	$0.11424^{\{8\}}$	$0.07727^{\{4\}}$	$0.11065^{\{7\}}$	$0.06906^{\{1\}}$
		$\hat{\rho}$	$0.07429^{\{2\}}$	$0.08027^{\{6\}}$	$0.11301^{\{8\}}$	$0.07695^{\{4\}}$	$0.07510^{\{3\}}$	$0.06959^{\{1\}}$	$0.10006^{\{7\}}$	$0.07885^{\{5\}}$
	MRE	$\hat{\beta}$	$0.06545^{\{6\}}$	$0.07092^{\{7\}}$	$0.06181^{\{4\}}$	$0.05865^{\{3\}}$	$0.07305^{\{8\}}$	$0.06388^{\{5\}}$	$0.05720^{\{1\}}$	$0.05818^{\{2\}}$
		$\hat{\varphi}$	$0.14665^{\{5\}}$	$0.15968^{\{6\}}$	$0.13883^{\{2\}}$	$0.14142^{\{4\}}$	$0.16656^{\{7\}}$	$0.14052^{\{3\}}$	$0.16836^{\{8\}}$	$0.13869^{\{1\}}$
		$\hat{\rho}$	$0.28442^{\{4\}}$	$0.30800^{\{6\}}$	$0.34751^{\{8\}}$	$0.23934^{\{1\}}$	$0.31801^{\{7\}}$	$0.27871^{\{3\}}$	$0.29436^{\{5\}}$	$0.26564^{\{2\}}$
	$\sum Ranks$		$43^{\{5\}}$	$57^{\{7\}}$	$43^{\{5\}}$	$25^{\{2\}}$	$63^{\{8\}}$	$32^{\{3\}}$	$43^{\{5\}}$	$18^{\{1\}}$
150	$BIAS$	$\hat{\beta}$	$0.15710^{\{5\}}$	$0.17322^{\{7\}}$	$0.15884^{\{6\}}$	$0.12565^{\{1\}}$	$0.18034^{\{8\}}$	$0.15570^{\{4\}}$	$0.13677^{\{3\}}$	$0.13282^{\{2\}}$
		$\hat{\varphi}$	$0.12101^{\{5\}}$	$0.13452^{\{6\}}$	$0.11753^{\{2\}}$	$0.11982^{\{4\}}$	$0.14030^{\{8\}}$	$0.11920^{\{3\}}$	$0.13563^{\{7\}}$	$0.11717^{\{1\}}$
		$\hat{\rho}$	$0.18741^{\{5\}}$	$0.20764^{\{6\}}$	$0.22152^{\{8\}}$	$0.12997^{\{1\}}$	$0.21430^{\{7\}}$	$0.18247^{\{4\}}$	$0.18219^{\{3\}}$	$0.15328^{\{2\}}$
	MSE	$\hat{\beta}$	$0.03503^{\{5\}}$	$0.04177^{\{7\}}$	$0.03595^{\{6\}}$	$0.02430^{\{1\}}$	$0.04524^{\{8\}}$	$0.03452^{\{4\}}$	$0.02731^{\{3\}}$	$0.02644^{\{2\}}$
		$\hat{\varphi}$	$0.02324^{\{5\}}$	$0.02892^{\{6\}}$	$0.02312^{\{4\}}$	$0.02206^{\{2\}}$	$0.03207^{\{8\}}$	$0.02259^{\{3\}}$	$0.02988^{\{7\}}$	$0.02172^{\{1\}}$
		$\hat{\rho}$	$0.04492^{\{4\}}$	$0.05253^{\{5\}}$	$0.07756^{\{8\}}$	$0.03102^{\{1\}}$	$0.05505^{\{6\}}$	$0.04331^{\{3\}}$	$0.05615^{\{7\}}$	$0.04073^{\{2\}}$
	MRE	$\hat{\beta}$	$0.05237^{\{5\}}$	$0.05774^{\{7\}}$	$0.05295^{\{6\}}$	$0.04188^{\{1\}}$	$0.06011^{\{8\}}$	$0.05190^{\{4\}}$	$0.04559^{\{3\}}$	$0.04427^{\{2\}}$
		$\hat{\varphi}$	$0.08067^{\{5\}}$	$0.08968^{\{6\}}$	$0.07835^{\{2\}}$	$0.07988^{\{4\}}$	$0.09353^{\{8\}}$	$0.07947^{\{3\}}$	$0.09042^{\{7\}}$	$0.07811^{\{1\}}$
		$\hat{\rho}$	$0.24988^{\{5\}}$	$0.27685^{\{6\}}$	$0.29536^{\{8\}}$	$0.17330^{\{1\}}$	$0.28573^{\{7\}}$	$0.24330^{\{4\}}$	$0.24292^{\{3\}}$	$0.20438^{\{2\}}$
	$\sum Ranks$		$44^{\{5\}}$	$56^{\{7\}}$	$50^{\{6\}}$	$16^{\{2\}}$	$68^{\{8\}}$	$32^{\{3\}}$	$43^{\{4\}}$	$15^{\{1\}}$

Table A1. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
300	$BIAS$	$\hat{\beta}$	$0.14168^{\{6\}}$	$0.15813^{\{7\}}$	$0.14054^{\{5\}}$	$0.09908^{\{1\}}$	$0.16119^{\{8\}}$	$0.13741^{\{4\}}$	$0.12501^{\{3\}}$	$0.11371^{\{2\}}$
		$\hat{\varphi}$	$0.08811^{\{4\}}$	$0.09818^{\{8\}}$	$0.08407^{\{1\}}$	$0.08413^{\{2\}}$	$0.09579^{\{6\}}$	$0.08833^{\{5\}}$	$0.09734^{\{7\}}$	$0.08436^{\{3\}}$
		$\hat{\rho}$	$0.17485^{\{5\}}$	$0.19747^{\{7\}}$	$0.19203^{\{6\}}$	$0.10694^{\{1\}}$	$0.20235^{\{8\}}$	$0.17178^{\{4\}}$	$0.16102^{\{3\}}$	$0.13399^{\{2\}}$
	MSE	$\hat{\beta}$	$0.02768^{\{5\}}$	$0.03321^{\{7\}}$	$0.02827^{\{6\}}$	$0.01618^{\{1\}}$	$0.03429^{\{8\}}$	$0.02637^{\{4\}}$	$0.02269^{\{3\}}$	$0.01979^{\{2\}}$
		$\hat{\varphi}$	$0.01218^{\{4\}}$	$0.01508^{\{8\}}$	$0.01106^{\{3\}}$	$0.01086^{\{2\}}$	$0.01461^{\{6\}}$	$0.01225^{\{5\}}$	$0.01491^{\{7\}}$	$0.01085^{\{1\}}$
		$\hat{\rho}$	$0.03901^{\{4\}}$	$0.04688^{\{6\}}$	$0.05677^{\{8\}}$	$0.01980^{\{1\}}$	$0.04891^{\{7\}}$	$0.03848^{\{3\}}$	$0.03914^{\{5\}}$	$0.02954^{\{2\}}$
	MRE	$\hat{\beta}$	$0.04723^{\{6\}}$	$0.05271^{\{7\}}$	$0.04685^{\{5\}}$	$0.03303^{\{1\}}$	$0.05373^{\{8\}}$	$0.04580^{\{4\}}$	$0.04167^{\{3\}}$	$0.03790^{\{2\}}$
		$\hat{\varphi}$	$0.05874^{\{4\}}$	$0.06545^{\{8\}}$	$0.05604^{\{1\}}$	$0.05609^{\{2\}}$	$0.06386^{\{6\}}$	$0.05889^{\{5\}}$	$0.06489^{\{7\}}$	$0.05624^{\{3\}}$
		$\hat{\rho}$	$0.23313^{\{5\}}$	$0.26329^{\{7\}}$	$0.25604^{\{6\}}$	$0.14259^{\{1\}}$	$0.26980^{\{8\}}$	$0.22904^{\{4\}}$	$0.21469^{\{3\}}$	$0.17866^{\{2\}}$
	$\sum Ranks$		$43^{\{6\}}$	$65^{\{7.5\}}$	$41^{\{4.5\}}$	$12^{\{1\}}$	$65^{\{7.5\}}$	$38^{\{3\}}$	$41^{\{4.5\}}$	$19^{\{2\}}$
400	$BIAS$	$\hat{\beta}$	$0.13391^{\{5\}}$	$0.15251^{\{7\}}$	$0.13510^{\{6\}}$	$0.09080^{\{1\}}$	$0.15666^{\{8\}}$	$0.13047^{\{4\}}$	$0.12093^{\{3\}}$	$0.10884^{\{2\}}$
		$\hat{\varphi}$	$0.07563^{\{4\}}$	$0.08448^{\{6\}}$	$0.07213^{\{1\}}$	$0.07391^{\{2\}}$	$0.08568^{\{8\}}$	$0.07616^{\{5\}}$	$0.08505^{\{7\}}$	$0.07470^{\{3\}}$
		$\hat{\rho}$	$0.16832^{\{5\}}$	$0.19117^{\{7\}}$	$0.18504^{\{6\}}$	$0.10058^{\{1\}}$	$0.19771^{\{8\}}$	$0.16261^{\{4\}}$	$0.15459^{\{3\}}$	$0.12992^{\{2\}}$
	MSE	$\hat{\beta}$	$0.02444^{\{5\}}$	$0.03037^{\{7\}}$	$0.02610^{\{6\}}$	$0.01392^{\{1\}}$	$0.03182^{\{8\}}$	$0.02339^{\{4\}}$	$0.02103^{\{3\}}$	$0.01810^{\{2\}}$
		$\hat{\varphi}$	$0.00897^{\{4\}}$	$0.01125^{\{7\}}$	$0.00811^{\{1\}}$	$0.00826^{\{2\}}$	$0.01171^{\{8\}}$	$0.00902^{\{5\}}$	$0.01124^{\{6\}}$	$0.00858^{\{3\}}$
		$\hat{\rho}$	$0.03630^{\{5\}}$	$0.04431^{\{6\}}$	$0.05245^{\{8\}}$	$0.01813^{\{1\}}$	$0.04707^{\{7\}}$	$0.03456^{\{4\}}$	$0.03418^{\{3\}}$	$0.02775^{\{2\}}$
	MRE	$\hat{\beta}$	$0.04464^{\{5\}}$	$0.05084^{\{7\}}$	$0.04503^{\{6\}}$	$0.03027^{\{1\}}$	$0.05222^{\{8\}}$	$0.04349^{\{4\}}$	$0.04031^{\{3\}}$	$0.03628^{\{2\}}$
		$\hat{\varphi}$	$0.05042^{\{4\}}$	$0.05632^{\{6\}}$	$0.04808^{\{1\}}$	$0.04928^{\{2\}}$	$0.05712^{\{8\}}$	$0.05077^{\{5\}}$	$0.05670^{\{7\}}$	$0.04980^{\{3\}}$
		$\hat{\rho}$	$0.22443^{\{5\}}$	$0.25489^{\{7\}}$	$0.24672^{\{6\}}$	$0.13411^{\{1\}}$	$0.26361^{\{8\}}$	$0.21682^{\{4\}}$	$0.20612^{\{3\}}$	$0.17322^{\{2\}}$
	$\sum Ranks$		$42^{\{6\}}$	$60^{\{7\}}$	$41^{\{5\}}$	$12^{\{1\}}$	$71^{\{8\}}$	$39^{\{4\}}$	$38^{\{3\}}$	$21^{\{2\}}$

Table A2. Simulation results for $\boldsymbol{\Theta} = (\beta = 0.75, \varphi = 1.00, \rho = 0.30)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	$BIAS$	$\hat{\beta}$	$0.05497^{\{2\}}$	$0.05640^{\{4\}}$	$0.12286^{\{8\}}$	$0.05995^{\{7\}}$	$0.05658^{\{5\}}$	$0.05258^{\{1\}}$	$0.05589^{\{3\}}$	$0.05901^{\{6\}}$
		$\hat{\varphi}$	$0.18671^{\{5\}}$	$0.19952^{\{8\}}$	$0.18265^{\{4\}}$	$0.19242^{\{6\}}$	$0.19256^{\{7\}}$	$0.17964^{\{3\}}$	$0.17932^{\{2\}}$	$0.17890^{\{1\}}$
		$\hat{\rho}$	$0.45325^{\{7\}}$	$0.45293^{\{6\}}$	$0.36028^{\{1\}}$	$0.46707^{\{8\}}$	$0.43161^{\{2\}}$	$0.44144^{\{4\}}$	$0.43901^{\{3\}}$	$0.44752^{\{5\}}$
	MSE	$\hat{\beta}$	$0.00458^{\{2\}}$	$0.00484^{\{4\}}$	$0.06117^{\{8\}}$	$0.00539^{\{7\}}$	$0.00502^{\{5\}}$	$0.00426^{\{1\}}$	$0.00482^{\{3\}}$	$0.00523^{\{6\}}$
		$\hat{\varphi}$	$0.05358^{\{4\}}$	$0.05969^{\{8\}}$	$0.05481^{\{5\}}$	$0.05910^{\{7\}}$	$0.05786^{\{6\}}$	$0.05004^{\{3\}}$	$0.04942^{\{2\}}$	$0.04926^{\{1\}}$
		$\hat{\rho}$	$0.28156^{\{7\}}$	$0.27771^{\{6\}}$	$0.20268^{\{1\}}$	$0.31328^{\{8\}}$	$0.25651^{\{3\}}$	$0.27391^{\{5\}}$	$0.25203^{\{2\}}$	$0.27031^{\{4\}}$
	MRE	$\hat{\beta}$	$0.07329^{\{2\}}$	$0.07520^{\{4\}}$	$0.16382^{\{8\}}$	$0.07994^{\{7\}}$	$0.07544^{\{5\}}$	$0.07011^{\{1\}}$	$0.07452^{\{3\}}$	$0.07868^{\{6\}}$
		$\hat{\varphi}$	$0.18671^{\{5\}}$	$0.19952^{\{8\}}$	$0.18265^{\{4\}}$	$0.19242^{\{6\}}$	$0.19256^{\{7\}}$	$0.17964^{\{3\}}$	$0.17932^{\{2\}}$	$0.17890^{\{1\}}$
		$\hat{\rho}$	$1.51082^{\{7\}}$	$1.50977^{\{6\}}$	$1.20093^{\{1\}}$	$1.55691^{\{8\}}$	$1.43871^{\{2\}}$	$1.47146^{\{4\}}$	$1.46337^{\{3\}}$	$1.49174^{\{5\}}$
	$\sum Ranks$		$41^{\{5\}}$	$54^{\{7\}}$	$40^{\{4\}}$	$64^{\{8\}}$	$42^{\{6\}}$	$25^{\{2\}}$	$23^{\{1\}}$	$35^{\{3\}}$
150	$BIAS$	$\hat{\beta}$	$0.04009^{\{2\}}$	$0.04205^{\{5\}}$	$0.11198^{\{8\}}$	$0.04709^{\{7\}}$	$0.04048^{\{3\}}$	$0.03960^{\{1\}}$	$0.04131^{\{4\}}$	$0.04590^{\{6\}}$
		$\hat{\varphi}$	$0.12091^{\{3\}}$	$0.13461^{\{8\}}$	$0.12843^{\{6\}}$	$0.12374^{\{4\}}$	$0.13198^{\{7\}}$	$0.11710^{\{2\}}$	$0.12416^{\{5\}}$	$0.11212^{\{1\}}$
		$\hat{\rho}$	$0.35004^{\{3\}}$	$0.37236^{\{7\}}$	$0.29516^{\{1\}}$	$0.37953^{\{8\}}$	$0.36006^{\{6\}}$	$0.34178^{\{2\}}$	$0.35325^{\{4\}}$	$0.35771^{\{5\}}$
	MSE	$\hat{\beta}$	$0.00237^{\{2\}}$	$0.00258^{\{5\}}$	$0.06064^{\{8\}}$	$0.00343^{\{7\}}$	$0.00244^{\{3\}}$	$0.00233^{\{1\}}$	$0.00249^{\{4\}}$	$0.00313^{\{6\}}$
		$\hat{\varphi}$	$0.02473^{\{3\}}$	$0.02959^{\{6\}}$	$0.03219^{\{8\}}$	$0.03017^{\{7\}}$	$0.02797^{\{5\}}$	$0.02330^{\{2\}}$	$0.02487^{\{4\}}$	$0.02248^{\{1\}}$
		$\hat{\rho}$	$0.18034^{\{4\}}$	$0.19931^{\{7\}}$	$0.14295^{\{1\}}$	$0.23276^{\{8\}}$	$0.18784^{\{6\}}$	$0.17404^{\{2\}}$	$0.17747^{\{3\}}$	$0.18218^{\{5\}}$
	MRE	$\hat{\beta}$	$0.05345^{\{2\}}$	$0.05606^{\{5\}}$	$0.14931^{\{8\}}$	$0.06279^{\{7\}}$	$0.05397^{\{3\}}$	$0.05280^{\{1\}}$	$0.05507^{\{4\}}$	$0.06119^{\{6\}}$
		$\hat{\varphi}$	$0.12091^{\{3\}}$	$0.13461^{\{8\}}$	$0.12843^{\{6\}}$	$0.12374^{\{4\}}$	$0.13198^{\{7\}}$	$0.11710^{\{2\}}$	$0.12416^{\{5\}}$	$0.11212^{\{1\}}$
		$\hat{\rho}$	$1.16680^{\{3\}}$	$1.24122^{\{7\}}$	$0.98386^{\{1\}}$	$1.26510^{\{8\}}$	$1.20022^{\{6\}}$	$1.13928^{\{2\}}$	$1.17749^{\{4\}}$	$1.19237^{\{5\}}$
	$\sum Ranks$		$25^{\{2\}}$	$58^{\{7\}}$	$47^{\{6\}}$	$60^{\{8\}}$	$46^{\{5\}}$	$15^{\{1\}}$	$37^{\{4\}}$	$36^{\{3\}}$
300	$BIAS$	$\hat{\beta}$	$0.03340^{\{2\}}$	$0.03573^{\{5\}}$	$0.10479^{\{8\}}$	$0.03867^{\{7\}}$	$0.03516^{\{4\}}$	$0.03338^{\{1\}}$	$0.03402^{\{3\}}$	$0.03861^{\{6\}}$
		$\hat{\varphi}$	$0.08292^{\{4\}}$	$0.09920^{\{7\}}$	$0.10216^{\{8\}}$	$0.08232^{\{3\}}$	$0.09493^{\{6\}}$	$0.07933^{\{2\}}$	$0.09348^{\{5\}}$	$0.07566^{\{1\}}$
		$\hat{\rho}$	$0.27273^{\{3\}}$	$0.30794^{\{8\}}$	$0.24152^{\{1\}}$	$0.29007^{\{6\}}$	$0.29730^{\{7\}}$	$0.26593^{\{2\}}$	$0.28359^{\{5\}}$	$0.28155^{\{4\}}$
	MSE	$\hat{\beta}$	$0.00165^{\{1\}}$	$0.00182^{\{5\}}$	$0.05921^{\{8\}}$	$0.00250^{\{7\}}$	$0.00177^{\{4\}}$	$0.00166^{\{2\}}$	$0.00175^{\{3\}}$	$0.00234^{\{6\}}$
		$\hat{\varphi}$	$0.01235^{\{3\}}$	$0.01759^{\{7\}}$	$0.02490^{\{8\}}$	$0.01466^{\{4\}}$	$0.01592^{\{6\}}$	$0.01128^{\{2\}}$	$0.01490^{\{5\}}$	$0.01074^{\{1\}}$

Table A2. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
		$\hat{\rho}$	$0.11123^{\{3\}}$	$0.14005^{\{7\}}$	$0.09957^{\{1\}}$	$0.14753^{\{8\}}$	$0.13195^{\{6\}}$	$0.10660^{\{2\}}$	$0.12083^{\{5\}}$	$0.12015^{\{4\}}$
	MRE	$\hat{\beta}$	$0.04453^{\{2\}}$	$0.04764^{\{5\}}$	$0.13972^{\{8\}}$	$0.05156^{\{7\}}$	$0.04688^{\{4\}}$	$0.04451^{\{1\}}$	$0.04536^{\{3\}}$	$0.05148^{\{6\}}$
		$\hat{\varphi}$	$0.08292^{\{4\}}$	$0.09920^{\{7\}}$	$0.10216^{\{8\}}$	$0.08232^{\{3\}}$	$0.09493^{\{6\}}$	$0.07933^{\{2\}}$	$0.09348^{\{5\}}$	$0.07566^{\{1\}}$
		$\hat{\rho}$	$0.90912^{\{3\}}$	$1.02648^{\{8\}}$	$0.80507^{\{1\}}$	$0.96689^{\{6\}}$	$0.99099^{\{7\}}$	$0.88642^{\{2\}}$	$0.94529^{\{5\}}$	$0.93849^{\{4\}}$
	$\sum$ *Ranks*		$25^{\{2\}}$	$59^{\{8\}}$	$51^{\{6.5\}}$	$51^{\{6.5\}}$	$50^{\{5\}}$	$16^{\{1\}}$	$39^{\{4\}}$	$33^{\{3\}}$
400	*BIAS*	$\hat{\beta}$	$0.03047^{\{1\}}$	$0.03410^{\{5\}}$	$0.10898^{\{8\}}$	$0.03605^{\{7\}}$	$0.03313^{\{4\}}$	$0.03063^{\{2\}}$	$0.03138^{\{3\}}$	$0.03548^{\{6\}}$
		$\hat{\varphi}$	$0.07037^{\{3\}}$	$0.08384^{\{7\}}$	$0.09425^{\{8\}}$	$0.07172^{\{4\}}$	$0.08313^{\{6\}}$	$0.06830^{\{2\}}$	$0.08171^{\{5\}}$	$0.06501^{\{1\}}$
		$\hat{\rho}$	$0.24256^{\{3\}}$	$0.28045^{\{8\}}$	$0.22193^{\{1\}}$	$0.26182^{\{6\}}$	$0.27334^{\{7\}}$	$0.24062^{\{2\}}$	$0.25646^{\{5\}}$	$0.25503^{\{4\}}$
	MSE	$\hat{\beta}$	$0.00140^{\{1\}}$	$0.00165^{\{5\}}$	$0.06368^{\{8\}}$	$0.00227^{\{7\}}$	$0.00156^{\{4\}}$	$0.00143^{\{2\}}$	$0.00151^{\{3\}}$	$0.00207^{\{6\}}$
		$\hat{\varphi}$	$0.00870^{\{3\}}$	$0.01319^{\{7\}}$	$0.02278^{\{8\}}$	$0.01143^{\{4\}}$	$0.01232^{\{6\}}$	$0.00830^{\{2\}}$	$0.01149^{\{5\}}$	$0.00759^{\{1\}}$
		$\hat{\rho}$	$0.08852^{\{3\}}$	$0.11744^{\{7\}}$	$0.08065^{\{1\}}$	$0.12744^{\{8\}}$	$0.11016^{\{6\}}$	$0.08787^{\{2\}}$	$0.10026^{\{5\}}$	$0.09939^{\{4\}}$
	MRE	$\hat{\beta}$	$0.04062^{\{1\}}$	$0.04547^{\{5\}}$	$0.14531^{\{8\}}$	$0.04807^{\{7\}}$	$0.04418^{\{4\}}$	$0.04084^{\{2\}}$	$0.04184^{\{3\}}$	$0.04730^{\{6\}}$
		$\hat{\varphi}$	$0.07037^{\{3\}}$	$0.08384^{\{7\}}$	$0.09425^{\{8\}}$	$0.07172^{\{4\}}$	$0.08313^{\{6\}}$	$0.06830^{\{2\}}$	$0.08171^{\{5\}}$	$0.06501^{\{1\}}$
		$\hat{\rho}$	$0.80852^{\{3\}}$	$0.93485^{\{8\}}$	$0.73977^{\{1\}}$	$0.87275^{\{6\}}$	$0.91113^{\{7\}}$	$0.80206^{\{2\}}$	$0.85486^{\{5\}}$	$0.85011^{\{4\}}$
	$\sum$ *Ranks*		$21^{\{2\}}$	$59^{\{8\}}$	$51^{\{6\}}$	$53^{\{7\}}$	$50^{\{5\}}$	$18^{\{1\}}$	$39^{\{4\}}$	$33^{\{3\}}$

Table A3. Simulation results for $\mathbf{\Theta} = (\beta = 1.00, \varphi = 1.50, \rho = -0.50)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	*BIAS*	$\hat{\beta}$	$0.05035^{\{3\}}$	$0.05552^{\{8\}}$	$0.04297^{\{1\}}$	$0.05536^{\{7\}}$	$0.05254^{\{4\}}$	$0.04891^{\{2\}}$	$0.05339^{\{5\}}$	$0.05514^{\{6\}}$
		$\hat{\varphi}$	$0.29167^{\{4\}}$	$0.28304^{\{1\}}$	$0.31787^{\{7\}}$	$0.29207^{\{5\}}$	$0.30329^{\{6\}}$	$0.28430^{\{2\}}$	$0.34093^{\{8\}}$	$0.28602^{\{3\}}$
		$\hat{\rho}$	$0.33131^{\{3\}}$	$0.32834^{\{2\}}$	$0.35874^{\{6\}}$	$0.35066^{\{4\}}$	$0.31656^{\{1\}}$	$0.35480^{\{5\}}$	$0.36839^{\{7\}}$	$0.36906^{\{8\}}$
	MSE	$\hat{\beta}$	$0.00408^{\{3\}}$	$0.00493^{\{6\}}$	$0.00299^{\{1\}}$	$0.00516^{\{8\}}$	$0.00438^{\{4\}}$	$0.00386^{\{2\}}$	$0.00464^{\{5\}}$	$0.00504^{\{7\}}$
		$\hat{\varphi}$	$0.12644^{\{4\}}$	$0.12748^{\{5\}}$	$0.15880^{\{6\}}$	$0.12547^{\{3\}}$	$0.15880^{\{7\}}$	$0.12211^{\{2\}}$	$0.20389^{\{8\}}$	$0.11612^{\{1\}}$
		$\hat{\rho}$	$0.21337^{\{2\}}$	$0.22110^{\{4\}}$	$0.22018^{\{3\}}$	$0.24269^{\{6\}}$	$0.20040^{\{1\}}$	$0.23352^{\{5\}}$	$0.27493^{\{8\}}$	$0.26648^{\{7\}}$
	MRE	$\hat{\beta}$	$0.05035^{\{3\}}$	$0.05552^{\{8\}}$	$0.04297^{\{1\}}$	$0.05536^{\{7\}}$	$0.05254^{\{4\}}$	$0.04891^{\{2\}}$	$0.05339^{\{5\}}$	$0.05514^{\{6\}}$
		$\hat{\varphi}$	$0.19444^{\{4\}}$	$0.18870^{\{1\}}$	$0.21191^{\{7\}}$	$0.19471^{\{5\}}$	$0.20220^{\{6\}}$	$0.18953^{\{2\}}$	$0.22729^{\{8\}}$	$0.19068^{\{3\}}$
		$\hat{\rho}$	$-0.66262^{\{6\}}$	$-0.65668^{\{7\}}$	$-0.71747^{\{3\}}$	$-0.70131^{\{5\}}$	$-0.63312^{\{8\}}$	$-0.70959^{\{4\}}$	$-0.73679^{\{2\}}$	$-0.73812^{\{1\}}$
	$\sum$ *Ranks*		$32^{\{2\}}$	$42^{\{5.5\}}$	$35^{\{3\}}$	$50^{\{7\}}$	$41^{\{4\}}$	$26^{\{1\}}$	$56^{\{8\}}$	$42^{\{5.5\}}$
150	*BIAS*	$\hat{\beta}$	$0.02975^{\{3\}}$	$0.03327^{\{8\}}$	$0.02726^{\{1\}}$	$0.02983^{\{4\}}$	$0.03186^{\{6\}}$	$0.02862^{\{2\}}$	$0.03122^{\{5\}}$	$0.03264^{\{7\}}$
		$\hat{\varphi}$	$0.23920^{\{4\}}$	$0.23366^{\{1\}}$	$0.27688^{\{8\}}$	$0.23413^{\{2\}}$	$0.23592^{\{3\}}$	$0.24001^{\{5\}}$	$0.26743^{\{7\}}$	$0.25938^{\{6\}}$
		$\hat{\rho}$	$0.28846^{\{3\}}$	$0.29146^{\{4\}}$	$0.30545^{\{6\}}$	$0.26143^{\{1\}}$	$0.28262^{\{2\}}$	$0.29478^{\{5\}}$	$0.30600^{\{7\}}$	$0.31586^{\{8\}}$
	MSE	$\hat{\beta}$	$0.00158^{\{3\}}$	$0.00192^{\{7\}}$	$0.00141^{\{1\}}$	$0.00176^{\{5\}}$	$0.00171^{\{4\}}$	$0.00146^{\{2\}}$	$0.00178^{\{6\}}$	$0.00210^{\{8\}}$
		$\hat{\varphi}$	$0.08421^{\{4\}}$	$0.08040^{\{1\}}$	$0.11430^{\{8\}}$	$0.08243^{\{2\}}$	$0.08393^{\{3\}}$	$0.08503^{\{5\}}$	$0.10408^{\{7\}}$	$0.09454^{\{6\}}$
		$\hat{\rho}$	$0.15845^{\{2\}}$	$0.17227^{\{6\}}$	$0.16797^{\{5\}}$	$0.13798^{\{1\}}$	$0.16104^{\{4\}}$	$0.16081^{\{3\}}$	$0.19270^{\{8\}}$	$0.19160^{\{7\}}$
	MRE	$\hat{\beta}$	$0.02975^{\{3\}}$	$0.03327^{\{8\}}$	$0.02726^{\{1\}}$	$0.02983^{\{4\}}$	$0.03186^{\{6\}}$	$0.02862^{\{2\}}$	$0.03122^{\{5\}}$	$0.03264^{\{7\}}$
		$\hat{\varphi}$	$0.15947^{\{4\}}$	$0.15577^{\{1\}}$	$0.18459^{\{8\}}$	$0.15609^{\{2\}}$	$0.15728^{\{3\}}$	$0.16001^{\{5\}}$	$0.17829^{\{7\}}$	$0.17292^{\{6\}}$
		$\hat{\rho}$	$-0.57691^{\{6\}}$	$-0.58292^{\{5\}}$	$-0.61090^{\{3\}}$	$-0.52286^{\{8\}}$	$-0.56525^{\{7\}}$	$-0.58956^{\{4\}}$	$-0.61200^{\{2\}}$	$-0.63173^{\{1\}}$
	$\sum$ *Ranks*		$32^{\{2\}}$	$41^{\{5.5\}}$	$41^{\{5.5\}}$	$29^{\{1\}}$	$38^{\{4\}}$	$33^{\{3\}}$	$54^{\{7\}}$	$56^{\{8\}}$
300	*BIAS*	$\hat{\beta}$	$0.02057^{\{4\}}$	$0.02308^{\{8\}}$	$0.01927^{\{1\}}$	$0.02033^{\{3\}}$	$0.02295^{\{7\}}$	$0.02022^{\{2\}}$	$0.02178^{\{5\}}$	$0.02237^{\{6\}}$
		$\hat{\varphi}$	$0.21577^{\{3\}}$	$0.21204^{\{2\}}$	$0.24170^{\{8\}}$	$0.19276^{\{1\}}$	$0.21722^{\{5\}}$	$0.21650^{\{4\}}$	$0.23983^{\{7\}}$	$0.23761^{\{6\}}$
		$\hat{\rho}$	$0.24968^{\{3\}}$	$0.25177^{\{4\}}$	$0.25882^{\{6\}}$	$0.20385^{\{1\}}$	$0.25595^{\{5\}}$	$0.24951^{\{2\}}$	$0.26351^{\{7\}}$	$0.27189^{\{8\}}$
	MSE	$\hat{\beta}$	$0.00080^{\{3\}}$	$0.00098^{\{7\}}$	$0.00078^{\{2\}}$	$0.00084^{\{4\}}$	$0.00095^{\{6\}}$	$0.00078^{\{1\}}$	$0.00094^{\{5\}}$	$0.00107^{\{8\}}$
		$\hat{\varphi}$	$0.07045^{\{3\}}$	$0.07018^{\{2\}}$	$0.08680^{\{8\}}$	$0.06161^{\{1\}}$	$0.07305^{\{5\}}$	$0.07172^{\{4\}}$	$0.08598^{\{7\}}$	$0.07932^{\{6\}}$
		$\hat{\rho}$	$0.11550^{\{3\}}$	$0.12742^{\{5\}}$	$0.11994^{\{4\}}$	$0.08816^{\{1\}}$	$0.13031^{\{6\}}$	$0.11490^{\{2\}}$	$0.14299^{\{8\}}$	$0.13345^{\{7\}}$
	MRE	$\hat{\beta}$	$0.02057^{\{4\}}$	$0.02308^{\{8\}}$	$0.01927^{\{1\}}$	$0.02033^{\{3\}}$	$0.02295^{\{7\}}$	$0.02022^{\{2\}}$	$0.02178^{\{5\}}$	$0.02237^{\{6\}}$
		$\hat{\varphi}$	$0.14385^{\{3\}}$	$0.14136^{\{2\}}$	$0.16113^{\{8\}}$	$0.12850^{\{1\}}$	$0.14481^{\{5\}}$	$0.14433^{\{4\}}$	$0.15989^{\{7\}}$	$0.15841^{\{6\}}$
		$\hat{\rho}$	$-0.49937^{\{6\}}$	$-0.50355^{\{5\}}$	$-0.51764^{\{3\}}$	$-0.40769^{\{8\}}$	$-0.51189^{\{4\}}$	$-0.49902^{\{7\}}$	$-0.52702^{\{2\}}$	$-0.54379^{\{1\}}$
	$\sum$ *Ranks*		$32^{\{3\}}$	$43^{\{5\}}$	$41^{\{4\}}$	$23^{\{1\}}$	$50^{\{6\}}$	$28^{\{2\}}$	$53^{\{7\}}$	$54^{\{8\}}$
400	*BIAS*	$\hat{\beta}$	$0.01762^{\{4\}}$	$0.01995^{\{7\}}$	$0.01637^{\{1\}}$	$0.01695^{\{2\}}$	$0.02002^{\{8\}}$	$0.01738^{\{3\}}$	$0.01852^{\{6\}}$	$0.01808^{\{5\}}$
		$\hat{\varphi}$	$0.20161^{\{2\}}$	$0.20271^{\{3\}}$	$0.22685^{\{7\}}$	$0.17186^{\{1\}}$	$0.20517^{\{4\}}$	$0.20973^{\{5\}}$	$0.23420^{\{8\}}$	$0.22643^{\{6\}}$
		$\hat{\rho}$	$0.23165^{\{2\}}$	$0.23820^{\{3\}}$	$0.24259^{\{6\}}$	$0.18034^{\{1\}}$	$0.24118^{\{5\}}$	$0.24083^{\{4\}}$	$0.25448^{\{8\}}$	$0.24669^{\{7\}}$

Table A3. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
	MSE	$\hat{\beta}$	$0.00060^{\{4\}}$	$0.00074^{\{7\}}$	$0.00055^{\{1\}}$	$0.00058^{\{2\}}$	$0.00075^{\{8\}}$	$0.00058^{\{3\}}$	$0.00068^{\{5\}}$	$0.00068^{\{6\}}$
		$\hat{\varphi}$	$0.06316^{\{2\}}$	$0.06560^{\{3\}}$	$0.07633^{\{7\}}$	$0.05238^{\{1\}}$	$0.06776^{\{5\}}$	$0.06746^{\{4\}}$	$0.08393^{\{8\}}$	$0.07204^{\{6\}}$
		$\hat{\rho}$	$0.09863^{\{2\}}$	$0.11344^{\{6\}}$	$0.09926^{\{3\}}$	$0.07007^{\{1\}}$	$0.11544^{\{7\}}$	$0.10288^{\{5\}}$	$0.12934^{\{8\}}$	$0.10262^{\{4\}}$
	MRE	$\hat{\beta}$	$0.01762^{\{4\}}$	$0.01995^{\{7\}}$	$0.01637^{\{1\}}$	$0.01695^{\{2\}}$	$0.02002^{\{8\}}$	$0.01738^{\{3\}}$	$0.01852^{\{6\}}$	$0.01808^{\{5\}}$
		$\hat{\varphi}$	$0.13441^{\{2\}}$	$0.13514^{\{3\}}$	$0.15123^{\{7\}}$	$0.11457^{\{1\}}$	$0.13678^{\{4\}}$	$0.13982^{\{5\}}$	$0.15614^{\{8\}}$	$0.15095^{\{6\}}$
		$\hat{\rho}$	$-0.46331^{\{7\}}$	$-0.47639^{\{6\}}$	$-0.48518^{\{3\}}$	$-0.36067^{\{8\}}$	$-0.48237^{\{4\}}$	$-0.48167^{\{5\}}$	$-0.50896^{\{1\}}$	$-0.49337^{\{2\}}$
	$\sum$ *Ranks*		$29^{\{2\}}$	$45^{\{5\}}$	$36^{\{3\}}$	$19^{\{1\}}$	$53^{\{7\}}$	$37^{\{4\}}$	$58^{\{8\}}$	$47^{\{6\}}$

Table A4. Simulation results for $\mathbf{\Theta} = (\beta = 4.00, \varphi = 3.00, \rho = 0.50)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	*BIAS*	$\hat{\beta}$	$0.19731^{\{6\}}$	$0.19039^{\{5\}}$	$0.19888^{\{7\}}$	$0.20764^{\{8\}}$	$0.18884^{\{3\}}$	$0.18071^{\{1\}}$	$0.18159^{\{2\}}$	$0.18985^{\{4\}}$
		$\hat{\varphi}$	$0.53813^{\{3\}}$	$0.59920^{\{7\}}$	$0.51596^{\{1\}}$	$0.55371^{\{5\}}$	$0.59963^{\{8\}}$	$0.53084^{\{2\}}$	$0.57907^{\{6\}}$	$0.55163^{\{4\}}$
		$\hat{\rho}$	$0.30609^{\{1\}}$	$0.31791^{\{3\}}$	$0.36726^{\{8\}}$	$0.33360^{\{5\}}$	$0.31449^{\{2\}}$	$0.33390^{\{6\}}$	$0.33659^{\{7\}}$	$0.33122^{\{4\}}$
	MSE	$\hat{\beta}$	$0.05987^{\{7\}}$	$0.05419^{\{4\}}$	$0.05825^{\{6\}}$	$0.06508^{\{8\}}$	$0.05477^{\{5\}}$	$0.04877^{\{2\}}$	$0.04772^{\{1\}}$	$0.05335^{\{3\}}$
		$\hat{\varphi}$	$0.48370^{\{5\}}$	$0.59604^{\{7\}}$	$0.44791^{\{1\}}$	$0.47082^{\{2\}}$	$0.68934^{\{8\}}$	$0.47095^{\{3\}}$	$0.56015^{\{6\}}$	$0.47440^{\{4\}}$
		$\hat{\rho}$	$0.18164^{\{1\}}$	$0.19679^{\{4\}}$	$0.22272^{\{7\}}$	$0.21058^{\{6\}}$	$0.18211^{\{2\}}$	$0.19879^{\{5\}}$	$0.19280^{\{3\}}$	$0.22724^{\{8\}}$
	MRE	$\hat{\beta}$	$0.04933^{\{6\}}$	$0.04760^{\{5\}}$	$0.04972^{\{7\}}$	$0.05191^{\{8\}}$	$0.04721^{\{3\}}$	$0.04518^{\{1\}}$	$0.04540^{\{2\}}$	$0.04746^{\{4\}}$
		$\hat{\varphi}$	$0.17938^{\{3\}}$	$0.19973^{\{7\}}$	$0.17199^{\{1\}}$	$0.18457^{\{5\}}$	$0.19988^{\{8\}}$	$0.17695^{\{2\}}$	$0.19302^{\{6\}}$	$0.18388^{\{4\}}$
		$\hat{\rho}$	$0.61217^{\{1\}}$	$0.63581^{\{3\}}$	$0.73452^{\{8\}}$	$0.66720^{\{5\}}$	$0.62897^{\{2\}}$	$0.66780^{\{6\}}$	$0.67318^{\{7\}}$	$0.66244^{\{4\}}$
	$\sum$ *Ranks*		$33^{\{2\}}$	$45^{\{6\}}$	$46^{\{7\}}$	$52^{\{8\}}$	$41^{\{5\}}$	$28^{\{1\}}$	$40^{\{4\}}$	$39^{\{3\}}$
150	*BIAS*	$\hat{\beta}$	$0.15882^{\{5\}}$	$0.15311^{\{3\}}$	$0.17073^{\{8\}}$	$0.16591^{\{7\}}$	$0.15488^{\{4\}}$	$0.15046^{\{1\}}$	$0.15118^{\{2\}}$	$0.16532^{\{6\}}$
		$\hat{\varphi}$	$0.31194^{\{3\}}$	$0.35091^{\{8\}}$	$0.30593^{\{1\}}$	$0.32565^{\{4\}}$	$0.34698^{\{7\}}$	$0.30625^{\{2\}}$	$0.34351^{\{5\}}$	$0.34559^{\{6\}}$
		$\hat{\rho}$	$0.26994^{\{2\}}$	$0.27685^{\{4\}}$	$0.31188^{\{8\}}$	$0.26536^{\{1\}}$	$0.27985^{\{5\}}$	$0.27392^{\{3\}}$	$0.29630^{\{6\}}$	$0.29923^{\{7\}}$
	MSE	$\hat{\beta}$	$0.03785^{\{5\}}$	$0.03564^{\{3\}}$	$0.04215^{\{8\}}$	$0.04114^{\{7\}}$	$0.03605^{\{4\}}$	$0.03362^{\{2\}}$	$0.03280^{\{1\}}$	$0.03889^{\{6\}}$
		$\hat{\varphi}$	$0.16119^{\{3\}}$	$0.20217^{\{7\}}$	$0.15920^{\{2\}}$	$0.17820^{\{4\}}$	$0.19808^{\{6\}}$	$0.15711^{\{1\}}$	$0.18993^{\{5\}}$	$0.20356^{\{8\}}$
		$\hat{\rho}$	$0.13687^{\{3\}}$	$0.14656^{\{5\}}$	$0.16829^{\{7\}}$	$0.13373^{\{1\}}$	$0.14492^{\{4\}}$	$0.13668^{\{2\}}$	$0.15203^{\{6\}}$	$0.17620^{\{8\}}$
	MRE	$\hat{\beta}$	$0.03971^{\{5\}}$	$0.03828^{\{3\}}$	$0.04268^{\{8\}}$	$0.04148^{\{7\}}$	$0.03872^{\{4\}}$	$0.03762^{\{1\}}$	$0.03779^{\{2\}}$	$0.04133^{\{6\}}$
		$\hat{\varphi}$	$0.10398^{\{3\}}$	$0.11697^{\{8\}}$	$0.10198^{\{1\}}$	$0.10855^{\{4\}}$	$0.11566^{\{7\}}$	$0.10208^{\{2\}}$	$0.11450^{\{5\}}$	$0.11520^{\{6\}}$
		$\hat{\rho}$	$0.53987^{\{2\}}$	$0.55370^{\{4\}}$	$0.62376^{\{8\}}$	$0.53071^{\{1\}}$	$0.55970^{\{5\}}$	$0.54783^{\{3\}}$	$0.59260^{\{6\}}$	$0.59845^{\{7\}}$
	$\sum$ *Ranks*		$31^{\{2\}}$	$45^{\{5\}}$	$51^{\{7\}}$	$36^{\{3\}}$	$46^{\{6\}}$	$17^{\{1\}}$	$38^{\{4\}}$	$60^{\{8\}}$
300	*BIAS*	$\hat{\beta}$	$0.13729^{\{3\}}$	$0.13971^{\{6\}}$	$0.14968^{\{7\}}$	$0.13379^{\{1\}}$	$0.13866^{\{5\}}$	$0.13756^{\{4\}}$	$0.13547^{\{2\}}$	$0.15046^{\{8\}}$
		$\hat{\varphi}$	$0.21885^{\{3\}}$	$0.25206^{\{8\}}$	$0.21485^{\{1\}}$	$0.22846^{\{4\}}$	$0.24644^{\{7\}}$	$0.21803^{\{2\}}$	$0.24595^{\{6\}}$	$0.23913^{\{5\}}$
		$\hat{\rho}$	$0.23687^{\{2\}}$	$0.24633^{\{5\}}$	$0.26974^{\{8\}}$	$0.21418^{\{1\}}$	$0.24264^{\{3\}}$	$0.24323^{\{4\}}$	$0.25122^{\{6\}}$	$0.25767^{\{7\}}$
	MSE	$\hat{\beta}$	$0.02934^{\{4\}}$	$0.03081^{\{6\}}$	$0.03229^{\{8\}}$	$0.02882^{\{2\}}$	$0.02997^{\{5\}}$	$0.02891^{\{3\}}$	$0.02725^{\{1\}}$	$0.03224^{\{7\}}$
		$\hat{\varphi}$	$0.07862^{\{1\}}$	$0.10393^{\{8\}}$	$0.08288^{\{3\}}$	$0.09014^{\{4\}}$	$0.09928^{\{6\}}$	$0.07903^{\{2\}}$	$0.09770^{\{5\}}$	$0.10222^{\{7\}}$
		$\hat{\rho}$	$0.09966^{\{2\}}$	$0.11314^{\{6\}}$	$0.12169^{\{7\}}$	$0.08886^{\{1\}}$	$0.10770^{\{4\}}$	$0.10365^{\{3\}}$	$0.10801^{\{5\}}$	$0.12177^{\{8\}}$
	MRE	$\hat{\beta}$	$0.03432^{\{3\}}$	$0.03493^{\{6\}}$	$0.03742^{\{7\}}$	$0.03345^{\{1\}}$	$0.03466^{\{5\}}$	$0.03439^{\{4\}}$	$0.03387^{\{2\}}$	$0.03761^{\{8\}}$
		$\hat{\varphi}$	$0.07295^{\{3\}}$	$0.08402^{\{8\}}$	$0.07162^{\{1\}}$	$0.07615^{\{4\}}$	$0.08215^{\{7\}}$	$0.07268^{\{2\}}$	$0.08198^{\{6\}}$	$0.07971^{\{5\}}$
		$\hat{\rho}$	$0.47375^{\{2\}}$	$0.49266^{\{5\}}$	$0.53948^{\{8\}}$	$0.42837^{\{1\}}$	$0.48527^{\{3\}}$	$0.48645^{\{4\}}$	$0.50244^{\{6\}}$	$0.51533^{\{7\}}$
	$\sum$ *Ranks*		$23^{\{2\}}$	$58^{\{7\}}$	$50^{\{6\}}$	$19^{\{1\}}$	$45^{\{5\}}$	$28^{\{3\}}$	$39^{\{4\}}$	$62^{\{8\}}$
400	*BIAS*	$\hat{\beta}$	$0.12955^{\{2\}}$	$0.13443^{\{5\}}$	$0.14106^{\{7\}}$	$0.12071^{\{1\}}$	$0.13453^{\{6\}}$	$0.13057^{\{4\}}$	$0.13033^{\{3\}}$	$0.14511^{\{8\}}$
		$\hat{\varphi}$	$0.18975^{\{2\}}$	$0.21345^{\{6\}}$	$0.19049^{\{3\}}$	$0.19472^{\{4\}}$	$0.21206^{\{5\}}$	$0.18833^{\{1\}}$	$0.21728^{\{8\}}$	$0.21407^{\{7\}}$
		$\hat{\rho}$	$0.22180^{\{2\}}$	$0.22856^{\{4\}}$	$0.24961^{\{8\}}$	$0.18807^{\{1\}}$	$0.23376^{\{5\}}$	$0.22430^{\{3\}}$	$0.23983^{\{6\}}$	$0.24753^{\{7\}}$
	MSE	$\hat{\beta}$	$0.02645^{\{3\}}$	$0.02895^{\{6\}}$	$0.02922^{\{7\}}$	$0.02512^{\{2\}}$	$0.02869^{\{5\}}$	$0.02659^{\{4\}}$	$0.02508^{\{1\}}$	$0.02981^{\{8\}}$
		$\hat{\varphi}$	$0.05886^{\{2\}}$	$0.07213^{\{5\}}$	$0.06428^{\{3\}}$	$0.06470^{\{4\}}$	$0.07299^{\{6\}}$	$0.05827^{\{1\}}$	$0.07631^{\{7\}}$	$0.08197^{\{8\}}$
		$\hat{\rho}$	$0.08643^{\{2\}}$	$0.09501^{\{4\}}$	$0.10507^{\{7\}}$	$0.06945^{\{1\}}$	$0.09836^{\{6\}}$	$0.08732^{\{3\}}$	$0.09806^{\{5\}}$	$0.10945^{\{8\}}$
	MRE	$\hat{\beta}$	$0.03239^{\{2\}}$	$0.03361^{\{5\}}$	$0.03527^{\{7\}}$	$0.03018^{\{1\}}$	$0.03363^{\{6\}}$	$0.03264^{\{4\}}$	$0.03258^{\{3\}}$	$0.03628^{\{8\}}$
		$\hat{\varphi}$	$0.06325^{\{2\}}$	$0.07115^{\{6\}}$	$0.06350^{\{3\}}$	$0.06491^{\{4\}}$	$0.07069^{\{5\}}$	$0.06278^{\{1\}}$	$0.07243^{\{8\}}$	$0.07136^{\{7\}}$
		$\hat{\rho}$	$0.44360^{\{2\}}$	$0.45713^{\{4\}}$	$0.49922^{\{8\}}$	$0.37613^{\{1\}}$	$0.46751^{\{5\}}$	$0.44861^{\{3\}}$	$0.47965^{\{6\}}$	$0.49506^{\{7\}}$
	$\sum$ *Ranks*		$19^{\{1.5\}}$	$45^{\{4\}}$	$53^{\{7\}}$	$19^{\{1.5\}}$	$49^{\{6\}}$	$24^{\{3\}}$	$47^{\{5\}}$	$68^{\{8\}}$

Table A5. Simulation results for $\boldsymbol{\Theta} = (\beta = 3.00, \varphi = 1.50, \rho = 0.30)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	$BIAS$	$\hat{\beta}$	$0.18133^{\{3\}}$	$0.18433^{\{5\}}$	$0.18363^{\{4\}}$	$0.20934^{\{8\}}$	$0.18114^{\{2\}}$	$0.17863^{\{1\}}$	$0.19393^{\{6\}}$	$0.19953^{\{7\}}$
		$\hat{\varphi}$	$0.28939^{\{5\}}$	$0.30744^{\{8\}}$	$0.25388^{\{1\}}$	$0.29657^{\{7\}}$	$0.29633^{\{6\}}$	$0.27690^{\{3\}}$	$0.27375^{\{2\}}$	$0.28495^{\{4\}}$
		$\hat{\rho}$	$0.44199^{\{6\}}$	$0.43180^{\{4\}}$	$0.37852^{\{1\}}$	$0.45771^{\{8\}}$	$0.41271^{\{2\}}$	$0.43542^{\{5\}}$	$0.42886^{\{3\}}$	$0.44375^{\{7\}}$
	MSE	$\hat{\beta}$	$0.05036^{\{2\}}$	$0.05183^{\{4\}}$	$0.05243^{\{5\}}$	$0.06610^{\{8\}}$	$0.05121^{\{3\}}$	$0.04928^{\{1\}}$	$0.05784^{\{6\}}$	$0.06001^{\{7\}}$
		$\hat{\varphi}$	$0.12833^{\{5\}}$	$0.14235^{\{8\}}$	$0.10433^{\{1\}}$	$0.13678^{\{6\}}$	$0.13882^{\{7\}}$	$0.11821^{\{3\}}$	$0.11486^{\{2\}}$	$0.12357^{\{4\}}$
		$\hat{\rho}$	$0.27523^{\{6\}}$	$0.26400^{\{4\}}$	$0.22207^{\{1\}}$	$0.30744^{\{8\}}$	$0.24580^{\{3\}}$	$0.27214^{\{5\}}$	$0.24481^{\{2\}}$	$0.27798^{\{7\}}$
	MRE	$\hat{\beta}$	$0.06044^{\{3\}}$	$0.06144^{\{5\}}$	$0.06121^{\{4\}}$	$0.06978^{\{8\}}$	$0.06038^{\{2\}}$	$0.05954^{\{1\}}$	$0.06464^{\{6\}}$	$0.06651^{\{7\}}$
		$\hat{\varphi}$	$0.19293^{\{5\}}$	$0.20496^{\{8\}}$	$0.16925^{\{1\}}$	$0.19772^{\{7\}}$	$0.19755^{\{6\}}$	$0.18460^{\{3\}}$	$0.18250^{\{2\}}$	$0.18997^{\{4\}}$
		$\hat{\rho}$	$1.47331^{\{6\}}$	$1.43933^{\{4\}}$	$1.26174^{\{1\}}$	$1.52570^{\{8\}}$	$1.37570^{\{2\}}$	$1.45140^{\{5\}}$	$1.42954^{\{3\}}$	$1.47918^{\{7\}}$
	$\sum Ranks$		$41^{\{5\}}$	$50^{\{6\}}$	$19^{\{1\}}$	$68^{\{8\}}$	$33^{\{4\}}$	$27^{\{2\}}$	$32^{\{3\}}$	$54^{\{7\}}$
150	$BIAS$	$\hat{\beta}$	$0.13707^{\{4\}}$	$0.13737^{\{5\}}$	$0.13607^{\{2\}}$	$0.16400^{\{8\}}$	$0.13413^{\{1\}}$	$0.13626^{\{3\}}$	$0.14497^{\{6\}}$	$0.15859^{\{7\}}$
		$\hat{\varphi}$	$0.18302^{\{4\}}$	$0.20553^{\{8\}}$	$0.15974^{\{1\}}$	$0.18813^{\{6\}}$	$0.19877^{\{7\}}$	$0.17818^{\{2\}}$	$0.18784^{\{5\}}$	$0.17851^{\{3\}}$
		$\hat{\rho}$	$0.35844^{\{3\}}$	$0.37134^{\{6\}}$	$0.31888^{\{1\}}$	$0.38499^{\{8\}}$	$0.36011^{\{4\}}$	$0.35320^{\{2\}}$	$0.36516^{\{5\}}$	$0.37802^{\{7\}}$
	MSE	$\hat{\beta}$	$0.02754^{\{3\}}$	$0.02769^{\{4\}}$	$0.02822^{\{5\}}$	$0.04115^{\{8\}}$	$0.02682^{\{1\}}$	$0.02725^{\{2\}}$	$0.03052^{\{6\}}$	$0.03687^{\{7\}}$
		$\hat{\varphi}$	$0.05610^{\{4\}}$	$0.06732^{\{7\}}$	$0.04571^{\{1\}}$	$0.06803^{\{8\}}$	$0.06258^{\{6\}}$	$0.05307^{\{2\}}$	$0.05585^{\{3\}}$	$0.05768^{\{5\}}$
		$\hat{\rho}$	$0.19029^{\{4\}}$	$0.20259^{\{6\}}$	$0.16497^{\{1\}}$	$0.23762^{\{8\}}$	$0.19279^{\{5\}}$	$0.18623^{\{2\}}$	$0.18641^{\{3\}}$	$0.20861^{\{7\}}$
	MRE	$\hat{\beta}$	$0.04569^{\{4\}}$	$0.04579^{\{5\}}$	$0.04536^{\{2\}}$	$0.05467^{\{8\}}$	$0.04471^{\{1\}}$	$0.04542^{\{3\}}$	$0.04832^{\{6\}}$	$0.05286^{\{7\}}$
		$\hat{\varphi}$	$0.12202^{\{4\}}$	$0.13702^{\{8\}}$	$0.10649^{\{1\}}$	$0.12542^{\{6\}}$	$0.13251^{\{7\}}$	$0.11878^{\{2\}}$	$0.12523^{\{5\}}$	$0.11901^{\{3\}}$
		$\hat{\rho}$	$1.19479^{\{3\}}$	$1.23781^{\{6\}}$	$1.06293^{\{1\}}$	$1.28331^{\{8\}}$	$1.20038^{\{4\}}$	$1.17733^{\{2\}}$	$1.21721^{\{5\}}$	$1.26007^{\{7\}}$
	$\sum Ranks$		$33^{\{3\}}$	$55^{\{7\}}$	$15^{\{1\}}$	$68^{\{8\}}$	$36^{\{4\}}$	$20^{\{2\}}$	$44^{\{5\}}$	$53^{\{6\}}$
300	$BIAS$	$\hat{\beta}$	$0.11743^{\{3\}}$	$0.12266^{\{5\}}$	$0.11418^{\{1\}}$	$0.13973^{\{8\}}$	$0.11954^{\{4\}}$	$0.11657^{\{2\}}$	$0.12401^{\{6\}}$	$0.13797^{\{7\}}$
		$\hat{\varphi}$	$0.12724^{\{4\}}$	$0.14997^{\{8\}}$	$0.11519^{\{1\}}$	$0.12975^{\{5\}}$	$0.14593^{\{7\}}$	$0.12476^{\{3\}}$	$0.13767^{\{6\}}$	$0.12027^{\{2\}}$
		$\hat{\rho}$	$0.28494^{\{3\}}$	$0.31225^{\{8\}}$	$0.26669^{\{1\}}$	$0.30087^{\{5\}}$	$0.30512^{\{7\}}$	$0.28132^{\{2\}}$	$0.29796^{\{4\}}$	$0.30306^{\{6\}}$
	MSE	$\hat{\beta}$	$0.02038^{\{2\}}$	$0.02152^{\{5\}}$	$0.02059^{\{3\}}$	$0.03159^{\{8\}}$	$0.02072^{\{4\}}$	$0.02014^{\{1\}}$	$0.02274^{\{6\}}$	$0.02895^{\{7\}}$
		$\hat{\varphi}$	$0.02980^{\{4\}}$	$0.03939^{\{8\}}$	$0.02593^{\{1\}}$	$0.03634^{\{6\}}$	$0.03689^{\{7\}}$	$0.02872^{\{3\}}$	$0.03275^{\{5\}}$	$0.02804^{\{2\}}$
		$\hat{\rho}$	$0.12405^{\{3\}}$	$0.14682^{\{7\}}$	$0.11700^{\{1\}}$	$0.15961^{\{8\}}$	$0.14086^{\{6\}}$	$0.12131^{\{2\}}$	$0.13154^{\{4\}}$	$0.13645^{\{5\}}$
	MRE	$\hat{\beta}$	$0.03914^{\{3\}}$	$0.04089^{\{5\}}$	$0.03806^{\{1\}}$	$0.04658^{\{8\}}$	$0.03985^{\{4\}}$	$0.03886^{\{2\}}$	$0.04134^{\{6\}}$	$0.04599^{\{7\}}$
		$\hat{\varphi}$	$0.08483^{\{4\}}$	$0.09998^{\{8\}}$	$0.07679^{\{1\}}$	$0.08650^{\{5\}}$	$0.09729^{\{7\}}$	$0.08318^{\{3\}}$	$0.09178^{\{6\}}$	$0.08018^{\{2\}}$
		$\hat{\rho}$	$0.94979^{\{3\}}$	$1.04084^{\{8\}}$	$0.88895^{\{1\}}$	$1.00289^{\{5\}}$	$1.01706^{\{7\}}$	$0.93775^{\{2\}}$	$0.99318^{\{4\}}$	$1.01020^{\{6\}}$
	$\sum Ranks$		$29^{\{3\}}$	$62^{\{8\}}$	$11^{\{1\}}$	$58^{\{7\}}$	$53^{\{6\}}$	$20^{\{2\}}$	$47^{\{5\}}$	$44^{\{4\}}$
400	$BIAS$	$\hat{\beta}$	$0.11102^{\{3\}}$	$0.11592^{\{5\}}$	$0.10638^{\{1\}}$	$0.12815^{\{7\}}$	$0.11325^{\{4\}}$	$0.11058^{\{2\}}$	$0.11710^{\{6\}}$	$0.12966^{\{8\}}$
		$\hat{\varphi}$	$0.10711^{\{5\}}$	$0.12898^{\{8\}}$	$0.09874^{\{1\}}$	$0.10597^{\{4\}}$	$0.12609^{\{7\}}$	$0.10543^{\{3\}}$	$0.11885^{\{6\}}$	$0.09877^{\{2\}}$
		$\hat{\rho}$	$0.26267^{\{3\}}$	$0.29093^{\{8\}}$	$0.24002^{\{1\}}$	$0.26530^{\{4\}}$	$0.28482^{\{7\}}$	$0.25982^{\{2\}}$	$0.27314^{\{5\}}$	$0.27768^{\{6\}}$
	MSE	$\hat{\beta}$	$0.01820^{\{2\}}$	$0.01915^{\{5\}}$	$0.01836^{\{3\}}$	$0.02792^{\{8\}}$	$0.01845^{\{4\}}$	$0.01819^{\{1\}}$	$0.02056^{\{6\}}$	$0.02637^{\{7\}}$
		$\hat{\varphi}$	$0.02138^{\{4\}}$	$0.03036^{\{8\}}$	$0.01843^{\{1\}}$	$0.02503^{\{6\}}$	$0.02868^{\{7\}}$	$0.02091^{\{3\}}$	$0.02485^{\{5\}}$	$0.01930^{\{2\}}$
		$\hat{\rho}$	$0.10448^{\{3\}}$	$0.12788^{\{7\}}$	$0.09496^{\{1\}}$	$0.13013^{\{8\}}$	$0.12326^{\{6\}}$	$0.10293^{\{2\}}$	$0.11174^{\{4\}}$	$0.11552^{\{5\}}$
	MRE	$\hat{\beta}$	$0.03701^{\{3\}}$	$0.03864^{\{5\}}$	$0.03546^{\{1\}}$	$0.04272^{\{7\}}$	$0.03775^{\{4\}}$	$0.03686^{\{2\}}$	$0.03903^{\{6\}}$	$0.04322^{\{8\}}$
		$\hat{\varphi}$	$0.07141^{\{5\}}$	$0.08598^{\{8\}}$	$0.06583^{\{1\}}$	$0.07064^{\{4\}}$	$0.08406^{\{7\}}$	$0.07029^{\{3\}}$	$0.07924^{\{6\}}$	$0.06584^{\{2\}}$
		$\hat{\rho}$	$0.87556^{\{3\}}$	$0.96975^{\{8\}}$	$0.80007^{\{1\}}$	$0.88433^{\{4\}}$	$0.94938^{\{7\}}$	$0.86607^{\{2\}}$	$0.91047^{\{5\}}$	$0.92560^{\{6\}}$
	$\sum Ranks$		$31^{\{3\}}$	$62^{\{8\}}$	$11^{\{1\}}$	$52^{\{6\}}$	$53^{\{7\}}$	$20^{\{2\}}$	$49^{\{5\}}$	$46^{\{4\}}$

Table A6. Simulation results for $\mathbf{\Theta} = (\beta = 0.75, \varphi = 1.00, \rho = 0.75)^\intercal$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	$BIAS$	$\hat{\beta}$	$0.06142^{\{5\}}$	$0.06764^{\{6\}}$	$0.11605^{\{8\}}$	$0.05800^{\{3\}}$	$0.06824^{\{7\}}$	$0.05977^{\{4\}}$	$0.05632^{\{1\}}$	$0.05682^{\{2\}}$
		$\hat{\varphi}$	$0.13095^{\{4\}}$	$0.14246^{\{6\}}$	$0.14229^{\{5\}}$	$0.12720^{\{2\}}$	$0.14842^{\{8\}}$	$0.12724^{\{3\}}$	$0.14615^{\{7\}}$	$0.12543^{\{1\}}$
		$\hat{\rho}$	$0.16416^{\{4\}}$	$0.17940^{\{6\}}$	$0.25527^{\{8\}}$	$0.13633^{\{1\}}$	$0.19025^{\{7\}}$	$0.16635^{\{5\}}$	$0.16016^{\{3\}}$	$0.14958^{\{2\}}$
	MSE	$\hat{\beta}$	$0.00579^{\{5\}}$	$0.00701^{\{6\}}$	$0.05061^{\{8\}}$	$0.00499^{\{3\}}$	$0.00718^{\{7\}}$	$0.00550^{\{4\}}$	$0.00493^{\{2\}}$	$0.00487^{\{1\}}$
		$\hat{\varphi}$	$0.02952^{\{4\}}$	$0.03479^{\{5\}}$	$0.03645^{\{6\}}$	$0.02497^{\{2\}}$	$0.04033^{\{8\}}$	$0.02791^{\{3\}}$	$0.03691^{\{7\}}$	$0.02489^{\{1\}}$
		$\hat{\rho}$	$0.03690^{\{2\}}$	$0.04179^{\{6\}}$	$0.10474^{\{8\}}$	$0.03670^{\{1\}}$	$0.04500^{\{7\}}$	$0.03768^{\{4\}}$	$0.03923^{\{5\}}$	$0.03767^{\{3\}}$
	MRE	$\hat{\beta}$	$0.08190^{\{5\}}$	$0.09019^{\{6\}}$	$0.15474^{\{8\}}$	$0.07733^{\{3\}}$	$0.09099^{\{7\}}$	$0.07969^{\{4\}}$	$0.07510^{\{1\}}$	$0.07576^{\{2\}}$
		$\hat{\varphi}$	$0.13095^{\{4\}}$	$0.14246^{\{6\}}$	$0.14229^{\{5\}}$	$0.12720^{\{2\}}$	$0.14842^{\{8\}}$	$0.12724^{\{3\}}$	$0.14615^{\{7\}}$	$0.12543^{\{1\}}$
		$\hat{\rho}$	$0.21888^{\{4\}}$	$0.23920^{\{6\}}$	$0.34036^{\{8\}}$	$0.18177^{\{1\}}$	$0.25366^{\{7\}}$	$0.22180^{\{5\}}$	$0.21354^{\{3\}}$	$0.19944^{\{2\}}$
	$\sum Ranks$		$37^{\{5\}}$	$53^{\{6\}}$	$64^{\{7\}}$	$18^{\{2\}}$	$66^{\{8\}}$	$35^{\{3\}}$	$36^{\{4\}}$	$15^{\{1\}}$
150	$BIAS$	$\hat{\beta}$	$0.04553^{\{5\}}$	$0.05020^{\{6\}}$	$0.10670^{\{8\}}$	$0.03905^{\{1\}}$	$0.05099^{\{7\}}$	$0.04491^{\{4\}}$	$0.04183^{\{3\}}$	$0.03943^{\{2\}}$
		$\hat{\varphi}$	$0.07400^{\{4\}}$	$0.08116^{\{6\}}$	$0.08981^{\{8\}}$	$0.07141^{\{1\}}$	$0.08194^{\{7\}}$	$0.07305^{\{3\}}$	$0.08097^{\{5\}}$	$0.07259^{\{2\}}$
		$\hat{\rho}$	$0.16079^{\{5\}}$	$0.17614^{\{6\}}$	$0.21832^{\{8\}}$	$0.11539^{\{1\}}$	$0.18218^{\{7\}}$	$0.16065^{\{4\}}$	$0.15011^{\{3\}}$	$0.13265^{\{2\}}$
	MSE	$\hat{\beta}$	$0.00304^{\{5\}}$	$0.00365^{\{6\}}$	$0.05054^{\{8\}}$	$0.00231^{\{1\}}$	$0.00369^{\{7\}}$	$0.00298^{\{4\}}$	$0.00267^{\{3\}}$	$0.00237^{\{2\}}$
		$\hat{\varphi}$	$0.00863^{\{4\}}$	$0.01042^{\{6\}}$	$0.01799^{\{8\}}$	$0.00790^{\{1\}}$	$0.01084^{\{7\}}$	$0.00840^{\{3\}}$	$0.01039^{\{5\}}$	$0.00819^{\{2\}}$
		$\hat{\rho}$	$0.03325^{\{4\}}$	$0.03707^{\{6\}}$	$0.07496^{\{8\}}$	$0.02565^{\{1\}}$	$0.03905^{\{7\}}$	$0.03378^{\{5\}}$	$0.03225^{\{3\}}$	$0.03174^{\{2\}}$
	MRE	$\hat{\beta}$	$0.06071^{\{5\}}$	$0.06694^{\{6\}}$	$0.14227^{\{8\}}$	$0.05207^{\{1\}}$	$0.06799^{\{7\}}$	$0.05988^{\{4\}}$	$0.05577^{\{3\}}$	$0.05257^{\{2\}}$
		$\hat{\varphi}$	$0.07400^{\{4\}}$	$0.08116^{\{6\}}$	$0.08981^{\{8\}}$	$0.07141^{\{1\}}$	$0.08194^{\{7\}}$	$0.07305^{\{3\}}$	$0.08097^{\{5\}}$	$0.07259^{\{2\}}$
		$\hat{\rho}$	$0.21439^{\{5\}}$	$0.23485^{\{6\}}$	$0.29109^{\{8\}}$	$0.15385^{\{1\}}$	$0.24291^{\{7\}}$	$0.21421^{\{4\}}$	$0.20015^{\{3\}}$	$0.17687^{\{2\}}$
	$\sum Ranks$		$41^{\{5\}}$	$54^{\{6\}}$	$72^{\{8\}}$	$9^{\{1\}}$	$63^{\{7\}}$	$34^{\{4\}}$	$33^{\{3\}}$	$18^{\{2\}}$
300	$BIAS$	$\hat{\beta}$	$0.03949^{\{5\}}$	$0.04420^{\{6\}}$	$0.10805^{\{8\}}$	$0.03078^{\{1\}}$	$0.04495^{\{7\}}$	$0.03892^{\{4\}}$	$0.03599^{\{3\}}$	$0.03314^{\{2\}}$
		$\hat{\varphi}$	$0.05221^{\{3\}}$	$0.05732^{\{5\}}$	$0.07326^{\{8\}}$	$0.05089^{\{1\}}$	$0.05734^{\{6\}}$	$0.05195^{\{2\}}$	$0.05802^{\{7\}}$	$0.05233^{\{4\}}$
		$\hat{\rho}$	$0.15209^{\{5\}}$	$0.17061^{\{6\}}$	$0.20094^{\{8\}}$	$0.09675^{\{1\}}$	$0.17504^{\{7\}}$	$0.15004^{\{4\}}$	$0.13943^{\{3\}}$	$0.12193^{\{2\}}$
	MSE	$\hat{\beta}$	$0.00222^{\{5\}}$	$0.00269^{\{6\}}$	$0.05487^{\{8\}}$	$0.00150^{\{1\}}$	$0.00273^{\{7\}}$	$0.00218^{\{4\}}$	$0.00199^{\{3\}}$	$0.00173^{\{2\}}$
		$\hat{\varphi}$	$0.00425^{\{4\}}$	$0.00512^{\{5\}}$	$0.01547^{\{8\}}$	$0.00395^{\{1\}}$	$0.00517^{\{6\}}$	$0.00419^{\{2\}}$	$0.00523^{\{7\}}$	$0.00422^{\{3\}}$
		$\hat{\rho}$	$0.02992^{\{5\}}$	$0.03471^{\{6\}}$	$0.06064^{\{8\}}$	$0.01877^{\{1\}}$	$0.03615^{\{7\}}$	$0.02964^{\{4\}}$	$0.02835^{\{3\}}$	$0.02651^{\{2\}}$
	MRE	$\hat{\beta}$	$0.05265^{\{5\}}$	$0.05893^{\{6\}}$	$0.14407^{\{8\}}$	$0.04103^{\{1\}}$	$0.05994^{\{7\}}$	$0.05190^{\{4\}}$	$0.04799^{\{3\}}$	$0.04418^{\{2\}}$
		$\hat{\varphi}$	$0.05221^{\{3\}}$	$0.05732^{\{5\}}$	$0.07326^{\{8\}}$	$0.05089^{\{1\}}$	$0.05734^{\{6\}}$	$0.05195^{\{2\}}$	$0.05802^{\{7\}}$	$0.05233^{\{4\}}$
		$\hat{\rho}$	$0.20279^{\{5\}}$	$0.22747^{\{6\}}$	$0.26792^{\{8\}}$	$0.12900^{\{1\}}$	$0.23338^{\{7\}}$	$0.20005^{\{4\}}$	$0.18591^{\{3\}}$	$0.16257^{\{2\}}$
	$\sum Ranks$		$40^{\{5\}}$	$51^{\{6\}}$	$72^{\{8\}}$	$9^{\{1\}}$	$60^{\{7\}}$	$30^{\{3\}}$	$39^{\{4\}}$	$23^{\{2\}}$
400	$BIAS$	$\hat{\beta}$	$0.03796^{\{5\}}$	$0.04301^{\{6\}}$	$0.10524^{\{8\}}$	$0.02802^{\{1\}}$	$0.04366^{\{7\}}$	$0.03731^{\{4\}}$	$0.03431^{\{3\}}$	$0.03156^{\{2\}}$
		$\hat{\varphi}$	$0.04636^{\{4\}}$	$0.05120^{\{6\}}$	$0.06532^{\{8\}}$	$0.04447^{\{1\}}$	$0.05126^{\{7\}}$	$0.04599^{\{2\}}$	$0.05046^{\{5\}}$	$0.04600^{\{3\}}$
		$\hat{\rho}$	$0.14784^{\{5\}}$	$0.16776^{\{6\}}$	$0.19146^{\{8\}}$	$0.08614^{\{1\}}$	$0.17154^{\{7\}}$	$0.14593^{\{4\}}$	$0.13598^{\{3\}}$	$0.11994^{\{2\}}$
	MSE	$\hat{\beta}$	$0.00204^{\{5\}}$	$0.00250^{\{6\}}$	$0.05407^{\{8\}}$	$0.00126^{\{1\}}$	$0.00253^{\{7\}}$	$0.00199^{\{4\}}$	$0.00180^{\{3\}}$	$0.00157^{\{2\}}$
		$\hat{\varphi}$	$0.00339^{\{4\}}$	$0.00411^{\{6\}}$	$0.01368^{\{8\}}$	$0.00302^{\{1\}}$	$0.00415^{\{7\}}$	$0.00334^{\{3\}}$	$0.00398^{\{5\}}$	$0.00328^{\{2\}}$
		$\hat{\rho}$	$0.02832^{\{5\}}$	$0.03372^{\{6\}}$	$0.05533^{\{8\}}$	$0.01591^{\{1\}}$	$0.03493^{\{7\}}$	$0.02810^{\{4\}}$	$0.02664^{\{3\}}$	$0.02606^{\{2\}}$
	MRE	$\hat{\beta}$	$0.05061^{\{5\}}$	$0.05734^{\{6\}}$	$0.14032^{\{8\}}$	$0.03736^{\{1\}}$	$0.05822^{\{7\}}$	$0.04975^{\{4\}}$	$0.04574^{\{3\}}$	$0.04207^{\{2\}}$
		$\hat{\varphi}$	$0.04636^{\{4\}}$	$0.05120^{\{6\}}$	$0.06532^{\{8\}}$	$0.04447^{\{1\}}$	$0.05126^{\{7\}}$	$0.04599^{\{2\}}$	$0.05046^{\{5\}}$	$0.04600^{\{3\}}$
		$\hat{\rho}$	$0.19713^{\{5\}}$	$0.22368^{\{6\}}$	$0.25528^{\{8\}}$	$0.11485^{\{1\}}$	$0.22872^{\{7\}}$	$0.19457^{\{4\}}$	$0.18130^{\{3\}}$	$0.15992^{\{2\}}$
	$\sum Ranks$		$42^{\{5\}}$	$54^{\{6\}}$	$72^{\{8\}}$	$9^{\{1\}}$	$63^{\{7\}}$	$31^{\{3\}}$	$33^{\{4\}}$	$20^{\{2\}}$

Table A7. Simulation results for $\boldsymbol{\Theta} = (\beta = 4.00, \varphi = 1.50, \rho = -0.50)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	*BIAS*	$\hat{\beta}$	$0.19941^{\{3\}}$	$0.21840^{\{8\}}$	$0.17472^{\{1\}}$	$0.21077^{\{7\}}$	$0.20550^{\{4\}}$	$0.19064^{\{2\}}$	$0.20992^{\{5\}}$	$0.21016^{\{6\}}$
		$\hat{\varphi}$	$0.28798^{\{4\}}$	$0.28301^{\{1\}}$	$0.33258^{\{7\}}$	$0.29792^{\{5\}}$	$0.30321^{\{6\}}$	$0.28759^{\{3\}}$	$0.33837^{\{8\}}$	$0.28640^{\{2\}}$
		$\hat{\rho}$	$0.34341^{\{3\}}$	$0.33981^{\{2\}}$	$0.35972^{\{7\}}$	$0.35511^{\{4\}}$	$0.33009^{\{1\}}$	$0.35690^{\{5\}}$	$0.35994^{\{8\}}$	$0.35754^{\{6\}}$
	MSE	$\hat{\beta}$	$0.06480^{\{3\}}$	$0.07612^{\{8\}}$	$0.04953^{\{1\}}$	$0.07600^{\{7\}}$	$0.06683^{\{4\}}$	$0.05870^{\{2\}}$	$0.07189^{\{5\}}$	$0.07457^{\{6\}}$
		$\hat{\varphi}$	$0.12522^{\{3\}}$	$0.12791^{\{4\}}$	$0.17628^{\{7\}}$	$0.12862^{\{5\}}$	$0.15457^{\{6\}}$	$0.12281^{\{2\}}$	$0.20293^{\{8\}}$	$0.11691^{\{1\}}$
		$\hat{\rho}$	$0.22530^{\{3\}}$	$0.23190^{\{5\}}$	$0.22421^{\{2\}}$	$0.24546^{\{6\}}$	$0.21202^{\{1\}}$	$0.23186^{\{4\}}$	$0.26740^{\{8\}}$	$0.25446^{\{7\}}$
	MRE	$\hat{\beta}$	$0.04985^{\{3\}}$	$0.05460^{\{8\}}$	$0.04368^{\{1\}}$	$0.05269^{\{7\}}$	$0.05137^{\{4\}}$	$0.04766^{\{2\}}$	$0.05248^{\{5\}}$	$0.05254^{\{6\}}$
		$\hat{\varphi}$	$0.19199^{\{4\}}$	$0.18867^{\{1\}}$	$0.22172^{\{7\}}$	$0.19862^{\{5\}}$	$0.20214^{\{6\}}$	$0.19173^{\{3\}}$	$0.22558^{\{8\}}$	$0.19093^{\{2\}}$
		$\hat{\rho}$	$-0.68682^{\{6\}}$	$-0.67962^{\{7\}}$	$-0.71944^{\{2\}}$	$-0.71022^{\{5\}}$	$-0.66018^{\{8\}}$	$-0.71380^{\{4\}}$	$-0.71989^{\{1\}}$	$-0.71507^{\{3\}}$
	$\sum$ *Ranks*		$32^{\{2\}}$	$44^{\{6\}}$	$35^{\{3\}}$	$51^{\{7\}}$	$40^{\{5\}}$	$27^{\{1\}}$	$56^{\{8\}}$	$39^{\{4\}}$
150	*BIAS*	$\hat{\beta}$	$0.11797^{\{3\}}$	$0.13223^{\{8\}}$	$0.11049^{\{1\}}$	$0.12274^{\{4\}}$	$0.12651^{\{6\}}$	$0.11606^{\{2\}}$	$0.12425^{\{5\}}$	$0.12750^{\{7\}}$
		$\hat{\varphi}$	$0.23713^{\{2\}}$	$0.23233^{\{1\}}$	$0.27845^{\{8\}}$	$0.23981^{\{4\}}$	$0.23865^{\{3\}}$	$0.24046^{\{5\}}$	$0.26720^{\{7\}}$	$0.25377^{\{6\}}$
		$\hat{\rho}$	$0.29037^{\{4\}}$	$0.29029^{\{3\}}$	$0.31321^{\{8\}}$	$0.27463^{\{1\}}$	$0.28863^{\{2\}}$	$0.29799^{\{5\}}$	$0.30608^{\{6\}}$	$0.30912^{\{7\}}$
	MSE	$\hat{\beta}$	$0.02505^{\{3\}}$	$0.03014^{\{6\}}$	$0.02288^{\{1\}}$	$0.03015^{\{7\}}$	$0.02705^{\{4\}}$	$0.02428^{\{2\}}$	$0.02847^{\{5\}}$	$0.03212^{\{8\}}$
		$\hat{\varphi}$	$0.08375^{\{2\}}$	$0.08023^{\{1\}}$	$0.11875^{\{8\}}$	$0.08631^{\{5\}}$	$0.08505^{\{3\}}$	$0.08578^{\{4\}}$	$0.10424^{\{7\}}$	$0.09118^{\{6\}}$
		$\hat{\rho}$	$0.16210^{\{2\}}$	$0.17202^{\{5\}}$	$0.17579^{\{6\}}$	$0.15454^{\{1\}}$	$0.16595^{\{4\}}$	$0.16544^{\{3\}}$	$0.19422^{\{8\}}$	$0.18503^{\{7\}}$
	MRE	$\hat{\beta}$	$0.02949^{\{3\}}$	$0.03306^{\{8\}}$	$0.02762^{\{1\}}$	$0.03069^{\{4\}}$	$0.03163^{\{6\}}$	$0.02901^{\{2\}}$	$0.03106^{\{5\}}$	$0.03187^{\{7\}}$
		$\hat{\varphi}$	$0.15809^{\{2\}}$	$0.15489^{\{1\}}$	$0.18563^{\{8\}}$	$0.15987^{\{4\}}$	$0.15910^{\{3\}}$	$0.16031^{\{5\}}$	$0.17813^{\{7\}}$	$0.16918^{\{6\}}$
		$\hat{\rho}$	$-0.58073^{\{5\}}$	$-0.58058^{\{6\}}$	$-0.62641^{\{1\}}$	$-0.54927^{\{8\}}$	$-0.57727^{\{7\}}$	$-0.59599^{\{4\}}$	$-0.61216^{\{3\}}$	$-0.61824^{\{2\}}$
	$\sum$ *Ranks*		$26^{\{1\}}$	$39^{\{5\}}$	$42^{\{6\}}$	$38^{\{3.5\}}$	$38^{\{3.5\}}$	$32^{\{2\}}$	$53^{\{7\}}$	$56^{\{8\}}$
300	*BIAS*	$\hat{\beta}$	$0.08181^{\{4\}}$	$0.09370^{\{8\}}$	$0.07593^{\{1\}}$	$0.08024^{\{2\}}$	$0.09115^{\{7\}}$	$0.08054^{\{3\}}$	$0.08865^{\{6\}}$	$0.08623^{\{5\}}$
		$\hat{\varphi}$	$0.21483^{\{3\}}$	$0.21354^{\{2\}}$	$0.24236^{\{7\}}$	$0.19327^{\{1\}}$	$0.21701^{\{4\}}$	$0.21705^{\{5\}}$	$0.24614^{\{8\}}$	$0.23201^{\{6\}}$
		$\hat{\rho}$	$0.25242^{\{2\}}$	$0.26083^{\{5\}}$	$0.25855^{\{4\}}$	$0.20942^{\{1\}}$	$0.26085^{\{6\}}$	$0.25558^{\{3\}}$	$0.27432^{\{8\}}$	$0.26633^{\{7\}}$
	MSE	$\hat{\beta}$	$0.01234^{\{3\}}$	$0.01623^{\{8\}}$	$0.01135^{\{1\}}$	$0.01368^{\{4\}}$	$0.01512^{\{5\}}$	$0.01195^{\{2\}}$	$0.01517^{\{6\}}$	$0.01621^{\{7\}}$
		$\hat{\varphi}$	$0.07106^{\{3\}}$	$0.07067^{\{2\}}$	$0.08789^{\{7\}}$	$0.06249^{\{1\}}$	$0.07322^{\{5\}}$	$0.07216^{\{4\}}$	$0.09094^{\{8\}}$	$0.07671^{\{6\}}$
		$\hat{\rho}$	$0.11818^{\{4\}}$	$0.13503^{\{7\}}$	$0.11524^{\{2\}}$	$0.09389^{\{1\}}$	$0.13334^{\{6\}}$	$0.11799^{\{3\}}$	$0.15275^{\{8\}}$	$0.12946^{\{5\}}$
	MRE	$\hat{\beta}$	$0.02045^{\{4\}}$	$0.02343^{\{8\}}$	$0.01898^{\{1\}}$	$0.02006^{\{2\}}$	$0.02279^{\{7\}}$	$0.02013^{\{3\}}$	$0.02216^{\{6\}}$	$0.02156^{\{5\}}$
		$\hat{\varphi}$	$0.14322^{\{3\}}$	$0.14236^{\{2\}}$	$0.16157^{\{7\}}$	$0.12885^{\{1\}}$	$0.14467^{\{4\}}$	$0.14470^{\{5\}}$	$0.16409^{\{8\}}$	$0.15467^{\{6\}}$
		$\hat{\rho}$	$-0.50484^{\{7\}}$	$-0.52166^{\{4\}}$	$-0.51710^{\{5\}}$	$-0.41883^{\{8\}}$	$-0.52171^{\{3\}}$	$-0.51115^{\{6\}}$	$-0.54865^{\{1\}}$	$-0.53265^{\{2\}}$
	$\sum$ *Ranks*		$33^{\{2\}}$	$46^{\{5\}}$	$35^{\{4\}}$	$21^{\{1\}}$	$47^{\{6\}}$	$34^{\{3\}}$	$59^{\{8\}}$	$49^{\{7\}}$
400	*BIAS*	$\hat{\beta}$	$0.06820^{\{4\}}$	$0.07883^{\{8\}}$	$0.06392^{\{1\}}$	$0.06560^{\{2\}}$	$0.07708^{\{7\}}$	$0.06773^{\{3\}}$	$0.07384^{\{6\}}$	$0.07078^{\{5\}}$
		$\hat{\varphi}$	$0.20115^{\{2\}}$	$0.20177^{\{3\}}$	$0.22520^{\{7\}}$	$0.16871^{\{1\}}$	$0.20472^{\{5\}}$	$0.20318^{\{4\}}$	$0.23019^{\{8\}}$	$0.21883^{\{6\}}$
		$\hat{\rho}$	$0.22947^{\{2\}}$	$0.23919^{\{4\}}$	$0.24121^{\{7\}}$	$0.17550^{\{1\}}$	$0.23958^{\{5\}}$	$0.23290^{\{3\}}$	$0.25067^{\{8\}}$	$0.24112^{\{6\}}$
	MSE	$\hat{\beta}$	$0.00886^{\{3\}}$	$0.01156^{\{8\}}$	$0.00856^{\{1\}}$	$0.00895^{\{4\}}$	$0.01094^{\{6\}}$	$0.00882^{\{2\}}$	$0.01068^{\{5\}}$	$0.01101^{\{7\}}$
		$\hat{\varphi}$	$0.06324^{\{2\}}$	$0.06479^{\{4\}}$	$0.07616^{\{7\}}$	$0.05074^{\{1\}}$	$0.06661^{\{5\}}$	$0.06395^{\{3\}}$	$0.08156^{\{8\}}$	$0.06859^{\{6\}}$
		$\hat{\rho}$	$0.09541^{\{2\}}$	$0.11173^{\{7\}}$	$0.09931^{\{4\}}$	$0.06719^{\{1\}}$	$0.11126^{\{6\}}$	$0.09655^{\{3\}}$	$0.12658^{\{8\}}$	$0.10136^{\{5\}}$
	MRE	$\hat{\beta}$	$0.01705^{\{4\}}$	$0.01971^{\{8\}}$	$0.01598^{\{1\}}$	$0.01640^{\{2\}}$	$0.01927^{\{7\}}$	$0.01693^{\{3\}}$	$0.01846^{\{6\}}$	$0.01769^{\{5\}}$
		$\hat{\varphi}$	$0.13410^{\{2\}}$	$0.13451^{\{3\}}$	$0.15013^{\{7\}}$	$0.11247^{\{1\}}$	$0.13648^{\{5\}}$	$0.13546^{\{4\}}$	$0.15346^{\{8\}}$	$0.14589^{\{6\}}$
		$\hat{\rho}$	$-0.45894^{\{7\}}$	$-0.47837^{\{5\}}$	$-0.48243^{\{2\}}$	$-0.35100^{\{8\}}$	$-0.47916^{\{4\}}$	$-0.46581^{\{6\}}$	$-0.50135^{\{1\}}$	$-0.48224^{\{3\}}$
	$\sum$ *Ranks*		$28^{\{2\}}$	$50^{\{6.5\}}$	$37^{\{4\}}$	$21^{\{1\}}$	$50^{\{6.5\}}$	$31^{\{3\}}$	$58^{\{8\}}$	$49^{\{5\}}$

Table A8. Simulation results for $\Theta = (\beta = 1.00, \varphi = 3.00, \rho = 0.50)^\intercal$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
50	$BIAS$	$\hat{\beta}$	$0.05217^{\{5\}}$	$0.05013^{\{3\}}$	$0.05380^{\{7\}}$	$0.05685^{\{8\}}$	$0.05031^{\{4\}}$	$0.04808^{\{1\}}$	$0.04818^{\{2\}}$	$0.05261^{\{6\}}$
		$\hat{\varphi}$	$0.55469^{\{4\}}$	$0.60149^{\{7\}}$	$0.51898^{\{1\}}$	$0.55215^{\{3\}}$	$0.60654^{\{8\}}$	$0.53352^{\{2\}}$	$0.58679^{\{6\}}$	$0.56767^{\{5\}}$
		$\hat{\rho}$	$0.30496^{\{1\}}$	$0.31201^{\{3\}}$	$0.37313^{\{8\}}$	$0.31920^{\{4\}}$	$0.31150^{\{2\}}$	$0.32611^{\{5\}}$	$0.32877^{\{6\}}$	$0.34013^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00418^{\{6\}}$	$0.00381^{\{3\}}$	$0.00433^{\{7\}}$	$0.00481^{\{8\}}$	$0.00388^{\{4\}}$	$0.00348^{\{2\}}$	$0.00345^{\{1\}}$	$0.00411^{\{5\}}$
		$\hat{\varphi}$	$0.50311^{\{5\}}$	$0.58984^{\{7\}}$	$0.45919^{\{1\}}$	$0.46431^{\{2\}}$	$0.64673^{\{8\}}$	$0.47109^{\{3\}}$	$0.55797^{\{6\}}$	$0.49498^{\{4\}}$
		$\hat{\rho}$	$0.17741^{\{2\}}$	$0.18830^{\{5\}}$	$0.22517^{\{7\}}$	$0.18830^{\{4\}}$	$0.17725^{\{1\}}$	$0.19142^{\{6\}}$	$0.18123^{\{3\}}$	$0.23202^{\{8\}}$
	MRE	$\hat{\beta}$	$0.05217^{\{5\}}$	$0.05013^{\{3\}}$	$0.05380^{\{7\}}$	$0.05685^{\{8\}}$	$0.05031^{\{4\}}$	$0.04808^{\{1\}}$	$0.04818^{\{2\}}$	$0.05261^{\{6\}}$
		$\hat{\varphi}$	$0.18490^{\{4\}}$	$0.20050^{\{7\}}$	$0.17299^{\{1\}}$	$0.18405^{\{3\}}$	$0.20218^{\{8\}}$	$0.17784^{\{2\}}$	$0.19560^{\{6\}}$	$0.18922^{\{5\}}$
		$\hat{\rho}$	$0.60991^{\{1\}}$	$0.62403^{\{3\}}$	$0.74626^{\{8\}}$	$0.63840^{\{4\}}$	$0.62300^{\{2\}}$	$0.65222^{\{5\}}$	$0.65754^{\{6\}}$	$0.68026^{\{7\}}$
	$\sum Ranks$		$33^{\{2\}}$	$41^{\{4.5\}}$	$47^{\{7\}}$	$44^{\{6\}}$	$41^{\{4.5\}}$	$27^{\{1\}}$	$38^{\{3\}}$	$53^{\{8\}}$
150	$BIAS$	$\hat{\beta}$	$0.04014^{\{3\}}$	$0.04036^{\{4\}}$	$0.04379^{\{7\}}$	$0.04415^{\{8\}}$	$0.04083^{\{5\}}$	$0.03964^{\{2\}}$	$0.03950^{\{1\}}$	$0.04368^{\{6\}}$
		$\hat{\varphi}$	$0.31400^{\{3\}}$	$0.34791^{\{8\}}$	$0.30381^{\{1\}}$	$0.32969^{\{4\}}$	$0.34077^{\{5\}}$	$0.30551^{\{2\}}$	$0.34184^{\{7\}}$	$0.34172^{\{6\}}$
		$\hat{\rho}$	$0.26626^{\{1\}}$	$0.27031^{\{2\}}$	$0.31823^{\{8\}}$	$0.27126^{\{3\}}$	$0.27361^{\{5\}}$	$0.27298^{\{4\}}$	$0.28920^{\{6\}}$	$0.30066^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00245^{\{3\}}$	$0.00252^{\{4\}}$	$0.00280^{\{7\}}$	$0.00286^{\{8\}}$	$0.00254^{\{5\}}$	$0.00236^{\{2\}}$	$0.00226^{\{1\}}$	$0.00273^{\{6\}}$
		$\hat{\varphi}$	$0.16162^{\{3\}}$	$0.19460^{\{7\}}$	$0.15770^{\{2\}}$	$0.17996^{\{4\}}$	$0.19011^{\{6\}}$	$0.15318^{\{1\}}$	$0.18479^{\{5\}}$	$0.19678^{\{8\}}$
		$\hat{\rho}$	$0.13092^{\{1\}}$	$0.13597^{\{4\}}$	$0.17326^{\{7\}}$	$0.13837^{\{5\}}$	$0.13418^{\{3\}}$	$0.13202^{\{2\}}$	$0.14247^{\{6\}}$	$0.17356^{\{8\}}$
	MRE	$\hat{\beta}$	$0.04014^{\{3\}}$	$0.04036^{\{4\}}$	$0.04379^{\{7\}}$	$0.04415^{\{8\}}$	$0.04083^{\{5\}}$	$0.03964^{\{2\}}$	$0.03950^{\{1\}}$	$0.04368^{\{6\}}$
		$\hat{\varphi}$	$0.10467^{\{3\}}$	$0.11597^{\{8\}}$	$0.10127^{\{1\}}$	$0.10990^{\{4\}}$	$0.11359^{\{5\}}$	$0.10184^{\{2\}}$	$0.11395^{\{7\}}$	$0.11391^{\{6\}}$
		$\hat{\rho}$	$0.53252^{\{1\}}$	$0.54062^{\{2\}}$	$0.63645^{\{8\}}$	$0.54253^{\{3\}}$	$0.54723^{\{5\}}$	$0.54596^{\{4\}}$	$0.57840^{\{6\}}$	$0.60133^{\{7\}}$
	$\sum Ranks$		$21^{\{1.5\}}$	$43^{\{4\}}$	$48^{\{7\}}$	$47^{\{6\}}$	$44^{\{5\}}$	$21^{\{1.5\}}$	$40^{\{3\}}$	$60^{\{8\}}$
300	$BIAS$	$\hat{\beta}$	$0.03532^{\{2\}}$	$0.03630^{\{5\}}$	$0.03803^{\{7\}}$	$0.03550^{\{3\}}$	$0.03674^{\{6\}}$	$0.03521^{\{1\}}$	$0.03555^{\{4\}}$	$0.03864^{\{8\}}$
		$\hat{\varphi}$	$0.22390^{\{3\}}$	$0.24926^{\{7\}}$	$0.22025^{\{1\}}$	$0.23219^{\{4\}}$	$0.24465^{\{5\}}$	$0.22079^{\{2\}}$	$0.25151^{\{8\}}$	$0.24770^{\{6\}}$
		$\hat{\rho}$	$0.23791^{\{2\}}$	$0.24489^{\{4\}}$	$0.27019^{\{8\}}$	$0.21707^{\{1\}}$	$0.24846^{\{5\}}$	$0.24188^{\{3\}}$	$0.26118^{\{6\}}$	$0.26536^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00197^{\{3\}}$	$0.00211^{\{5\}}$	$0.00213^{\{8\}}$	$0.00198^{\{4\}}$	$0.00212^{\{6\}}$	$0.00194^{\{2\}}$	$0.00187^{\{1\}}$	$0.00213^{\{7\}}$
		$\hat{\varphi}$	$0.08176^{\{2\}}$	$0.10013^{\{6\}}$	$0.08863^{\{3\}}$	$0.09276^{\{4\}}$	$0.09749^{\{5\}}$	$0.07952^{\{1\}}$	$0.10237^{\{7\}}$	$0.10912^{\{8\}}$
		$\hat{\rho}$	$0.10079^{\{2\}}$	$0.10737^{\{4\}}$	$0.12529^{\{7\}}$	$0.09202^{\{1\}}$	$0.10838^{\{5\}}$	$0.10156^{\{3\}}$	$0.11626^{\{6\}}$	$0.12984^{\{8\}}$
	MRE	$\hat{\beta}$	$0.03532^{\{2\}}$	$0.03630^{\{5\}}$	$0.03803^{\{7\}}$	$0.03550^{\{3\}}$	$0.03674^{\{6\}}$	$0.03521^{\{1\}}$	$0.03555^{\{4\}}$	$0.03864^{\{8\}}$
		$\hat{\varphi}$	$0.07463^{\{3\}}$	$0.08309^{\{7\}}$	$0.07342^{\{1\}}$	$0.07740^{\{4\}}$	$0.08155^{\{5\}}$	$0.07360^{\{2\}}$	$0.08384^{\{8\}}$	$0.08257^{\{6\}}$
		$\hat{\rho}$	$0.47581^{\{2\}}$	$0.48978^{\{4\}}$	$0.54039^{\{8\}}$	$0.43415^{\{1\}}$	$0.49693^{\{5\}}$	$0.48376^{\{3\}}$	$0.52235^{\{6\}}$	$0.53072^{\{7\}}$
	$\sum Ranks$		$21^{\{2\}}$	$47^{\{4\}}$	$50^{\{6.5\}}$	$25^{\{3\}}$	$48^{\{5\}}$	$18^{\{1\}}$	$50^{\{6.5\}}$	$65^{\{8\}}$
400	$BIAS$	$\hat{\beta}$	$0.03383^{\{4\}}$	$0.03454^{\{5\}}$	$0.03536^{\{7\}}$	$0.03151^{\{1\}}$	$0.03495^{\{6\}}$	$0.03379^{\{3\}}$	$0.03371^{\{2\}}$	$0.03698^{\{8\}}$
		$\hat{\varphi}$	$0.18897^{\{3\}}$	$0.21294^{\{8\}}$	$0.18795^{\{2\}}$	$0.19640^{\{4\}}$	$0.20916^{\{5\}}$	$0.18616^{\{1\}}$	$0.21055^{\{7\}}$	$0.20938^{\{6\}}$
		$\hat{\rho}$	$0.22345^{\{2\}}$	$0.23084^{\{4\}}$	$0.24631^{\{7\}}$	$0.19021^{\{1\}}$	$0.23377^{\{5\}}$	$0.22661^{\{3\}}$	$0.24268^{\{6\}}$	$0.24849^{\{8\}}$
	MSE	$\hat{\beta}$	$0.00180^{\{4\}}$	$0.00193^{\{6\}}$	$0.00184^{\{5\}}$	$0.00165^{\{1\}}$	$0.00195^{\{7\}}$	$0.00178^{\{3\}}$	$0.00168^{\{2\}}$	$0.00195^{\{8\}}$
		$\hat{\varphi}$	$0.05782^{\{2\}}$	$0.07178^{\{7\}}$	$0.06285^{\{3\}}$	$0.06684^{\{4\}}$	$0.06952^{\{5\}}$	$0.05562^{\{1\}}$	$0.07164^{\{6\}}$	$0.07948^{\{8\}}$
		$\hat{\rho}$	$0.08669^{\{2\}}$	$0.09421^{\{4\}}$	$0.10194^{\{7\}}$	$0.07373^{\{1\}}$	$0.09490^{\{5\}}$	$0.08683^{\{3\}}$	$0.09792^{\{6\}}$	$0.10963^{\{8\}}$
	MRE	$\hat{\beta}$	$0.03383^{\{4\}}$	$0.03454^{\{5\}}$	$0.03536^{\{7\}}$	$0.03151^{\{1\}}$	$0.03495^{\{6\}}$	$0.03379^{\{3\}}$	$0.03371^{\{2\}}$	$0.03698^{\{8\}}$
		$\hat{\varphi}$	$0.06299^{\{3\}}$	$0.07098^{\{8\}}$	$0.06265^{\{2\}}$	$0.06547^{\{4\}}$	$0.06972^{\{5\}}$	$0.06205^{\{1\}}$	$0.07018^{\{7\}}$	$0.06979^{\{6\}}$
		$\hat{\rho}$	$0.44691^{\{2\}}$	$0.46168^{\{4\}}$	$0.49261^{\{7\}}$	$0.38042^{\{1\}}$	$0.46754^{\{5\}}$	$0.45321^{\{3\}}$	$0.48537^{\{6\}}$	$0.49698^{\{8\}}$
	$\sum Ranks$		$26^{\{3\}}$	$51^{\{7\}}$	$47^{\{5\}}$	$18^{\{1\}}$	$49^{\{6\}}$	$21^{\{2\}}$	$44^{\{4\}}$	$68^{\{8\}}$

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