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Solution of Some Impulsive Differential Equations via Coupled Fixed Point

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Abstract: In this article, we employ the notion of coupled fixed points on a complete b-metric space endowed with a graph to give sufficient conditions to guarantee a solution of system of differential equations with impulse effects. We derive recisely some new coupled fixed point theorems under some conditions and then apply our results to achieve our goal.

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1. Introduction

The fixed point theory is one of the best tools in applied sciences that can be used to determine an existence solution for such an integral equation or differential equation.

In 2006, Bhaskar and Lakshmikantham [1] applied coupled fixed points to provide sufficient conditions to solve some differential equations by introducing and proving many exciting results for coupled fixed points. Many of the results were obtained in this motivated subject; for example, see [2–11].

In recent years, some authors have employed graphs to obtain new types of fixed point theory. Jachymski's paper [12] is one of the best articles in fixed point endowed with graphs. In this direction, see [13–22].

Alfuraidan and Khamsi [23] and Chifu and Petrusel [24] have very recently employed a directed graph to gain some new coupled fixed point results.

The connotation of b-metric spaces was started by Czerwik [25] as a generalization of metric spaces. In the 1960's, Milman and Myshkis [26,27] initiated and studied differential equations with impulses. Mathematically, this type of equations is used to describe an evolution of a real process with a short-term perturbation; it is sometimes convenient to neglect the duration of the perturbation and consider these perturbations to be "instantaneous." For such an idealization, it becomes necessary to study dynamical systems with discontinuous trajectories. As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth.

In our paper, we apply the directed graphs with the connotation of b-metric spaces to derive new coupled fixed point results. Additionally, we employ our results to assure the solutions of such impulsive differential equations are exist under certain conditions. We start with the notion of b-metric space.

Definition 1. *[25] Given $s \geq 1$. On a set M, define a map $d \colon M \times M \to \mathbb{R}^+$, such that:*

(i) $d(e, m) = 0 \iff e = m$,

(*ii*) $d(e, m) = d(m, e)$, *and*
(*iii*) $d(e, l) \leq s[d(e, m) + d(m, l)]$

hold $\forall\, e, m, l \in M$.

Subsequently, we refer the pair (M, d) to a b-metric space.

On M, let $\Delta = \{(s, s) \colon s \in M\}$. On the directed graph $G = (V(G), E(G))$, assume that all loops are in $E(G)$ and G has no parallel edges.

A finite sequence $\{t_j\}_{j=0}^r$ in $V(G)$ with $t_0 = t$, $t_r = u$ and $(t_{j-1}, t_j) \in E(G)$, for all $j = 1, 2, \ldots, r$, is called a path from the vertex t to the vertex u.
For the vertex u, we put

$$[u]_G = \{t \in M \colon \exists \text{ a path from } u \text{ to } t\}.$$

If each two vertices of G can be connected by a path, then G is called connected; that is, $V(G) = [u]_G$ for all $u \in M$.

By reversing the direction of each edge of the directed graph G, we obtained a directed graph, which is denoted by G^{-1}, with $V(G^{-1}) := V(G)$.

By ignoring the directions of the edges of the directed graph G, we obtained the undirected graph $\widetilde{G}$ with $V(\widetilde{G}) := V(G)$ and

$$E(\widetilde{G}) := E(G^{-1}) \cup E(G).$$

Throughout this paper, (M, d) stands to a b-metric space that is endowed with directed graph G, such that the set $V(G) = M$ and $\Delta \subseteq E(G)$. Further, we endow the product space $M \times M$ by another graph that is also denoted by G, such that

$$((t, j), (v, m)) \in E(G) \Leftrightarrow (t, v) \in E(G) \text{ and } (m, j) \in E(G),$$

for any $(t, j), (v, m) \in M \times M$.

Definition 2. *Ref. [1] The pair $(e, l) \in M \times M$ is called a coupled fixed point of $T \colon M \times M \to M$ if*
$$T(e, l) = e \text{ and } T(l, e) = l.$$

Definition 3. *Ref. [23] Endowed the complete metric space (M, d) with the direct graph G. The mapping $T \colon M \times M \to M$ possess the mixed G-monotone property if*

$$(t_1, t_2) \in E(G) \Rightarrow (T(t_1, l), T(t_2, l)) \in E(G),$$

for all $t_1, t_2, l \in M$, and

$$(m_1, m_2) \in E(G) \Rightarrow (T(l, m_2), T(t, m_1)) \in E(G),$$

for all $l, m_1, m_2 \in M$.

Seshagiri Rao and Kalyani [8] gave the following result:

Theorem 1. *Ref. [8] Endowed the set M with partial order $\preceq$. On $(M, d, \preceq)$, let the continuous map $T \colon M \times M \to M$ with a strict mixed monotone property on M satisfy:*

$$d(T(l, w), T(m, v)) \leq \alpha \frac{d(l, T(l, w))[1 + d(m, T(m, v))]}{1 + d(l, m)}$$
$$+ \beta[d(l, T(l, w)) + d(m, T(m, v))] + \gamma d(l, m),$$

where $\alpha, \beta, \gamma \in [0, 1)$, such that $1 > \alpha + 2\beta + \gamma$. If there exist two points $t_0, w_0 \in M$ with $t_0 \preceq T(t_0, w_0)$ and $T(w_0, t_0) \preceq w_0$, then T possess a coupled fixed point $(t, w) \in M \times M$.

2. Main Result

Let (M, d, G) stands to a complete b-metric space endowed with directed graph G and $T: M \times M \to M$ possess the mixed G-monotone property.

Theorem 2. *On (M, d, G), suppose that T is continuous. Assume that $\exists\, \alpha, \beta, \gamma \in [0, 1)$ with*

$$\sum_{i=0}^{\infty} s^i \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^i < \infty$$

such that

$$
d(T(l, w), T(m, v)) \leq \alpha \frac{d(l, T(l, w))[1 + d(m, T(m, v))]}{1 + d(l, m)} \\
+ \beta[d(l, T(l, w)) + d(m, T(m, v))] + \gamma d(l, m),
$$

holds $\forall\ (l, w), (m, v) \in M \times M$ with $((l, w), (m, v)) \in E(G)$. If $\exists\, l_0, w_0 \in M$ such that $((l_0, w_0), (T(l_0, w_0), T(w_0, l_0))) \in E(G)$, then T possess a coupled fixed point $(l^*, w^*) \in M \times M$.

Proof. Set $l_1 = T(l_0, w_0)$ and $w_1 = T(w_0, l_0)$. The assumption implies that

$$((l_0, w_0), (l_1, w_1)) \in E(G).$$

Hence

$$
d(l_2, l_1) = d(T(l_1, w_1), T(l_0, w_0)) \leq \alpha \frac{d(l_1, T(l_1, w_1))[1 + d(l_0, T(l_0, w_0))]}{1 + d(l_1, l_0)} \\
+ \beta[d(l_0, T(l_0, w_0)) + d(l_1, T(l_1, w_1))] + \gamma d(l_0, l_1).
$$

So,

$$d(l_2, l_1) \leq \frac{\beta + \gamma}{1 - \alpha - \beta} d(l_1, l_0).$$

Similarly, because $((w_1, l_1), (w_0, l_0)) \in E(G)$, then

$$d(w_2, w_1) \leq \frac{\beta + \gamma}{1 - \alpha - \beta} d(w_1, w_0).$$

Further, for $n = 1, 2, ...$, we let

$$l_{n+1} = T(l_n, w_n), \text{ and } w_{n+1} = T(w_n, l_n).$$

Referring to the fact that T possess the mixed G-monotone property on M, we have

$$((l_n, w_n), (l_{n+1}, w_{n+1})) \in E(G) \text{ and } ((w_{n+1}, l_{n+1}), (w_n, l_n)) \in E(G).$$

Afterwards,

$$d(l_{n+1}, l_n) \leq \frac{\beta + \gamma}{1 - \alpha - \beta} d(l_n, l_{n-1}),$$

and

$$d(w_{n+1}, w_n) \leq \frac{\beta + \gamma}{1 - \alpha - \beta} d(w_n, w_{n-1}).$$

Therefore, for $n \in \mathbb{N}$ we get

$$d(l_{n+1}, l_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^n d(l_1, l_0),$$

and

$$d(w_{n+1}, w_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^n d(w_1, w_0).$$

For $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we gain

$$d(l_n, l_{n+p}) \leq s d(l_n, l_{n+1}) + s^2 d(l_{n+1}, l_{n+2}) + \cdots + s^n d(l_{n+p-1}, l_{n+p})$$

$$= \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i d(l_i, l_{i+1}) \leq \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^i d(l_0, l_1).$$

By assumption, we get $\lim_{n \to \infty} d(l_n, l_{n+p}) = 0$.

By the same process, we obtain

$$d(w_n, w_{n+p}) \leq \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^i d(w_0, w_1).$$

Subsequently, $\lim_{n \to \infty} d(w_n, w_{n+p}) = 0$.

This implies that $\{l_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are Cauchy. The completeness of M implies that $\exists l^*, w^* \in M$ with

$$\lim_{n \to \infty} l_n = l^* \text{ and } \lim_{n \to \infty} w_n = w^*.$$

The continuity of T implies that

$$l^* = \lim_{n \to \infty} l_n = \lim_{n \to \infty} T(l_{n-1}, w_{n-1}) = T \left(\lim_{n \to \infty} l_{n-1}, \lim_{n \to \infty} w_{n-1} \right) = T(l^*, w^*),$$

$$w^* = \lim_{n \to \infty} w_n = \lim_{n \to \infty} T(w_{n-1}, l_{n-1}) = T \left(\lim_{n \to \infty} w_{n-1}, \lim_{n \to \infty} l_{n-1} \right) = T(w^*, l^*),$$

i.e., T possess (l^*, w^*) as a couple fixed point. $\square$

The continuity of T in Theorem 2 can be discarded by adding some new conditions. Now, assume that (M, d, G) possess property $(^*)$; that is,

(*i*)   for any $\{l_n\}_{n \in \mathbb{N}}$ in M such that $(l_n, l_{n+1}) \in E(G)$ and $\lim_{n \to \infty} l_n = l$, then $(l_n, l) \in E(G)$,
          and

(*ii*)   for any $\{l_n\}_{n \in \mathbb{N}}$ in M, such that $(l_{n+1}, l_n) \in E(G)$ and $\lim_{n \to \infty} l_n = l$, then $(l, l_n) \in E(G)$.

Theorem 3. *Endowed (M, d, G) with the property $(^*)$. Suppose $\exists \alpha, \beta, \gamma \in [0, 1)$ with*

$$\sum_{i=0}^{\infty} s^i \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^i < \infty$$

such that

$$d(T(l, w), T(m, v)) \leq \alpha \frac{d(l, T(l, w))[1 + d(m, T(m, v))]}{1 + d(l, m)}$$
$$+ \beta[d(l, T(l, w)) + d(m, T(m, v))] + \gamma d(l, m),$$

holds $\forall (l, w), (m, v) \in M \times M$ with $((l, w), (m, v)) \in E(G)$. If $\exists l_0, w_0 \in M$ such that $((l_0, w_0), (T(l_0, w_0), T(w_0, l_0))) \in E(G)$, then T possess a coupled fixed point $(l^*, w^*) \in M \times M$.

Proof. By referring to the proof of Theorem 2, we only need to show that $l^* = T(l^*, w^*)$ and $w^* = T(w^*, l^*)$.

Accordingly, $\lim_{n\to\infty} l_{n+1} = \lim_{n\to\infty} T(l_n, w_n) = l^*$, $\lim_{n\to\infty} w_{n+1} = \lim_{n\to\infty} T(w_n, l_n) = w^*$ and $(l_n, l_{n+1}) \in E(G)$ and $(w_{n+1}, w_n) \in E(G)$, the property (*) implies that

$$(l_n, l^*) \in E(G) \text{ and } (w^*, w_n) \in E(G).$$

Hence,

$$((l_n, w_n), (l^*, w^*)) \in E(G).$$

Thus, we get

$$
\begin{aligned}
d(T(l_n, w_n), T(l^*, w^*)) &\le \alpha \frac{d(l_n, T(l_n, w_n))[1 + d(l^*, T(l^*, w^*))]}{1 + d(l_n, l^*)} \\
&+ \beta[d(l_n, T(l_n, w_n)) + d(l^*, T(l^*, w^*))] + \gamma d(l_n, l^*) \\
&= \alpha \frac{d(l_n, l_{n+1})[1 + d(l^*, T(l^*, w^*))]}{1 + d(l_n, l^*)} \\
&+ \beta[d(l_n, l_{n+1}) + d(l^*, T(l^*, w^*))] + \gamma d(l_n, l^*).
\end{aligned}
$$

By the same way, we have

$$
\begin{aligned}
d(T(w_n, l_n), T(w^*, l^*)) &\le \alpha \frac{d(w_n, w_{n+1})[1 + d(w^*, T(w^*, l^*))]}{1 + d(w_n, w^*)} \\
&+ \beta[d(w_n, w_{n+1}) + d(w^*, T(w^*, l^*))] + \gamma d(w_n, w^*).
\end{aligned}
$$

Letting $n \to \infty$, we arrive

$$\lim_{n\to\infty} d(T(l_n, w_n), T(l^*, w^*)) = 0 \text{ and } \lim_{n\to\infty} d(T(w_n, l_n), T(w^*, l^*)) = 0.$$

Therefore,

$$\lim_{n\to\infty} l_{n+1} = T(l^*, w^*) \text{ and } \lim_{n\to\infty} w_{n+1} = T(w^*, l^*).$$

Accordingly, $l^* = T(l^*, w^*)$ and $w^* = T(w^*, l^*)$, i.e., T possess (l^*, w^*) as a couple fixed point. □

Remark 1. *Suppose that T satisfies the hypotheses of Theorem 2 (Theorem 3). If the coupled fixed point (l^*, w^*) of T satisfies $((l^*, w^*), (l_0, w_0)) \in E(G)$, then (l^*, w^*) is unique. Indeed, if we suppose that there is another coupled fixed point (u, v). By referring to the proof of Theorem 2 or Theorem 3, we construct two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ such $l_{n+1} = T(l_n, w_n)$ and $w_{n+1} = T(w_n, l_n)$ for $n \in \mathbb{N}$ with $\lim_{n\to\infty} l_n = l^*$ and $\lim_{n\to\infty} w_n = w^*$. Because T possess the mixed G-monotone, then $((u, v), (l_n, w_n)) \in E(G)$. Therefore,*

$$
\begin{aligned}
d(u, l_{n+1}) &= d(T(u, v), T(l_n, w_n)) \\
&\le \alpha \frac{d(u, T(u, v))[1 + d(l_n, l_{n+1})]}{1 + d(u, l_n)} + \beta[d(u, T(u, v)) + d(l_n, l_{n+1})] \\
&+ \gamma d(u, l_n) = \beta d(l_n, l_{n+1}) + \gamma d(u, l_n),
\end{aligned}
$$

and

$$
\begin{aligned}
d(v, w_{n+1}) &= d(T(v, u), T(w_n, l_n)) \\
&\le \alpha \frac{d(v, T(v, u))[1 + d(w_n, w_{n+1})]}{1 + d(v, w_n)} + \beta[d(v, T(v, u)) + d(w_n, w_{n+1})] \\
&+ \gamma d(v, w_n) = \beta d(w_n, w_{n+1}) + \gamma d(v, w_n).
\end{aligned}
$$

On letting $n \to \infty$, we arrive to

$$\lim_{n \to \infty} l_{n+1} = u \text{ and } \lim_{n \to \infty} w_{n+1} = v.$$

Thus,

$$l^* = u \text{ and } w^* = v.$$

Theorem 4. *Suppose that T satisfies the hypothesis of Theorem 2 (Theorem 3). If $(l^*, w^*) \in E(G)$, then $l^* = w^*$.*

Proof. Since $(l^*, w^*) \in E(G)$, we have $((l^*, w^*), (w^*, l^*)) \in E(G)$. Thus,

$$d(l^*, w^*) = d(T(l^*, w^*), T(w^*, l^*)) \leq \alpha \frac{d(l^*, T(l^*, w^*))[1 + d(w^*, T(w^*, l^*))]}{1 + d(l^*, w^*)}$$
$$+ \beta[d(l^*, T(l^*, w^*)) + d(w^*, T(w^*, l^*))] + \gamma d(l^*, w^*) = 0,$$

and hence $l^* = w^*$. $\square$

Referring to the fact that every metric space is a b-metric, we derive the next results:

Corollary 1. *Endowed the complete metric space (M, d) with the direct graph G. Suppose that the continuous mapping $T: M \times M \to M$ possesses the mixed G-monotone property on M. Assume $\exists \, \alpha, \beta, \gamma \in [0, 1)$ with $1 > \alpha + \beta + \gamma$, such that*

$$d(T(l, w), T(m, v)) \leq \alpha \frac{d(l, T(l, w))[1 + d(m, T(m, v))]}{1 + d(l, m)}$$
$$+ \beta[d(l, T(l, w)) + d(m, T(m, v))] + \gamma d(l, m),$$

holds $\forall \, (l, w), (m, v) \in M \times M$ with $((l, w), (m, v)) \in E(G)$. If there exists $l_0, w_0 \in M$ such that $((l_0, w_0), (T(l_0, w_0), T(w_0, l_0))) \in E(G)$, then T possess a coupled fixed point $(l^*, w^*) \in M \times M$.

Corollary 2. *Endowed the complete metric space (M, d) with the direct graph G. Suppose that (X, d, G) possess property$(*)$. Suppose that $T: M \times M \to M$ satisfies the mixed G-monotone property on M. Additionally, assume $\exists \, \alpha, \beta, \gamma \in [0, 1)$ with $1 > \alpha + \beta + \gamma 1$, such that*

$$d(T(l, w), T(m, v)) \leq \alpha \frac{d(l, T(l, w))[1 + d(m, T(m, v))]}{1 + d(l, m)}$$
$$+ \beta[d(l, T(l, w)) + d(m, T(m, v))] + \gamma d(l, m),$$

holds $\forall \, (l, w), (m, v) \in M \times M$ with $((l, w), (m, v)) \in E(G)$. If $\exists \, l_0, w_0 \in M$ such that $((l_0, w_0), (T(l_0, w_0), T(w_0, l_0))) \in E(G)$, then T possess a coupled fixed point $(l^*, w^*) \in M \times M$.

3. Application

The development of the theory of impulsive differential equations gives an opportunity for some real-world processes and phenomena to be more accurately modeled; see the monographs [28–31]. Coupled fixed point theory plays a basic role in applications of many branches of mathematics, especially in differential equations, stochastics, and statistics [32,33]. For this reason, we will use our results to prove the existence of solutions for differential equations with impulse effects.

Let consider the following system of differential equations with impulse effects:

$$w'(\tau) = f(\tau, w(\tau), z(\tau)), \quad z'(\tau) = f(\tau, z(\tau), w(\tau)), \tag{1}$$

$$w(\tau^+) - w(\tau^-) = I(w(\tau), z(\tau)), \quad z(\tau^+) - z(\tau^-) = I(z(\tau), w(\tau)), \tag{2}$$

$$w(0) = w_0, \quad z(0) = z_0, \tag{3}$$

where $0 < \tau < 1$, $J := [0,1]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $I \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $w(\tau^+) = \lim\limits_{h \to 0^+} w(\tau + h)$ and $w(\tau^-) = \lim\limits_{h \to 0^+} w(\tau - h)$.

In order to define a solutions for Problems (1)–(3), consider the space of piecewise continuous functions:

$$PC([0,1], \mathbb{R}) = \{z \colon [0,1] \to \mathbb{R}, \quad z \in C(J \backslash \{\tau\}, \mathbb{R}); \text{ such that } z(\tau^-) \text{ and } z(\tau^+)$$
$$\text{exist and satisfy } z(\tau^-) = z(\tau)\}.$$

Define d on $PC([0,1])$ by

$$d(w, z) = (\sup_{t \in J} |w(t) - z(t)|)^2.$$

Assumption 1. *Assume the following assertions:*

1. $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
2. $\forall\, w, z, u, v \in PC([0,1])$, *with $w \leq u$ and $z \leq y$, we have*

$$f(t, w(t), z(t)) \leq f(t, u(t), y(t)) \text{ and } I(w(t), z(t)) \leq I(u(t), v(t)) \quad \forall\, t \in [0,1];$$

3. $\exists\, \alpha, \beta, \gamma \in [0,1)$ *with* $\sum\limits_{i=0}^{\infty} 2^i \left(\frac{\beta+\gamma}{1-\alpha-\beta} \right)^i < \infty$ *such that*

$$|f(t, w(t), z(t)) - f(t, u(t), v(t))|^2 \leq \frac{\alpha}{2} \frac{|w(t) - f(t, w(t), z(t))|^2 [1 + |u(t) - f(t, u(t), v(t))|^2]}{1 + |w(t) - u(t)|^2}$$
$$+ \frac{\beta}{2} \Big[|w(t) - f(t, w(t), z(t))|^2 + |u(t) - f(t, u(t), v(t))|^2 \Big],$$

and

$$|I(w(t), z(t)) - I(u(t), v(t))|^2 \leq \frac{\gamma}{2} \Big(|w(t) - u(t)|^2 \Big)$$

and for each $t \in J$, $w, z, u, v \in PC([0,1])$, $w \leq u$ and $v \leq z$.

We shall obtain the unique solution of Equations (1)–(3). This problem is equivalent to the integral equations:

$$\begin{cases} w(t) = w_0 + \int_0^t f(s, w(s), z(s)) ds + I(w(\tau), z(\tau)), \\ z(t) = z_0 + \int_0^t f(s, z(s), w(s)) ds + I(z(\tau), w(\tau)), \end{cases} \quad t \in J. \tag{4}$$

Consider, on $PC([0;1], \mathbb{R}) \times PC([0;1], \mathbb{R})$, the partial order relation:

$$(w_1, z_1) \leq (w_2, z_2) \Leftrightarrow w_1(t) \leq w_2(t) \text{ and } z_1(t) \geq z_2(t), \quad t \in J,$$

and define for $t \in J$,

$$T(w, z)(t) = w_0 + \int_0^t f(s, w(s), z(s)) ds + I(w(\tau), z(\tau)), \quad t \in J.$$

Note that, if $(w, z) \in PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R})$ is a couple fixed point of T, then we have

$$w(t) = T(w, z)(t) \text{ and } z(t) = T(z, w)(t),$$

for all $t \in J$, and (w, z) is a solution of (4).

Theorem 5. *Assume that the Assumption 1 holds. Assume that there exists $(u_0, v_0) \in PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R})$ such that*

$$u_0(t) \leq u_0(0) + \int_0^t f(s, u_0(s), v_0(s))ds + I(u(\tau), v(\tau))$$

and

$$v_0(t) \geq v_0(0) + \int_0^t f(s, v_0(s), u_0(s))ds + I(v(\tau), u(\tau)), \quad t \in [0, 1].$$

Subsequently, the system (1)–(3) possess a solution.

Proof. We prove that the integral system (4) has a solution by showing that the operator $T \colon M \times M \to M$ has a coupled fixed point in $M \times M$. To do this, we have to show that T satisfies the conditions of Theorem 2 or Theorem 3.

Consider the graph G with $V(G) = PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R})$, and

$$E(G) = \{(w, z) \in PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R}), \quad w \leq z\},$$

and we endow the product space $PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R})$ by another graph also denoted by G, such that

$$((w, z), (u, v)) \in E(G) \Leftrightarrow (w, u) \in E(G) \text{ and } (v, z) \in E(G),$$

for any $(w, z), (u, v) \in PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R})$.

By using Assumption 1, we obtain for all $w, z, w_1, w_2, z_1, z_2 \in PC([0; 1], \mathbb{R})$, if $(w_1, w_2) \in E(G)$, then

$$T(w_1, z)(t) = w_0 + \int_0^t f(s, w_1(s), z(s))ds + I(w_1(\tau), z(\tau))$$

$$\leq w_0 + \int_0^t f(s, w_2(s), z(s))ds + I(w_2(\tau), z(\tau)) = T(w_2, z)(t).$$

Thus $(T(w_1, z), T(w_2, z)) \in E(G)$.
Also, if $(z_1, z_2) \in E(G)$ we have

$$T(w, z_2)(t) = w_0 + \int_0^t f(s, w(s), z_2(s))ds + I(w(\tau), z_2(\tau))$$

$$\leq w_0 + \int_0^t f(s, w(s), z_1(s))ds + I(w(\tau), z_1(\tau)) = T(w, z_1)(t).$$

Subsequently, $(T(w, z_2), T(w, z_1)) \in E(G)$.
Thus, $T(w, z)$ possesses the mixed G-monotone property.

Now, let us consider $(w, z), (u, v) \in PC([0; 1], \mathbb{R}) \times PC([0; 1], \mathbb{R})$ such that $((w, z), (u, v)) \in E(G)$, then

$$|T(w,z)(t) - T(u,v)(t)|^2 = |\int_0^t f(t,w(s),z(s))ds + I(w(\tau),z(\tau))$$

$$- \int_0^t f(t,u(s),v(s))ds - I(u(\tau),v(\tau))|^2$$

$$\leq 2\int_0^t |f(t,w(s),y(s)) - f(t,u(s),v(s))|^2 ds$$

$$+ 2|I(w(\tau),z(\tau)) - I(u(\tau),v(\tau))|^2$$

$$\leq \int_0^t \alpha \frac{|w(s) - f(s,w(s),z(s))|^2[1 + |u(s) - f(s,u(s),v(s))|^2]}{1 + |w(s) - u(s)|^2}$$

$$+ \beta[|w(s) - f(s,w(s),z(s))|^2 + |u(s) - f(s,u(s),v(s))|^2]ds$$

$$+ \gamma|x(\tau) - u(\tau)|^2.$$

Therefore,

$$d(T(w,z),T(u,v)) \leq \alpha \frac{d(w,T(w,z))[1 + d(u,T(u,v))]}{1 + d(w,u)}$$

$$+ \beta[d(w,T(w,z)) + d(u,T(u,v))] + \gamma d(w,u).$$

Now, by hypotheses we can conclude that

$$((u_0,v_0),(T(u_0,v_0),T(v_0,u_0))) \in E(G).$$

Because T is a continuous mapping and (X,d,G) possesses the property $(^*)$, which shows that all hypotheses of Theorem 2 and Theorem 3 are satisfied. Thus, $T(x,y)$ has a coupled fixed point in $PC([0;1],\mathbb{R}) \times PC([0;1],\mathbb{R})$. □

4. Conclusions

In this work, we employed the notion of coupled fixed point to formulate and prove many fixed point theorems for mapping satisfying certain conditions over a complete b-metric space endowed with a directed graph. On a complete $b-$metric space endowed with a directed graph (M,d,G) we precisely proved the mapping $T\colon M \times M \to M$ has a coupled fixed point under some conditions on M and T. Our results have been applied to provide sufficient conditions to guarantee an existence solution of such impulse differential equations.

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