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A Model for the Maxwell Equations Coupled with Matter Based on Solitons †

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† To the memory of my friend Antonio Ambrosetti.

Abstract: We present a simple model of interaction of the Maxwell equations with a *matter field* defined by the Klein–Gordon equation. A simple linear interaction and a nonlinear perurbation produces solutions to the equations containing hylomorphic solitons, namely stable, solitary waves, whose existence is related to the ratio energy/charge. These solitons, at low energy, behave as poinwise charged particles in an electromagnetic field. The basic points are the following ones: (i) the *matter field* is described by the Nonlinear Klein–Gordon equation with a suitable nonlinear term; (ii) the interaction is not described by the equivariant derivative, but by a very simple coupling which preseves the invariance under the Poincaré group; (iii) the existence of soliton can be proved using the tecniques of nonlinear analysis and, in particular, the Mountain Pass Theorem; (iv) a suitable choice of the parameters produces solitons with a prescribed electric charge and mass/energy; (v) thanks to the point (ii), the dynamics of these solitons at low energies is the same of classical charged particles.

1. Introduction

In classical mechanics, the coupling of the electromagnetic field with matter is given by the following equation

$$\frac{d}{dt}\left(\frac{m\dot{\xi}}{\sqrt{1-|\dot{\xi}|^2}}\right) = q\left(\mathbf{E} + \dot{\xi} \times \mathbf{H}\right) \tag{1}$$

where m denotes the rest mass of a particle, $\xi = \xi(t)$ is its position in space and $\dot{\xi}$ is the time derivative of ξ. Unfortunately, this equation is not consistent with the Maxwell equations. One of the main reasons for this inconsistency comes from the fact that the Maxwell equations are relativistic invariants, and hence the inertial mass of a charged material point is infinite by the Maxwell equation, the electrostatic energy of a pointwise material point is given by

$$E = \int \mathbf{E}^2 dx = \int \left|\nabla \frac{1}{|x|}\right|^2 dx = +\infty$$

Then, if you couple the Lorentz equation with the Maxwell equations, in order to accelerate a material point (which produces a electric field), it would necessarily be an infinite force since the inertial mass, by the Lorentz inariance of the equations, equals the energy. If the material point is replaced by a sort of ball, other problems are present, such as the self-interaction of the field produced by the particle and the difficulty of a relativistic

description of a solid body. As far as I know, there is not a satisfactory description of the dynamics of a microscopic charged "ball" in an e.m. field. With the advent of quantum mechanics, this problem has lost its relevance and quantum models have been sought to describe this interaction.

Here, we recall the formally simplest of them, since it has some relevance for this paper. It is given by the interaction of the Klein–Gordon equation (which describes a spinless boson field) with the e.m. field. In this case, the action functional is given by

$$\mathbb{A}_{\mathrm{W}} := \frac{1}{2} \iint \left(|(\partial_t + iq\varphi)\psi|^2 - |(\nabla - iq\mathbf{A})\psi|^2 + m^2|\psi|^2 \right) dx\, dt. \tag{2}$$

where ψ is the wave-function of the boson field, $(\varphi, \mathbf{A})$ is the gauge potential, m and q is the mass and the electric charge of a particle.

Despite the fact that quantum electrodynamics (QED) is a well-established theory, we think that the study of the possibility of a consistent, not-quantistic electrodynamics (CED) is still a relevant issue that might shed new light on the unsolved problems in QED.

Here, we propose a model based on the idea that charged particles can be described by solitons, which can be seen as bumps of a "matter field". The idea is not new; in the last half century, since the pioneering work of Rosen in '68 [1], a lot of papers have been written. The original point of this paper is the introduction of a very simple interaction between the "matter field" and the e.m. field which produces "almost" identical particles, which obey the known laws of CED (see Theorem 3).

Finally, we recall that the existence and the qualitative properties of solitons produced by the interaction of the Nonlinear Klein–Gordon equation with the e.m. have been largely studied in the case of the interaction (2) (see, e.g., [2–8]). The new (and simpler) interaction introduced in this paper (see (9)) produces new phenomena. This interaction is not realistic from the physical perspective, but it can be regarded as a first step towards more sophisticated models.

2. The Model

2.1. The Basic Equations

The action of the electromagnetic field is defined by the Lagrangian density

$$\mathcal{L}_{\mathrm{F}}[\varphi, \mathbf{A}] = \frac{1}{2} \left(|\partial_t \mathbf{A} + \nabla \varphi|^2 - |\nabla \times \mathbf{A}|^2 \right) \tag{3}$$

Now, we need to choose an equation to describe the matter field. The simplest semilinear equation invariant for the Poincaré group is the following

$$\Box \psi + W'(|\psi|)\frac{\psi}{|\psi|} = 0 \tag{4}$$

where

$$\Box := \partial_t^2 - \Delta$$

represents the d'Alembert operator; ψ takes values in $\mathbb{C}$ and

$$W(s) = \frac{1}{2}s^2 + N(s); \tag{5}$$

$$N \in C^2\left(\mathbb{R}^3\right), \quad N(0) = N'(0) = 0.$$

The use of the complex variable is important since it gives to the field ψ an internal degree of freedom, represented by a phase shift given by

$$\psi(t, x) \mapsto e^{i\theta}\psi(t, x) \tag{6}$$

Usually, people refer to Equation (4) as to the nonlinear Klein–Gordon equation since,

its linearization gives the Klein–Gordon equation

$$\Box \psi + \psi = 0.$$

Equation (4) has a variational structure and its Lagrangian density can be written as follows:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{M}}[u, S] &= \frac{1}{2}\left(|\partial_t \psi|^2 - |\nabla \psi|^2\right) - W(|\psi|) \\
&= \frac{1}{2}\left[|\partial_t u|^2 - |\nabla u|^2 + (\partial_t S)^2 u^2 - |\nabla S|^2 u^2\right] - W(u)
\end{aligned}
\tag{7}
$$

where we have set

$$\psi(t, x) = u(t, x)e^{iS(t,x)}; \quad u \geq 0. \tag{8}$$

We want to couple the matter field ψ with the electromagnetic field in the most simple and natural way, making sure that the Lorentz invariance of the equations be satisfied. Since the Lagrangian of the electromagnetic field depends on the 4-vector $(\varphi, \mathbf{A})$, it must be coupled with a 4-vector, determined by ψ. There are two possible candidates, which are linear in u and invariant for the transformation $\psi(t, x) \mapsto \psi(t, x)e^{i\theta}$:

$$(\partial_t u, \nabla u)$$

$$(\partial_t S, \nabla S)u$$

They lead to the following interaction Lagrangian densities

$$\mathcal{L}_0[u, \varphi, \mathbf{A}] = \beta(\varphi \partial_t u + \mathbf{A} \cdot \nabla u)$$

$$\mathcal{L}_{\mathrm{I}}[u, S, \varphi, \mathbf{A}] = \beta(\partial_t S \varphi + \mathbf{A} \cdot \nabla S)u \tag{9}$$

where β is the interaction constant which, in order to fix the ideas, we assume positive; on the contrary, the sign "+" in the above definitions is necessary; a "−" would violate the time-reversal property and the equations would lose the invariance for the Poincaré group. Since $\mathcal{L}_{\mathrm{I}}$ is not locally gauge invariant, we assume the Lorentz condition

$$\partial_t \varphi + \nabla \cdot \mathbf{A} = 0. \tag{10}$$

In this paper, we will examine the case $\mathcal{L}_{\mathrm{I}}$ (with $\beta > 0$) which provides a very rich model.

Therefore, we will study the equations relative to the following Lagrangian density

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{M}} + \mathcal{L}_{\mathrm{I}} + \mathcal{L}_{\mathrm{F}} \\
&= \frac{1}{2}\left[|\partial_t u|^2 - |\nabla u|^2\right] dx dt - W(u) + \frac{1}{2}\left[(\partial_t S)^2 - |\nabla S|^2\right]u^2 \\
&\quad + \beta(\mathbf{A} \cdot \nabla S + \varphi \partial_t S)u \\
&\quad + \frac{1}{2}\left(|\partial_t \mathbf{A} + \nabla \varphi|^2 - |\nabla \times \mathbf{A}|^2\right)
\end{aligned}
$$

Making the variation of the action functional

$$\mathbb{A} = \iint \mathcal{L} \, dx \, dt$$

with respect to u, S, φ and $\mathbf{A}$, we obtain the following system of equations:

$$\Box u + W'(u) - \left[(\partial_t S)^2 - |\nabla S|^2\right]u = \beta(\mathbf{A} \cdot \nabla S + \varphi \partial_t S) \tag{11}$$

$$\partial_t\left(\partial_t S \, u^2 - \beta \varphi u\right) - \nabla \cdot \left(\nabla S \, u^2 + \beta \mathbf{A} u\right) = 0 \tag{12}$$

$$\nabla \cdot (\partial_t \mathbf{A} + \nabla \varphi) = \beta \partial_t S u \tag{13}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \partial_t (\partial_t \mathbf{A} + \nabla \varphi) = \beta \nabla S u. \tag{14}$$

We can express these equations with new variables in order to make the equations independent of β and to obtain the Maxwell equations

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi \tag{15}$$

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{16}$$

$$\rho = -\beta \partial_t S u \tag{17}$$

$$\mathbf{j} = \beta \nabla S u \tag{18}$$

Therefore, we obtain

$$\nabla \cdot \mathbf{E} = \rho \tag{19}$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{E} = \mathbf{j} \tag{20}$$

and (15) and (16) give rise to the first couple of Maxwell equations:

$$\nabla \times \mathbf{E} + \partial_t \mathbf{H} = 0 \tag{21}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{22}$$

The equations of the matter field become:

$$\Box u + W'(u) + \frac{\mathbf{j}^2 - \rho^2 + \varphi \rho + \mathbf{A} \cdot \mathbf{j}}{u} = 0 \tag{23}$$

$$\partial_t (\rho u - \varphi u) + \nabla \cdot (\mathbf{j} u + \mathbf{A} u) = 0. \tag{24}$$

Notice that the Equations (23) and (24) depend only on gauge-independent variables $(u, \rho, \mathbf{j}, \mathbf{E}, \mathbf{H})$ since the dependence from φ and $\mathbf{A}$ in these equations can be eliminated via the GAUSS equations, which gives φ and $\mathbf{A}$ by the appropriate Green functions. The action can be rewritten as follows

$$\begin{aligned}
\mathbb{A} \;=\; & \frac{1}{2} \iint \left(|\partial_t u|^2 - |\nabla u|^2 \right) dx dt - \iint W(u) dx dt \\
& + \frac{1}{2} \iint \left(\rho^2 - \mathbf{j}^2 \right) dx dt \\
& + \iint (\mathbf{A} \cdot \mathbf{j} - \varphi \rho) dx dt + \frac{1}{2} \iint \left(\mathbf{E}^2 - \mathbf{H}^2 \right) dx dt.
\end{aligned} \tag{25}$$

Remark 1. *As we have already remarked, the Lagrangian (9) is not invariant for the local action of the gauge group of the electromagnetic field; however, it is invariant for the global action of a gauge transformation, namely, invariant for the 4-parameters group*

$$(u, S, \varphi, \mathbf{A}) \mapsto (u, S - at - \mathbf{b} \cdot x, \ \varphi + a, \ \mathbf{A} + \mathbf{b}); \ (a, \mathbf{b}) \in \mathbb{R}^4 \tag{26}$$

Moreover, we have the invariance (6), which can be rewritten as follows

$$T_\theta S = S + \theta; \ \theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \tag{27}$$

and it is typical of the (4) equation. Notice that ,in this model, the invariance (27) is independent of the gauge invariance (26), and hence leads to a different conservation law (see Section 2.2). Finally, we remark that the position (17) and (18) are appropriate if we put ourselves in a gauge, where $(u, S, \varphi, \mathbf{A})$ vanish at infinity, or to be more precise, if $u, S \in H^1(\mathbb{R}^3)$, $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\mathbf{A} \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$ (see (41)).

2.2. Conservation Laws

Let us examine the main integral of motion that will be used in the following. We assume that $(u, S, \varphi, \mathbf{A})$ is a solution of (11)–(14) and that all the quantities $|\partial_t u|^2$, $|\nabla u|^2$, $(\partial_t S)^2$, etc., are integrable; under these assumptions, we will compute some integral of motion relevant to this paper.

Conservation of Energy. Energy, by definition, is the quantity preserved by the time invariance of the Lagrangian. We have the following result

Proposition 1. *The energy takes the following form*

$$E = E_{\mathrm{M}} + E_{\mathrm{F}} + E_{\mathrm{I}} \tag{28}$$

where

$$
\begin{aligned}
E_{\mathrm{M}}[u, S] &= \frac{1}{2} \int \left[|\partial_t u|^2 + |\nabla u|^2 \right] dx + \frac{1}{2} \int \left[(\partial_t S)^2 + |\nabla S|^2 \right] u^2 dx + \int W(u) dx \\
&= \frac{1}{2} \int \left[|\partial_t u|^2 + |\nabla u|^2 \right] dx + \frac{1}{2} \int \frac{\rho^2 + \mathbf{j}^2}{\beta^2} \, dx + \int W(u) dx
\end{aligned} \tag{29}
$$

*is the **matter field energy**,*

$$
\begin{aligned}
E_{\mathrm{F}}[\varphi, \mathbf{A}] &= \frac{1}{2} \int \left(|\partial_t \mathbf{A} + \nabla \varphi|^2 + |\nabla \times \mathbf{A}|^2 \right) dx \\
&= \frac{1}{2} \int \left(\mathbf{E}^2 + \mathbf{H}^2 \right) dx.
\end{aligned}
$$

*is the **e.m. field energy**, and*

$$
\begin{aligned}
E_{\mathrm{I}}[u, S, \varphi, \mathbf{A}] &= -\beta \int (\varphi \partial_t S + \mathbf{A} \cdot \nabla S) u \, dx \\
&= \int (\varphi \rho - \mathbf{A} \cdot \mathbf{j}) \, dx.
\end{aligned} \tag{30}
$$

*is the **interaction energy**.*

The names *matter field energy, e.m. field energy,* and *interaction energy* are motivated by the fact that E_{M} depend only on the matter variables (u, S), E_{F} depend on the e.m. field variables $(\varphi, \mathbf{A})$ and only E_{I} depend on all four variables.

Proof. By Noether's Theorem, we have that the energy density E_{M} relative to the Lagrangian density $\mathcal{L}_{\mathrm{M}}$ is given by

$$
\begin{aligned}
E_{\mathrm{M}} &= \frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial(\partial_t u)} \cdot \partial_t u + \frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial(\partial_t S)} \cdot \partial_t S - \mathcal{L}_{\mathrm{M}} \\
&= (\partial_t u)^2 + (\partial_t S)^2 u^2 - \frac{1}{2} \left[|\partial_t u|^2 - |\nabla u|^2 - W(u) + (\partial_t S)^2 u^2 - |\nabla S|^2 u^2 \right] \\
&= \frac{1}{2} \left[|\partial_t u|^2 + |\nabla u|^2 + (\partial_t S)^2 u^2 + |\nabla S|^2 u^2 \right] + W(u) \\
&= \frac{1}{2} \left[|\partial_t u|^2 + |\nabla u|^2 + \frac{\rho^2 + \mathbf{j}^2}{\beta^2} \right] + W(u)
\end{aligned}
$$

The computation of energy density relative to the Lagrangian density $\mathcal{L}_F$ is the usual one and we report it for completeness:

$$
\begin{aligned}
\frac{\partial \mathcal{L}_F}{\partial(\partial_t \mathbf{A})} - \mathcal{L}_F &= (\partial_t \mathbf{A} + \nabla \varphi) \cdot \partial_t \mathbf{A} - \frac{1}{2}(\partial_t \mathbf{A} + \nabla \varphi)^2 + \frac{1}{2}(\nabla \times \mathbf{A})^2 \\
&= -\mathbf{E} \cdot (-\mathbf{E} + \nabla \varphi) - \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{H}^2 \\
&= \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{H}^2 - \mathbf{E} \cdot \nabla \varphi = \mathcal{E}_F - \mathbf{E} \cdot \nabla \varphi
\end{aligned}
$$

The energy density relative to the Lagrangian density $\mathcal{L}_I$ is given by

$$
\begin{aligned}
\frac{\partial \mathcal{L}_I}{\partial(\partial_t S)} \cdot \partial_t S - \mathcal{L}_I &= \beta \varphi u \partial_t S - \beta(\mathbf{A} \cdot \nabla S + \partial_t S \varphi)u \\
&= -\beta \mathbf{A} \cdot \nabla S u = -\mathbf{A} \cdot \mathbf{j}
\end{aligned}
$$

If we set

$$
\mathcal{E}_I = -\mathbf{E} \cdot \nabla \varphi - \mathbf{A} \cdot \mathbf{j}
$$

we have that

$$
\begin{aligned}
E_I &= \int \mathcal{E}_I dx = \int (-\mathbf{E} \cdot \nabla \varphi - \mathbf{A} \cdot \mathbf{j}) dx \\
&= \int (\nabla \cdot \mathbf{E}\, \varphi - \mathbf{A} \cdot \mathbf{j}) dx = \int (\rho \varphi - \mathbf{A} \cdot \mathbf{j}) dx \\
&= -\beta \int (\varphi \partial_t S + \mathbf{A} \cdot \nabla S)u\, dx
\end{aligned}
$$

Then

$$
E = E_M + E_F + E_I.
$$

$\square$

Conservation of Momentum. Momentum, by definition, is the quantity, which is preserved by virtue of the space invariance of the Lagrangian. Here, we will compute only the matter-field moment, since it is the only part needed in the rest of this paper.

Proposition 2. *The momentum takes the following form*

$$
\mathbf{P} = \mathbf{P}_M + \mathbf{P}_F + \mathbf{P}_I \tag{31}
$$

where

$$
\begin{aligned}
\mathbf{P}_M &= \int \left(\partial_t u \nabla u + \partial_t S \nabla S u^2 \right) dx \tag{32} \\
&= \int (\partial_t u \nabla u + \rho \mathbf{j}) dx \tag{33}
\end{aligned}
$$

*is the **matter field** momentum.*

Proof. By Noether's Theorem, we have that the momentum densities $\mathcal{P}_M$ relative to the Lagrangian densities $\mathcal{L}_M$ is given by

$$
\mathcal{P}_M = \frac{\partial \mathcal{L}_M}{\partial(\partial_t u)} \nabla u + \frac{\partial \mathcal{L}_M}{\partial(\partial_t S)} \nabla S = \partial_t u \nabla u + \partial_t S \nabla S u^2
$$

$\square$

Conservation of electric charge: Even if our equations are not invariant for the whole gauge group; nevertheless, the electric charge is preserved, as it is a consequence of (26).

In fact, by (19) and (20), we obtain the continuity equation

$$\partial_t \rho = \nabla \cdot (\partial_t \mathbf{E}) = \nabla \cdot (\nabla \times \mathbf{H} - \mathbf{j}) = -\nabla \cdot \mathbf{j}$$

Therefore, the total electric charge

$$Q[u, S] = \int \rho dx = -\beta \int \partial_t S u \, dx \tag{34}$$

is preserved.

Conservation of hylenic charge: Following [9,10] the hylenic charge, by definition, is the quantity which is preserved by the invariance for the transformation (6). It is defined as follows

$$H[u, S, \varphi] = \int \left(\frac{\partial_t S u^2}{\beta} - \varphi u \right) dx = \int (\rho - \varphi) u \, dx \tag{35}$$

By Equation (12), we see directly that the hylenic charge is preserved.

2.3. The Cauchy Problem

In order to study the Cauchy problem, it is more convenient to use the variable ψ rather that (u, S). To this end, we introduce the following operators:

$$\begin{aligned}
\mathfrak{D}_t(\psi) &= \operatorname{Im}\left(\frac{\partial_t \psi}{\psi} \right) = \operatorname{Im}\left(\frac{\partial_t \left(u e^{iS} \right)}{u e^{iS}} \right) \\
&= \operatorname{Im}\left(\frac{\partial_t u e^{iS} + i u \partial_t S e^{iS}}{u e^{iS}} \right) = \operatorname{Im}(\partial_t u + i \partial_t S) = \partial_t S
\end{aligned}$$

and

$$\mathfrak{D}_x(\psi) = \operatorname{Im}\left(\frac{\nabla \psi}{\psi} \right) = \nabla S$$

Since we have assumed the Lorentz condition, the Equations (11)–(14), using (5), can be rewritten as follows

$$\Box \psi + \psi = -N'(|\psi|) \frac{\psi}{|\psi|} + \mathbf{A} \cdot \mathfrak{D}_x(\psi) - \varphi \mathfrak{D}_t(\psi) \tag{36}$$

$$\Box \varphi = \mathfrak{D}_t(\psi) |\psi| \tag{37}$$

$$\Box \mathbf{A} = \mathfrak{D}_x(\psi) |\psi|. \tag{38}$$

We make the following (redundant) assumptions on N

$$N, \ N' \text{ and } N'' \text{ are bounded;} \tag{39}$$

$$N(s) \geq -\frac{1}{2}(1 - \delta)s^2; \ 0 < \delta < 1. \tag{40}$$

Theorem 1. *If (39) and (40) hold and β is sufficiently small, the Cauchy problem relative to Equations (36)–(38) has a unique weak solution.*

Proof. The proof of this theorem follows standard arguments and we will just provide a sketch. The function space to work in is $H^1 \times \left(\mathcal{D}^{1,2} \right)^4$ where

$$H^1 = \left\{ \psi \in L^2(\mathbb{R}^3, \mathbb{C}) \mid \int \left(|\nabla \psi|^2 + |\psi|^2 \right) dx < +\infty \right\}$$

$$\mathcal{D}^{1,2} = \left\{ f \in L^6(\mathbb{R}^3) \mid \int |\nabla f|^2 dx < +\infty \right\}. \tag{41}$$

We set
$$U(t, x) = (\psi(t, x), \varphi(t, x), \mathbf{A}(t, x));$$

where $\psi \in H^1(\mathbb{R}^3, \mathbb{C})$, $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R})$, and $\mathbf{A} \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3) = \left[\mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^1)\right]^3$. So, we end with the Cauchy problem

$$\Box U + P_1 U = F(U) \tag{42}$$

where P_1 is the projection of U on the first component, i.e., $P_1 U = \psi$.

We equip the phase space

$$X := \left[H^1 \times \left(\mathcal{D}^{1,2}\right)^4\right] \times \left(L^2\right)^6, \tag{43}$$

with its natural norm given by

$$\|U\|^2 = \|\partial_t \psi\|_{L^2}^2 + \|\psi\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2 + \|\nabla \varphi\|_{L^2}^2 + \|\partial_t \mathbf{A}\|_{L^2}^2 + \|\nabla \mathbf{A}\|_{L^2}^2$$

It is well known that a sufficient condition for the Cauchy problem to have a unique solution for the initial data in X is:

- the energy inequality holds: there exists two positive constants c_1 and c_2, such that

$$c_1 \|U\|^2 \leq E[U] \leq c_2 \|U\|^2$$

  This inequality can be proved if β is sufficiently small and if (39) holds;
- $F : X \to X'$ is locally compact; this fact holds, since the embedding

$$X_{loc} \to \left(L_{loc}^6\right)^6$$

  is compact;

  Under these conditions, the proof goes as follows:

1. We take a sequence of approximate solutions; for example, we can use the Faedo–Galerkin procedure;
2. We take the weak limit of the approximated solutions which exists thank to the second energy inequality;
3. We pass to the limit in the weak formulation of the equations; we can take the limit in the nonlinear part F, since it is locally compact;
4. We can prove the uniqueness thanks to the first energy inequality and the Gronwall's inequality.

□

Remark 2. *The optimal conditions for the existence of solutions and the study of the their regularity is not the aim of this paper and it is a question that, for the moment, is left open.*

3. q-Solitons

Roughly speaking, a *solitary wave* is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A *soliton* is a solitary wave which exhibits some form of stability so that it has a particle-like behavior (see, e.g., [10–13]).

It is well known that Equation (4) presents solitons under suitable assumptions on W. It was largely studied during the 1970s and the 1980s. The first rigorous result for the finite energy solution was due to Strauss [14], and later Beresticky and Lions [15] gave sufficient and "almost necessary" condition for the existence. In [10], there is a detailed analysis of the case in which $W \geq 0$. If we couple (4) with the Maxwell equation via the interaction (2), the solitons usually are called Q-balls (Coleman [16]). The first rigorous result about the existence of Q-balls was established in 2002 [17]. Afterwards, their stability

was proved in [18]. A detailed analysis of Q-balls and the references to the large literature can be found in [10]; in all these papers, the interaction between the solitons and the e.m. field is established by the Lagrangian (2).

In this section, we analyze the existence and the properties of solitons when the interaction with the Maxwell equation is simply given by the Lagrangian (9) and not by (2). They will be called q-solitons. The main difference between q-solitons and the Q-balls is that the former behave like single particles while the latter behave like a swarm of particles (see [10], Sections 4.1.2 and 5.1.5).

3.1. Existence of Stationary Waves

Let us prove the existence of some particular solution of Equations (11)–(14); first, we look for stationary solutions, namely solutions where ψ is a stationary wave, i.e.,

$$\psi(t, x) = u(x)e^{-i\omega t}; \; u \geq 0 \tag{44}$$

We make the following ansaz

$$u = u(x); \; S = -\omega t; \; \varphi = \varphi(x); \; \mathbf{A} = 0. \tag{45}$$

Replacing these variables in (11)–(14), Equations (12) and (14) are identically satisfied, while Equations (11) and (13) become

$$-\Delta u + W'(u) - \omega^2 u + \beta \omega \varphi = 0 \tag{46}$$

$$-\Delta \varphi = \beta \omega u \tag{47}$$

These two equations have nontrivial solutions, provided that suitable conditions on $W \in C^2$ are satisfied: we write W as follows,

$$W(s) = \frac{1}{2}s^2 + N(s), \tag{48}$$

In the model of our interest, N must be considered as a small perturbation of the parabola $1/2s^2$. However, in order to get an existence result , it is sufficient to make the following assumptions on N

- (N-1) $N(0) = N'(0) = N''(0) = 0$;
- (N-2) $\inf_{s \in \mathbb{R}^+} N(s) := N_{\inf} < 0$, ($N_{\inf}$ is allowed also to be $-\infty$);
- (N-3) There exist $C > 0$ and $2 < p < 6$, such that $|N'(s)| \leq C(1 + s^{p-1})$.

We will show that, at least for β small, the above assumptions guarantee the existence of nontrivial solutions to Equations (46) and (47). In most of the literature relative to (4), we usually have the following choice of N :

$$N(s) = \frac{1}{p}|s|^p, \; 2 < p < 6. \tag{49}$$

This assumption implies the existence of nontrivial solutions also for Equations (46) and (47) for every $\beta > 0$. However, in our model, it is more interesting (see Theorem 3) to choose a "bump-like" N, such as

$$N(s) = -\varepsilon^2 s^3 \exp\left(-\left|\frac{s-1}{\varepsilon}\right|^2\right). \tag{50}$$

or a "bell" function such as

$$N(s) = \begin{cases} -\left[(s-1)^2 - \varepsilon^2\right]^2 & if \; |s-1| < \varepsilon \\ 0 & if \; |s-1| \geq \varepsilon \end{cases} \tag{51}$$

where ε is a small parameter which makes $W(s) \geq 0$. Its relevance will be discussed in Theorem 3.

We define the following bilinear form

$$a_\omega(u, u) = \frac{1}{2} \int \left[|\nabla u|^2 + \left(1 - \omega^2\right) u^2 \right] dx + \beta^2 \omega^2 \iint \frac{u(x)u(y)}{|x - y|^2} \, dx dy \tag{52}$$

Notice that

$$\iint \frac{u(x)u(y)}{|x - y|^2} \, dx dy = \int (G * u)u \, dx$$

where

$$G(x) = \frac{1}{4\pi |x|^2}$$

is the Green function relative to the Poisson equation

$$- \Delta \phi = u \tag{53}$$

namely $(G * u)(x) = (-\Delta)^{-1} : (\mathcal{D}^{1,2}) \to \mathcal{D}^{1,2}$.

Now, let us introduce a number ω_{inf}, which is very relevant in this study of solitons

$$\omega_{\text{inf}} := \inf \left\{ \omega > 0 \mid \exists u \in H_c^1, \, a_\omega(u, u) + \int N(u) \, dx < 0 \right\}; \tag{54}$$

ω_{inf} depends on β and the shape of N. For example, if $N(u)$ is given by (49), then it is immediately necessary to check that $\omega_{\text{inf}} = 0$ for every $\beta > 0$. If $N(u)$ is given by (50), ω_{inf} depends on β (see Corollary 1).

We have the following theorem.

Theorem 2. *If (N-1) and (N-3) hold and if $\omega_{\text{inf}} < 1$, then for every $\omega \in (\omega_{\text{inf}}, 1)$, Equations (46) and (47) have nontrivial solutions in $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.*

Proof. The couple of Equations (46) and (47) can be easily solved by standard variational methods; we will here provide a sketch of the proof, avoiding standard estimates which are well known among people working in nonlinear analysis.

Set

$$H_{\text{rad}}^1 \left(\mathbb{R}^3 \right) = \left\{ u \in H^1 \left(\mathbb{R}^3 \right) \mid u = u(|x|) \right\};$$

and

$$V := \left\{ u \in H_{\text{rad}}^1 \left(\mathbb{R}^3 \right) \mid a_\omega(u, u) < +\infty \right\}$$

Since $\omega_{\sup} < 1$, V is a Hilbert space equipped with scalar product $a_\omega(u, v)$ and norm $\|u\|_V := \sqrt{a_\omega(u, v)}$. Moreover, by the definition of $\omega_{\sup}$, $\exists \delta > 0$,

$$\|u\|_V^2 \geq \delta \|u\|_{H^1}^2$$

Then, using the Gagliardo–Nirenberg–Sobolev estimate, we can see that

$$V = H_{\text{rad}}^1 + \left[\mathcal{D}_{\text{rad}}^{1,2}(\mathbb{R}^3) \right]' \subset H_{\text{rad}}^1 + L^{6/5} \tag{55}$$

Now, we define, on V, the following functional:

$$\begin{aligned} J[u] &= \|u\|_V^2 + \int N(u) dx \\ &= \frac{1}{2} \int \left[|\nabla u|^2 + \left(1 - \omega^2\right) u^2 \right] dx + \omega^2 \beta^2 \int [(G * u)u + N(u)] dx \end{aligned} \tag{56}$$

By (N-3), (55) and standard arguments, J is a differentiable functional in V. So we have to prove two facts: (1) the critical points of J solve Equations (46) and (47) and (2) if $\omega \in (\omega_{\text{inf}}, 1)$, J has at least a nontrivial critical point.

(1) We have that $\forall v \in V$

$$dJ[u](v) = \int \left[\nabla u \nabla v + \left(1 - \omega^2\right) uv \right] dx + \omega^2 \beta^2 \int \left[(G * u) + N'(u) \right] v dx$$

and by well known arguments, we have that

$$-\Delta u + (1 - \omega^2)u + \omega^2 \beta^2 (G * u) + N'(u) = 0.$$

Taking account of (48)

$$-\Delta u + W'(u) + \omega^2 \beta^2 (G * u) - \omega^2 u = 0;$$

finally, setting $\varphi = \omega \beta (G * u)$, we get Equation (46) while Equation (47) follows from the definition of φ.

(2) The simplest way to prove the existence of critical points of J is the use of the Mountain Pass theorem of Ambrosetti and Rabinowitz [19]. Following standard arguments, it is easy to prove that J satisfies the Palais–Smale condition (for a very similar result, see [17], Lemma 4.3). The interesting fact is to check the conditions, which guarantee the geometry of the Mountain Pass theorem, namely, that

$$\exists r > 0, \ \|u\|_V^2 = r \Rightarrow J[u] \geq b > 0 \tag{57}$$

and

$$\exists \bar{u}, \ \|\bar{u}\|_{H^1}^2 > r, \ J[\bar{u}] \leq 0 \tag{58}$$

By the definition of $\|\cdot\|_V$

$$\|u\|_V^2 \geq \delta \|u\|_{H^1}^2, \ \ \delta > 0.$$

If $r > 0$, is sufficiently small, by (48), (N-1), (N-3) and standard computations, $\exists C, \eta > 0$ such that

$$\int |N(u)| dx \leq C \|u\|_{H^1}^{2+\eta} \leq \frac{C}{\delta} \|u\|_{H^1}^{2+\eta}$$

then, if $\|u\|_V = r$

$$
\begin{aligned}
J[u] &= \frac{1}{2} \|u\|_V^2 + \int |N(u)| dx \\
&\geq \delta \|u\|_V^2 - \frac{C}{\delta} \|u\|_V^{2+\eta} \\
&= \left[\delta - \frac{Cr^\eta}{\delta} \right] r^2
\end{aligned}
$$

Then if r is sufficiently small $J[u] \geq b > 0$ and (57) is proved. (58) holds by the definition (54) of ω_{inf}. $\square$

Corollary 1. *If (N-1), (N-2) and (N-3) hold, then there exists $\beta_0 > 0$, such that for every $\beta \in (0, \beta_0)$ Equations (46) and (47) have nontrivial solutions in $H^1(\mathbb{R}^N)$.*

Proof. By Theorem 2, it is sufficient to prove that

$$\omega_{\text{inf}} < 1.$$

By (*N*-2), we can choose a point s_1 such that

$$N(s_1) = -h^2.$$

We set

$$u_r = \begin{cases} s_1 & if \ |x| < r \\ 0 & if \ |x| > r+1 \\ \frac{|x|}{r} s_1 - [(r+1)|x| - 1]s_1 & if \ r < |x| < r+1 \end{cases}, \tag{59}$$

$$\bar{\omega} = \sqrt{1 - \frac{h^2}{s_1^2}},$$

and

$$F[u] := \frac{1}{2} \int |\nabla u|^2 dx + \int \left(W(u) - \frac{1}{2}\bar{\omega}^2 u^2 \right) dx$$

Let us compute $F[u_r]$:

$$\begin{aligned} F[u_r] &= \frac{1}{2} \int_{B_{r+1}\backslash B_r} |\nabla u_r|^2 dx + \int_{B_{r+1}\backslash B_r} \left(W(u) - \frac{1}{2}\bar{\omega}^2 u_r^2 \right) dx \\ &+ \int_{B_r} \left(W(u_r) - \frac{1}{2}\bar{\omega}^2 u_r^2 \right) dx \end{aligned}$$

The first part can be estimated as follows:

$$\begin{aligned} \frac{1}{2} \int_{B_{r+1}\backslash B_r} |\nabla u_r|^2 dx + \int_{B_{r+1}\backslash B_r} \left(W(u) - \frac{1}{2}\bar{\omega}^2 u^2 \right) dx &\leq C \cdot meas(B_{r+1}\backslash B_r) \\ &\leq C_1 r^2 \end{aligned}$$

For the second part, we have that

$$\begin{aligned} \int_{B_r} \left(W(u_r) - \frac{1}{2}\bar{\omega}^2 u_r^2 \right) dx &= \int_{B_r} \left[\frac{1}{2}u_r^2 - \frac{1}{2}\bar{\omega}^2 u_r^2 + N(u_r) \right] dx \\ &= \int_{B_r} \left[\frac{1}{2}\left(1 - \bar{\omega}^2\right) s_1^2 + N(u_r(s_1)) \right] dx \\ &\leq \int_{B_r} \left[\frac{1}{2} \left[1 - \left(1 - \frac{h^2}{s_1^2} \right) \right] s_1^2 - h^2 \right] dx \\ &= \int_{B_r} \left[\frac{1}{2}h^2 - h^2 \right] dx = \frac{4}{3}\pi r^3 \frac{h^2}{2} = \frac{2}{3}\pi r^3 h^2 \end{aligned}$$

Then, we have that

$$F[u_r] \leq C_1 r^2 - \frac{2}{3}\pi r^3 h^2.$$

Therefore, we can choose a $\bar{r}$ so large that

$$F[u_{\bar{r}}] < -\bar{r}^3 h^2 < -1$$

and β so small that

$$\beta^2 \bar{\omega}^2 \int (G * u_{\bar{r}}) u_{\bar{r}} dx \leq 1.$$

Then

$$a_{\bar{\omega}}(u_{\bar{r}}, u_{\bar{r}}) + \int N(u_{\bar{r}}) \, dx = F[u_{\bar{r}}] + \frac{1}{2}\beta^2 \bar{\omega}^2 \int (G * u_{\bar{r}}) u_{\bar{r}} dx \leq -\frac{1}{2}$$

and hence

$$\omega_{\text{inf}} < \bar{\omega} = \sqrt{1 - \frac{h^2}{s_1^2}} < 1.$$

□

3.2. Stationary q-Solitons

Using the equivariant Mountain Pass theorem and exploiting the fact that the functional (56) is even, it is possible to prove that Equations (11)–(14), have an infinite number of radially symmetric solutions of the form (44) and (45), namely solitary waves. We call **ground state solution**, the radially symmetric solution $u_0 > 0$, which minimizes the following quantity

$$\Lambda[u, \omega] = \frac{E[u, \omega]}{|H[u, \omega]|} = \frac{\int \left(\frac{1}{2}|\nabla u| - \frac{1}{2}\omega^2 u^2 + W(u) + \beta^2 \omega^2 (G * u)\right) dx}{\left|\int (-\beta^{-2}\omega u^2 - \omega(G * u)u) dx\right|}$$

in $H^1 \times \mathbb{R}^+$. Clearly, at least for a generic W, this solution is unique and it corresponds to the critical value determined by the Mountain Pass Theorem, having chosen ω, which minimizes $\Lambda[u, \omega]$. Notice that $\Lambda[u, \omega]$ is the ratio of the matter energy (29) and the hylenic charge (35). If $W \geq 0$, the ground state solution, is a soliton in the sense that it is orbitally stable (see, e.g., [18] or [10]).

By Theorem 1, Equations (11)–(14) define a dynamical system whose phase space is given by (43). A generic point of the phase space, at time t, can be represented as follows:

$$\begin{bmatrix} (u(t, \cdot), S(t, \cdot), \varphi(t, \cdot), \mathbf{A}(t, \cdot)) \\ (\partial_t u(t, \cdot), \partial_t S(t, \cdot), \partial_t \varphi(t, \cdot), \partial_t \mathbf{A}(t, \cdot)) \end{bmatrix}$$

From now on, σ_0 we will denote the ground state solution of Equations (46) and (47), namely

$$\sigma_0(x) = \begin{bmatrix} (u_0(x), \theta, 0, 0) \\ (0, -\omega_0 + \theta, 0, 0) \end{bmatrix} \tag{60}$$

where θ is a possible phase shift which is not relevant and, from now on, it will be neglected. Such a function will be called a *q***-soliton**. We have chosen this name to emphasize the comparison with the Q-balls which are stable configurations of the equations determined by the action $\mathbb{A}_W + \mathbb{A}_F$ (see (2) and (3) and the discussion at the beginning of Section 3). Roughly speaking, a Q-ball behaves like a swarm of charged particles kept close to each other by the gluing force determined by $N(s)$ (see [9] or [10] Section 5.1.5). Instead, as we will see in this and the following sections, a *q***-soliton** behaves like a single particle of "matter" condensed by the gluing force determined by $N(s)$.

The *q*-soliton, has a positive electric charge $\rho_0 = \omega u_0$ (and hence, by (47), $\varphi_0(x) > 0$). However, the Equations (11)–(14), have a solution with negative charge, given by

$$\sigma_0^-(x) = \begin{bmatrix} (u_0(x), 0, 0, 0) \\ (0, \omega_0, 0, 0) \end{bmatrix}$$

Then, (11)–(14), have at least two orbitally stable solutions determined by a *q*-soliton $\sigma_0(x)$ and a *q*-antisoliton $\sigma_0^-(x)$:

$$U(t, x) = \begin{bmatrix} (u_0(x), 0, \varphi_0(x), 0) \\ (0, -\omega_0 t, 0, 0) \end{bmatrix}; \quad U^-(t, x) = \begin{bmatrix} (u_0(x), 0, -\varphi_0(x), 0) \\ (0, \omega_0 t, 0, 0) \end{bmatrix}$$

Generally, they are unique up to space–time translations and phase shift. Rotations do not produce new solutions, since u_0 is radially symmetric.

The "shape" of a soliton is determined by the nonlinear $N(s)$. In [20], there is a detailed analysis of this topic in the case $\beta = 0$. Clearly, this analysis can be extended to the q-soliton when β is small. The next theorem examines some properties of the q-solitons in the case, in which $N(s)$ is a small bump such as (51)

Theorem 3. *For every $\varepsilon > 0$, we can choose N, such that*

- $W \geq 0$;
- $1 - \varepsilon < \omega_{\inf} < 1$.
- *if u_0 is the Mountain Pass solution of Equations (46) and (47), then*

$$1 - \varepsilon \leq \|u_0\|_{L^\infty} \leq 1 + \varepsilon$$

Proof. We choose N to be a bell function such as (51), so that

$$\min N = N(1) \geq -\varepsilon^2$$

and

$$supp(N) = [1 - \varepsilon, \ 1 + \varepsilon]$$

Then, the first inequality is trivially verified. In order to prove the second inequality, we see that

$$N(u) \geq -\frac{1}{2}\varepsilon^2$$

and so, if we put $\bar{\omega}^2 = 1 - \varepsilon^2$, we have that

$$
\begin{aligned}
a_{\bar{\omega}}(u, u) + \int N(u)\, dx &\geq \frac{1}{2}\int \left[|\nabla u|^2 + \left(1 - \bar{\omega}^2\right)u^2\right]\, dx + \int N(u)dx \\
&= \frac{1}{2}\int \left[|\nabla u|^2 + \varepsilon^2 u^2\right]\, dx - \frac{1}{2}\varepsilon^2 \int u^2 dx \geq 0
\end{aligned}
$$

Therefore, by the definition of $\omega_{\inf}$ (54), we have that $\omega_{\inf}^2 > \bar{\omega}^2$ and hence $\omega_{\inf} > \sqrt{1 - \varepsilon^2} > 1 - \varepsilon$.

The third inequality follows applying to Equation (46) the maximum principle. The details of the proof can bo found in [20]. $\square$

Remark 3. *The picture which comes out from Corollary 1 and Theorem 3 is the following: given the free electromagnetic field and the free matter field relative to KG, we obtain q-solitons, provided that*

- *The interaction between them is given by a Lagrangian of type (9) with β **very small**;*
- *KG is perturbed by a nonlinear term $N(s)$ (negative in some point) **small** with respect to 1 and **large** with respect to β.*

If we want to analyze the properties of a q-soliton considered as a model for physical particles, it is useful to rewrite Equation (11) with dimensional constants. We get the following equation (which is satisfied by σ_0 if suitably rescaled)

$$\partial_t^2 u - c^2 \Delta u + \alpha^2 u + \frac{c^2}{\ell^2}N'(u) - \left[(\partial_t S)^2 - c^2|\nabla S|^2\right]u = \eta\beta(\mathbf{A} \cdot \nabla S - \varphi \partial_t S) \qquad (61)$$

In this equation,

- c is the speed of light which makes the equation invariant for the Lorentz transformations with the parameter c;
- u has the dimension of

$$\frac{\{\text{mass}\}^{\frac{1}{2}}}{\{\text{space}\}},$$

this fact can be deduced ,e.g., by the fact that, by Theorem 1,

$$\frac{1}{2}\int\left[|\partial_t u|^2 + c^2|\nabla u|^2\right]dx$$

has the dimension of energy;

- α has the dimension of a frequency; if we linearize Equation (61) with $\beta = 0$, we get KG

$$\partial_t^2 u - c^2\partial_x^2 u + \alpha^2 u = 0. \tag{62}$$

which has the following dispersion relations

$$\omega_{\mathrm{KG}} = \alpha\sqrt{1 + \frac{c^2}{\alpha^2}k_{\mathrm{KG}}^2}$$

where ω_{KG} and k_{KG} are the frequency and the wave number of the small perturbations of the matter field. Since $\omega_0 < \alpha < \omega_{\mathrm{KG}}$, the oscillations of the q-soliton, having frequency ω_0, do not excite dispersive waves in the surrounding matter field. This fact partially explains the stability of the soliton;

- If we give to N' the same dimension of u, ℓ has the dimension of a length and it is of the order of the radius of the soliton in the sense that

$$u_0(x),\ \nabla u_0(x) \cong 0 \ \text{ for every } x \geq r_0 := k\ell$$

where $\cong$ means that the quantity is exponentially small and k is a dimensionless variable which depends on N.

- Here, S is supposed to be dimensionless;
- $\eta\beta$ represents the strength of the interaction of the matter field with the electromagnetic field; by (17)

$$\dim\beta = \frac{\{\text{electric charge}\}\cdot\{\text{time}\}\cdot\{\text{space}\}}{\{\text{mass}\}^{\frac{1}{2}}}$$

and if we give to φ the dimension of an electric field, i.e.,

$$\{\text{electric charge}\}/\{\text{space}\},$$

then

$$\dim\eta = \frac{\{\text{mass}\}}{\{\text{electric charge}\}^2\cdot\{\text{space}\}^2\cdot\{\text{time}\}^2};$$

using these variables, then $a_\omega(u, u)$ defined by (52) becomes

$$a_\omega(u, u) = \int\left[c^2|\nabla u|^2 + \left(\alpha^2 - \omega^2\right)u^2\right]dx + \omega^2\eta^2\beta^2\iint\frac{u(x)u(y)}{|x-y|^2}dxdy; \tag{63}$$

hence, if $\eta\beta$ is too large with respect to the other constants, then, by (54), if $\omega_{\inf} \geq \alpha$ and there are no solitons. Actually, there is a competition between the gluing force which increases with $c\ell^{-1}N$ and the electric force, which increases with $\eta\beta$. The gluing force tends to concentrate the matter field while the electric force tends to spread it.

By this discussion, it shows that

$$W(u) = \frac{1}{2}\alpha^2 u + \frac{c^2}{\ell^2}N(u)$$

represents the potential of the "nuclear force", which is repellent when u is small and attractive when the values of u are in range, where $N(s)$ is negative. $N(s)$ is responsible for the nonlinear behavior of the matter field, and hence of the existence of q-solitons. By these considerations, Theorem 3 and Remark 3, a q-soliton is a good model for physical

particles if, in the dimensionless equation

$$\beta^2 \ll \max N(s) \ll 1$$

Additionally, the condition

$$W(u) \geq 0 \tag{64}$$

is suitable for a physical model.

We denote by

$$\{\sigma_0, \varphi_0\} = \left[\begin{array}{c} (u_0(x), 0, \varphi_0, 0) \\ (0, -\omega_0, 0, 0) \end{array} \right] \tag{65}$$

the equilibrium configuration containing a q-soliton. The energy of this configuration is given by

$$
\begin{aligned}
E[\{\sigma_0, \varphi_0\}] &= E_{\mathrm{M}}[\{\sigma_0, \varphi_0\}] + E_{\mathrm{I}}[\{\sigma_0, \varphi_0\}] + E_{\mathrm{F}}[\{\sigma_0, \varphi_0\}] \\
&= E[\sigma_0] + E_{\mathrm{I}}[\{\sigma_0, \varphi_0\}] + E_{\mathrm{F}}[\varphi_0]
\end{aligned}
$$

where

$$
\begin{aligned}
E[\sigma_0] &= \int \left[\frac{1}{2} |\nabla u_0|^2 - \frac{1}{2} \omega_0^2 u_0^2 dx \right] dx + \int W(u_0) dx; \tag{66} \\
E_{\mathrm{I}}[\{\sigma_0, \varphi_0\}] &= \omega_0 \beta \int \varphi_0 u_0 dx; \\
E_{\mathrm{F}}[\varphi_0] &= \frac{1}{2} \int |\nabla \varphi_0|^2 dx.
\end{aligned}
$$

All these terms are positive; $E[\sigma_0]$ is positive by (29) and (64); $E_{\mathrm{I}}[\{\sigma_0, \varphi_0\}]$ is positive since, by Equation (47) ω_0 and $\varphi_0(x)$ have same sign. Thus, this term is also positive for anti-solitons. The energy $E[\sigma_0] + E_{\mathrm{I}}[\{\sigma_0, \varphi_0\}]$ is concentrated around 0 in a region of radius r_0. In fact, since u decays exponentially, from the physical perspective, it can be considered null for $|x|$ larger that a suitable r_0. The field energy $E_{\mathrm{F}}[\varphi_0]$ is not concentrated; by Equation (47), it decays as $|x|^{-4}$. Finally, notice that the e.m. field energy of a soliton does not diverge as the energy of a pointwise particle would.

3.3. Travelling q-Solitons

The action functional is invariant for the group of the Lorentz boosts:

$$t' = \frac{t - vx_1}{\sqrt{1 - v^2}}; \ x' = \left(\frac{x_1 - vt}{\sqrt{1 - v^2}}, \ x_2, \ x_3 \right). \tag{67}$$

Hence, if $u(t, x)$, $S(t, x)$, $\varphi(t, x)$, $\mathbf{A}(t, x)$ is a solution to (11)–(14), also $u(t', x')$, $S(t', x')$, $\varphi'(t', x')$, $\mathbf{A}'(t, x)$ is a solution.

Since $(\varphi, \mathbf{A})$ is a 4-vector, it transforms as follows

$$\varphi'(t, x) = \frac{\varphi(t', x') - vA_1(t', x')}{\sqrt{1 - v^2}}$$

$$\mathbf{A}'(t, x) = \left(\frac{A_1(t', x') - v\varphi(t', x')}{\sqrt{1 - v^2}}, \ A_2(t', x'), \ A_3(t', x') \right)$$

As usual, we set

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

If σ_0 denotes the stationary q-soliton defined by (60) and if $\mathbf{v} = (v, 0, 0)$, we obtain the following family of solutions:

$$u_{\mathbf{v}}(t, x) := u_0(\gamma(x_1 - vt), x_2, x_3) = u_0(x') \tag{68}$$

$$S_{\mathbf{v}}(t,x) := -\omega_0\gamma(t - vx_1) = -\omega_0 t' = \mathbf{k} \cdot x - \omega_{\mathbf{v}} t \tag{69}$$

where

$$\mathbf{k} = (k,0,0) = (\gamma\omega_0 v, 0, 0); \tag{70}$$

$$\omega_{\mathbf{v}} = \gamma\omega_0; \tag{71}$$

$$\varphi_{\mathbf{v}}(t,x) = \gamma\varphi_0(x') \tag{72}$$

$$\mathbf{A}_{\mathbf{v}}(t,x) = -\gamma(v\varphi_0(x'),\ 0,\ 0) \tag{73}$$

If $\mathbf{v} = (v,0,0)$, $|v| < 1$, we define a moving solitons as follows

$$\sigma_{\mathbf{v}}(x) = \begin{bmatrix} (u_{\mathbf{v}}(0,x), S_{\mathbf{v}}(0,x), 0, 0) \\ (\partial_t u_{\mathbf{v}}(0,x), \partial_t S_{\mathbf{v}}(0,x), 0, 0) \end{bmatrix} = \begin{bmatrix} (u_0(x'), \mathbf{k},\ 0,\ 0) \\ (\partial_t u_0(x'), -\gamma\omega_0,\ 0,\ 0) \end{bmatrix}_{t=0}.$$

The configuration

$$\sigma_{\mathbf{v}}(x) + \begin{bmatrix} (0, 0, \gamma\varphi_0(x'), -\gamma(v\varphi_0(x'))) \\ (0, 0, \gamma\partial_t\varphi_0(x'), -\gamma v\partial_t\varphi_0(x')) \end{bmatrix}_{t=0}$$

is the initial condition of the solution (68), (69), (72), (73) of Equations (11)–(14).

If $\mathbf{v}$ is any vector with $|\mathbf{v}| < 1$, $R \in O(3)$ is a rotation such that $R\mathbf{v} = (|\mathbf{v}|, 0, 0)$, we set,

$$\sigma_{\mathbf{v}}(x) = \sigma_{R^{-1}\mathbf{v}}(x)$$

Definition 1. *A moving q-soliton with velocity* $\mathbf{v} \in \mathbb{R}^3$ *in the point* $\bar{x} \in \mathbb{R}^3$ *is a function of the form*

$$\sigma_{\mathbf{v}}(x - \bar{x}).$$

The evolution of a free moving soliton is given by

$$\sigma_{\mathbf{v}}(x - \mathbf{v}t - \bar{x}) = \sigma_{R_{\mathbf{v}}^{-1}}(x - R(\mathbf{v}t - \bar{x})). \tag{74}$$

3.4. Mechanical Properties of q-Solitons

First, we will investigate the intrinsic quantities of a moving q-soliton. Since these properties are independent of R and $\bar{x}$, we will just consider $\sigma_{\mathbf{v}}$ with $\mathbf{v} = (v,0,0)$ and $\bar{x} = 0$.

The simplest quantity to describe of a q-soliton is the electric charge. It is defined by (34) and, in this case, is

$$q[\sigma_0] = \omega_0\beta \int u_0 dx \tag{75}$$

This depends only on the soliton and not on the configuration of the surrounding field. Moreover, it has the following property:

Proposition 3. *The electric charge of a moving soliton is independent of the motion:*

$$q[\sigma_{\mathbf{v}}] := q[\sigma_0]$$

Proof. By (34) and (71), making a change in variable $x_1 = 1/\gamma x_1' + vt$, we have that

$$\begin{aligned} q[\sigma_{\mathbf{v}}] &= -\beta \int \partial_t S_{\mathbf{v}}\, dx = \beta\omega_{\mathbf{v}} \int u_{\mathbf{v}}(0,x)\, dx \\ &= \gamma\omega_0\beta \int u_{\mathbf{v}}(x')dx = \gamma\omega_0\beta \int u_{\mathbf{v}}(x')\frac{1}{\gamma}dx' \\ &= \omega_0\beta \int u_0 dx. \end{aligned}$$

$\square$

From now on, q will denote the charge of a q-soliton. Next, let us consider the mass:

Definition 2. *If the momentum of the matter field,*

$$\mathbf{P}[\sigma_{\mathbf{v}}] = \mathbf{P}_{\mathrm{M}}[\{\sigma_{\mathbf{v}}, \varphi_{\mathbf{v}}\}] = \int \left(\partial_t u_{\mathbf{v}} \nabla u_{\mathbf{v}} + \partial_t S_{\mathbf{v}} \nabla S u_{\mathbf{v}}^2 \right) dx$$

(see Proposition 2) is proportional to **v**, *the constant of proportionality is called the* **mass** *the moving q-soliton, namely*

$$\mathbf{P}[\sigma_{\mathbf{v}}] = m[\sigma_{\mathbf{v}}]\mathbf{v}$$

Remark 4. *Notice that this definition of mass is intrinsic to the Equations (11)–(14) and it is independent of any physical interpretation; it can be interpreted as a "physical" mass whenever x and t are interpreted as variables of the physical space-time.*

Let us explicitly compute the momentum:

Theorem 4. *The* **momentum** *of a q-soliton takes the following form:*

$$\mathbf{P}_{\mathrm{M}}[\sigma_{\mathbf{v}}] = \gamma \mathbf{v} \left[\frac{1}{3} \int |\nabla u_0|^2 dx + \omega_0^2 \int u_0^2 dx \right] \tag{76}$$

Proof. By Proposition 2,

$$\mathbf{P}[\sigma_{\mathbf{v}}] = \int \left(\partial_t u_{\mathbf{v}} \nabla u_{\mathbf{v}} + \partial_t S_{\mathbf{v}} \nabla S \, u_{\mathbf{v}}^2 \right) dx;$$

assuming $\mathbf{v} = (1, 0, 0)$, by (68)–(71)

$$
\begin{aligned}
\mathbf{P}_1[\sigma_{\mathbf{v}}] &= \int \left(\partial_t u_{\mathbf{v}} \partial_{x_1} u_{\mathbf{v}} + \partial_t S_{\mathbf{v}} \partial_{x_1} S \, u_{\mathbf{v}}^2 \right) dx \\
&= \int \partial_t u_0(x') \partial_{x_1} u_0(x') dx + k\omega_{\mathbf{v}} \int u_0^2(x') dx \\
&= \int \left[\partial_{x_1'} u_0(x') \partial_t x_1' \right] \left[\partial_{x_1'} u_0(x') \partial_{x_1} x_1' \right] dx + v\gamma^2 \omega_0^2 \int u_0^2(x') dx \\
&= v\gamma^2 \int \left[\partial_{x_1'} u_0(x') \right]^2 dx + v\gamma^2 \omega_0^2 \int u_0^2(x') dx
\end{aligned}
$$

Making a change of variable $x_1 = 1/\gamma x_1' + vt$, we get

$$
\begin{aligned}
\mathbf{P}_1[\sigma_{\mathbf{v}}] &= v\gamma^2 \int \left[\partial_{x_1'} u_0(x') \right]^2 \frac{1}{\gamma} dx' + v\gamma \omega_0^2 \int u_0^2(x') \frac{1}{\gamma} dx' \\
&= v\gamma \left[\int [\partial_{x_1} u_0(x)]^2 dx + \omega_0^2 \int u_0^2(x) dx \right]
\end{aligned}
$$

Since u_0 is radially symmetric,

$$\int \partial_{x_1} u_0^2 dx = \frac{1}{3} \int |\nabla u_0|^2 dx$$

then

$$\mathbf{P}_1[\sigma_{\mathbf{v}}] = v\gamma \left[\frac{1}{3} \int |\nabla u_0|^2 dx + \omega_0^2 \int u_0^2(x) dx \right]$$

It is immediate to see that $\mathbf{P}_2[\sigma_{\mathbf{v}}] = \mathbf{P}_3[\sigma_{\mathbf{v}}] = 0$, and hence we obtain the conclusion. $\square$

Therefore, we have obtained the following result.

Corollary 2. *The **mass** of a q-soliton is well defined and it takes the following value:*

$$m[\sigma_{\mathbf{v}}] = \gamma m[\sigma_0] = \gamma \left[\frac{1}{3} \int |\nabla u_0|^2 dx + \omega_0^2 \int u_0^2 dx \right] \tag{77}$$

From now on, m will denote the rest mass of a q-soliton.

We define the energy of a moving soliton as follows:

$$E[\sigma_{\mathbf{v}}] = E_{\mathrm{M}}[\{\sigma_{\mathbf{v}}, \varphi_{\mathbf{v}}\}]$$

The next proposition describes how the energy transforms in a moving soliton:

Theorem 5. *The energy of a q-soliton is given by*

$$E[\sigma_{\mathbf{v}}] := \gamma m + \frac{1}{\gamma} \left(\frac{5}{3} \omega_0 \beta \int \varphi_0 u_0 dx \right) = \gamma m - \frac{1}{\gamma} \left(\frac{5}{3} \int \varphi_0 \rho_0 dx \right)$$

where $\rho_0(x) = \omega_0 \beta u_0(x)$ (see (34)).

Remark 5. *In a theory with $\beta = 0$, the energy of a soliton coincides with its mass γm, and hence it transforms as the time-component of a time-like vector. If $\beta \neq 0$, part of the energy transforms differently. This fact is not so surprising since the energy of a q-soliton includes the energy of the self-interaction of the soliton with the e.m. field generated by itself. The energy-momentum of the e.m. field does not transform as the energy of a space-like vector since it is a light-like vector. Hence, there is a small term of the order β, which transforms differently. Since this term is related to the interaction of the matter field with the e.m. field it might be related to a sort of not quantistic counterpart of the fine-structure constant; however, this point needs further investigation.*

In order to prove Theorem 5, we need the following lemma, which is a variant of the Pohozaev–Derrik theorem [21,22]:

Lemma 1. *If u is any solution of Equations (46) and (47), then*

$$\int W(u)dx = \frac{1}{2}\omega^2 \int u^2 dx - \frac{1}{6} \int |\nabla u|^2 dx - \frac{5}{3}\omega^2 \beta^2 \int (G * u)u dx$$

Proof. Let

$$J[u] = \int \left[\frac{1}{2}|\nabla u|^2 + W(u)dx - \frac{1}{2}\omega^2 u^2 dx + \omega^2 \beta^2 (G * u)u \right] dx \tag{78}$$

be the functional J defined by (56). Then, if u is a solution of Equations (46) and (47), we have that $dJ[u] = 0$. Now, let us consider the "curve" $\lambda \longmapsto u_\lambda$ in $V = H^1 + \mathcal{D}^{1.2}$ defined by

$$u_\lambda = u\left(\frac{x}{\lambda}\right)$$

Then,

$$\left(\frac{d}{d\lambda} J[u_\lambda] \right)_{\lambda=1} = 0.$$

Making the change in variable $x \longmapsto x\lambda^{-1}$, we get that

$$J[u_\lambda] = \frac{\lambda}{2} \int |\nabla u|^2 dx + \lambda^3 \int W(u)dx - \frac{1}{2}\lambda^3 \int \omega^2 u^2 dx + \lambda^5 \beta^2 \int \omega^2 (G * u)u dx$$

Then,

$$
\begin{aligned}
0 &= \frac{d}{d\lambda} J[u_\lambda]_{\lambda=1} \\
&= \left[\frac{1}{2} \int |\nabla u|^2 dx + 3\lambda^3 \int W(u) dx - \frac{3}{2}\lambda^2 \int \omega^2 u^2 dx + 5\lambda^4 \beta^2 \int \omega^2 (G * u) u dx \right]_{\lambda=1} \\
&= \frac{1}{2} \int |\nabla u|^2 dx + 3 \int W(u) dx - \frac{3}{2} \int \omega^2 u^2 dx + 5\beta^2 \int \omega^2 (G * u) u dx
\end{aligned}
$$

Hence

$$
\int W(u) dx = \frac{1}{2} \int \omega^2 u^2 dx - \frac{1}{6} \int |\nabla u|^2 dx - \frac{5}{3}\omega^2 \beta^2 \int (G * u) u dx.
$$

□

Corollary 3. *Given a stationary q-soliton σ_0, we have that*

$$
E[\sigma_0] = \frac{1}{3} \int |\nabla u_0|^2 dx + \omega_0^2 \int u_0^2 dx - \frac{5}{3}\beta^2 \int (G * u_0) u_0 dx
$$

Proof. Replacing W in (66), and using Lemma 1, we get

$$
\begin{aligned}
E[\sigma_0] &= \frac{1}{2} \int \left[|\nabla u_0|^2 + \omega_0^2 u_0^2 \right] dx + \frac{1}{2} \int W(u_0) dx \\
&= \frac{1}{2} \int \left[|\nabla u_0|^2 + \omega_0^2 u_0^2 \right] dx + \frac{1}{2} \int \omega^2 u^2 dx \\
&\quad - \frac{1}{6} \int |\nabla u|^2 dx - \frac{5}{3}\omega^2 \beta^2 \int (G * u_\lambda) u_\lambda dx \\
&= \left(\frac{1}{2} - \frac{1}{6} \right) \int |\nabla u_0|^2 dx + \left(\frac{1}{2} + \frac{1}{2} \right) \omega_0^2 \int u_0^2 dx - \frac{5}{3}\beta^2 \int (G * u_0) u_0 dx
\end{aligned}
$$

□

Proof of Theorem 5. By Proposition 1 and (68)–(73) we have that

$$
\begin{aligned}
E[\sigma_{\mathbf{v}}] &= \frac{1}{2} \int |\partial_t u_0(x')|^2 dx + \frac{1}{2} \int |\nabla u_0(x')|^2 dx \\
&\quad + \frac{1}{2} \left(k^2 + \omega_{\mathbf{v}}^2 \right) \int u_0(x')^2 dx + \int W(u_0(x')) dx
\end{aligned}
$$

making the change of the integration variable $x_1 = 1/\gamma x_1' + vt$, we get

$$
\begin{aligned}
E[\sigma_{\mathbf{v}}] &= \frac{1}{2\gamma} \int |\partial_t u_0(x')|^2 dx' + \frac{1}{2\gamma} \int |\nabla u_0(x')|^2 dx' \\
&\quad + \frac{1}{2\gamma} \left(k^2 + \omega_{\mathbf{v}}^2 \right) \int u_0(x')^2 dx' + \frac{1}{\gamma} \int W(u_0(x')) dx'
\end{aligned}
$$

Let us compute each piece individually

$$
\begin{aligned}
A &= \frac{1}{2\gamma} \int |\partial_t u_0(x')|^2 dx' = \frac{1}{2} \int \left| \partial_{x_1'} u_0(x') \partial_t x_1' \right|^2 dx' \\
&= \frac{1}{2} \int \left| \partial_{x_1'} u_0(x') \gamma^2 v^2 \right|^2 dx' = \frac{v^2 \gamma}{2} \int \left| \partial_{x_1'} u_0(x') \right|^2 dx' \\
&= \frac{v^2 \gamma}{2} \int |\partial_{x_1} u_0(x)|^2 dx
\end{aligned}
$$

Since u is radially symmetric,

$$\int |\partial_{x_i} u_0(x)|^2 dx = \frac{1}{3} \int |\nabla u_0|^2 dx \tag{79}$$

Then,

$$A = \frac{v^2 \gamma}{6} \int |\nabla_{x'} u_0|^2 dx$$

Let us compute the second piece using (79) again

$$
\begin{aligned}
B &= \frac{1}{2\gamma} \int |\nabla u_0(x')|^2 dx' \\
&= \frac{1}{2\gamma} \int \left[\left| \partial_{x'_1} u_0(x') \partial_{x_1} x'_1 \right|^2 \right] dx' + \frac{1}{2\gamma} \int \left[\left| \partial_{x'_2} u_0(x') \right|^2 + \left| \partial_{x'_3} u_0(x') \right|^2 \right] dx' \\
&= \frac{1}{2\gamma} \int \left| \partial_{x'_1} u_0(x') \gamma \right|^2 dx' + \frac{1}{2\gamma} \int \left[\left| \partial_{x'_2} u_0(x') \right|^2 + \left| \partial_{x'_3} u_0(x') \right|^2 \right] dx' \\
&= \frac{\gamma}{6} \int |\partial_{x_1} u_0(x)|^2 dx + \frac{1}{2\gamma} \int \left[|\partial_{x_2} u_0(x)|^2 + |\partial_{x_3} u_0(x)|^2 \right] dx \\
&= \frac{\gamma}{6} \int |\nabla u_0|^2 dx + \frac{1}{3\gamma} \int |\nabla u_0|^2 dx = \left(\frac{\gamma}{6} + \frac{1}{3\gamma} \right) \int |\nabla u_0|^2 dx
\end{aligned}
$$

In order to compute the third piece, we need (70) and (71):

$$
\begin{aligned}
C &= \frac{1}{2\gamma} \left(k^2 + \omega_{\mathbf{v}}^2 \right) \int u_0(x)^2 dx = \frac{1}{2\gamma} \left[(\gamma \omega_0 v)^2 + (\gamma \omega_0)^2 \right] \int u_0(x)^2 dx \\
&= \frac{1}{2} \omega_0^2 \gamma \left(v^2 + 1 \right) \int u_0(x)^2 dx
\end{aligned}
$$

The computation of the fourth piece uses Lemma 1:

$$
\begin{aligned}
\frac{1}{\gamma} \int W(u_0(x')) dx' &= \frac{1}{\gamma} \int W(u_0(x)) dx \\
&= -\frac{1}{6\gamma} \int |\nabla u_0|^2 dx + \frac{1}{2\gamma} \omega_0^2 \int u_0^2 dx - \frac{5}{3} \omega_0^2 \beta^2 \int (G * u_0) u_0 dx \\
&= E + F + G
\end{aligned}
$$

Then,

$$E[\sigma_{\mathbf{v}}] = A + B + C + E + F = (A + B + E) + (C + F) + G$$

We have that

$$\gamma^2 v^2 + \gamma^2 + 1 = \frac{v^2 + 1}{1 - v^2} + 1 = \frac{2}{1 - v^2} = 2\gamma^2$$

then,

$$
\begin{aligned}
A + B + E &= \left(\frac{v^2 \gamma}{6} + \frac{\gamma}{6} + \frac{1}{3\gamma} - \frac{1}{6\gamma} \right) \int |\nabla u_0|^2 dx \\
&= \frac{1}{6\gamma} \left(v^2 \gamma^2 + \gamma^2 + 1 \right) \int |\nabla u_0|^2 dx \\
&= \frac{\gamma}{3} \int |\nabla u_0|^2 dx
\end{aligned}
$$

and

$$
\begin{aligned}
C + F &= \left[\frac{1}{2}\omega_0^2\gamma\left(v^2+1\right) + \frac{1}{2\gamma}\omega_0^2 \right] \int u_0(x)^2 dx \\
&= \frac{1}{2}\frac{\omega_0^2}{\gamma}\left[\gamma^2\left(v^2+1\right) + 1 \right] \int u_0(x)^2 dx \\
&= \frac{1}{2}\frac{\omega_0^2}{\gamma}2\gamma^2 \int u_0(x)^2 dx = \gamma\omega_0^2 \int u_0(x)^2 dx
\end{aligned}
$$

Concluding, using Corollary 3, we have that

$$
E[\sigma_\mathbf{v}] = \frac{\gamma}{3}\int |\nabla u_0|^2 dx + \gamma\omega_0^2 \int u_0(x)^2 dx - \frac{5}{3\gamma}\omega_0^2\beta^2 \int (G * u_0)u_0 dx
$$

□

Placing a stationary q-soliton in an generic electromagnetic field with gauge potential $(\varphi, \mathbf{A})$ and using the notation (60), we get the following configuration,

$$
\sigma_0 + \left[\begin{array}{c} (0, 0, \varphi, \mathbf{A}) \\ (0, 0, \partial_t\varphi, \partial_t\mathbf{A}) \end{array} \right] = \left[\begin{array}{c} (u_0, 0, \varphi, \mathbf{A}) \\ (0, -\omega_0, \partial_t\varphi, \partial_t\mathbf{A}) \end{array} \right];
$$

By Proposition 1, the energy of this configuration is

$$
\begin{aligned}
E[\{\sigma_0, \varphi, \mathbf{A}, \partial_t\mathbf{A}\}] &= E_\mathrm{M}[\{\sigma_0, \varphi\}] + E_\mathrm{I}[\{\sigma_0, \varphi, \mathbf{A}, \partial_t\mathbf{A}\}] + E_\mathrm{F}[\{\varphi, \mathbf{A}, \partial_t\mathbf{A}\}] \\
&= E[\sigma_0] + \omega_0\beta \int \varphi\, u_0 dx + \frac{1}{2}\int\left(\mathbf{E}^2 + \mathbf{H}^2\right) dx
\end{aligned}
$$

If the soliton is small with respect to $\nabla\varphi$, (namely, if $r_0 = k\ell$ is small), then, by (75),

$$
\omega_0 \int \varphi(x)u_0(x)\, dx \cong \omega_0\varphi(0)\beta \int u_0(x)\, dx = q\varphi(0)
$$

where "$\cong$" means that the accuracy of this approximation is good if the quantities involved are large with respect to β (and to the radius of the soliton). In fact, the field $\varphi_0(x)$ produced by the q-soliton, is of the order of $\beta \ll 1$, and hence, if $\varphi \approx 1$, we have that $\varphi - \varphi_0 \cong \varphi$ and $\nabla(\varphi - \varphi_0) \cong \nabla\varphi$. Then

$$
E[\{\sigma_0, \varphi, \mathbf{A}\}] \cong E[\sigma_0] + q\varphi(0) + \frac{1}{2}\int\left(\mathbf{E}^2 + \mathbf{H}^2\right) dx
$$

Therefore, thanks to Proposition 1 and our analysis if a soliton is placed in an e.m. field, we can distinguish the **soliton energy** $E[\sigma_0]$, the **potential energy** $q\varphi(0)$ and the **e.m. field energy** $\frac{1}{2}\int(\mathbf{E}^2 + \mathbf{H}^2)dx$. This distinction is crucial for the study of the dynamics of the soliton (see Section 3.5). Finally, we remark that, the potential energy $q\varphi(0)$ is localized within the radius of the soliton. This fact eliminates one of the difficulties posed by the dualism particle-field where the localization of the potential energy of a particle is a meaningless problem.

If the q-soliton is moving, extending the above arguments, we have the following result:

Proposition 4. *If the q-soliton is small with respect to $\nabla\varphi$ and $\nabla\mathbf{A}$ and $\beta \ll 1$, then*

$$
\begin{aligned}
E[\{\sigma_\mathbf{v}, \varphi, \mathbf{A}\}] &= E_\mathrm{M}[\{\sigma_\mathbf{v}, \varphi_\mathbf{v}\}] + E_\mathrm{F}[\{\sigma_\mathbf{v}, \varphi_\mathbf{v}, \mathbf{A}\}] \\
&\cong \gamma m + q[\varphi(0) + \mathbf{v}\cdot\mathbf{A}(0)]
\end{aligned} \tag{80}
$$

Proof. By Theorem 5, $E_\mathrm{M}[\{\sigma_\mathbf{v}, \varphi_\mathbf{v}\}] = \gamma E[\sigma_0] \cong \gamma m$. Then, by Proposition 1 and (70),

we get

$$
\begin{aligned}
E[\{\sigma_{\mathbf{v}}, \varphi, \mathbf{A}\}] &= \gamma m - \int (\varphi \partial_t S_{\mathbf{v}} + \mathbf{A} \cdot \nabla S_{\mathbf{v}}) u_{\mathbf{v}} dx \\
&= \gamma m + \omega_0 \int \varphi(x) u_{\mathbf{v}}(x) \gamma dx + \omega_0 \mathbf{v} \cdot \int \mathbf{A}(x) u_{\mathbf{v}}(x) \gamma dx
\end{aligned}
$$

and, using (67), (68), (72), (73) and causing a change in variables, we have that

$$
\begin{aligned}
E[\{\sigma_{\mathbf{v}}, \varphi, \mathbf{A}\}] &= \gamma m + \omega_0 \int \varphi(x) u_0(x') dx' + \omega_0 \mathbf{v} \cdot \int \mathbf{A}(x) u_0(x') dx' \\
&= \gamma m + \omega_0 \int \varphi(L^{-1}x) u_0(x) dx + \omega_0 \mathbf{v} \cdot \int \mathbf{A}(L^{-1}x) u_0(x) dx
\end{aligned}
$$

where L denotes the Lorentz boost defined by (67), namely $Lx = x'$. If the soliton is small with respect to $\nabla \varphi$ and $\nabla \mathbf{A}$, then, using the definition (75) of q,

$$
\omega_0 \int \varphi(L^{-1}x) u_0(x) dx \cong \varphi(L^{-1}0) \omega_0 \int u_0(x) dx = q\varphi(0)
$$

and similarly

$$
\omega_0 \mathbf{v} \cdot \int \mathbf{A}(L^{-1}x) u_0(x) dx \cong \mathbf{v} \cdot \mathbf{A}(L^{-1}0) \omega_0 \int u_0(x) dx = q\varphi \mathbf{v} \cdot \mathbf{A}(0)
$$

$\square$

Notice that (80) is the energy of the soliton, namely the matter field energy plus the interaction energy contained in the radius of the soliton; the total energy of a configuration which contains a soliton also depends on $\partial_t \mathbf{A}$ and, by Proposition 1 and Proposition 4, it takes the following form:

$$
E_{tot}[\{\sigma_{\mathbf{v}}, \varphi, \mathbf{A}, \partial_t \mathbf{A}\}] \cong \gamma m + q[\varphi(0) + \mathbf{v} \cdot \mathbf{A}(0)] + \frac{1}{2} \int \left(\mathbf{E}^2 + \mathbf{H}^2 \right) dx.
$$

Now, let us examine a configuration containing several solitons

$$
\sigma_{\mathbf{v}_k, \bar{x}_k} := \sigma_{\mathbf{v}}(\cdot - \bar{x}_k), \quad k = 1, .., N
$$

where $\sigma_{\mathbf{v}_k}(\cdot - \bar{x}_k)$ was defined by Definition 1. We assume that

$$
|\bar{x}_k - \bar{x}_h| \geq 2r_0, \quad k \neq h \tag{81}
$$

where r_0 denote the radius of the solitons. We remember that u decays exponentially, so the matter field is essentially null out of a neighborhood of each soliton, and hence

$$
E\left[\sum_{k=1}^{N} \sigma_{\mathbf{v}_k, \bar{x}_k} \right] \cong \sum_{k=1}^{N} E\left[\sigma_{\mathbf{v}_k, \bar{x}_k} \right] \cong m \sum_{k=1}^{N} \gamma_k \tag{82}
$$

where

$$
\gamma_k = \frac{1}{\sqrt{1 - |\mathbf{v}_k|^2}}
$$

Notice that, in the configuration (82), the q-antisolitons can be included. They have the same mass of solitons, but opposite electric charges.

If we embed this configuration in an external e.m. field, the total energy takes the following form

$$E_{tot}\left[\left\{\sum_{k=1}^{N}\sigma_{\mathbf{v}_k,\bar{x}_k},\varphi,\mathbf{A},\partial_t\mathbf{A}\right\}\right]$$

$$\cong\ m\sum_{k=1}^{N}\gamma_k+q\sum_{k=1}^{N}[\varphi(\bar{x}_k)+\mathbf{v}_k\cdot\mathbf{A}(\bar{x}_k)]+\frac{1}{2}\int\left(\mathbf{E}^2+\mathbf{H}^2\right)dx.$$

3.5. Dynamics of q-Solitons

Now, let us examine the dynamics of a solitons in the presence of an "external" electromagnetic field. More exactly, we want to examine the behavior of the solution of the Cauchy problem with the following initial conditions

$$U_0=\sum_{k=1}^{N}\sigma_{\mathbf{v}_k,\bar{x}_k}+\left[\begin{array}{c}(0,0,\varphi_0,\mathbf{A}_0)\\(0,0,\varphi_1,\mathbf{A}_1)\end{array}\right]\tag{83}$$

where $\mathbf{v}_k\in\mathbb{R}^3$ is such that $|\mathbf{v}_k|<1$.

It is well known that, thanks to the invariance of the hylenic ratio, the soliton is orbitally stable (see, e.g., [10]). This means that if the perturbation field generated by $(\varphi_0(x),\mathbf{A}_0(x)),(\varphi_1(x),\mathbf{A}_1(x))$ is small (with respect to β^{-1}) around the soliton, then the solution of the Cauchy problem has the following form

$$U_0(t,x)=\sum_{k=1}^{N}\sigma_{\mathbf{v}_k(t),\bar{x}_k(t)}+\left[\begin{array}{c}(u_p(t,x),\ S_p(t,x),\varphi(t,x),\mathbf{A}(t,x))\\(\partial_t u_p(t,x),\partial_t S_p(t,x),\partial_t\varphi(t,x),\partial_t\mathbf{A}(t,x))\end{array}\right]\tag{84}$$

where

- $u_p(t,x)$, $S_p(t,x)$ are essentially null thanks to the orbital stability of the soliton and they will be neglected;
- $\sum_{k=1}^{N}\sigma_{\mathbf{v}_k(t),\bar{x}_k(t)}$ is the configuration of the q-solitons and its structure is determined by a N function $\xi_k:\mathbb{R}\to\mathbb{R}^3$ such that $\xi_k(t)=\bar{x}_k(t);\ \dot{\xi}_k(t)=\mathbf{v}_k(t);$

  Our aim is to investigate the dynamics of the q-solitons under the following assumptions:

- *(A-1)* $\beta\ll1$—as we have seen, this condition implies that the Cauchy problem is well posed and that the energy of a q-solitons equals its mass (Theorem 5);
- *(A-2)* The solitons are far from each other (i.e., (81) holds) during the time interval considered; this happens if

  – (i) This assumption is satisfied by the initial condition (83);
  – (ii) All the q-solitons have the same charge (namely, there are not q-antisolitons), so that, during the evolution, the q-solitons repel each other;
  – (iii) The e.m. field is not locally too strong, so that the q-solitons cannot collide;

- *(A-3)* $\left|\ddot{\xi}_k(t)\right|\ll1$; this fact avoids the q-soliton to produce a strong radiation and, from the technical point of view, it simplifies the computations. Clearly, this happens if the e.m. field is not too strong.

We will show, that, under these assumptions, the q-solitons behave as classical particles. To this aim, we analyze the action functional relative to the configuration (84)

$$\mathbb{A}\ =\ \iint(\mathcal{L}_{\mathrm{M}}+\mathcal{L}_{\mathrm{I}}+\mathcal{L}_{\mathrm{F}})dxdt\tag{85}$$

$$=\ \iint\mathcal{L}_{\mathrm{M}}\left[\sum_{k=1}^{N}\sigma_{\dot{\xi}_k(t),\xi_k(t)}\right]dxdt$$

$$+\iint\mathcal{L}_{\mathrm{I}}\left[\left\{\sum_{k=1}^{N}\sigma_{\dot{\xi}_k(t),\xi_k(t)},\varphi,\mathbf{A}\right\}\right]dxdt+\iint\mathcal{L}_{\mathrm{F}}[\{\varphi,\mathbf{A}\}]dxdt$$

Since we have assumed *(A-2)*, then

$$\int \mathcal{L}_{\mathrm{M}} dx \cong \sum_{k=1}^{N} \int \mathcal{L}_{\mathrm{I}} \left[\sigma_{\dot{\xi}_k(t),\xi_k(t)}; \varphi, \mathbf{A} \right] dx$$

and

$$\int \mathcal{L}_{\mathrm{I}} dx \cong \sum_{k=1}^{N} \int \mathcal{L}_{\mathrm{I}} \left[\left\{ \sigma_{\dot{\xi}_k(t),\xi_k(t)}, \varphi, \mathbf{A} \right\} \right] dx$$

Let us compute each piece of the action separately

Lemma 2. *Under the assumptions (A-1),(A-2),(A-3), we have that*

$$\int \mathcal{L}_{\mathrm{M}} \left[\sigma_{\dot{\xi}_k(t),\xi_k(t)} \right] dx \cong -m \sqrt{1 - \left| \dot{\xi}_k(t) \right|^2}.$$

Proof. Since the Lagrangian $\mathcal{L}_{\mathrm{M}}$ does not depend explicitly on t and x,, we can choose a reference frame where, for a fixed t,

$$\xi_k(t) = 0 \quad and \quad \dot{\xi}_k(t) = (v_k, 0, 0)$$

so that

$$\int \mathcal{L}_{\mathrm{M}} \left[\sigma_{\dot{\xi}_k(t),\xi_k(t)} \right] dx = \int \mathcal{L}_{\mathrm{M}} \left[\sigma_{v_k}(x) \right] dx$$

We recall that by (68) and (69),

$$\sigma_{v_k}(x) = \left[\begin{array}{c} (u_0(x'), -\gamma_k\omega_0, 0, 0) \\ (0, \gamma_k\omega_0 v_k, 0, 0) \end{array} \right]$$

where we have set

$$\gamma_k = \frac{1}{\sqrt{1 - \left| \dot{\xi}_k(t) \right|^2}}.$$

Then, by (7), and (68)–(71)

$$\begin{aligned}
\int \mathcal{L}_{\mathrm{M}} \left[\sigma_{\dot{\xi}_k(t),\xi_k(t)} \right] dx &= \int \mathcal{L}_{\mathrm{M}} \left[\sigma_{v_k} \right] dx \\
&= \frac{1}{2} \int |\partial_t u_0(x')|^2 dx - \int |\nabla u_0(x')|^2 dx \\
&\quad + \frac{1}{2} \left(k_k^2 - \omega_{\mathbf{v}_k}^2 \right) \int u_0(x')^2 dx - \int W(u_0(x')) dx dt
\end{aligned}$$

If we assume that $\ddot{\xi}_k$ is not too large (i.e., *(A-2)*),

$$\partial_t x_1' = \partial_t \frac{x' - \dot{\xi}_k(t) t}{\sqrt{1 - \left| \dot{\xi}_k(t) \right|^2}} \cong v_k \gamma_k;$$

then, arguing as in the proof of Theorem 5 and using similar notations for each k,

$$\int \mathcal{L}_{\mathrm{M}} \left[\sigma_{\dot{\xi}_k(t),\xi_k(t)} \right] dx = A_k - B_k + C_k^a - C_k^b - E_k - F_k - G_k$$

where

$$C_k^a = \frac{1}{2} \omega_0^2 \gamma_k v_k^2 \int u_0(x)^2 dx; \quad C_k^b = \frac{1}{2} \omega_0^2 \gamma_k \int u_0(x)^2 dx.$$

Continuing with our computation,

$$
\begin{aligned}
A_k - B_k - E_k &= \left[\frac{\gamma_k v_k^2}{6} - \frac{\gamma_k}{6} - \frac{1}{3\gamma_k} + \frac{1}{6\gamma_k} \right] \int |\nabla u_0|^2 dx \\
&= \frac{1}{6\gamma_k} \left[v_k^2 \gamma_k^2 - \gamma_k^2 - 1 \right] \int |\nabla u_0|^2 dx \\
&= \frac{1}{6\gamma_k} \left[\frac{v_k^2 - 1}{1 - v_k^2} - 1 \right] \int |\nabla u_0|^2 dx = -\frac{1}{3\gamma_k} \int |\nabla u_0|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
C_k^a - C_k^b - F_k &= \left[\frac{1}{2} \omega_0^2 \gamma_k \left(v_k^2 - 1 \right) - \frac{1}{2\gamma_k} \omega_0^2 \right] \int u_0^2 dx \\
&= \left[-\frac{1}{2\gamma_k} \omega_0^2 - \frac{1}{2\gamma_k} \right] \int u_0(x)^2 dx \\
&= -\frac{\omega_0^2}{\gamma_k} \int u_0(x)^2 dx
\end{aligned}
$$

The term G_k will be ignored, since we have assumed $\beta \ll 1$ (i.e., *(A-2)*). Then, by ([77](#)),

$$
\begin{aligned}
\int \mathcal{L}_{\mathrm{M}} \left[\sigma_{\dot{\xi}_k(t), \xi_k(t)} \right] dx &= -\frac{1}{3\gamma_k} \int |\nabla u_0|^2 dx - \frac{\omega_0^2}{\gamma_k} \int u_0(x)^2 dx \\
&= -\frac{1}{\gamma_k} m = -m \sqrt{1 - |\dot{\xi}_k(t)|^2}
\end{aligned}
$$

$\square$

Now let us compute $\int \mathcal{L}_I dx$.

Lemma 3. *If (A-1) and (A-2) hold, then*

$$
\int \mathcal{L}_I \left[\sigma_{\dot{\xi}_k(t), \xi_k(t)}; \varphi, \mathbf{A} \right] dx \cong q \left[\varphi(t, \xi_k(t)) - \mathbf{A}(t, \xi_k(t)) \cdot \dot{\xi}_k(t) \right] dt.
$$

Proof. As in the previous lemma, we can choose a reference frame where, for a fixed t, $\dot{\xi}_k(t) = \mathbf{v}$ and $\xi_k(t) = 0$. Then, following the same arguments as used in the proof of Proposition [4](#), we have that

$$
\int \mathcal{L}_I [\sigma_{\mathbf{v}, \bar{x}}; \varphi, \mathbf{A}] dx \cong q(\mathbf{v} \cdot \mathbf{A}(\bar{x}) - \varphi(\bar{x})).
$$

$\square$

The above lemmas give the following result

Theorem 6. *Let $U_0(t, x)$ be the solution of the Cauchy problem relative to Equations ([11](#))–([14](#)) with the initial condition ([83](#)). Then If (A-1), (A-2) and (A-3) hold, we have that*

$$
\frac{d}{dt} \left(\frac{m \dot{\xi}_k}{\sqrt{1 - |\dot{\xi}_k|^2}} \right) \cong q \left(\mathbf{E} + \dot{\xi}_k \times \mathbf{H} \right) \tag{86}
$$

$$
\nabla \cdot \mathbf{E} = \sum_{k=1}^{N} \rho_0(x - \xi_k)
$$

$$
\nabla \times \mathbf{H} - \partial_t \mathbf{E} = \sum_{k=1}^{N} \mathbf{j}_0(x - \xi_k)
$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{H} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Proof. The action (85) becomes

$$
\begin{aligned}
\mathbb{A} &= \mathbb{A}_{\mathrm{M}} + \mathbb{A}_{\mathrm{I}} + \mathbb{A}_{\mathrm{F}} \\
&= \sum_{k=1}^{N} -m \int \left[\sqrt{1 - |\dot{\xi}_k(t)|^2} + q_k \left[\mathbf{A}(t, \xi_k(t)) \cdot \dot{\xi}_k(t) - \varphi(t, \xi_k(t)) \right] \right] dt \\
&\quad + \iint \mathcal{L}_{\mathrm{F}}[\varphi, \mathbf{A}] \, dx \, dt
\end{aligned}
$$

Making the variation in $\mathbb{A}_{\mathrm{M}} + \mathbb{A}_{\mathrm{I}}$ with respect to ξ_k, we get the Lorentz Equation (86); making the variation in $\mathbb{A}$ given by (25), we obtain the Maxwell equations. $\square$

Remark 6. *Theorem 6 states that Equations (11)–(14) provide a model for material particles which, at low energies, agree with the well known physics. It is interesting to investigate the predictions of this model when the assumptions (A-2), (A-3) are violated. If (A-2)-(ii) is violated, there are antisolitons which attract solitons since they have opposite charges; then (A-2)-(i) will eventually be violated and the two particles will annihilate. Since our equation is invariant for time-reversal, the creation of an particle–antiparticle couple might occur; of course, this can happen only if there is sufficient energy, namely if (A-2),(iii) does not hold. If (A-3) does not hold, a numerical computation of the radiation when $\ddot{\xi}$ is large gives a spectrum which can probably be compared with the experimental data.*

4. Conclusive Remarks

More than 50 years ago, De Broglie wrote:

Des considèrations sur lesquelles je reviendrai me conduisent aujourd'hui a penser que le corpuscule doit etre assimilé non pas à un véritable singularité punctuelle de u, mais à un trés petite région singuliére de l'espace où u prenderait une très grande valeur et obéirait à une équation non linéaire dont l'équation linéaire de la Mécanique ondulatoire ne serait qu'une forme approximative valable en dehors de la region singulière. L'idée que l'équation de propagation de u, contrairement è l'équation classique du Ψ, est en principe non linéaire m'apparait meme maintenant comme tout à fait essentielle. ([23], Chap. IX, 1, p. 95.)

The development of the nonlinear analysis of the last half century allows the construction of models of particles in line with the ideas of De Broglie. The model presented here is strongly based on a Classical Field Theory and "*a priori*" has nothing to do with Quantum Mechanics (QM), in contrast with the ideas of De Broglie. Nevertheless, it is interesting to notice that it presents some features which are considered peculiar to QM.

The first thing to note is the fact that particle-like solutions of nonlinear equations with positive energy, in dimension 3, seem possible only if they have at least one internal degree of freedom, namely ψ takes values in $\mathbb{C}$ and not in $\mathbb{R}$ (see, e.g., Derrik theorem [21]). This fact implies that

$$\psi(t, x) = u(t, x) e^{i(\mathbf{k} \cdot x - \omega t)}$$

presents an undulatory aspect, as was desired by De Broglie. Furthermore, since the energy/momentum $(E, \mathbf{p})$ of the particle and the wave number $(\omega, \mathbf{k})$ are 4-vectors, they must be proportional and hence

$$E = \hbar \omega \quad and \quad \mathbf{p} = \hbar \mathbf{k}$$

where $\hbar$ is a constant depending on the parameters of the problem. Therefore, we can say that Equations (11)–(14) present one kind of intrinsic Plank constant. However, this

similarity does not imply the De Broglie pilot wave theory or the Bohmian mechanics, since the "interference" or the "entanglement" phenomena cannot be reproduced by this model.

The second remarkable fact is the existence of anti-particles, and the fact that an antiparticle is produced by time-reversion (or by charge inversion).

Another peculiarity is that the q-solitons are equal to each other and two of them cannot be in the same position. This fact implies that they are forced to follow the Bose–Einstein statistics, which is also considered a quantum phenomenon.

However, we do not think that the q-solitons could be considered a model for elementary particles. They are just an example (and probably the simplest one) showing the possibility of a classical theory of electrodynamics and the fact that some quantum phenomena are consequences of a consistent field theory, independently of the quantization.

Nevertheless, it is possible to implement the ideas presented here to build a "classical" model of elementary particles. It is necessary to take ψ-functions with spinor values and a Lagrangian with a suitable symmetry. For example, in [24], a $U(1) \times SU(2)$ symmetry is considered.

The final conclusion is the following: if the Maxwell equations are weakly coupled in the simplest way with a linear equation, invariant for the Poincaré group, then a small nonlinear perturbation (see Theorem 3) is sufficient to produce not only a consistent electrodynamics theory, but also solitons which share some characteristics with quantum particles.

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