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New Aspects for Oscillation of Differential Systems with Mixed Delays and Impulses

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Abstract: Oscillation and symmetry play an important role in many applications such as engineering, physics, medicine, and vibration in flight. In this work, we obtain sufficient and necessary conditions for the oscillation of the solutions to a second-order differential equation with impulses and mixed delays when the neutral coefficient lies within $[0,1)$. Furthermore, an examination of the validity of the proposed criteria has been demonstrated via particular examples.

Keywords: lebesgue's dominated convergence theorem; neutral; oscillation; nonoscillation; nonlinear

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1. Introduction

As is well known, impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. They appear in the study of many real world problems (see, for instance, [1–3]). We also stress that the modeling of several phenomena is suitably formulated by evolutive partial differential equations and, moreover, moment problem approaches appear as a natural instrument in control theory of neutral type systems; see [4–7], respectively.

Next, we list some recent improvements of oscillation theory for impulsive differential systems.

In [8], the authors considered the impulsive system

$$\begin{cases} (z(\eta)-b(\eta)z(\eta-\vartheta))'+c(\eta)z(\eta-\varsigma_1)-v(\eta)z(\eta-\varsigma_2)=0, & \varsigma_1\geq\varsigma_2>0 \\ z(\varphi_i^+)=I_i(z(\varphi_i)), & i\in\mathbb{N} \end{cases} \tag{1}$$

and obtained several sufficient conditions that ensure the oscillation of the solutions of (1) when $b(\eta)\in PC([\eta_0,\infty),\mathbb{R}_+)$ and $b_i\leq\frac{I_i(u)}{u}\leq 1$. In [9], the authors considered the problem

$$\begin{cases} (z(\eta)-b(\eta)z(\eta-\vartheta))'+c(\eta)|z(\eta-\varsigma)|^\lambda \operatorname{sgn} z(\eta-\varsigma)=0, & \eta\geq\eta_0 \\ z(\varphi_i^+)=b_i z(\varphi_i), & i\in\mathbb{N} \end{cases} \tag{2}$$

assuming that $b(\eta)\in PC([\eta_0,\infty),\mathbb{R}_+)$ (that is, $b(\eta)$ is piece-wise continuous in $[\eta_0,\infty)$) established sufficient conditions for the oscillation of (2). In [10], Shen and Wang considered the impulsive system

$$\begin{cases} z'(\eta)+c(\eta)z(\eta-\varsigma)=0, & \eta\neq\varphi_i, & \eta\geq\eta_0 \\ z(\varphi_i^+)-z(\varphi_i^-)=I_i(z(\varphi_i)), & i\in\mathbb{N} \end{cases} \tag{3}$$

where $r, I_i \in C(\mathbb{R}, \mathbb{R})$ for $i \in \mathbb{N}$ and obtained sufficient conditions for the oscillation of (3).

Oscillatory and non-oscillatory behaviors of a second-order impulsive differential system of neutral type with constant delays and constant coefficients were studied by Tripathy and Santra in [11], where the authors considered the problem

$$\begin{cases} (z(\eta) - bz(\eta - \vartheta))'' + rz(\eta - \varsigma) = 0, & \eta \neq \varphi_i, \quad i \in \mathbb{N} \\ \Delta(z(\varphi_i) - bz(\varphi_i - \vartheta))' + \tilde{c}z(\varphi_i - \varsigma) = 0, & i \in \mathbb{N} \end{cases}. \tag{4}$$

Other sufficient and necessary conditions for the oscillation of second-order impulsive systems of neutral type were found in [12], where Tripathy and Santra studied systems of the form

$$\begin{cases} (a(\eta)(z(\eta) + b(\eta)z(\eta - \vartheta))')' + c(\eta)g(z(\eta - \varsigma)), & \eta \neq \varphi_i, \quad i \in \mathbb{N} \\ \Delta(a(\varphi_i)(z(\varphi_i) + b(\varphi_i)z(\varphi_i - \vartheta))') + c(\varphi_i)g(z(\varphi_i - \varsigma)) = 0, & i \in \mathbb{N} \end{cases}. \tag{5}$$

In [13], the authors found some new sufficient conditions to ensure the oscillation of the impulsive system

$$\begin{cases} (z(\eta) - b(\eta)z(\eta - \vartheta))' + c(\eta)g(z(\eta - \varsigma)) = 0, & \eta \neq \varphi_i, \quad \eta \geq \eta_0 \\ z(\varphi_i^+) = I_i(z(\varphi_i)), & i \in \mathbb{N} \\ z(\varphi_i^+ - \varsigma) = I_i(z(\varphi_i - \varsigma)), & i \in \mathbb{N} \end{cases} \tag{6}$$

for $|b(\eta)| < +\infty$.

In [14], the authors established conditions, both sufficient and necessary, for the oscillation of the following highly nonlinear impulsive differential system of neutral type

$$\begin{cases} (a(\eta)(d'(\eta))^{\mu})' + \sum_{k=1}^{m} c_k(\eta)g_k(z(\varsigma_k(\eta))) = 0, & \eta \geq \eta_0, \quad \eta \neq \varphi_i, \quad i \in \mathbb{N} \\ \Delta(a(\varphi_i)(d'(\varphi_i))^{\mu}) + \sum_{k=1}^{m} \tilde{c}_k(\varphi_i)g_k(z(\varsigma_k(\varphi_i))) = 0, \end{cases} \tag{7}$$

where

$$d(\eta) = z(\eta) + b(\eta)z(\vartheta(\eta)), \quad \Delta z(a) = \lim_{s \to a^+} z(s) - \lim_{s \to a^-} z(s), \quad -1 \leq b(\eta) \leq 0.$$

In [15], the authors obtained oscillation and non-oscillation properties for the solutions to the following class of forced nonlinear neutral impulsive differential systems

$$\begin{cases} (a(\eta)(z(\eta) + b(\eta)z(\eta - \vartheta))')' + c(\eta)g(z(\eta - \varsigma)) = f(\eta), & \eta \neq \varphi_i, \quad i \in \mathbb{N} \\ \Delta(a(\varphi_i)(z(\varphi_i) + b(\varphi_i)z(\varphi_i - \vartheta))') + \tilde{c}(\varphi_i)g(z(\varphi_i - \varsigma)) = \tilde{f}(\varphi_i), & i \in \mathbb{N} \end{cases} \tag{8}$$

for different values of $b(\eta)$ and obtained sufficient conditions for the existence of positive bounded solutions of the above system.

In their recent work [16], Tripathy and Santra studied the following second-order neutral impulsive differential system

$$\begin{cases} (a(\eta)(d'(\eta))^{\mu})' + \sum_{k=1}^{m} c_k(\eta)z^{\mu_k}(\varsigma_k(\eta)) = 0, & \eta \geq \eta_0, \quad \eta \neq \varphi_i \\ \Delta(a(\varphi_i)(d'(\varphi_i))^{\mu}) + \sum_{k=1}^{m} \tilde{c}_k(\varphi_i)z^{\mu_k}(\varsigma_k(\varphi_i)) = 0, & i \in \mathbb{N} \end{cases} \tag{9}$$

where $d(\eta) = z(\eta) + b(\eta)z(\vartheta(\eta))$ and $-1 < b(\eta) \leq 0$ and obtained some new oscillation results.

For further details on recent results related to the oscillation theory for ordinary differential equations and for neutral impulsive differential system, we refer the reader to the papers [12,17–33]. In previous studies, most authors studied the oscillation of solutions of the neutral impulsive differential system when the neutral coefficient lies in $(-1, 0]$, but only a few studied oscillation of solution of the neutral differential system when the neutral coefficient lies in $[0,1)$.

Motivated by the above works, in this study, we aim to establish sufficient and necessary conditions for the oscillation of solutions to the following second-order nonlinear impulsive differential system when the neutral coefficient lies within $[0,1)$:

$$\begin{cases} \left(a(\eta)(d'(\eta))^{\mu}\right)' + c(\eta)z^{\nu}(\varsigma(\eta)) = 0, \quad \eta \geq \eta_0, \quad \eta \neq \varphi_i,\ i \in \mathbb{N}, \\ \Delta\left(a(\varphi_i)(d'(\varphi_i))^{\mu}\right) + \tilde{c}(\varphi_i)z^{\nu}(\varsigma(\varphi_i)) = 0\,, \end{cases} \tag{10}$$

where

$$d(\eta) = z(\eta) + b(\eta)z(\vartheta(\eta)), \quad \Delta z(g) = \lim_{h\to g^+} z(h) - \lim_{h\to g^-} z(h),$$

and the functions $b, c, \tilde{c}, \varsigma, \vartheta$ are continuous and satisfy the following:

Hypothesis 1 (H1). $\varsigma \in C([0,\infty),\mathbb{R})$, $\vartheta \in C^2([0,\infty),\mathbb{R})$, $\varsigma(\eta) < \eta$, $\vartheta(\eta) < \eta$, $\lim_{\eta\to\infty} \varsigma(\eta) = \infty$, $\lim_{\eta\to\infty} \vartheta(\eta) = \infty$.

Hypothesis 2 (H2). $\varsigma \in C([0,\infty),\mathbb{R})$, $\vartheta \in C^2([0,\infty),\mathbb{R})$, $\varsigma(\eta) > \eta$, $\vartheta(\eta) < \eta$, $\lim_{\eta\to\infty} \vartheta(\eta) = \infty$.

Hypothesis 3 (H3). *$a \in C^1([0,\infty),\mathbb{R})$ with $a(\eta) > 0$; $c, \tilde{c} \in C([0,\infty),\mathbb{R})$ with $c(\eta), \tilde{c}(\eta) \geq 0$ for $\eta \geq 0$.*

Hypothesis 4 (H4). *$b \in C^2([0,\infty),\mathbb{R}_+)$ with $0 \leq b(\eta) \leq b < 1$.*

Hypothesis 5 (H5). *$\lim_{\eta\to\infty} A(\eta) = \infty$ where $A(\eta) = \int_0^{\eta} a^{-1/\mu}(s)\,\mathrm{d}s$.*

Hypothesis 6 (H6). *The sequence $\{\varphi_i\}$ satisfies $0 < \varphi_1 < \varphi_2 < \cdots \to \infty$ as $i \to \infty$; and μ and ν are the quotient of two odd positive integers.*

A solution $z(\eta)$ to (10) is said to be eventually positive (or eventually negative) if there exist $\eta_1 > 0$ such that $z(\eta) > 0$ (or $z(\eta) < 0$) for $\eta \geq \eta_1$.

A differential equation involving an impulse effect is called an impulsive differential equation.

2. Some Preliminaries

In this section, we are providing two important lemmas to use in main results.

Lemma 1. *Under assumptions (H1)–(H6) for $\eta \geq \eta_0$, and z being an eventually positive solution of (10), we have*

$$d(\eta) > 0, \quad 0 < d'(\eta), \quad \textit{and} \quad 0 \geq \left(a(\eta)(d'(\eta))^{\mu}\right)' \quad \textit{for every} \quad \eta \geq \eta_1\,. \tag{11}$$

Proof. Let z be an eventually positive solution. Therefore, $d(\eta) > 0$ and there exists $\eta_0 \geq 0$ such that $z(\eta) > 0$, $z(\varsigma(\eta)) > 0$, $z(\vartheta(\eta)) > 0$ for $\eta \geq \eta_0$. Then, (10) gives

$$\begin{aligned} \left(a(\eta)(d'(\eta))^{\mu}\right)' &= -c(\eta)z^{\nu}(\varsigma(\eta)) \leq 0 \quad \text{for } \eta \neq \varphi_i, \\ \Delta\left(a(\varphi_i)(d'(\varphi_i))^{\mu}\right) &= -\tilde{c}(\varphi_i)z^{\nu}(\varsigma(\varphi_i)) \leq 0 \quad \text{for } i = 1, 2, \cdots \end{aligned} \tag{12}$$

which shows that $a(\eta)\big(d'(\eta)\big)^{\mu}$ is non-increasing for $\eta \geq \eta_0$, including jumps of discontinuity. Next, we claim that for $d > 0$, $a(\eta)\big(d'(\eta)\big)^{\mu}$ is positive for $\eta \geq \eta_1 > \eta_0$. If not, let $a(\eta)\big(d'(\eta)\big)^{\mu} \leq 0$ for $\eta \geq \eta_1$; we can choose $c_1 > 0$ such that

$$a(\eta)\big(d'(\eta)\big)^{\mu} \leq -c_1\,,$$

that is,

$$d'(\eta) \leq (-c_1)^{1/\mu} a^{-1/\mu}(\eta)\,.$$

Integrating both sides from η_1 to η, we get

$$d(\eta) - d(\eta_1) - \sum_{i=1}^{\infty} d'(\varphi_i) \leq (-c_1)^{1/\mu}\big(A(\eta) - A(\eta_1)\big).$$

Taking the limit of both sides as $\eta \to \infty$, we have $\lim_{\eta\to\infty} d(\eta) \leq -\infty$, which leads to a contradiction to $d(\eta) > 0$. Hence, $a(\eta)\big(d'(\eta)\big)^{\mu} > 0$ for $\eta \geq \eta_1$, i.e., $d'(\eta) > 0$ for $\eta \geq \eta_1$. Thus, the proof is completed. □

Lemma 2. *Under assumptions (H1)–(H6) for $\eta \geq \eta_0$, and with z being an eventually positive solution of* (10)*, we have*

$$(1-b)d(\eta) \leq z(\eta) \quad for \quad \eta \geq \eta_1. \tag{13}$$

Proof. Let z be an eventually positive solution of (10). Therefore, $d(\eta) > 0$ and there exists $\eta \geq \eta_1 > \eta_0$ such that $z(\eta) = d(\eta) - b(\eta)z(\vartheta(\eta)) \geq d(\eta) - b(\eta)d(\vartheta(\eta)) \geq d(\eta) - b(\eta)d(\eta) = \big(1 - b(\eta)\big)d(\eta) \geq (1-b)d(\eta)$. Hence d satisfies (13) for $\eta \geq \eta_1$. □

Remark 1. *Lemmas 1 and 2 hold for $\mu > \nu$ and $\mu < \nu$.*

3. Oscillation Theorems

In this section, we provide main results to find the sufficient and necessary conditions for the oscillation of solutions to the impulsive system (10).

Theorem 1. *Under assumptions (H2)–(H6) for $\eta \geq \eta_0$ and $\nu > \mu$, each solution of* (10) *is oscillatory if and only if*

$$\int_0^{\infty} a^{-1/\mu}(h)\Big[\int_h^{\infty} c(g)\,\mathrm{d}g + \sum_{\varphi_i \geq h} \tilde{c}(\varphi_i)\Big]^{1/\mu} \mathrm{d}h = \infty\,. \tag{14}$$

Proof. Let z be an eventually positive solution of (10). Therefore, $d(\eta) > 0$ and there exists $\eta_0 \geq 0$ such that $z(\eta) > 0$, $z(\varsigma(\eta)) > 0$, $z(\vartheta(\eta)) > 0$ for $\eta \geq \eta_0$. Thus, Lemmas 1 and 2 hold for $\eta \geq \eta_1$. Using Lemma 1 and for $\eta_2 > \eta_1$, we have $d'(\eta) > 0$ for $\eta \geq \eta_2$. Thus, for $\eta_3 > \eta_2$ and $c > 0$, we have $d(\eta) \geq c$ where $\eta \geq \eta_3$. Again, by Lemma 2, we have $z(\eta) \geq (1-b)d(\eta)$ for $\eta \geq \eta_3$, and (10) becomes

$$\begin{aligned} \Big(a(\eta)\big(d'(\eta)\big)^{\mu}\Big)' + c(\eta)\Big((1-b)d\big(\varsigma(\eta)\big)\Big)^{\nu} \leq 0 \quad & \text{for } \eta \neq \varphi_i, \\ \Delta\Big(a(\varphi_i)\big(d'(\varphi_i)\big)^{\mu}\Big) + \tilde{c}(\varphi_i)\Big((1-b)d\big(\varsigma(\varphi_i)\big)\Big)^{\nu} \leq 0 \quad & \text{for } i = 1, 2, 3, \ldots. \end{aligned} \tag{15}$$

Integrating (15) from η to $+\infty$, we get

$$[a(s)\big(d'(h)\big)^{\mu}]_{\eta}^{\infty} + \int_{\eta}^{\infty} c(g)\Big((1-b)d\big(\varsigma(g)\big)\Big)^{\nu} \mathrm{d}g + \sum_{\varphi_i \geq \eta} \tilde{c}(\varphi_i)\Big((1-b)d\big(\varsigma(\varphi_i)\big)\Big)^{\nu} \leq 0\,.$$

Since $a(\eta)\big(d'(\eta)\big)^{\mu}$ is positive and non-decreasing. Therefore, $\lim_{\eta\to\infty} a(\eta)\big(d'(\eta)\big)^{\mu}$ finitely exists and is positive.

$$a(\eta)\big(d'(\eta)\big)^{\mu} \geq \int_{\eta}^{\infty} c(g)\Big((1-b)d\big(\varsigma(g)\big)\Big)^{\nu}\,\mathrm{d}g + \sum_{\varphi_i\geq\eta}\tilde{c}(\varphi_i)\Big((1-b)d\big(\varsigma(\varphi_i)\big)\Big)^{\nu},$$

that is,

$$\begin{aligned} d'(\eta) &\geq a^{-1/\mu}(\eta)\Big[\int_{\eta}^{\infty} c(g)\Big((1-b)d\big(\varsigma(g)\big)\Big)^{\nu}\,\mathrm{d}g + \sum_{\varphi_i\geq\eta}\tilde{c}(\varphi_i)\Big((1-b)d\big(\varsigma(\varphi_i)\big)\Big)^{\nu}\Big]^{1/\mu} \\ &= (1-b)^{\nu/\mu}a^{-1/\mu}(\eta)\Big[\int_{\eta}^{\infty} c(g)d^{\nu}\big(\varsigma(g)\big)\,\mathrm{d}g + \sum_{\varphi_i\geq\eta}\tilde{c}(\varphi_i)d^{\nu}\big(\varsigma(\varphi_i)\big)\Big]^{1/\mu}. \end{aligned} \tag{16}$$

Using (H2) and with $d(\eta)$ being non-decreasing, we have

$$d'(\eta) \geq (1-b)^{\nu/\mu}a^{-1/\mu}(\eta)\Big[\int_{\eta}^{\infty} c(g)\,\mathrm{d}g + \sum_{\varphi_i\geq\eta}\tilde{c}(\varphi_i)\Big]^{1/\mu} d^{\nu/\mu}(\eta),$$

that is,

$$\frac{d'(\eta)}{d^{\nu/\mu}(\eta)} \geq (1-b)^{\nu/\mu}a^{-1/\mu}(\eta)\Big[\int_{\eta}^{\infty} c(g)\,\mathrm{d}g + \sum_{\varphi_i\geq\eta}\tilde{c}(\varphi_i)\Big]^{1/\mu}.$$

Since $\nu > \mu$. Integrating both sides from η_3 to $+\infty$, we get

$$(1-b)^{\nu/\mu}\int_{\eta_3}^{\infty} a^{-1/\mu}(h)\Big[\int_{h}^{\infty} c(g)\,\mathrm{d}g + \sum_{\varphi_i\geq\eta}\tilde{c}(\varphi_i)\Big]^{1/\mu}\mathrm{d}h \leq \int_{\eta_3}^{\infty}\frac{d'(g)}{d^{\nu/\mu}(g)}\,\mathrm{d}g < \infty,$$

which contradicts (14). Thus, the proof of the sufficient part is completed.

Next, we are going to prove the necessary part of the theorem. For this, we assume that (14) does not hold. Hence, for every $\varepsilon > 0$ there exists $\eta \geq \eta_0$ such that

$$\int_{\eta}^{\infty} a^{-1/\mu}(h)\Big[\int_{h}^{\infty} c(g)\,\mathrm{d}g + \sum_{\varphi_i\geq h}\tilde{c}(\varphi_i)\Big]^{1/\mu}\mathrm{d}h < \varepsilon \quad \text{for} \quad \eta \geq T,$$

where $2\varepsilon = \Big[\frac{1}{1-b}\Big]^{-\nu/\mu} > 0$. We define

$$S = \Big\{z \in C([0,\infty)) : \frac{1}{2} \leq z(\eta) \leq \frac{1}{1-b}\ \text{for}\ \eta \geq T\Big\}$$

and $\varphi : S \to S$ as

$$(\varphi z)(\eta) = \begin{cases} 0 & \text{if } \eta \leq T, \\ \frac{1+b}{2(1-b)} - b(\eta)z(\vartheta(\eta)) & \\ + \int_T^{\eta} a^{-1/\mu}(h)\Big[\int_h^{\infty} c(g)z^{\nu}(\varsigma(g))\,\mathrm{d}g + \sum_{\varphi_i\geq h}\tilde{c}(\varphi_i)z^{\nu}(\varsigma(\varphi_i))\Big]^{1/\mu}\mathrm{d}h & \text{if } \eta > T. \end{cases}$$

Next, we prove that $(\varphi z)(\eta) \in S$. For $z(\eta) \in S$,

$$\begin{aligned} (\varphi z)(\eta) &\leq \frac{1+b}{2(1-b)} + \int_T^{\eta} a^{-1/\mu}(s)\Big[\int_s^{\infty} c(\psi)\Big(\frac{1}{1-b}\Big)^{\nu}\mathrm{d}\psi + \sum_{\varphi_i\geq s}\tilde{c}(\varphi_i)\Big(\frac{1}{1-b}\Big)^{\nu}\Big]^{1/\mu}\mathrm{d}s \\ &\leq \frac{1+b}{2(1-b)} + \Big(\frac{1}{1-b}\Big)^{\nu/\mu}.\varepsilon = \frac{1+b}{2(1-b)} + \frac{1}{2} = \frac{1}{1-b} \end{aligned}$$

and further, for $z(\eta) \in S$

$$(\varphi z)(\eta) \geq \frac{1+b}{2(1-b)} - b(\eta) \times \frac{1}{1-b} + 0 \geq \frac{1+b}{2(1-b)} - \frac{a}{1-b} = \frac{1}{2}.$$

Hence, φ maps from S to S.

Now, we define a sequence in S by

$$\begin{aligned} u_0(\eta) &= 0 \quad \text{for } \eta \geq \eta_0, \\ u_1(\eta) &= (\varphi z_0)(\eta) = \begin{cases} 0 & \text{if } \eta < T \\ \frac{1}{2} & \text{if } \eta \geq T \end{cases}, \\ u_{n+1}(\eta) &= (\varphi z_n)(\eta) \quad \text{for } n \geq 1, \eta \geq T. \end{aligned}$$

Here we see $u_1(\eta) \geq u_0(\eta)$ for each fixed η and $\frac{1}{2} \leq u_{n-1}(\eta) \leq u_n(\eta) \leq \frac{1}{1-b}, \quad \eta \geq T$ for $n \geq 1$. Thus, u_n converges point-wise to a function z. Hence, z is a fixed point of φ in S by using Lebesgue's Dominated Convergence Theorem, which proves that there has an eventually positive solution. Thus, the theorem is proved. □

Theorem 2. *Under assumptions (H1), (H3)–(H6) for $\eta \geq \eta_0$ and $\nu < \mu$, every solution of* (10) *oscillates if and only if*

$$\left[\int_0^\infty c(g)[(1-b)A(\varsigma(g))]^\nu \,\mathrm{d}g + \sum_{i=1}^\infty \tilde{c}(\varphi_i)[(1-b)A(\varsigma(\varphi_i))]^\nu\right] = \infty. \tag{17}$$

Proof. Let $z(\eta)$ be an eventually positive solution of (10). Then, similar to the proof of Theorem 1, we conclude that (16) holds for $\eta \geq \eta_2$, where $\eta_2 > \eta_1 > \eta_0$. Using (H5), there exists $\eta_3 > \eta_2$ for which $A(\eta) - A(\eta_3) \geq \frac{1}{2}A(\eta)$ for $\eta \geq \eta_3$. Integrating (16) from η_3 to η, we have

$$\begin{aligned} d(\eta) - d(\eta_3) \geq& \int_{\eta_3}^{\eta} a^{-1/\mu}(h)\Big[\int_h^\infty c(g)\Big((1-b)d(\varsigma(g))\Big)^\nu \mathrm{d}g \\ &+ \sum_{\varphi_i \geq h} \tilde{c}(\varphi_i)\Big((1-b)d(\varsigma(\varphi_i))\Big)^\nu\Big]^{1/\mu} \mathrm{d}h \\ \geq& \int_{\eta_3}^{\eta} a^{-1/\mu}(h)\Big[\int_\eta^\infty c(g)\Big((1-b)d(\varsigma(g))\Big)^\nu \mathrm{d}g \\ &+ \sum_{\varphi_i \geq \eta} \tilde{c}(\varphi_i)\Big((1-b)d(\varsigma(\varphi_i))\Big)^\nu\Big]^{1/\mu} \mathrm{d}h, \end{aligned}$$

that is,

$$\begin{aligned} d(\eta) \geq& (a(\eta) - a(\eta_3))\Big[\int_\eta^\infty c(g)\Big((1-b)d(\varsigma(g))\Big)^\nu \mathrm{d}g \\ &+ \sum_{\varphi_i \geq \eta} \tilde{c}(\varphi_i)\Big((1-b)d(\varsigma(\varphi_i))\Big)^\nu\Big]^{1/\mu} \end{aligned} \tag{18}$$

$$\geq \frac{1}{2}A(\eta)\Big[\int_\eta^\infty c(g)\Big((1-b)d(\varsigma(g))\Big)^\nu \mathrm{d}g + \sum_{\varphi_i \geq \eta} \tilde{c}(\varphi_i)\Big((1-b)d(\varsigma(\varphi_i))\Big)^\nu\Big]^{1/\mu}. \tag{19}$$

Hence,

$$d(\eta) \geq \frac{1}{2}A(\eta)\nabla^{1/\mu}(\eta) \quad \text{for} \quad \eta \geq \eta_3$$

where

$$\nabla(\eta) = \int_{\eta}^{\infty} c(g)\Big((1-b)d\big(\varsigma(g)\big)\Big)^{\nu} \mathrm{d}g + \sum_{\varphi_i \ge \eta} \tilde{c}(\varphi_i)\Big((1-b)d\big(\varsigma(\varphi_i)\big)\Big)^{\nu}.$$

Now,

$$\begin{aligned} \nabla'(\eta) &= -c(\eta)\Big((1-b)d\big(\varsigma(\eta)\big)\Big)^{\nu} \\ &\le -\frac{1}{2^{\nu}} c(\eta)[(1-b)A\big(\varsigma(\eta)\big)]^{\nu} \nabla^{\nu/\mu}\big(\varsigma(\eta)\big) \le 0 \end{aligned} \tag{20}$$

and

$$\Delta\Big(\nabla(\varphi_i)\Big) = -\frac{1}{2^{\nu}} c(\varphi_i)[(1-b)A\big(\varsigma(\varphi_i)\big)]^{\nu} \nabla^{\nu/\mu}\big(\varsigma(\varphi_i)\big) \le 0. \tag{21}$$

From (21), it is clear that $\nabla(\eta)$ is non-increasing in $[\eta_4, \infty)$ and $\lim_{\eta\to\infty} \nabla(\eta)$ exists. Using (20) and (H1), we find

$$\begin{aligned} \Big[\nabla^{1-\nu/\mu}(\eta)\Big]' &= (1-\nu/\mu)\nabla^{-\nu/\mu}(\eta)\nabla'(\eta) \\ &\le -\frac{1-\nu/\mu}{2^{\nu}} c(\eta)[(1-b)A\big(\varsigma(\eta)\big)]^{\nu} \nabla^{\nu/\mu}\big(\varsigma(\eta)\big)\nabla^{-\nu/\mu}(\eta) \\ &\le -\frac{1-\nu/\mu}{2^{\nu}} c(\eta)[(1-b)A\big(\varsigma(\eta)\big)]^{\nu}. \end{aligned} \tag{22}$$

To estimate the discontinuity of $\nabla^{1-\nu/\mu}$, we use a Taylor polynomial of order 1 from the function $h(u) = \nabla^{1-\nu/\mu}$, with $0 < \nu < \mu$, about $u = \tilde{a}$:

$$\tilde{b}^{1-\nu/\mu} - \tilde{a}^{1-\nu/\mu} \le (1-\nu/\mu)\tilde{a}^{-\nu/\mu}(\tilde{b} - \tilde{a}).$$

Then

$$\Delta\Big(\nabla^{1-\nu/\mu}(\varphi_i)\Big) \le (1-\nu/\mu)\nabla^{-\nu/\mu}(\varphi_i)\Delta\Big(\nabla(\varphi_i)\Big) \le -\frac{1-\nu/\mu}{2^{\nu}} c(\varphi_i)[(1-b)A\big(\varsigma(\varphi_i)\big)]^{\nu}.$$

Integrating (22) from η_3 to η, we have

$$\Big[\nabla^{1-\nu/\mu}(h)\Big]_{\eta_4}^{\eta} - \sum_{\varphi_i \ge \eta} \Delta[\nabla^{1-\nu/\mu}(\varphi_i)] \le -\frac{1-\nu/\mu}{2^{\nu}} \int_{\eta_3}^{\eta} c(h)[(1-b)A\big(\varsigma(h)\big)]^{\nu} \mathrm{d}h,$$

that is,

$$\begin{aligned} \frac{1-\nu/\mu}{2^{\nu}}\Big[\int_0^{\infty} c(h)[(1-b)A\big(\varsigma(h)\big)]^{\nu}\,\mathrm{d}h + \sum_{i=1}^{\infty} \tilde{c}(\varphi_i)[(1-b)A\big(\varsigma(\varphi_i)\big)]^{\nu}\Big] \\ \le -\Big[\nabla^{1-\nu/\mu}(h)\Big]_{\eta_3}^{\eta} < \nabla^{1-\nu/\mu}(\eta_3) < \infty \end{aligned}$$

which contradicts (17). Thus, the proof is completed. □

Example 1. *Consider the neutral differential system*

$$\begin{cases} \Big(\big((z(\eta) + e^{-\eta} z(\vartheta(\eta)))'\big)^{1/3}\Big)' + \eta(z(\eta+2))^{5/3} = 0, \\ \Big(\big((z(3^i) + e^{-3^i} z(\vartheta(3^i)))'\big)^{1/3}\Big)' + (1+3^i)(z(3^i+2))^{5/3} = 0. \end{cases} \tag{23}$$

Here $\nu = 5/3 > \mu = 1/3$, $a(\eta) = 1$, $0 < b(\eta) = e^{-\eta} < 1$, $\varsigma(\eta) = \eta + 2$, $\varphi_i = 3^i$ for $i \in \mathbb{N}$. To check (14), we have

$$\int_{\eta_0}^{\infty}\Big[\frac{1}{a(h)}\Big[\int_h^{\infty} c(g)\,dg + \sum_{\varphi_i \geq h}\tilde{c}(\varphi_i)\Big]\Big]^{1/\mu} dh \geq \int_{\eta_0}^{\infty}\Big[\frac{1}{a(h)}\Big[\int_h^{\infty} c(g)\,dg\Big]\Big]^{1/\mu} dh$$
$$\geq \int_2^{\infty}\Big[\int_h^{\infty} g\,dg\Big]^3 dh = \infty.$$

Therefore, each conditions of Theorem 1 hold. Hence, every solution of (23) oscillates.

Example 2. *Consider the neutral differential system*

$$\begin{cases} \Big(e^{-\eta}\big((z(\eta) + e^{-\eta}z(\vartheta(\eta)))'\big)^3\Big)' + \frac{1}{\eta+1}(z(\eta-2))^{7/3} = 0, \\ \Big(e^{-i}\big((z(i) + e^{-i}z(\vartheta(i)))'\big)^3\Big)' + \frac{1}{i+4}(z(i-2))^{7/3} = 0. \end{cases} \tag{24}$$

Here, $\nu = 7/3 < \mu = 3$, $a(\eta) = e^{-\eta}$, $0 < b(\eta) = e^{-\eta} < 1$, $\varsigma(\eta) = \eta - 2$, $\varphi_i = i$ for $i \in \mathbb{N}$, $A(\eta) = \int_0^{\eta} e^{s/3}ds = 3(e^{\eta/3} - 1)$. To check (17), we have

$$\frac{1}{2^{\nu}}\Big[\int_0^{\infty} c(g)[(1-b)A(\varsigma(g))]^{\nu}\,dg + \sum_{i=1}^{\infty}\tilde{c}(\varphi_i)[(1-b)A(\varsigma(\varphi_i))]^{\nu}\Big]$$
$$\geq \frac{1}{(2)^{7/3}}\int_0^{\infty} c(g)[(1-b)A(\varsigma(g))]^{\nu}\,dg$$
$$= \frac{1}{(2)^{7/3}}\int_0^{\infty}\frac{1}{g+1}\Big[(1-b)3\big(e^{(g-2)/3} - 1\big)\Big]^{7/3} dg = \infty.$$

So, all conditions of Theorem 2 are satisfied. Hence, each solution of (24) oscillates.

4. Conclusions

In this work, we tried to establish some new sufficient and necessary conditions for the oscillation of solutions of second-order nonlinear neutral impulsive differential systems with mixed delays of the form (10). Our study is restricted to only when the neutral coefficients $b(\eta)$ lies in $[0, 1)$. Still, the problem is open for $-\infty < b(\eta) \leq -1$ and $1 \leq b(\eta) < \infty$. It would be of interest to examine the oscillation of (10) with different neutral coefficients; see, e.g., the papers [34–40] for more details. Furthermore, it is also interesting to analyze the oscillation of (10) with a nonlinear neutral term; see, e.g., the paper [41] for more details.

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