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Dynamics of an Eco-Epidemic Predator–Prey Model Involving Fractional Derivatives with Power-Law and Mittag–Leffler Kernel

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Abstract: In this paper, we consider a fractional-order eco-epidemic model based on the Rosenzweig–MacArthur predator–prey model. The model is derived by assuming that the prey may be infected by a disease. In order to take the memory effect into account, we apply two fractional differential operators, namely the Caputo fractional derivative (operator with power-law kernel) and the Atangana–Baleanu fractional derivative in the Caputo (ABC) sense (operator with Mittag–Leffler kernel). We take the same order of the fractional derivative in all equations for both senses to maintain the symmetry aspect. The existence and uniqueness of solutions of both eco-epidemic models (i.e., in the Caputo sense and in ABC sense) are established. Both models have the same equilibrium points, namely the trivial (origin) equilibrium point, the extinction of infected prey and predator point, the infected prey free point, the predator-free point and the co-existence point. For a model in the Caputo sense, we also show the non-negativity and boundedness of solution, perform the local and global stability analysis and establish the conditions for the existence of Hopf bifurcation. It is found that the trivial equilibrium point is a saddle point while other equilibrium points are conditionally asymptotically stable. The numerical simulations show that the solutions of the model in the Caputo sense strongly agree with analytical results. Furthermore, it is indicated numerically that the model in the ABC sense has quite similar dynamics as the model in the Caputo sense. The essential difference between the two models is the convergence rate to reach the stable equilibrium point. When a Hopf bifurcation occurs, the bifurcation points and the diameter of the limit cycles of both models are different. Moreover, we also observe a bistability phenomenon which disappears via Hopf bifurcation.

Keywords: Atangana–Baleanu; Caputo; eco-epidemiology; Rosenzweig–MacArthur

1. Introduction

The long history of mathematical biology reveals that predator–prey modeling plays an imperative role in scientific research. Since the classical Lotka–Volterra, as the fundamental predator–prey model, have been proposed [1–3], the theoretical ecology has been constantly developed. The Lotka–Volterra model has been modified by a lot of researchers to contrive the relevant model which corresponds to the actual phenomena, such as the functional response [4–9], the Allee effect [10–14], the impact of competition [15–17] and so forth. All of these modifications affect the density of populations as the result of interactions between two or more populations. From the biological point of view, the population density also depends on the epidemiological frameworks, which leads to the increment of the death rate caused by the disease in the population. Eco-epidemiology describes the occurrence of ecological and epidemiological circumstances simultaneously [18–23].

For instance, in a biotope that involves pests and its natural enemies, we observe that the eco-epidemiological problem occurs and described by the interaction between pest and its predator. One or both populations may be infected by a disease caused by microbiological pathogens such as parasites, viruses, fungi and bacteria, for further see refs. [24–28] and some references therein. For the real-world example, to suppress the growth rate of rats (*Rattus sp.*) in agricultural landscape, the farmers use barn owls (*tyto alba*) and some pathogens such as viruses (*paramyxoviridae* and *pneumovirus* family), bacterias (*klebsiella pneumoniae*, *mycoplasma pulmonis*, *citrobacter rodentium* and *streptococcus pneumoniae*), and parasites (*giardia muris*, *spironucleus muris*, *oxyuriasis* and *acariasis*) [29–33].

Regarding to the description above, some researchers have successfully constructed and studied the eco-epidemiological problem in a deterministic model. Mondal et al. [21], Wang et al. [34] and Suryanto et al. [35] study the dynamics of the interaction between two populations in a predator–prey relationship where the prey is infected by a disease and the predator is hunting the infected prey. In facts, many natural phenomena in the ecological system show that predation still occurs although the infected prey does not exist. This means both susceptible and infected prey are regarded to be predated. Based on this assumption, Sahoo [19], Saifuddin et al. [20], Panigoro et al. [23], Upadhyay et al. [36] and Nugraheni et al. [37] study the eco-epidemic model with the predation existing on both susceptible and infected prey. The fundamental differences of their models lie on the infectious transmission behavior, the predator functional response, the existence of the Allee effect and the operator of the derivative. Here, we study the eco-epidemic model formulated under the following assumptions.

(a) In the presence of disease, the prey is divided into two compartments, namely susceptible prey $S(t)$ and infected prey $I(t)$. The susceptible prey becomes infected when the individuals have contact with the infected prey. Since the density of prey and predator are assumed large enough, the infection rate due to this contact is bilinear which is symbolized by b.

(b) In the presence of the predator–prey relationship, the interaction between susceptible prey, infected prey and predator is following the Rosenzweig–MacArthur model [38] with a few adjustments. The susceptible prey growth logistically with intrinsic growth rate r and environmental carrying capacity K. The infected prey competes for food with the susceptible prey but has no contribution to the growth rate of susceptible prey. Both susceptible prey and infected prey are predated following Holling type-II with the attack rate of predator on susceptible prey m_s, the attack rate of predator on infected prey m_i, the half-saturation constant of predator for susceptible prey k_s and the half-saturation constant of predator for infected prey k_i. Since both predations contribute to the predator birth, the conversion efficiency consists of two parts, i.e., the conversion efficiency of predator on susceptible prey b_s and the conversion efficiency of predator on infected prey b_i. It is also assumed that both infected prey and predator are reduced due to mortality following exponential decay where d is the death rate of infected prey, and a is the death rate of predator.

Based on above assumptions, we have the following eco-pidemic model.

$$
\begin{aligned}
\frac{dS}{dt} &= rS\left(1 - \frac{S+I}{K}\right) - bSI - \frac{m_s SP}{k_s + S}, \\
\frac{dI}{dt} &= bSI - dI - \frac{m_i IP}{k_i + I}, \\
\frac{dP}{dt} &= \left(\frac{b_s S}{k_s + S} + \frac{b_i I}{k_i + I} - a\right)P,
\end{aligned}
\tag{1}
$$

For simplicity, model (1) is transformed into a non-dimensional system by introducing transformation of variables $(S, I, P, t) \rightarrow \left(\frac{S}{K}, \frac{I}{K}, \frac{m_s P}{rK}, rt\right)$ to obtain the following eco-epidemic model.

$$\frac{dS}{dt} = \left[1 - S - (1 + \hat{\eta})I - \frac{P}{\kappa + S} \right] S,$$

$$\frac{dI}{dt} = \left[\hat{\eta}S - \hat{\delta} - \frac{\hat{m}P}{\omega + I} \right] I, \tag{2}$$

$$\frac{dP}{dt} = \left[\frac{\hat{\mu}S}{\kappa + S} + \frac{\hat{\beta}I}{\omega + I} - \hat{q} \right] P,$$

where $\hat{\eta} = \frac{bK}{r}$, $\hat{\delta} = \frac{d}{r}$, $\hat{m} = \frac{m_i}{m_s}$, $\hat{\mu} = \frac{b_s}{r}$, $\hat{\beta} = \frac{b_i}{r}$, $\kappa = \frac{k_s}{K}$, $\omega = \frac{k_i}{K}$ and $\hat{q} = \frac{a}{r}$.

To approach the superlative shape of the eco-epidemiological model, the ordinary calculus is considered less effective in describing the complex ecological phenomena that involves the system memory and hereditary biological properties of complex multiple timescale dynamics, see refs. [39–41]. To overcome such problem, many researchers apply fractional calculus because it is considered to have the ability to represent biological conditions related to the memory effects more powerfully and accurately than the classical calculus [42–48]. Particularly, the fractional-order derivatives, as part of the fundamental theory of fractional calculus, have nonlocal properties which are naturally connected to the biological systems. It means that the current state of population density depends on all earlier states [49–51]. If we revisited the evolution of fractional-order derivative, the Riemann–Liouville [52] and Caputo [53] operators have been widely applied to the biological modeling. To investigate the behavior of the fractional-order dynamical system, the theoretical aspect of the Caputo operator is the most complete tool compared to others, see refs. [52,54,55]. However, the kernels of the first two definitions of fractional operators are single and local [56–59]. Therefore, Caputo operator is not sufficient enough to express better nonlocal dynamics. To cover the limitation of Caputo operator, in 2015, Caputo and Fabrizio proposed a new fractional operator, which is called the Caputo–Fabrizio derivative [60]. The non-singular and exponential kernel of this fractional derivative is the novelty of their result and has been successfully applied in several fields [40,61–63]. One year later, Atangana and Baleanu introduced a new fractional operator with a nonlocal and non-singular kernel. Such an operator is well-known as the Atangana–Baleanu operator, which has all the advantages of the Caputo–Fabrizio operator but it uses the Mittag–Leffler function as its kernel [64]. Most researchers reveal that the Atangana–Baleanu operator gives better results and claim that the effect of memory is represented efficiently, see refs. [9,40,51,65].

For a better approach in epidemiological modeling, the fractional-order derivative is utilized in a similar way with [48,66] which replace the first-order derivative $\frac{d}{dt}$ at the left-hand side of model (2) with the fractional-order derivative $\mathcal{D}_t^\alpha$. Therefore, we obtain

$$\mathcal{D}_t^\alpha S = \left[1 - S - (1 + \hat{\eta})I - \frac{P}{\kappa + S} \right] S,$$

$$\mathcal{D}_t^\alpha I = \left[\hat{\eta}S - \hat{\delta} - \frac{\hat{m}P}{\omega + I} \right] I, \tag{3}$$

$$\mathcal{D}_t^\alpha P = \left[\frac{\hat{\mu}S}{\kappa + S} + \frac{\hat{\beta}I}{\omega + I} - \hat{q} \right] P.$$

Pay close attention to the dimension of the equations in model (3), where the fractional-order derivatives have the dimensions of $(\text{time})^{-\alpha}$ while the parameters $\hat{\eta}$, $\hat{\delta}$, $\hat{m}$, $\hat{\mu}$, $\hat{\beta}$ and $\hat{q}$ have the dimensions of $(\text{time})^{-1}$. This circumstance means the inconsistency of physical

dimensions in the model (3) and can be surmounted by rescaling the parameters as in the following model.

$$
\mathcal{D}_t^\alpha S = \left[1 - S - (1 + \hat{\eta}^\alpha) I - \frac{P}{\kappa + S} \right] S,
$$

$$
\mathcal{D}_t^\alpha I = \left[\hat{\eta}^\alpha S - \hat{\delta}^\alpha - \frac{\hat{m}^\alpha P}{\omega + I} \right] I, \tag{4}
$$

$$
\mathcal{D}_t^\alpha P = \left[\frac{\hat{\mu}^\alpha S}{\kappa + S} + \frac{\hat{\beta}^\alpha I}{\omega + I} - \hat{q}^\alpha \right] P.
$$

By applying new parameters $\hat{\eta} = \eta, \hat{\delta} = \delta, \hat{m} = m, \hat{\mu} = \mu, \hat{\beta} = \beta$ and $\hat{q} = q$, we achieve

$$
\mathcal{D}_t^\alpha S = \left[1 - S - (1 + \eta) I - \frac{P}{\kappa + S} \right] S,
$$

$$
\mathcal{D}_t^\alpha I = \left[\eta S - \delta - \frac{mP}{\omega + I} \right] I, \tag{5}
$$

$$
\mathcal{D}_t^\alpha P = \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q \right] P.
$$

We note that model (5) consists of three fractional differential equations. The order of the fractional derivative in all equations is set to be the same to maintain their symmetrical aspect.

Notice that we have to assume in model (5) that the disease transmission follows a bilinear incidence rate. Previously, Nugraheni et al. [37] have studied the same model but with saturated incidence rate. However, Nugraheni et al. [37] have only presented some numerical simulations of model (5) with Caputo sense without any analytical studies. In this article, we start our work by constructing the fractional-order eco-epidemic model consisting of the model assumptions, the first-order modeling, its non-dimensional form and the fractional-order modeling including the replacement of the first-order derivative with fractional-order derivative and the time dimension adjustment for some parameters to prevent the inconsistency of physical dimensions. Furthermore, we present the complete dynamics of model (5) with Caputo operator including the local and global stability, the existence of Hopf bifurcation and their appropriated numerical simulations. We also use the Atangana–Baleanu in Caputo sense as the fractional-order operator of model (5) numerically by previously showing the existence and uniqueness of solution of the model. We compare numerically the difference between model (5) with Caputo and Atangana–Baleanu operators, especially the difference of the dynamical behaviors when the Hopf bifurcation occurs. All of these analytical results and numerical simulations have never been done in [37], which is the novelty of our work.

This paper is organized as follows. In Section 2, we present some fundamental concepts which consist of definitions, theorems and lemmas that are associated with Caputo and Atangana–Baleanu derivatives and dynamical systems. Further, in Section 3, we investigate the dynamics of model (5) with Caputo derivative. The investigation includes the existence, uniqueness, non-negativity, boundedness of the solutions, the existence of equilibrium points, their local and global stability, as well as the occurrence of Hopf bifurcation. The existence and uniqueness of solutions of model (5) with Atangana–Baleanu derivative in Caputo sense are discussed in Section 4. To support our theoretical findings, we demonstrate some numerical simulations in Section 5. Finally, we present some conclusions in Section 6.

2. Fundamental Concepts

In this section, we present the important results from previous research such as definitions, theorems and lemmas associated with the fractional-order differential equation.

Definition 1 ([55]). *Suppose $0 < \alpha \leq 1$. The Caputo fractional derivative of order$-\alpha$ is defined by*

$$^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \tag{6}$$

where $t \geq 0$, $f \in C^n([0, +\infty), \mathbb{R})$, and Γ is the Gamma function.

Definition 2 ([64]). *Suppose $0 < \alpha \leq 1$. The Atangana–Baleanu fractional integral and derivative in Caputo sense of order$-\alpha$ (ABC sense) are respectively defined by*

$$^{ABC}\mathcal{I}_t^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{\Gamma(\alpha)B(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds,$$

$$^{ABC}\mathcal{D}_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-s)^{\alpha} \right] f'(s) \, ds,$$

where $t \geq 0$, $f \in C^n([0, +\infty), \mathbb{R})$, E_α is the Mittag–Leffler function defined by $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$, and $B(\alpha)$ is a normalization function with $B(0) = B(1) = 1$. In this paper, we define $B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Theorem 1 ([64]). *By using the inverse Laplace transform and convolution theorem, the unique solution of the time fractional differential equation*

$$^{ABC}\mathcal{D}_t^{\alpha} f(t) = \varphi(t) \tag{7}$$

can be written as

$$f(t) = \frac{1-\alpha}{B(\alpha)} \varphi(t) + \frac{\alpha}{\Gamma(\alpha)B(\alpha)} \int_0^t \varphi(s)(t-s)^{\alpha-1} \, ds. \tag{8}$$

Lemma 1 ([67]). *Let $0 < \alpha \leq 1$. Suppose that $f(t) \in C[a,b]$ and $^{C}\mathcal{D}_t^{\alpha} f(t) \in C[a,b]$. If $^{C}\mathcal{D}_t^{\alpha} f(t) \geq 0$, $\forall t \in (a,b)$, then $f(t)$ is a non-decreasing function for each $t \in [a,b]$. If $^{C}\mathcal{D}_t^{\alpha} f(t) \leq 0$, $\forall t \in (a,b)$, then $f(t)$ is a non-increasing function for each $t \in [a,b]$.*

Theorem 2 (Matignon condition [55,68]). *Consider a Caputo fractional-order system*

$$^{C}\mathcal{D}_t^{\alpha} \vec{x} = \vec{f}(\vec{x}); \; \vec{x}(0) = \vec{x}_0; \; \alpha \in (0,1]. \tag{9}$$

A point $\vec{x}^*$ is called an equilibrium point of Equation (9) if it satisfies $\vec{f}(\vec{x}^*) = 0$. This equilibrium point is locally asymptotically stable if all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial \vec{f}}{\partial \vec{x}}$ evaluated at $\vec{x}^*$ satisfy $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$. If there exists at least one eigenvalue that satisfies $|\arg(\lambda_k)| > \frac{\alpha\pi}{2}$ while $|\arg(\lambda_l)| < \frac{\alpha\pi}{2}$, $k \neq l$, then $\vec{x}^*$ is a saddle-point.

Lemma 2 ([69]). *Consider a Caputo fractional-order system*

$$^{C}\mathcal{D}_t^{\alpha} x(t) = f(t, x(t)), \; t > 0, \; x(0) \geq 0, \; \alpha \in (0,1], \tag{10}$$

where $f : (0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \subseteq \mathbb{R}^n$. A unique solution of Equation (10) on $(0, \infty) \times \Omega$ exists if $f(t, x(t))$ satisfies the locally Lipschitz condition with respect to x.

Lemma 3 (Standard comparison theorem for Caputo fractional-order derivative [42]). *Let $x(t) \in C([0, +\infty))$. If $x(t)$ satisfies*

$$^{C}\mathcal{D}_t^{\alpha} x(t) + \lambda x(t) \leq \mu, \; x(0) = x_0,$$

where $\alpha \in (0,1]$, $(\lambda, \mu) \in \mathbb{R}^2$ and $\lambda \neq 0$, then

$$x(t) \leq \left(x_0 - \frac{\mu}{\lambda}\right) E_\alpha[-\lambda t^\alpha] + \frac{\mu}{\lambda}.$$

Lemma 4 ([70]). *Let $x(t) \in C(\mathbb{R}_+)$, $x^* \in \mathbb{R}_+$, and its Caputo fractional derivatives of order α exist for any $\alpha \in (0,1]$. Then, for any $t > 0$, we have*

$$^C\mathcal{D}_t^\alpha \left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*}\right] \leq \left(1 - \frac{x^*}{x(t)}\right)^C\mathcal{D}_t^\alpha x(t).$$

Lemma 5 (Generalized Lasalle Invariance Principle [71]). *Suppose Ω is a bounded closed set and every solution of system*

$$^C\mathcal{D}_t^\alpha x(t) = f(x(t)), \tag{11}$$

which starts from a point in Ω remains in Ω for all time. If $\exists V(x) : \Omega \to \mathbb{R}$ with continuous first order partial derivatives satisfies following condition:

$$^C\mathcal{D}_t^\alpha V|_{Eq.(11)} \leq 0,$$

then every solution $x(t)$ originating in Ω tends to M as $t \to \infty$, where M is the largest invariant set of E and $E := \left\{x|^C\mathcal{D}_t^\alpha V|_{Eq.(11)} = 0\right\}$.

3. Eco-Epidemic Model in the Caputo Sense

In this section, we consider a fractional-order eco-epidemic model (5) with the fractional derivative in the Caputo sense as defined in Definition 1:

$$
\begin{aligned}
^C\mathcal{D}_t^\alpha S &= \left[1 - S - (1 + \eta)I - \frac{P}{\kappa + S}\right]S = F_1(X), \\
^C\mathcal{D}_t^\alpha I &= \left[\eta S - \delta - \frac{mP}{\omega + I}\right]I = F_2(X), \\
^C\mathcal{D}_t^\alpha P &= \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q\right]P = F_3(X),
\end{aligned}
\tag{12}
$$

where $X = (S, I, P)$. In the following sub-sections, we investigate the dynamics of model (12).

3.1. Existence and Uniqueness

In this section, we investigate the sufficient condition for the existence and uniqueness of solution of model (12).

Theorem 3. *Consider model (12) with positive initial condition $S_0 \geq 0$, $I_0 \geq 0$, $P_0 \geq 0$ and $\alpha \in (0,1]$, $F : [0, \infty) \to \mathbb{R}^3$, where $F(X) = (F_1(X), F_2(X), F_3(X))$, $X \equiv X(t)$ and $\Psi = \left\{(S, I, P) \in \mathbb{R}_+^3 : \max\{|S|, |I|, |P|\} \leq \gamma\right\}$ for sufficiently large γ. The model (12) with positive initial condition has a unique solution.*

Proof. We use a similar approach as in [8]. For any $X = (S, I, P)$, $\bar{X} = (\bar{S}, \bar{I}, \bar{P})$, $X, \bar{X} \in \Psi$, it follows from model (12) that

$$
\begin{aligned}
\|F(X) - F(\bar{X})\| &= |F_1(X) - F_1(\bar{X})| + |F_2(X) - F_2(\bar{X})| + |F_3(X) - F_3(\bar{X})| \\
&= \left|(S - \bar{S}) - (S^2 - \bar{S}^2) - (1 + \eta)(SI - \bar{S}\bar{I}) - \left(\frac{SP}{\kappa + S} - \frac{\bar{S}\bar{P}}{\kappa + \bar{S}}\right)\right| + \\
&\quad \left|\eta(SI - \bar{S}\bar{I}) - \delta(I - \bar{I}) - m\left(\frac{IP}{\omega + I} - \frac{\bar{I}\bar{P}}{\omega + \bar{I}}\right)\right| + \\
&\quad \left|\mu\left(\frac{SP}{\kappa + S} - \frac{\bar{S}\bar{P}}{\kappa + \bar{S}}\right) + \beta\left(\frac{IP}{\omega + I} - \frac{\bar{I}\bar{P}}{\omega + \bar{I}}\right) - q(P - \bar{P})\right|
\end{aligned}
$$

$$\leq \left|S - \bar{S}\right| + \left|S^2 - \bar{S}^2\right| + (1 + \eta)\left|SI - \bar{S}\bar{I}\right| + \left|\frac{SP}{\kappa + S} - \frac{\bar{S}\bar{P}}{\kappa + \bar{S}}\right| +$$

$$\eta\left|SI - \bar{S}\bar{I}\right| + \delta|I - \bar{I}| + m\left|\frac{IP}{\omega + I} - \frac{\bar{I}\bar{P}}{\omega + \bar{I}}\right| + \mu\left|\frac{SP}{\kappa + S} - \frac{\bar{S}\bar{P}}{\kappa + \bar{S}}\right| +$$

$$\beta\left|\frac{IP}{\omega + I} - \frac{\bar{I}\bar{P}}{\omega + \bar{I}}\right| + q|P - \bar{P}|$$

$$\leq L_1\left|S - \bar{S}\right| + L_2|I - \bar{I}| + L_3|P - \bar{P}|$$

$$\leq L\|X - \bar{X}\|$$

where

$$L_1 = 1 + \left(3 + 2\eta + \frac{1 + \mu}{\kappa}\right)\gamma,$$

$$L_2 = \delta + \left(1 + 2\eta + \frac{m + \beta}{\omega}\right)\gamma,$$

$$L_3 = q + \left(\frac{1 + \mu}{\kappa} + \frac{m + \beta}{\omega}\right)\gamma + \left(\frac{1 + \mu}{\kappa^2} + \frac{m + \beta}{\omega^2}\right)\gamma^2,$$

$$L = \max\{L_1, L_2, L_3\}.$$

Hence, $F(X)$ satisfies the Lipschitz condition. According to Lemma 2, the existence and uniqueness of model (12) with initial value $S_0 \geq 0$, $I_0 \geq 0$ and $P_0 \geq 0$ is established, and the theorem is well proven. □

3.2. Non-Negativity and Boundedness

Model (12) describes an eco-epidemiological model in fractional-order differential equations. Therefore, the solution of model (12) must be bounded and non-negative, as it is performed in the following theorem.

Theorem 4. *All solutions of model* (12) *with non-negative initial values are non-negative and uniformly bounded.*

Proof. We start by proving that for non-negative initial condition, $S(t) \geq 0$ for $t \to \infty$. Suppose that is not true, then there exists $t_1 > 0$ such that

$$\begin{cases} S(t) & > & 0, \ 0 \leq t < t_1, \\ S(t_1) & = & 0, \\ S(t_1^+) & < & 0. \end{cases} \tag{13}$$

Employing (13) and the first equation of model (12), we obtain

$$\left.{}^C\mathcal{D}_t^\alpha S(t_1)\right|_{S(t_1)=0} = 0. \tag{14}$$

Based on Lemma 1, we have $S(t_1^+) = 0$, which contradicts to the fact $S(t_1^+) < 0$. Thus, $S(t) \geq 0$ for all $t \geq 0$. Using the similar procedure, we conclude $I(t) \geq 0$ and $P(t) \geq 0$ for all $t > 0$.

Now, we show the boundedness of solutions by adopting similar manner as in [8]. We first define a function

$$V(t) = S + I + \zeta P.$$

Then, for each $\xi > 0$, we obtain

$$^C\mathcal{D}_t^\alpha V(t) + \xi V(t) = \left(S - S^2 - (1 + \eta)SI - \frac{SP}{\kappa + S}\right) + \left(\eta SI - \delta I - \frac{mIP}{\omega + I}\right) +$$

$$\zeta\left(\frac{\mu SP}{\kappa + S} + \frac{\beta IP}{\omega + I} - qP\right) + \xi(S + I + \zeta P)$$

$$= (1 + \xi)S - S^2 - SI + \beta\left(\zeta - \frac{m}{\beta}\right)\frac{IP}{\omega + I} + \mu\left(\zeta - \frac{1}{\mu}\right)\frac{SP}{\kappa + S} +$$

$$(\xi - \delta)I + \zeta(\xi - q)P.$$

By choosing $\xi < \min\{\delta, q\}$ and $\zeta < \min\left\{\frac{m}{\beta}, \frac{1}{\mu}\right\}$, we have

$$^C\mathcal{D}_t^\alpha V(t) + \xi V(t) \leq (1 + \xi)S - S^2$$

$$= (1 + \xi)S - S^2 - \left(\frac{1 + \xi}{2}\right)^2 + \left(\frac{1 + \xi}{2}\right)^2$$

$$= -\left(S^2 - (1 + \xi)S + \left(\frac{1 + \xi}{2}\right)^2\right) + \left(\frac{1 + \xi}{2}\right)^2$$

$$\leq \frac{(1 + \xi)^2}{4}.$$

The standard comparison theorem for fractional-order derivative (see Lemma 3) gives

$$V(t) \leq \left(V(0) - \frac{(1 + \xi)^2}{4\xi}\right)E_\alpha[-\xi t^\alpha] + \frac{(1 + \xi)^2}{4\xi},$$

from which we have that $V(t)$ is convergent to $\frac{(1+\xi)^2}{4\xi}$ for $t \to \infty$. Therefore, all solutions of model (12) with non-negative initial conditions are confined to the region Φ, where

$$\Phi := \left\{(S, I, P) \in \mathbb{R}_+^3 : V(t) \leq \frac{(1 + \xi)^2}{4\xi} + \varepsilon, \ \varepsilon > 0\right\}. \tag{15}$$

Therefore, the proof of non-negativity and boundedness of solutions are completely presented. □

3.3. The Existence of Equilibrium Point

From model (12), we can determine the nullclines of the susceptible prey, infected prey and predator, which are respectively denoted by N_S, N_I and N_P and are defined by the following sets

$$N_S := \left\{(S, I, P) : S = 0 \lor S + (1 + \eta)I + \frac{P}{\kappa + S} = 1\right\},$$

$$N_I := \left\{(S, I, P) : I = 0 \lor S = \frac{\delta}{\eta} + \frac{mP}{\eta(\omega + I)}\right\},$$

$$N_P := \left\{(S, I, P) : P = 0 \lor \frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} = q\right\}.$$

Since we are only interested in solutions that satisfy biological conditions, we only consider equilibrium points that satisfy $N_S \cap N_I \cap N_P \subset \mathbb{R}_+^3$. We can obviously identify that the infected prey and predator are extinct if the susceptible prey is zero. Therefore, the following lemma shows that the origin is the only equilibrium point when $N_S = \{(S, I, P) : S = 0\}$.

Lemma 6. *If $N_S := \{(S, I, P) : S = 0\}$ then the origin $E_0 = (0, 0, 0)$ is the only equilibrium point of model (12).*

Proof. For $N_S := \{(S, I, P) : S = 0\}$, the equilibrium point is defined by $N_I \cap N_P \cap \mathbb{R}^3_+$, where

$$N_I := \left\{ (S, I, P) : I = 0 \vee \frac{\delta}{\eta} + \frac{mP}{\eta(\omega + I)} = 0 \right\},$$

$$N_P := \left\{ (S, I, P) : P = 0 \vee \frac{\beta I}{\omega + I} = q \right\}.$$

Since $\frac{\delta}{\eta} + \frac{mP}{\eta(\omega+I)} \neq 0$, then $I = 0$ is the only nullcline of I. By substituting $I = 0$ to N_P, we have $N_P = \{(S, I, P) : P = 0 \vee q = 0\}$. $P = 0$ is the only nullcline of P because $q \neq 0$. Thus, $E_0 = (0, 0, 0)$ is the only equilibrium point. $\square$

Now, we will investigate the equilibrium point when $S \neq 0$. Notice that if $N_I := \{(S, I, P) : I = 0\}$, then $N_S \cap N_P \cap \mathbb{R}^3_+$ is the equilibrium point of model (12) where

$$N_S := \left\{ (S, I, P) : S + \frac{P}{\kappa + S} = 1 \right\}, \text{ and } N_P := \left\{ (S, I, P) : P = 0 \vee \frac{\mu S}{\kappa + S} = q \right\}.$$

Immediately we recognize two equilibrium points as follows:

1. The extinction of infected prey and predator point: $E_1 = (1, 0, 0)$, which always exists.
2. The infected prey free point $E_2 = (\hat{S}, 0, \hat{P})$ where $\hat{S} = \frac{q\kappa}{\mu - q}$ and $\hat{P} = (1 - \hat{S})(\kappa + \hat{S})$ which exists if $\mu > (1 + \kappa)q$. The condition $\mu > (1 + \kappa)q$ is equivalent to condition that the conversion rate of susceptible prey predation into the birth rate of predator is larger than the sum of the death rate of predator and its multiplication with half-saturation constant of predation.

Furthermore, if $N_P = \{(S, I, P) : P = 0\}$, we obtain equilibrium points that satisfy $N_S \cap N_I \cap \mathbb{R}^3_+$ where

$$N_S := \{(S, I, P) : S + (1 + \eta)I = 1\}, \text{ and } N_I := \left\{ (S, I, P) : I = 0 \vee S = \frac{\delta}{\eta} \right\}.$$

Thus, we have the extinction of infected prey and predator point $E_1 = (1, 0, 0)$ and the predator-free point $E_3 = \left(\tilde{S}, \frac{1 - \tilde{S}}{1 + \eta}, 0 \right)$, where $\tilde{S} = \frac{\delta}{\eta}$. The point E_3 exists if $\tilde{S} \in (0, 1)$ or $\eta > \delta$, i.e., when the prey infection rate is greater than the infected prey death rate.

By considering the following nullclines

$$N_S := \left\{ (S, I, P) : S + (1 + \eta)I + \frac{P}{\kappa + S} = 1 \right\},$$

$$N_I := \left\{ (S, I, P) : S = \frac{\delta}{\eta} + \frac{mP}{\eta(\omega + I)} \right\}, \quad N_P := \left\{ (S, I, P) : \frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} = q \right\},$$

we obtain the co-existence equilibrium point $E^* = (S^*, I^*, P^*)$ that satisfies $N_S \cap N_I \cap N_P \cap \mathbb{R}^3_+$, i.e.,

$$S^* = \frac{-a_2 \pm \sqrt{D}}{2a_1}, \quad I^* = \frac{(1 - S^*)(\kappa + S^*)m - \omega(\eta S^* - \delta)}{(\kappa + S^*)(\eta + 1)m + (\eta S^* - \delta)}, \quad P^* = \frac{(\eta S^* - \delta)(\omega + I^*)}{m},$$

where

$$a_1 = (\beta + \mu)m - mq, \qquad\qquad a_3 = ((\eta + 1)\omega + 1)mq\kappa - (m\kappa + \delta\omega)\beta$$
$$a_2 = ((\eta + 1)\omega + 1)mq + (\eta\omega + m\kappa)\beta \qquad D = a_2^2 - 4a_1 a_3$$
$$\qquad - (((\eta + 1)\omega + 1)\mu + q\kappa + \beta)m$$

The existence of E^* is described by the following lemma.

Lemma 7. *Suppose that*

$$S_1^* = -\frac{a_2 - \sqrt{D}}{2a_1}, \qquad S_2^* = -\frac{a_2 + \sqrt{D}}{2a_1}, \qquad S_3^* = -\frac{a_2}{2a_1},$$

$$S_{1,2}^* \in \left(\frac{\delta}{\eta}, 1\right), \qquad m > \frac{\omega(\eta S_j^* - \delta)}{(1 - S_j^*)(\kappa + S_j^*)}, \ j = 1, 2, \qquad a_1 a_2 < 0.$$
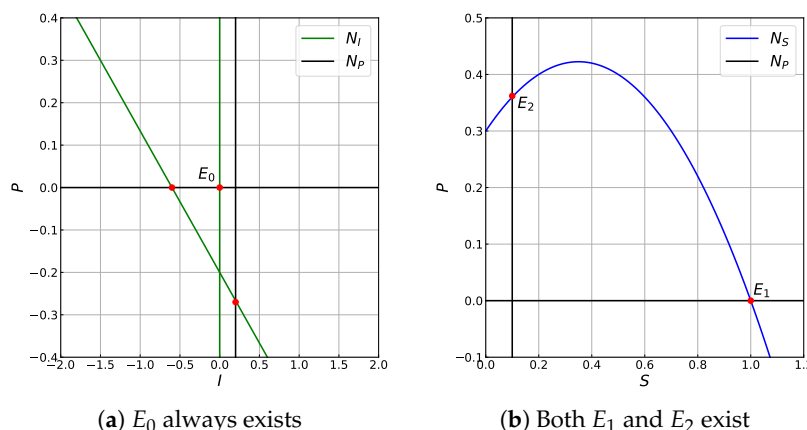
(i) *If $D < 0$, then the co-existence point does not exist.*
(ii) *if $D > 0$ and*

    (a) *$a_1 a_3 > 0$ then there are two co-existence points, i.e., $E_1^* = (S_1^*, I_1^*, P_1^*)$ and $E_2^* = (S_2^*, I_2^*, P_2^*)$.*
    (b) *$a_1 a_3 \leq 0$ then $E_1^* = (S_1^*, I_1^*, P_1^*)$ is the unique co-existence point.*

(iii) *If $D = 0$, then there is a unique co-existence point $E_3^* = (S_3^*, I_3^*, P_3^*)$.*

Proof. Notice if $S_j^* \in \left(\frac{\delta}{\eta}, 1\right)$ and $m > \frac{\omega(\eta S_j^* - \delta)}{(1 - S_j^*)(\kappa + S_j^*)}$ then $I_j^* > 0$ and $P_j^* > 0, j = 1, 2$.
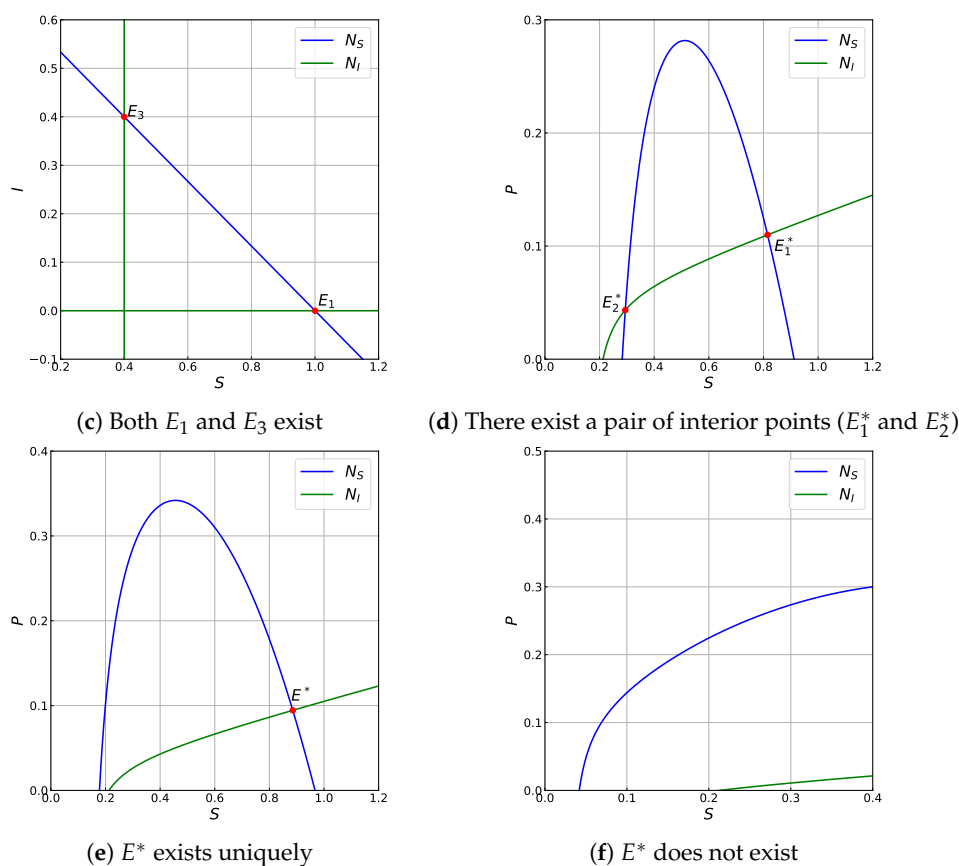
(i)   It is clear that if $D < 0$ then $S_j^* \notin \mathbb{R}$, and thus the co-existence point does not exist.
(ii)   if $D > 0$ then $S_j^* \in \mathbb{R}$. As a result that $a_1 a_2 < 0$, we have $S_1^* + S_2^* > 0$. Furthermore, if

    (a) $a_1 a_3 > 0$ then $S_1^* S_2^* > 0$. Therefore, we have $S_1^* > 0$ and $S_2^* > 0$ and $E_{1,2}^* \in \mathbb{R}_+^3$.
    (b) $a_1 a_3 \leq 0$ then $S_1^* S_2^* \leq 0$ so that $S_1^* > 0$ and $S_2^* \leq 0$.

(iii)   If $D = 0$ then S_3^* is the only solution for S_j^*. Furthermore, if $a_1 a_2 < 0$ then $S_3^* > 0$.

Thus, we have the lemma. $\quad\square$

To illustrate the existence of equilibrium point by utilizing the nullclines, we take $\eta = 0.95$, $\kappa = 0.3$, $\delta = 0.2$, $m = 0.6$, $\omega = 0.6$ $\mu = 0.4$, $\beta = 0.4$ and $q = 0.1$. We note that E_0 always exists. Besides E_0, there also exist E_1 and E_2 in $\mathbb{R}_+^3$, see Figure 1a,b. If we decrease η so that $\eta = 0.5$, then model (12) has a predator-free point E_3, see Figure 1c. Next, to show the existence of co-existence point (E^*), we choose parameter values: $\eta = 0.8$, $\delta = 0.17$, $m = 0.7$, $\omega = 0.1$, $\mu = 0.5$, $\beta = 0.3$ and $q = 0.4$. It can be seen in Figure 1d that model (12) with $\kappa = 0.6$ has two co-existence points. If we increase κ such that $\kappa = 0.4$, then we have a unique co-existence point, see Figure 1e. However, if we take $\kappa = 1$, then model (12) does not have co-existence point (see Figure 1f).

(a) E_0 always exists   **(b)** Both E_1 and E_2 exist

Figure 1. *Cont.*

(c) Both E_1 and E_3 exist

(d) There exist a pair of interior points (E_1^* and E_2^*)

(e) E^* exists uniquely

(f) E^* does not exist

Figure 1. The existence of equilibrium point of model (12) by utilizing the intersection of nullclines.

3.4. Local Stability of Equilibrium Points

In this part, we investigate the local stability properties of each equilibrium point of model (12). The local stability properties are studied by observing the eigenvalues of the Jacobian matrix at each equilibrium points, and the results are described in the following theorems.

Theorem 5. *The equilibrium point E_0 is always a saddle point.*

Proof. The Jacobian matrix of model (12) evaluated at E_0 is

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -q \end{bmatrix}.$$

The eigenvalues of this Jacobian matrix are $\lambda_1 = 1$, $\lambda_2 = -\delta$ and $\lambda_3 = -q$. Thus, $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$ and $|\arg(\lambda_2)| = |\arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$. Based on Matignon condition in Theorem 2, we conclude that E_0 is a saddle point. □

Theorem 6. *The equilibrium point E_1 is:*
(i) locally asymptotically stable if $\eta < \delta$ and $\mu < (1 + \kappa)q$.
(ii) a saddle point if $\eta > \delta$ or $\mu > (1 + \kappa)q$.

Proof. The Jacobian matrix at E_1 is

$$J(E_1) = \begin{bmatrix} -1 & -(\eta + 1) & -\frac{1}{\kappa+1} \\ 0 & \eta - \delta & 0 \\ 0 & 0 & \frac{\mu}{\kappa+1} - q \end{bmatrix}.$$

$J(E_1)$ has eigenvalues: $\lambda_1 = -1$, $\lambda_2 = \eta - \delta$ and $\lambda_3 = \frac{\mu}{\kappa+1} - q$. We have $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. Hence, the stability of E_1 depends on $\lambda_{2,3}$.

(i)   If $\eta < \delta$ and $\mu < (1+\kappa)q$, then $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ and $|\arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$. Due to Matignon condition at Theorem 2, E_1 is locally asymptotically stable.

(ii)   If $\eta > \delta$ then $|\arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$. In addition, if $\mu > (1+\kappa)q$ then $|\arg(\lambda_3)| = 0 < \frac{\alpha\pi}{2}$. Thus, Theorem 2 says that E_1 is a saddle point.

□

Theorem 7. *Suppose that:*

$$\hat{\eta} = \frac{\delta}{\hat{S}} + \frac{m(1-\hat{S})(\kappa+\hat{S})}{\omega\hat{S}}, \qquad \hat{\Delta} = \frac{4(1-\hat{S})\mu\kappa\hat{S}}{(\kappa+\hat{S})^2} - \left(\frac{\kappa-1+2\hat{S}}{\kappa+\hat{S}}\right)^2 \hat{S}^2,$$

$$\hat{\alpha} = \frac{2}{\pi}\tan^{-1}\left(\frac{(\kappa+\hat{S})\sqrt{\hat{\Delta}}}{(\kappa-1+2\hat{S})\hat{S}}\right).$$

The equilibrium point E_2 is:

(i)   locally asymptotically stable if $\eta < \hat{\eta}$ and

   (a) $\kappa > 1 - 2\hat{S}$, or;

   (b) $\kappa < 1 - 2\hat{S}$, $\hat{\Delta} > 0$ and $\alpha < \hat{\alpha}$.

(ii)   a saddle point if

   (a) $\eta > \hat{\eta}$ and $\kappa > 1 - 2\hat{S}$, or;

   (b) $\eta > \hat{\eta}$, $\kappa < 1 - 2\hat{S}$, $\hat{\Delta} > 0$, and $\alpha < \hat{\alpha}$, or;

   (c) $\eta < \hat{\eta}$, $\kappa < 1 - 2\hat{S}$, and $\alpha > \hat{\alpha}$.

Proof. The Jacobian matrix of model (12) evaluated at E_2 is

$$J(E_2) = \begin{bmatrix} -\hat{S} + \frac{(1-\hat{S})\hat{S}}{\kappa+\hat{S}} & -(1+\eta)\hat{S} & -\frac{\hat{S}}{\kappa+\hat{S}} \\ 0 & (\eta-\hat{\eta})\hat{S} & 0 \\ \frac{(1-\hat{S})\mu\kappa}{\kappa+\hat{S}} & \frac{\beta(1-\hat{S})(\kappa+\hat{S})}{\omega} & 0 \end{bmatrix},$$

which has eigenvalues: $\lambda_1 = (\eta - \hat{\eta})\hat{S}$ and $\lambda_{2,3} = -\frac{\hat{S}}{2}\left(\frac{\kappa-1+2\hat{S}}{\kappa+\hat{S}}\right) \pm \frac{\sqrt{\hat{\Delta}}}{2}$. Notice that if $\eta < \hat{\eta}$ then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$, else if $\eta > \hat{\eta}$ then $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$. Furthermore, if $\kappa > 1 - 2\hat{S}$ then $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$ for both $\hat{\Delta} \geq 0$ and $\hat{\Delta} < 0$. If $\kappa < 1 - 2\hat{S}$ and $\hat{\Delta} > 0$, then $\lambda_{2,3}$ is a pair of complex eigenvalues. Thus, $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$ is attained if $\alpha < \hat{\alpha}$. When $\kappa < 1 - 2\hat{S}$, and $\alpha > \hat{\alpha}$, we have $|\arg(\lambda_{2,3})| < \frac{\alpha\pi}{2}$ for both $\hat{\Delta} \leq 0$ and $\hat{\Delta} > 0$. Therefore, by Matignon condition in Theorem 2, the theorem is completely proven.   □

Theorem 8. *Suppose that:* $\tilde{q} = \frac{\mu\tilde{S}}{\kappa+\tilde{S}} + \frac{(1-\tilde{S})\beta}{\omega(1+\eta)+(1-\tilde{S})}$. *The predator-free point E_3 is locally asymptotically stable if $q > \tilde{q}$ and it is a saddle point if $q < \tilde{q}$.*

Proof. We compute the Jacobian matrix of model (12) evaluated at E_3 and obtain

$$J(E_3) = \begin{bmatrix} -\tilde{S} & -(1+\eta)\tilde{S} & -\frac{\tilde{S}}{\kappa+\tilde{S}} \\ \frac{(1-\tilde{S})\eta}{1+\eta} & 0 & -\frac{(1-\tilde{S})m}{\omega(1+\eta)+(1-\tilde{S})} \\ 0 & 0 & \tilde{q} - q \end{bmatrix}.$$

The eigenvalues of $J(E_3)$ are $\lambda_1 = \tilde{q} - q$ and $\lambda_{2,3} = \frac{-\tilde{S} \pm \sqrt{(\tilde{S}-4(1-\tilde{S})\eta)\tilde{S}}}{2}$. If $\eta \leq \frac{\tilde{S}}{4(1-\tilde{S})}$, then the eigenvalues $\lambda_{2,3}$ are always real and negative. Moreover, if $\eta > \frac{\tilde{S}}{4(1-\tilde{S})}$, then $\lambda_{2,3}$ are a pair of complex conjugates where $\text{Re}(\lambda_{2,3}) < 0$. Hence, the eigenvalues $\lambda_{2,3}$ always

satisfy $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$. Finally, $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ is achieved if $q > \tilde{q}$. Therefore, the predator-free point E_3 is locally asymptotically stable if $q > \tilde{q}$; otherwise it is a saddle point. □

Theorem 9. *Suppose that:*

$$d_1 = S^* - \left(\frac{S^* P^*}{(\kappa + S^*)^2} + \frac{mI^* P^*}{(\omega + I^*)^2} \right),$$

$$d_2 = \left[\frac{mS^* I^* (P^*)^2}{(\kappa + S^*)^2 (\omega + I^*)^2} + \frac{\beta mI^* P^*}{(\omega + I^*)^2} + \frac{\mu S^* P^*}{(\kappa + S^*)^2} + (\eta + 1)\eta S^* I^* \right] - \left[\frac{mS^* I^* P^*}{(\omega + I^*)^2} + \frac{m\beta (I^*)^2 P^*}{(\omega + I^*)^3} + \frac{\mu (S^*)^2 P^*}{(\kappa + S^*)^3} \right],$$

$$d_3 = \left[\frac{\eta}{\kappa + S^*} + \frac{\omega m}{(\omega + I^*)^2} \right] \frac{\beta S^* I^* P^*}{\omega + I^*} - \left[\frac{\beta \eta I^*}{\omega + I^*} + \frac{(\eta + 1)\mu\kappa m}{\kappa + S^*} \right] \frac{S^* I^* P^*}{(\kappa + S^*)(\omega + I^*)} - \left[\frac{\mu\kappa}{\kappa + S^*} + \frac{\beta\omega}{\omega + I^*} \right] \frac{mS^* I^* (P^*)^2}{(\kappa + S^*)^2 (\omega + I^*)^2},$$

$$\Delta^* = 18 d_1 d_2 d_3 + (d_1 d_2)^2 - 4 d_3 d_1^3 - 4 d_2^3 - 27 d_3^2.$$

The co-existence point $E^* = (S^*, I^*, P^*)$ is locally asymptotically stable if one of the following statements is satisfied.

(i) $\Delta^* > 0$, $d_1 > 0$, $d_3 > 0$, and $d_1 d_2 > c d_3$.
(ii) $\Delta^* < 0$, $d_1 \geq 0$, $d_2 \geq 0$, $d_3 > 0$, and $0 < \alpha < (2/3)$.
(iii) $\Delta^* < 0$, $d_1 < 0$, $d_2 < 0$, and $(2/3) < \alpha < 1$.
(iv) $\Delta^* < 0$, $d_1 > 0$, $d_2 > 0$, $d_1 d_2 = d_3$, and $0 < \alpha < 1$.

Proof. The Jacobian matrix of model (12) evaluated at E^* is,

$$J(E^*) = \begin{bmatrix} \frac{S^* P^*}{(\kappa + S^*)^2} - S^* & -(\eta + 1)S^* & -\frac{S^*}{\kappa + S^*} \\ \eta I^* & \frac{mI^* P^*}{(\omega + I^*)^2} & -\frac{mI^*}{\omega + I^*} \\ \left[1 - \frac{S^*}{\kappa + S^*} \right] \frac{\mu P^*}{\kappa + S^*} & \left[1 - \frac{I^*}{\omega + I^*} \right] \frac{\beta P^*}{\omega + I^*} & 0 \end{bmatrix}.$$

The characteristic equation of $J(E^*)$ is $\lambda^3 + d_1 \lambda^2 + d_2 \lambda + d_3 = 0$. Using the Routh–Hurwitz condition for a fractional-order dynamical system (See Proposition 1 in [72]), the locally stability conditions of co-existence point $E^* = (S^*, I^*, P^*)$ are proven. □

3.5. Global Stability of Equilibrium Points

The global asymptotic stability of the equilibrium point of model (12) is studied. The results are presented in the following theorems.

Theorem 10. $E_1 = (1, 0, 0)$ *is globally asymptotically stable if* $\max\left\{ \frac{\eta}{\delta}, \frac{\beta\kappa + \mu}{\kappa q} \right\} < 1$.

Proof. Consider a Lyapunov function $W_1(S, I, P) = [S - 1 - \ln S] + \frac{1+\eta}{\eta} I + \frac{1}{\mu} P$. By using Lemma 4, we get

$$^C\mathcal{D}_t^\alpha W_1(S, I, P) \leq \left(\frac{S-1}{S} \right) {}^C\mathcal{D}_t^\alpha S + \frac{1+\eta}{\eta} {}^C\mathcal{D}_t^\alpha I + \frac{1}{\mu} {}^C\mathcal{D}_t^\alpha P$$

$$= (S - 1) \left[1 - S - (1 + \eta)I - \frac{P}{\kappa + S} \right] + \frac{1+\eta}{\eta} \left[\eta S - \delta - \frac{mP}{\omega + I} \right] I$$

$$+ \frac{1}{\mu} \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q \right] P$$

$$= -(S - 1)^2 - \frac{(\delta - \eta)(1 + \eta)I}{\eta} + \frac{P}{\kappa + S} - \frac{(1 + \eta)mIP}{(\omega + I)\eta}$$

$$+ \frac{\beta IP}{(\omega + I)\mu} - \frac{qP}{\mu}$$

$$\leq - (S - 1)^2 - \frac{(\delta - \eta)(1 + \eta)I}{\eta} + \frac{P}{\kappa} + \frac{\beta P}{\mu} - \frac{qP}{\mu}$$

$$= - (S - 1)^2 - \frac{(\delta - \eta)(1 + \eta)I}{\eta} - \left(q - \left(\beta + \frac{\mu}{\kappa} \right) \right) \frac{P}{\mu}$$

Thus, $^{C}\mathcal{D}_t^\alpha W_1(S, I, P) \leq 0$ when $\max\left\{ \frac{\eta}{\delta}, \frac{\beta\kappa + \mu}{\kappa q} \right\} < 1$. According to Lemma 5, it follows that E_1 is globally asymptotically stable. $\square$

Theorem 11. *If* $\hat{S} < \min\left\{ \frac{\delta}{\eta}, \frac{((1+\eta)m\mu - \beta\eta)\kappa}{\beta\eta} \right\}$ *and* $\hat{P} < \kappa^2$ *then the infected prey extinction point* E_2 *is globally asymptotically stable.*

Proof. We define a Lyapunov function

$$W_2(S, I, P) = \left[S - \hat{S} - \hat{S} \ln \frac{S}{\hat{S}} \right] + \frac{1 + \eta}{\eta} I + \frac{\kappa + \hat{S}}{\mu\kappa} \left[P - \hat{P} - \hat{P} \ln \frac{P}{\hat{P}} \right].$$

Based on Lemma 4, we have

$$^{C}\mathcal{D}_t^\alpha W_2(S, I, P) \leq \left(\frac{S - \hat{S}}{S} \right) {}^{C}\mathcal{D}_t^\alpha S + \frac{1 + \eta}{\eta} {}^{C}\mathcal{D}_t^\alpha I + \frac{\kappa + \hat{S}}{\mu\kappa} \left(\frac{P - \hat{P}}{P} \right) {}^{C}\mathcal{D}_t^\alpha P$$

$$= (S - \hat{S})\left[1 - S - (1 + \eta)I - \frac{P}{\kappa + S} \right] + \frac{1 + \eta}{\eta} \left[\eta S - \delta - \frac{mP}{\omega + I} \right] I$$

$$+ \frac{\kappa + \hat{S}}{\mu\kappa}(P - \hat{P})\left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q \right]$$

$$= (S - \hat{S})\left[-(S - \hat{S}) - (1 + \eta)I - \frac{(\kappa + \hat{S})(P - \hat{P}) - \hat{P}(S - \hat{S})}{(\kappa + S)(\kappa + \hat{S})} \right]$$

$$+ \frac{1 + \eta}{\eta} \left[\eta S - \delta - \frac{mP}{\omega + I} \right] I$$

$$+ \frac{\kappa + \hat{S}}{\mu\kappa}(P - \hat{P})\left[\frac{\mu\kappa(S - \hat{S})}{(\kappa + S)(\kappa + \hat{S})} + \frac{\beta I}{\omega + I} \right]$$

$$= -(S - \hat{S})^2 + \frac{\hat{P}(S - \hat{S})^2}{(\kappa + S)(\kappa + \hat{S})} - \left(\frac{\delta}{\eta} - \hat{S} \right)(1 + \eta)I$$

$$- \left(\frac{(1 + \eta)m}{\eta} - \frac{(\kappa + \hat{S})\beta}{\mu\kappa} \right) \frac{IP}{\omega + I}$$

$$\leq - \left(1 - \frac{\hat{P}}{\kappa^2} \right)(S - \hat{S})^2 - \left(\frac{\delta}{\eta} - \hat{S} \right)(1 + \eta)I$$

$$- \left(\frac{(1 + \eta)m}{\eta} - \frac{(\kappa + \hat{S})\beta}{\mu\kappa} \right) \frac{IP}{\omega + I}$$

It is clear that $^{C}\mathcal{D}_t^\alpha W_2(E_2) \leq 0$ if $\hat{S} < \min\left\{ \frac{\delta}{\eta}, \frac{((1+\eta)m\mu - \beta\eta)\kappa}{\beta\eta} \right\}$ and $\hat{P} < \kappa^2$. Consequently, Lemma 5 says that E_2 is globally asymptotically stable. $\square$

Theorem 12. *If* $q > \beta + \frac{\mu\tilde{S}}{\kappa} + \frac{(1+\eta)\mu m\tilde{I}}{\eta\omega}$ *then the predator-free point* E_3 *is globally asymptotically stable.*

Proof. We first write $\tilde{I} = \frac{1-\tilde{S}}{1+\eta}$ and define a Lyapunov function

$$W_3(S, I, P) = \left[S - \tilde{S} - \tilde{S} \ln \frac{S}{\tilde{S}} \right] + \frac{1+\eta}{\eta} \left[I - \tilde{I} - \tilde{I} \ln \frac{I}{\tilde{I}} \right] + \frac{1}{\mu} P.$$

By using Lemma 4, we obtain

$$^{C}\mathcal{D}_t^{\alpha} W_3(S, I, P) \leq \left(\frac{S - \tilde{S}}{S} \right) {}^{C}\mathcal{D}_t^{\alpha} S + \frac{1+\eta}{\eta} \left(\frac{I - \tilde{I}}{I} \right) {}^{C}\mathcal{D}_t^{\alpha} I + \frac{1}{\mu} {}^{C}\mathcal{D}_t^{\alpha} P$$

$$= (S - \tilde{S}) \left[1 - S - (1+\eta)I - \frac{P}{\kappa + S} \right] + \frac{1+\eta}{\eta} (I - \tilde{I}) \left[\eta S - \delta - \frac{mP}{\omega + I} \right]$$

$$+ \frac{1}{\mu} \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q \right] P$$

$$= (S - \tilde{S}) \left[\tilde{S} + (1+\eta)\tilde{I} - S - (1+\eta)I - \frac{P}{\kappa + S} \right]$$

$$+ \frac{1+\eta}{\eta} (I - \tilde{I}) \left[\eta S - \eta \tilde{S} - \frac{mP}{\omega + I} \right] + \frac{1}{\mu} \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q \right] P$$

$$= -(S - \tilde{S})^2 + \frac{\tilde{S}P}{\kappa + S} - \left(\frac{(1+\eta)m}{\eta} - \frac{\beta}{\mu} \right) \frac{IP}{\omega + I} + \frac{(1+\eta)m\tilde{I}P}{(\omega + I)\eta} - \frac{qP}{\mu}$$

$$\leq -(S - \tilde{S})^2 + \frac{\tilde{S}P}{\kappa} + \frac{\beta P}{\mu} + \frac{(1+\eta)m\tilde{I}P}{\eta\omega} - \frac{qP}{\mu}$$

$$= -(S - \tilde{S})^2 - \left(\frac{q}{\mu} - \frac{\beta}{\mu} - \frac{\tilde{S}}{\kappa} - \frac{(1+\eta)m\tilde{I}}{\eta\omega} \right) P$$

If $q > \beta + \frac{\mu\tilde{S}}{\kappa} + \frac{(1+\eta)\mu m\tilde{I}}{\eta\omega}$, then we have $^{C}\mathcal{D}_t^{\alpha} W_3(S, I, P) \leq 0$. It follows from Lemma 5 that E_3 is globally asymptotically stable. □

Theorem 13. *Suppose that*

$$\varphi_1 = \frac{qP^*}{\mu} + \frac{(1+\eta)\delta I^*}{\eta},$$

$$\varphi_2 = \min \left\{ I^* + \eta I^* - 1, \frac{\delta}{\eta}, \frac{q\eta\omega - (1+\eta)m\kappa I^*}{\eta\omega} \right\}.$$

The co-existence point E^* is globally asymptotically stable if $\mu > \frac{\beta}{m}$ and $\varphi_1 < S^* < \varphi_2$.

Proof. Consider a positive Lyapunov function

$$W_4(E^*) = \left[S - S^* - S^* \ln \frac{S}{S^*} \right] + \frac{1+\eta}{\eta} \left[I - I^* - I^* \ln \frac{S}{I^*} \right] + \frac{1}{\mu} \left[P - P^* - P^* \ln \frac{P}{P^*} \right].$$

By utilizing Lemma 4, one has

$$^{C}\mathcal{D}_t^{\alpha} W_4(E^*) \leq \left(\frac{S - S^*}{S} \right) {}^{C}\mathcal{D}_t^{\alpha} S + \frac{1+\eta}{\eta} \left(\frac{I - I^*}{I} \right) {}^{C}\mathcal{D}_t^{\alpha} I + \frac{1}{\mu} \left(\frac{P - P^*}{P} \right) {}^{C}\mathcal{D}_t^{\alpha} P$$

$$= (S - S^*) \left[1 - S - (1+\eta)I - \frac{P}{\kappa + S} \right]$$

$$+ \frac{1+\eta}{\eta} (I - I^*) \left[\eta S - \delta - \frac{mP}{\omega + I} \right] + \frac{1}{\mu} (P - P^*) \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q \right]$$

$$= -S^2 - ((1+\eta)I^* - (1+S^*))S - \left(\frac{\delta}{\eta} - S^* \right) (1+\eta)I + \frac{S^*P}{\kappa + S}$$

$$+ \frac{(1+\eta)\delta I^*}{\eta} - \left(\frac{(1+\eta)m}{\eta} - \frac{\beta}{\mu} \right) \frac{IP}{\omega + I} + \frac{(1+\eta)mI^*P}{\eta(\omega + I)} - \frac{qP}{\mu}$$

$$- \frac{P^*S}{\kappa + S} - \frac{\beta P^* I}{\mu(\omega + I)} - S^* + \frac{qP^*}{\mu}$$

$$\leq -((1+\eta)I^* - (1+S^*))S - \left(\frac{\delta}{\eta} - S^*\right)(1+\eta)I - \left(m - \frac{\beta}{\mu}\right)\frac{IP}{\omega + I}$$

$$- \left(\frac{q}{\mu} - \frac{S^*}{\kappa} - \frac{(1+\eta)mI^*}{\eta\omega}\right)P - \left(S^* - \frac{qP^*}{\mu} - \frac{(1+\eta)\delta I^*}{\eta}\right).$$

Obviously, $^C\mathcal{D}_t^\alpha W_4(E^*) \leq 0$ whenever $\mu > \frac{\beta}{m}$ and $\varphi_1 < S^* < \varphi_2$. Thus, by applying Lemma 5, we can conclude that E^* is globally asymptotically stable. □

3.6. The Existence of Hopf Bifurcation

One of the interesting phenomena in studying the predator–prey model is the occurrence of Hopf bifurcation. This circumstance arises when the stability of an equilibrium point changes and a limit-cycle appears simultaneously as a parameter is varied [73,74]. In a system of first order differential equations, the occurrence of Hopf bifurcation is indicated by the appearance of purely imaginary eigenvalues of the Jacobian matrix. If we vary the bifurcation parameter, then the sign of the real part of the complex eigenvalues changes [75]; and therefore the stability properties of the equilibrium point also changes. In a fractional-order system, this bifurcation also occurs when the order of fractional derivative (α) is varied [76]. It is shown in [77,78] that a 3rd-dimensional fractional-order system undergoes a Hopf bifurcation around an equilibrium point if eigenvalues $\lambda_{1,2,3}$ of its Jacobian matrix evaluated at the equilibrium point satisfy the following conditions:

1. $\lambda_1 < 0$ and $\lambda_{2,3} = \theta \pm \omega i$ where $\theta > 0$;
2. $m(\alpha^*) = \alpha^*\pi/2 - \min_{1 \leq i \leq 3}|\arg(\lambda_i)| = 0$;
3. $\left.\frac{dm(\alpha)}{d\alpha}\right|_{\alpha=\alpha^*} \neq 0$.

When α crosses $\alpha^* = (2/\pi)\tan^{-1}(\omega/\theta)$, the equilibrium point changes its stability and is accompanied by the appearance of a stable limit-cycle. Since the fractional-order system has no periodic orbits [79], the limit-cycle is not a periodic solution, but it is a nearby solution that converges to periodic signals [76,80].

4. Eco-Epidemic Model in the Atangana–Baleanu Sense

If the fractional-order eco-epidemic model (5) is expressed in the Atangana–Baleanu derivative in Caputo (ABC) sense, then we obtain

$$\begin{aligned}
^{ABC}\mathcal{D}_t^\alpha S &= \left[1 - S - (1+\eta)I - \frac{P}{\kappa + S}\right]S, \\
^{ABC}\mathcal{D}_t^\alpha I &= \left[\eta S - \delta - \frac{mP}{\omega + I}\right]I, \\
^{ABC}\mathcal{D}_t^\alpha P &= \left[\frac{\mu S}{\kappa + S} + \frac{\beta I}{\omega + I} - q\right]P.
\end{aligned} \tag{16}$$

By Theorem 1, the solution of model (16) can be expressed in the following Volterra-type integral equation

$$\begin{aligned}
S(t) - S(0) &= \frac{1-\alpha}{B(\alpha)}G_1(t,S) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G_1(s,S)\,ds, \\
I(t) - I(0) &= \frac{1-\alpha}{B(\alpha)}G_2(t,I) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G_2(s,I)\,ds, \\
P(t) - P(0) &= \frac{1-\alpha}{B(\alpha)}G_3(t,P) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G_3(s,P)\,ds,
\end{aligned} \tag{17}$$

where

$$G_1(t,S) = \left[1 - S(t) - (1+\eta)I(t) - \frac{P(t)}{\kappa + S(t)} \right] S(t),$$

$$G_2(t,S) = \left[\eta S(t) - \delta - \frac{mP(t)}{\omega + I(t)} \right] I(t),$$

$$G_3(t,S) = \left[\frac{\mu S(t)}{\kappa + S(t)} + \frac{\beta I(t)}{\omega + I(t)} - q \right] P(t).$$

The existence and uniqueness of the solutions of model (16) will be investigated in the following sub-section.

Existence and Uniqueness

To prove the existence and uniqueness of solutions of model (16), we first show that the kernels $G_i(t,S), i = 1, 2, 3$ satisfy the Lipschitz condition. Suppose that S, $\bar{S}$, I, $\bar{I}$, P and $\bar{P}$ are functions that satisfy $\|S\| \leq a_1$, $\|\bar{S}\| \leq a_2$, $\|I\| \leq b_1$, $\|\bar{I}\| \leq b_2$, $\|P\| \leq c_1$ and $\|\bar{P}\| \leq c_2$. For the kernel $G_1(t,S) = \left(1 - S - (1+\eta)I - \frac{P}{\kappa+S} \right) S$ and two functions S and $\bar{S}$, we get

$$
\begin{aligned}
& \left\| G_1(t,S) - G_1(t,\bar{S}) \right\| \\
&= \left\| S - S^2 - (1+\eta)SI - \frac{SP}{\kappa + S} - \left(\bar{S} - \bar{S}^2 - (1+\eta)\bar{S}I - \frac{\bar{S}P}{\kappa + S} \right) \right\| \\
&= \left\| S - S^2 - (1+\eta)SI - \frac{SP}{\kappa + S} - \bar{S} + \bar{S}^2 + (1+\eta)\bar{S}I + \frac{\bar{S}P}{\kappa + \bar{S}} \right\| \\
&= \left\| (S - \bar{S}) - \left(S^2 - \bar{S}^2 \right) - \left((1+\eta)SI - (1+\eta)\bar{S}I \right) - \left(\frac{SP}{\kappa + S} - \frac{\bar{S}P}{\kappa + \bar{S}} \right) \right\| \\
&= \left\| (S - \bar{S}) - (S + \bar{S})(S - \bar{S}) - (1+\eta)I(S - \bar{S}) - \left(\frac{SP(\kappa + \bar{S}) - \bar{S}P(\kappa + S)}{(\kappa + S)(\kappa + \bar{S})} \right) \right\| \\
&\leq \|S - \bar{S}\| + (a_1 + a_2)\|S - \bar{S}\| + (1+\eta)b_1\|S - \bar{S}\| + \frac{c_1}{\kappa}\|S - \bar{S}\| \\
&= \left(1 + a_1 + a_2 + (1+\eta)b_1 + \frac{c_1}{\kappa} \right) \|S - \bar{S}\| \\
&= g_1 \|S - \bar{S}\|,
\end{aligned}
\tag{18}
$$

where $g_1 = 1 + a_1 + a_2 + (1+\eta)b_1 + \frac{c_1}{\kappa}$. Hence, the Lipschitz condition holds for $G_1(t,S)$. In a similar manner, we can show that

$$
\begin{aligned}
\|G_2(t,I) - G_2(t,\bar{I})\| &\leq g_2\|I - \bar{I}\|, \\
\|G_3(t,P) - G_3(t,\bar{P})\| &\leq g_3\|P - \bar{P}\|,
\end{aligned}
\tag{19}
$$

where $g_2 = a_1\eta + \delta + \frac{c_1 m}{\omega}$ and $g_3 = \frac{a_1 \mu}{\kappa} + \frac{b_1 \beta}{\omega} + q$. Hence, the Lipschitz condition also holds for kernels $G_2(t,I)$ and $G_3(t,P)$. Furthermore, $G_2(t,I)$ and $G_3(t,P)$ are contracted if $0 \leq g_2 < 1$ and $0 \leq g_3 < 1$, respectively.

Now, we investigate the existence of solutions of model (16) by employing the fixed-point theorem. For this purpose, we start by writing Equation (17) in the following recursive formulae

$$S_n(t) = \frac{1-\alpha}{B(\alpha)} G_1(t, S_{n-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G_1(s, S_{n-1})\, ds,$$

$$I_n(t) = \frac{1-\alpha}{B(\alpha)} G_2(t, I_{n-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G_2(s, I_{n-1})\, ds, \tag{20}$$

$$P_n(t) = \frac{1-\alpha}{B(\alpha)} G_3(t, P_{n-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G_3(s, P_{n-1})\, ds.$$

The associated initial conditions along with Equation (20) are $S_0(t) = S(0)$, $I_0(t) = I(0)$, and $P_0(t) = P(0)$. Next, from Equation (20), we have the difference expression of successive terms as follows.

$$
\begin{aligned}
\Phi_{1,n}(t) &= S_n(t) - S_{n-1}(t) \\
&= \frac{1-\alpha}{B(\alpha)}(G_1(t, S_{n-1}) - G_1(t, S_{n-2})) \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(G_1(s, S_{n-1}) - G_1(s, S_{n-2}))\, ds, \\
\Phi_{2,n}(t) &= I_n(t) - I_{n-1}(t) \\
&= \frac{1-\alpha}{B(\alpha)}(G_2(t, I_{n-1}) - G_2(t, I_{n-2})) \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(G_2(s, I_{n-1}) - G_2(s, I_{n-2}))\, ds, \\
\Phi_{3,n}(t) &= P_n(t) - P_{n-1}(t) \\
&= \frac{1-\alpha}{B(\alpha)}(G_3(t, P_{n-1}) - G_3(t, P_{n-2})) \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(G_3(s, P_{n-1}) - G_3(s, P_{n-2}))\, ds.
\end{aligned}
\tag{21}
$$

Based on Equation (21), we have that

$$
S_n(t) = \sum_{i=1}^n \Phi_{1,i}(t), \quad I_n(t) = \sum_{i=1}^n \Phi_{2,i}(t), \text{ and } P_n(t) = \sum_{i=1}^n \Phi_{3,i}(t).
\tag{22}
$$

By using (18) and (19), we can show that the norm of both sides in (21) fulfill the following relations

$$
\begin{aligned}
\|\Phi_{1,n}(t)\| &\le \frac{1-\alpha}{B(\alpha)} g_1 \|\Phi_{1,n-1}\| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} g_1 \int_0^t \|\Phi_{1,n-1}(s)\|(t-s)^{\alpha-1}\, ds, \\
\|\Phi_{2,n}(t)\| &\le \frac{1-\alpha}{B(\alpha)} g_2 \|\Phi_{2,n-1}\| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} g_2 \int_0^t \|\Phi_{2,n-1}(s)\|(t-s)^{\alpha-1}\, ds, \\
\|\Phi_{3,n}(t)\| &\le \frac{1-\alpha}{B(\alpha)} g_3 \|\Phi_{3,n-1}\| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} g_3 \int_0^t \|\Phi_{3,n-1}(s)\|(t-s)^{\alpha-1}\, ds.
\end{aligned}
\tag{23}
$$

Now, by applying (23), the existence and uniqueness of model (16) are shown by the following theorem.

Theorem 14. *Model (16) has a unique solution if we can find t_{max} such that*

$$
\frac{(1-\alpha)g_i}{B(\alpha)} + \frac{t_{max}^\alpha g_i}{B(\alpha)\Gamma(\alpha)} < 1, \ i = 1, 2, 3
\tag{24}
$$

Proof. We assume that $S(t)$, $I(t)$ and $P(t)$ are bounded functions, and hence the Lipschitz condition is satisfied. From Equation (23) we can get the following inequalities.

$$
\begin{aligned}
\|\Phi_{1,n}(t)\| &\le \|S_0\| \left(\frac{(1-\alpha)g_1}{B(\alpha)} + \frac{t^\alpha g_1}{B(\alpha)\Gamma(\alpha)} \right)^n, \\
\|\Phi_{2,n}(t)\| &\le \|I_0\| \left(\frac{(1-\alpha)g_2}{B(\alpha)} + \frac{t^\alpha g_2}{B(\alpha)\Gamma(\alpha)} \right)^n, \\
\|\Phi_{3,n}(t)\| &\le \|P_0\| \left(\frac{(1-\alpha)g_3}{B(\alpha)} + \frac{t^\alpha g_3}{B(\alpha)\Gamma(\alpha)} \right)^n.
\end{aligned}
\tag{25}
$$

Therefore, the existence and smoothness of the solution presented in Equation (22) are proven since $\|\Phi_{1,n}(t)\| \to 0$, $\|\Phi_{2,n}(t)\| \to 0$ and $\|\Phi_{3,n}(t)\| \to 0$ as $n \to \infty$ and $t = t_{max}$. To show that the functions which satisfy Equation (17) are the solutions of Equation (16), we suppose that

$$
\begin{aligned}
S(t) - S(0) &= S_n(t) - Y_{1,n}(t), \\
I(t) - I(0) &= I_n(t) - Y_{2,n}(t), \\
P(t) - P(0) &= P_n(t) - Y_{3,n}(t),
\end{aligned}
\tag{26}
$$

where $Y_{i,n}(t), i = 1, 2, 3$ are the remainder terms of series solutions. The norm of $Y_{1,n}(t)$ satisfies

$$
\begin{aligned}
\|Y_{1,n}(t)\| \leq &\frac{1-\alpha}{B(\alpha)} \|G_1(t,S) - G_1(t,S_{n-1})\| \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t \|G_1(s,S) - G_1(s,S_{n-1})\|(t-s)^{\alpha-1} \, ds, \\
\leq &\|S - S_{n-1}\| \left(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)} \right) g_1.
\end{aligned}
\tag{27}
$$

By applying this relation iteratively, we get at $t = t_{max}$

$$
\|Y_{1,n}(t)\| \leq a_1 \left(\frac{1-\alpha}{B(\alpha)} + \frac{t_{max}^\alpha}{B(\alpha)\Gamma(\alpha)} \right)^{n+1} g_1^{n+1}.
\tag{28}
$$

For $n \to \infty$, we obtain $\|Y_{1,n}(t)\| \to 0$. Applying the similar manner, we have $\|Y_{2,n}(t)\| \to 0$ and $\|Y_{3,n}(t)\| \to 0$. Hence, the functions which satisfy Equation (17) are the solutions of Equation (16).

Now, we show the uniqueness of solutions of Equation (16). For this aim, we suppose that $S^*(t)$, $I^*(t)$ and $P^*(t)$ are another solution of Equation (16). Then, we have

$$
\begin{aligned}
S(t) - S^*(t) = &\frac{1-\alpha}{B(\alpha)} (G_1(t,S) - G_1(t,S^*)) \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (G_1(s,S) - G_1(s,S^*))(t-s)^{\alpha-1} \, ds.
\end{aligned}
\tag{29}
$$

Taking the norm for both sides and using the same procedures as in (23) and (25), we obtain

$$
\|S(t) - S^*(t)\| \left(1 - \frac{(1-\alpha)g_1}{B(\alpha)} - \frac{t^\alpha g_1}{B(\alpha)\Gamma(\alpha)} \right) \leq 0.
\tag{30}
$$

For $t = t_{max}$, we have (24). Hence, $\|S(t) - S^*(t)\| = 0$ and consequently $S(t) = S^*(t)$. In the same way, we can show that $I(t) = I^*(t)$ and $P(t) = P^*(t)$. Hence, the uniqueness of the solution of Equation (16) is proven. □
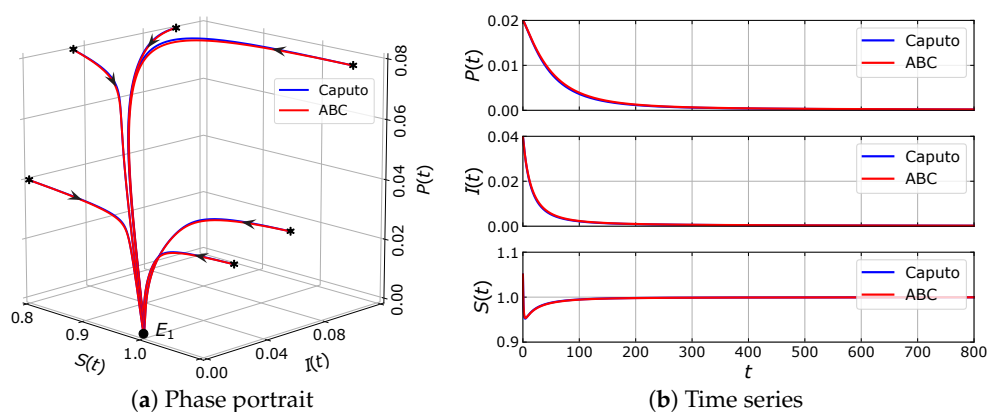
5. Numerical Simulations

In this section, we present some results of our numerical simulations for the fractional-order eco-epidemic models in both Caputo sense (12) and ABC sense (16). For this aim, we solve the model in Caputo sense (12) using the predictor–corrector scheme developed by Diethelm et al. [81], while the model in ABC sense (16) is solved by applying the predictor–corrector scheme proposed by Baleanu et al. [82]. Since the field data are not available, the simulations are performed by using some hypothetical parameter values.

We first perform simulation by setting the parameter values as follows:
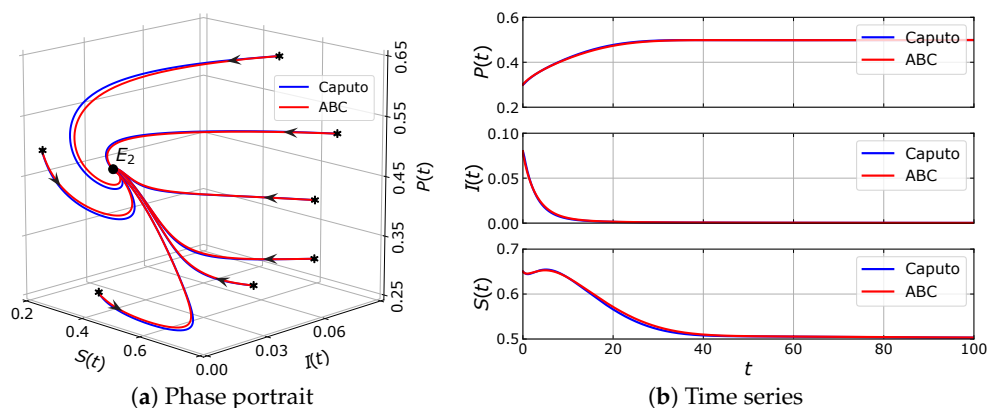
$$
\eta = 0.25, \ \kappa = 0.5, \ \delta = 0.3, \ m = 0.6, \ \omega = 0.6, \ \mu = 0.4, \ \beta = 0.4, \ q = 0.3, \ \alpha = 0.9.
\tag{31}
$$

Using these parameter values, the eco-epidemic model with fractional-order derivative in both Caputo sense (12) and ABC sense (16) have two equilibrium points, i.e., E_0 and

E_1. Based on the analysis for the model in Caputo sense (12), it is shown that E_0 is a saddle point and E_1 is asymptotically stable. This behavior is confirmed by our numerical simulation shown in Figure 2. If we modify the parameter values in (31) such that $\eta = 0.35$ and $q = 0.2$, then, in addition to E_0 and E_1, the model also has equilibrium points E_2 and E_3. All of these equilibrium points are unstable except E_2. The stability of E_2 is clearly observed in Figure 3. Now, some parameter values in (31) are replaced by $\eta = 0.95$, $\delta = 0.2$ and $q = 0.4$. Under these parameter values, the model has three equilibrium points, i.e., E_0, E_1 and E_3. The previous analysis for the model in Caputo sense shows that E_0 and E_1 are unstable, while E_3 is asymptotically stable. Such stability behavior can be seen in Figure 4. Furthermore, in Figures 2–4, the numerical solutions of model in the Caputo sense are compared to those of models in ABC sense. It is observed that the phase portraits and time series of both models have similar dynamical behavior. To see the difference between the solutions of the two models, we perform some simulations using the same parameter values as in Figures 2–4, but with varying value of α. The time series of solutions obtained from those simulations are plotted in Figures 5–7. In these simulations, although the value of α does not affect the stability of the equilibrium point, Figures 5–7 show that the value of α greatly affects the rate of convergence in reaching the equilibrium point. Indeed, when $\alpha = 1$, the eco-epidemic model with fractional derivative in the Caputo sense and model with fractional derivative in the ABC sense have solutions that coincide with each other.


(a) Phase portrait

(b) Time series

Figure 2. Numerical simulation of the eco-epidemic model with parameter values: $\eta = 0.25$, $\kappa = 0.5$, $\delta = 0.3$, $m = 0.6$, $\omega = 0.6$, $\mu = 0.4$, $\beta = 0.4$, $q = 0.3$ and $\alpha = 0.9$.

(a) Phase portrait

(b) Time series

Figure 3. Numerical simulation of the eco-epidemic model with parameter values: $\eta = 0.35$, $\kappa = 0.5$, $\delta = 0.3$, $m = 0.6$, $\omega = 0.6$, $\mu = 0.4$, $\beta = 0.4$, $q = 0.2$ and $\alpha = 0.9$.

(**a**) Phase portrait
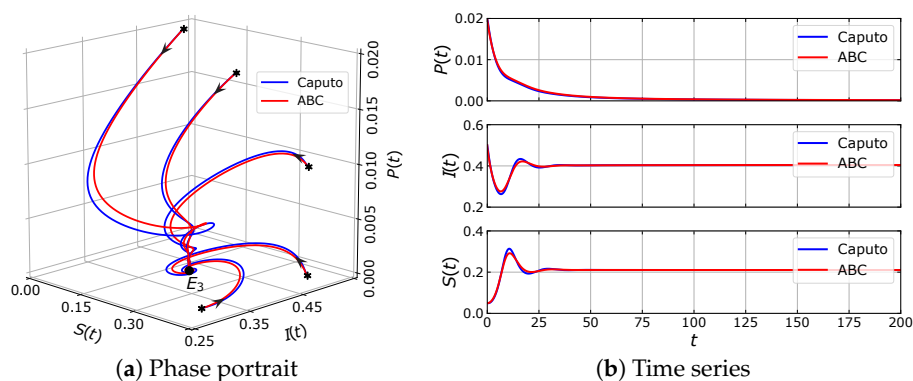
(**b**) Time series

Figure 4. Numerical simulation of the eco-epidemic model with parameter values: $\eta = 0.95$, $\kappa = 0.5$, $\delta = 0.2$, $m = 0.6$, $\omega = 0.6$, $\mu = 0.4$, $\beta = 0.4$, $q = 0.4$ and $\alpha = 0.9$.

(**a**) Caputo sense
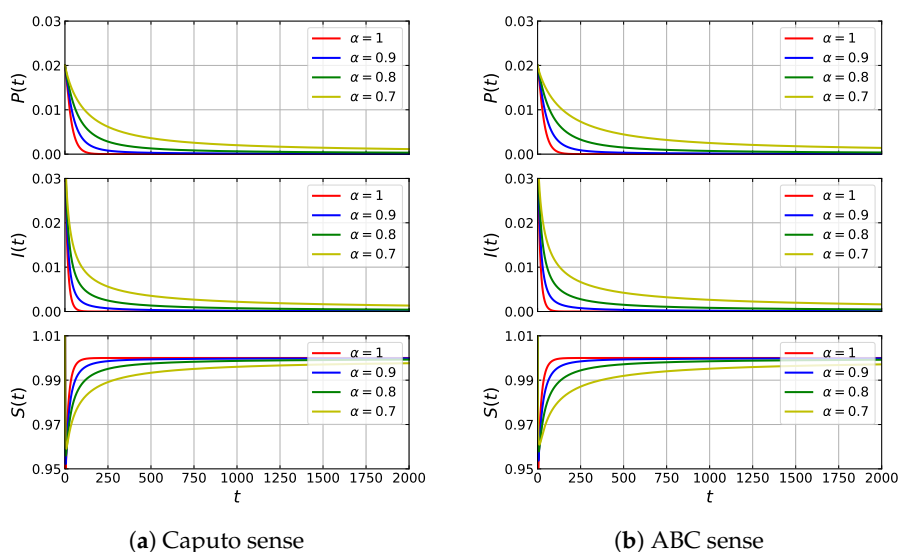
(**b**) ABC sense

Figure 5. Time series of solutions of the eco-epidemic model (5) with fractional derivative in the Caputo sense and fractional derivative in ABC sense. The parameter values are $\eta = 0.35$, $\kappa = 0.5$, $\delta = 0.3$, $m = 0.6$, $\omega = 0.6$, $\mu = 0.4$, $\beta = 0.4$ and $q = 0.2$.


(**a**) Caputo sense
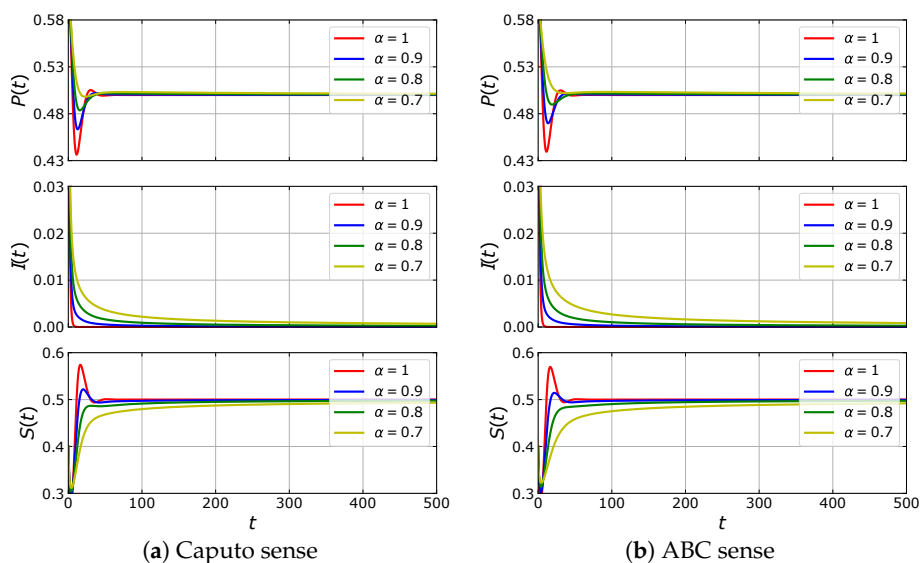
(**b**) ABC sense

Figure 6. Time series of solutions of the eco-epidemic model (5) with fractional derivative in the Caputo sense and fractional derivative in ABC sense. The parameter values are $\eta = 0.35$, $\kappa = 0.5$, $\delta = 0.3$, $m = 0.6$, $\omega = 0.6$, $\mu = 0.4$, $\beta = 0.4$ and $q = 0.2$.

(**a**) Caputo sense
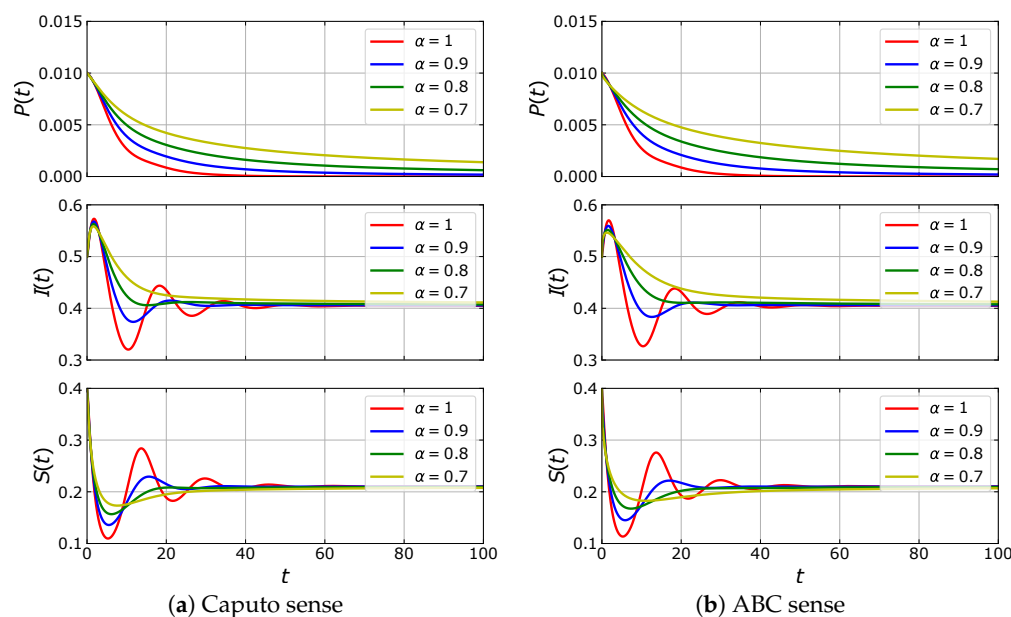
(**b**) ABC sense

Figure 7. Time series of solutions of the eco-epidemic model (5) with fractional derivative in the Caputo sense and fractional derivative in ABC sense. The parameter values are $\eta = 0.95$, $\kappa = 0.5$, $\delta = 0.2$, $m = 0.6$, $\omega = 0.6$, $\mu = 0.4$, $\beta = 0.4$ and $q = 0.4$.

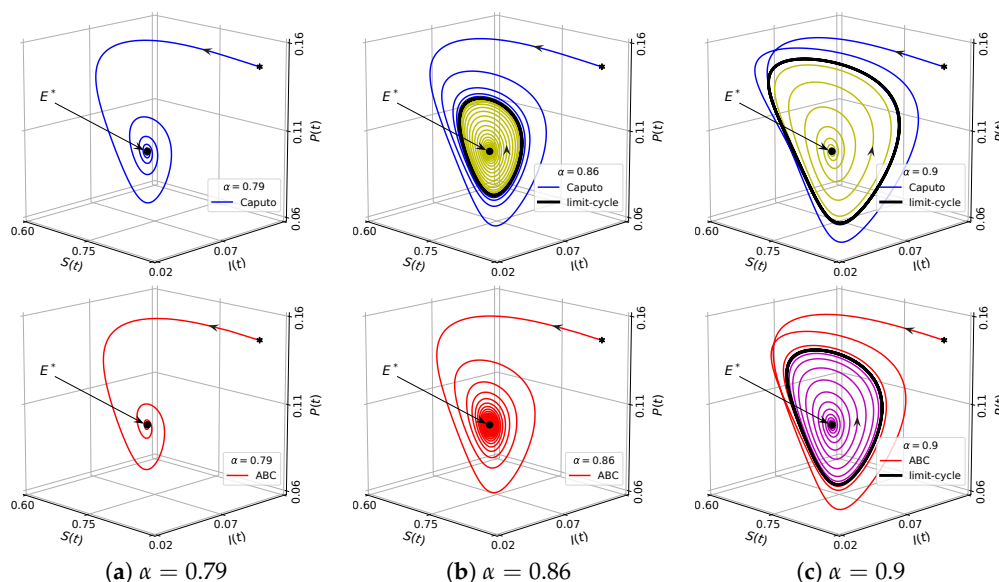Next, we perform simulation using the following parameter values:

$$\eta = 0.8, \ \kappa = 0.01, \ \delta = 0.17, \ m = 0.7, \ \omega = 0.1, \ \mu = 0.2, \ \beta = 0.27, \ q = 0.3. \tag{32}$$

Here, the model has equilibrium points: E_0, E_1, E_3 and E^*. By applying the stability analysis for the model in the Caputo sense, it can be shown that E_0, E_1 and E_3 are unstable; while the stability of E^* is determined by α. If $\alpha < \alpha^* \approx 0.85662$ then E^* is asymptotically stable. On the other hand, E^* becomes unstable if $\alpha > \alpha^*$, where in this case, the solution is convergent to a limit-cycle. In other words, there occurs a Hopf bifurcation controlled by α where the bifurcation point is at $\alpha = \alpha^*$. The Hopf bifurcation is indeed verified by our bifurcation diagram shown in Figure 8. We confirm numerically that both models with Caputo sense and ABC sense undergo the Hopf bifurcation, but with different bifurcation points. The bifurcation point of model with Caputo sense has smaller value of α than that of the model with ABC sense. To describe their dynamics, we select three values of $\alpha = 0.79, 0.86, 0.9$, each of which is denoted by the labels [a], [b], [c] in Figure 8, respectively. When $\alpha = 0.79$, both models with Caputo sense and ABC sense are convergent to E^* as in Figure 9a. From the time series in Figure 10a, the solution of model with ABC sense converges faster than model with Caputo sense. For $\alpha = 0.86$, E^* of model with Caputo sense losses its stability and the solution goes to the limit-cycle while E^* of model with ABC sense still maintains its stability, see Figures 9b and 10b. This circumstance confirms that the model with Caputo sense has undergone the Hopf bifurcation while model with ABC sense has not. When $\alpha = 0.9$, E^* of model with ABC sense losses its stability via Hopf bifurcation as in Figure 9c. The solution of both models converge to the limit-cycle where the diameter of limit-cycles obtained by model in ABC sense is smaller than those obtained by model in Caputo sense, see Figure 10c. To see the evolution of limit-cycle in more detail, we perform simulations using parameter values in (32) and $\alpha \in (0.8, 0.94)$. In Figure 11, we show the stable equilibrium point or limit-cycle in $(I, P)-$plane as function of α. As mentioned before the stable limit-cycle appears if $\alpha > \alpha^*$. It can be seen that the diameter of limit-cycle obtained by both models in Caputo sense and ABC sense are getting bigger when α is increased. We notice from Figure 8 that the model with Caputo sense has a smaller critical value of α. Therefore, there are situations where the model with Caputo sense has an α that passes its critical value (α^*) while the model with ABC sense does not. This situation shows that the model with the two fractional derivative operators

have different biological interpretations in determining the density of prey and predators. On one hand, the density of prey and predator obtained by the model with Caputo sense fluctuates periodically, whereas those obtained by the model with ABC sense converge to a constant value.
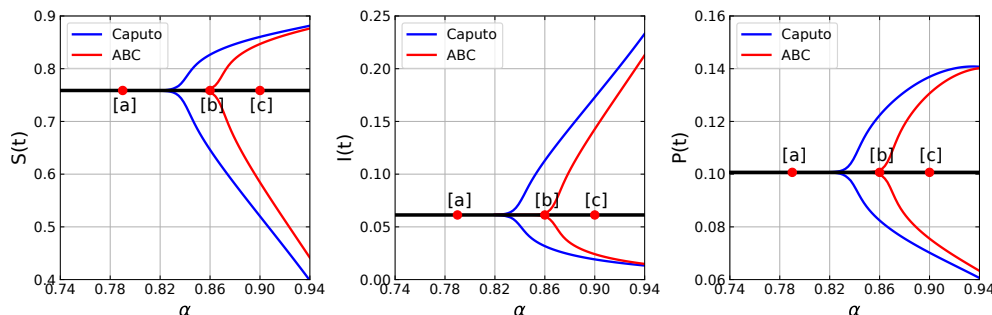
Figure 8. Bifurcation diagram of the eco-epidemic model with parameter values $\eta = 0.8$, $\kappa = 0.01$, $\delta = 0.17$, $m = 0.7$, $\omega = 0.1$, $\mu = 0.2$, $\beta = 0.27$ and $q = 0.3$.

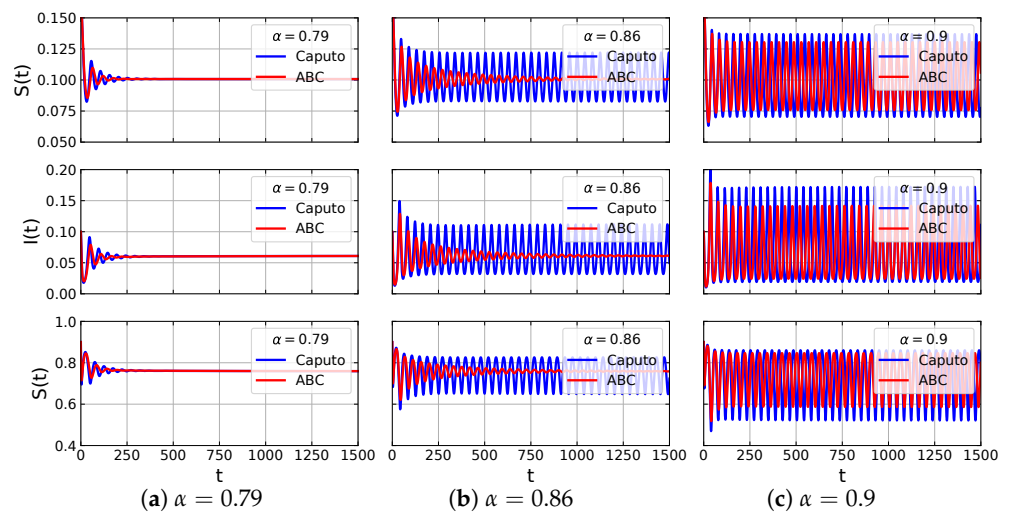(**a**) $\alpha = 0.79$ (**b**) $\alpha = 0.86$ (**c**) $\alpha = 0.9$

Figure 9. Phase portraits of the eco-epidemic model with parameter values: $\eta = 0.8$, $\kappa = 0.01$, $\delta = 0.17$, $m = 0.7$, $\omega = 0.1$, $\mu = 0.2$, $\beta = 0.27$ and $q = 0.3$. The top figures are solutions of model with Caputo sense while the bottom figures are ABC sense.
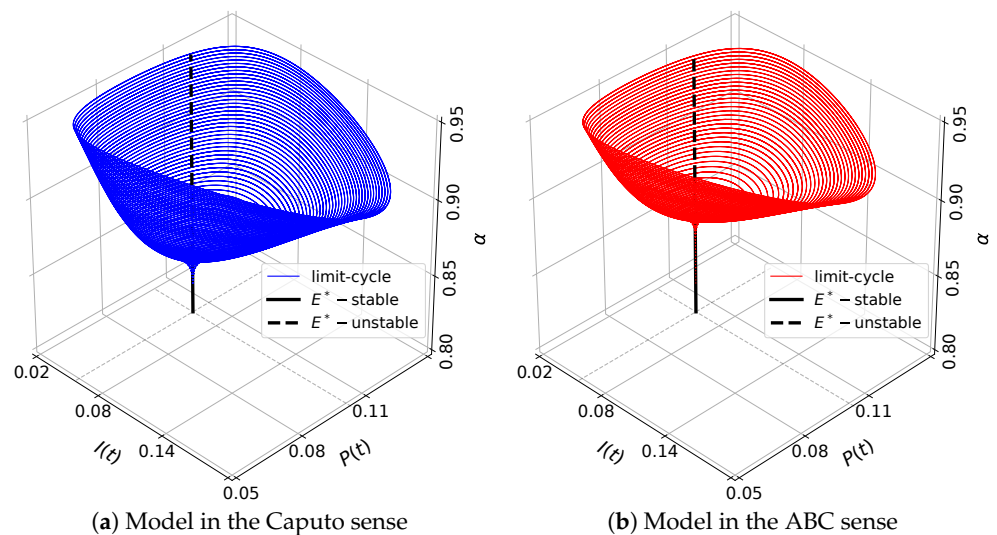
Finally, we take the following parameter values:

$$\eta = 0.8, \ \kappa = 0.6, \ \delta = 0.17, \ m = 0.7, \ \omega = 0.1, \ \mu = 0.5, \ \beta = 0.3, \ q = 0.4. \qquad (33)$$

For this case, the model has five equilibrium points, namely E_0, E_1, E_3, E_1^* and E_2^*. Using the results of a previous stability analysis, it is shown that E_0, E_1 and E_2^* are unstable, while E_3 is asymptotically stable regardless of the value of α. We also check that there exists Hopf bifurcation around E_1^*. In the latter case, E_1^* is stable for $\alpha < \alpha_1^* \approx 0.84730$. If $\alpha > \alpha_1^*$ then E_1^* loses its stability and there appears a limit-cycle. Hence, the model exhibits a bistability phenomenon for $\alpha < \alpha_1^*$, where, in this case, E_3 and E_1^* are locally asymptotically stable. To illustrate the dynamics of eco-epidemic model with parameter values in (33), we plot numerical solutions with two slightly different initial values in Figure 12. When we take $\alpha = 0.83 < \alpha_1^*$ the solutions of both models are respectively convergent to different equilibrium points, namely E_3 and E_1^*, see Figure 12a,b. Furthermore, when we increase the order of fractional derivative to $\alpha = 0.95$ then E_3 remains stable but the stability of E_1^* vanishes via Hopf bifurcation as in Figure 12c,d. Thus, a captivating circumstance has been

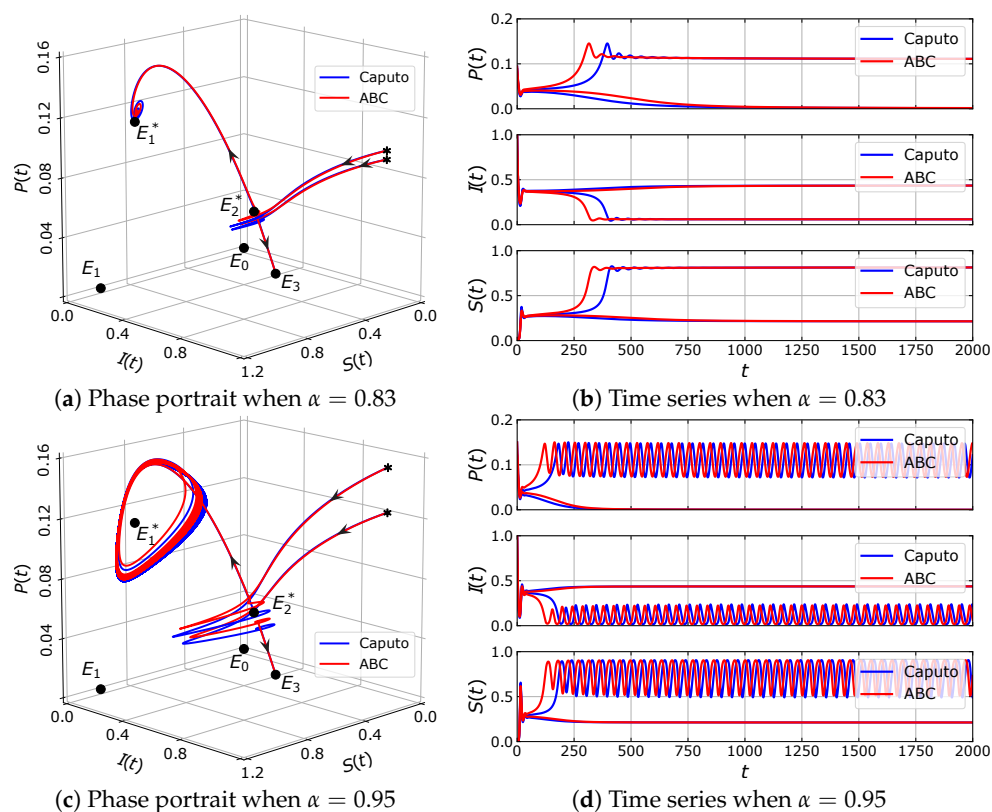shown, where the initial condition is very sensitive in determining the limiting behavior of the system.

(a) $\alpha = 0.79$ **(b)** $\alpha = 0.86$ **(c)** $\alpha = 0.9$

Figure 10. Time series of the eco-epidemic model with parameter values: $\eta = 0.8$, $\kappa = 0.01$, $\delta = 0.17$, $m = 0.7$, $\omega = 0.1$, $\mu = 0.2$, $\beta = 0.27$ and $q = 0.3$.

(a) Model in the Caputo sense   **(b)** Model in the ABC sense

Figure 11. The evolution of limit-cycle in $(I, \ P)-$plane as a function of α for eco-epidemic model with parameter values $\eta = 0.8$, $\kappa = 0.01$, $\delta = 0.17$, $m = 0.7$, $\omega = 0.1$, $\mu = 0.2$, $\beta = 0.27$ and $q = 0.3$.

(a) Phase portrait when $\alpha = 0.83$

(b) Time series when $\alpha = 0.83$

(c) Phase portrait when $\alpha = 0.95$

(d) Time series when $\alpha = 0.95$

Figure 12. Numerical simulation of the eco-epidemic model with parameter values: $\eta = 0.8$, $\kappa = 0.6$, $\delta = 0.17$, $m = 0.7$, $\omega = 0.1$, $\mu = 0.5$, $\beta = 0.3$ and $q = 0.4$.

6. Conclusions

We have presented the dynamics of fractional-order Rosenzweig–MacArthur eco-epidemic model using fractional derivative in both Caputo sense and ABC-sense. We have determined conditions for the existence and uniqueness of solutions for models in both Caputo sense and ABC sense. It is also shown that all solutions are non-negative and bounded in $\mathbb{R}_+^3$. The model has at most five types of equilibrium points, i.e., the origin, the extinction of infected prey and predator point, the infected prey free point, the predator-free point and the co-existence point. Based on the stability analysis for the model in the Caputo sense, it is found that the origin is a saddle point, meaning that the extinction of all populations will never happen. We also found that the other equilibrium points are conditionally asymptotically stable. Furthermore, the conditions for the existence of Hopf bifurcation have been established, where the bifurcation is driven by the order of the fractional derivative. Our theoretical results have been confirmed by numerical solutions of the model in the Caputo sense. In this article, the eco-epidemic model in the ABC sense has also been solved numerically. The comparison of our numerical results shows that model with both Caputo sense and ABC sense have the same dynamical behavior except around the interior equilibrium point. In other words, the dynamical behavior of the proposed model with both senses are symmetric around axial equilibrium points, but it is asymmetric around the interior point when a Hopf bifurcation occurs. We confirm numerically that the interior point of both models has a different bifurcation point when Hopf bifurcation occurs. For some values of the order$-\alpha$, the interior point of model with ABC sense is stable while the interior point of model with Caputo sense is unstable. Our numerical simulations also show that the proposed models may exhibit a bistable phenomenon. We finally notice that our simulations are based on some hypothetical parameter values. For further studies, it is recommended to compare the performance of both models, namely with Caputo sense and with ABC sense, by using real data of selected eco-epidemiological case.

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